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Behavior of Eigenfunctions near the Ideal Boundary of Hyperbolic Space

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Abstract.

The spectrum of the Laplacian on hyperbolic space is a proper subset of the positive reals. We study eigenfunctions, defined on the complements of compact sets, whose eigenvalues lie below the bottom of the spectrum. Such eigenfunctions may arise by perturbing the metric on compact subsets of the space. One divides the eigenfunctions by normalizing factors, so that the quotients have analytic boundary values on the ideal boundary at infinity. The renormalized eigenfunctions are approximated by special polynomials, in nontangential approach regions to the ideal boundary.

§1. Introduction

In [6] and [7], the authors studied asymptotic behavior of eigenfunctions, near infinity, for the Schrödinger operator $-\sum_{i=1}^{n} \partial^2/\partial x_i^2 + V$ in R^n. Let $\psi \in L^2(R^n)$ be a square integrable eigenfunction, of the Schrödinger operator, with eigenvalue $\lambda < 0$. If V decays rapidly, then a multiple, $\hat{\psi}(r,\theta) = \psi(r,\theta)/h(r)$, was shown to have analytic boundary values $A(\theta)$ on the sphere S^{n-1}, compactifying R^n at infinity. A detailed estimate was given for the asymptotic behavior of $\hat{\psi}$ near a zero of A. The two dimensional case was treated in [7], where applications were given to the structure of the nodal set of ψ. If $n > 2$, no such development seems feasible, as discussed in [6].

The present work gives analogous results for eigenfunctions of the Laplacian Δ on hyperbolic space H^n. One assumes that $\Delta\phi = \lambda\phi$, with $\lambda < (n-1)^2/4$ outside some compact subset. This reflects the fact [4], that the essential spectrum of Δ is now $[(n-1)^2/4, \infty)$. The case $n = 2$ was described in [3]. Here we proceed to generalize that work to arbitrary dimension n.

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§2. Boundary values at infinity

Let H be the simply connected, complete, hyperbolic space of dimension $n \geq 2$. That is, H has the Poincaré metric of constant curvature -1. Fixing $p \in H$, the exponential map $\exp : T_pH \to H$ is a diffeomorphism. We endow this manifold H with the corresponding geodesic polar coordinates. The metric is then given by

$$\begin{aligned} (ds)^2 &= (dr)^2 + g^2(r)(d\theta)^2, \\ g(r) &= \sinh r. \end{aligned}$$

Suppose that $r_0 > 0$ and set $H(r_0) = \{x \in H \mid r(x) > r_0\}$. We consider eigenfunctions $\phi \in L^2(H(r_0))$ of the Laplacian Δ, associated to the given Riemannian metric. Thus, one has $\Delta\phi = \lambda\phi$. Our concern is with the behavior of ϕ as $r \to \infty$. Thus, we feel free to choose r_0 sufficiently large. The eigenfunction ϕ need not satisfy any constraints on the compact boundary of $H(r_0)$, where $r(x) = r_0$.

It seems natural to employ separation of variables. The spherical harmonics $Y_{k,j}(\theta)$, for $k \geq 0$ and $1 \leq j \leq q(k)$, form a complete orthonormal basis for $L^2(S^{n-1})$. Each $Y_{k,j}(\theta)$ belongs to a $q(k)$-dimensional eigenspace of the spherical Laplacian, with corresponding eigenvalue λ_k. One may expand

$$\phi(r, \theta) = \sum_{k=0}^{\infty} \sum_{j=1}^{q(k)} \phi_{k,j}(r) Y_{k,j}(\theta).$$

A computation using the local defining formula for Δ gives

$$\Delta\phi = \sum_{k=0}^{\infty} \sum_{j=1}^{q(k)} \Delta_k \phi_{k,j}(r) Y_{k,j}(\theta),$$

where

$$\Delta_k \phi_{k,j} = -\phi''_{k,j} - (n-1)\frac{g'}{g}\phi'_{k,j} + \lambda_k g^{-2}\phi_{k,j}.$$

Here the $'$ denotes differentiation in r. Thus $\phi_{k,j} \in L^2((r_0, \infty), g^{n-1}(r)dr)$. So Δ is decomposed as a direct sum of the operators Δ_k, with multiplicity $q(k)$.

Now Δ_k is unitarily equivalent to $D_k = g^{(n-1)/2}\Delta_k g^{(1-n)/2}$ acting on $L^2((r_0, \infty),\ dr)$. A calculation yields

$$D_k\psi = -\psi'' + [\gamma(r) + \lambda_k g^{-2}]\psi.$$

Here $\gamma(r) = ((n-1)/2)f'' + ((n-1)/2)^2(f')^2$, with $g = e^f$. In particular $\gamma(r) = ((n-1)/2)^2 + 0(e^{-2r})$.

Set $\overline{\phi}_{k,j} = g^{(n-1)/2}\phi_{k,j}$. Since $\Delta\phi = \lambda\phi$, one has the corresponding equation $D_k\overline{\phi}_{k,j} = \lambda\overline{\phi}_{k,j}$. Therefore

$$-\overline{\phi}_{k,j}{}'' + [\gamma(r) - \lambda + \lambda_k g^{-2}]\overline{\phi}_{k,j} = 0. \tag{2.1}$$

The potential term $\gamma(r) - \lambda + \lambda_k g^{-2}$ decays rapidly to $((n-1)/2)^2 - \lambda$. The hypothesis $\overline{\phi}_{k,j} \in L^2((r_0, \infty), dr)$ and the method of asymptotic integrations [5, pp. 370–384] give

Lemma 2.2. *The equation* (2.1) *has square integrable solutions on* (r_0, ∞) *if and only if* $E = (n-1)^2/4 - \lambda$ *is positive. When* $E > 0$, *there is a one-dimensional space of square integrable solutions. Moreover, any non-zero* L^2 *solution satisfies, for* r *large,*

$$\overline{\phi}_{k,j} \sim b_{k,j}e^{-\sqrt{E}r}, \ \overline{\phi}_{k,j}{}' \sim -\sqrt{E}\ b_{k,j}e^{-\sqrt{E}r}.$$

Here $\sim$ *means that the ratio approaches one as* $r \to \infty$. *The constant* $b_{k,j}$ *is not zero.*

Assume $E > 0$, and let $\overline{h}_k$ be a solution of (2.1). Suppose $\overline{h}_k$ lies inside the one-dimensional space of square integrable solutions, as specified in Lemma 2.2. If $\overline{h}_k(r_1) > 0$ for some $r_1 > r_0$, then $\overline{h}_k(r) > 0$ for all $r \geq r_1$. Otherwise, let $r_2 > r_1$ be the first zero of $\overline{h}_k$. Clearly, $\overline{h}'_k(r_2) \leq 0$. By the uniqueness theorem, for second order ordinary differential equations, this forces $\overline{h}'_k(r_2) < 0$. Since $\overline{h}_k \in L^2((r_0, \infty),\ dr)$, the function $\overline{h}_k$ must have a negative minimum $r_3 > r_2$. However, if r_0 is sufficiently large, then the potential term $\gamma(r) - \lambda + \lambda_k g^{-2} > 0$ for all $r \geq r_0$. Consequently, solutions to (2.1) cannot have negative local interior minimums. This contradiction shows that $\overline{h}_k(r) > 0$ for all $r \geq r_1$.

Now we fix $r_1 > r_0$. Define $\overline{h}_k \in L^2((r_0, \infty), dr)$ by requiring $\overline{h}_k$ to satisfy (2.1) and the normalization $\overline{h}_k(r_1) = 1$. The remarks above show that this defines $\overline{h}_k$ uniquely. Moreover $\overline{\phi}_{k,j}(r) = \overline{\phi}_{k,j}(r_1)\overline{h}_k(r)$, since both $\overline{\phi}_{k,j}$ and $\overline{h}_k$ lie in the one-dimensional space of solutions, specified by Lemma 2.2. If $\overline{\phi} = \phi g^{(n-1)/2}$, we may write

$$\overline{\phi}(r, \theta) = \sum_{k=0}^{\infty}\sum_{j=1}^{q(k)} \overline{\phi}_{k,j}(r_1)\overline{h}_k(r)Y_{k,j}(\theta). \tag{2.3}$$

The function $\phi(r, \theta)$ is analytic because $(\Delta - \lambda)\phi = 0$ and the elliptic operator $\Delta - \lambda$ has analytic coefficients [8, p. 178]. By Proposition 4.5

of the appendix, we may write

$$|\overline{\phi}_{k,j}(r_1)| \leq c_1 e^{-c_2\sqrt{\lambda_k}} \tag{2.4}$$

for positive constants c_1 and c_2. Recall that the eigenvalues of the spherical Laplacian satisfy $\lambda_k = 0(k^2)$.

It is also necessary to control the dependence of the $\overline{h}_k$ upon k. This is provided by

Lemma 2.5. *For all k, one has $0 < \overline{h}_k(r) \leq \overline{h}_0(r)$, whenever $r \geq r_1$.*

Proof. We have already shown that $\overline{h}_k(r) > 0$. The difference $a_k = \overline{h}_0 - \overline{h}_k$ satisfies, since $\lambda_0 = 0$,

$$\begin{aligned} -a_k'' + [\gamma(r) - \lambda]a_k - \lambda_k g^{-2}\overline{h}_k &= 0, \\ \overline{a}_k(r_1) &= 0. \end{aligned}$$

If a_k is ever negative, then, since $a_k \in L^2((r_0,\infty), dr)$, the function a_k must have a negative minimum. At such a local minimum, the differential equation for a_k cannot hold. This contradiction proves the lemma. □

One now has the necessary preparations to study the asymptotic behavior of eigenfunctions. Set $\hat{\phi}(r,\theta) = \overline{\phi}(r,\theta)\overline{h}_0^{-1}(r)$, or equivalently, $\hat{\phi}(r,\theta) = g^{(n-1)/2}(r)\phi(r,\theta)\overline{h}_0^{-1}(r)$. The central result of this section is

Theorem 2.6. *As $r \to \infty$, one has $\hat{\phi}(r,\theta) \to A(\theta)$, uniformly in θ. The function $A(\theta)$ is real analytic.*

Proof. By (2.4) and Lemma 2.5, we have $|\overline{\phi}_{k,j}(r)| = |\overline{\phi}_{k,j}(r_1)|\overline{h}_k(r) \leq c_1 e^{-c_2\sqrt{\lambda_k}}\, \overline{h}_0(r)$. Now

$$\hat{\phi}(r,\theta) = \sum_{k=0}^{\infty}\sum_{j=1}^{q(k)} \overline{\phi}_{k,j}(r)\overline{h}_0^{-1}(r) Y_{k,j}(\theta).$$

Lemma 2.2 guarantees that $A_{k,j} = \lim_{r\to\infty} \overline{\phi}_{k,j}\overline{h}_0^{-1}$ exists. The given bound on the functions $\overline{\phi}_{k,j}$ allows one to interchange the limit in r with the infinite sum in j. Note that $\|Y_{k,j}\|_2 = 1$, and thus $\|Y_{k,j}\|_\infty \leq c_3\lambda_k^{(n-2)/4}$, by standard elliptic theory. Moreover, the multiplicity $q(k) = 0(\lambda_k^{(n-2)/2})$. □

Thus, one has

$$A(\theta) = \lim_{r\to\infty} \hat{\phi}(r,\theta) = \sum_{k=0}^{\infty}\sum_{j=1}^{q(k)} A_{k,j}Y_{k,j}(\theta).$$

The estimate $|A_{k,j}| \le c_1 e^{-c_2\sqrt{\lambda_k}}$ and Proposition 4.2 show that A is real analytic.

If $\phi_{k,j}$ is not identically zero, Lemma 2.2 guarantees that $A_{k,j} \ne 0$. In particular, $A(\theta)$ is not the zero function. We also have the expected

Corollary 2.7. *For any ℓ, $\lim_{r\to\infty} \nabla_\theta^\ell \hat{\phi}(r,\theta) = \nabla_\theta^\ell A$, where ∇_θ denotes the covariant derivative of S^{n-1} with its standard metric.*

Proof. The rapid decay $|\overline{\phi}_{k,j}(r)|\overline{h}_0^{-1}(r) \le c_1 e^{-c_2\sqrt{\lambda_k}}$, and standard elliptic estimates for the derivatives of eigenfunctions, allow one to write

$$\nabla_\theta^\ell \hat{\phi}(r,\theta) = \sum_{k=0}^{\infty}\sum_{j=1}^{q(k)} \overline{\phi}_{k,j}(r)\overline{h}_0^{-1}(r)\nabla_\theta^\ell Y_{k,j}(\theta).$$

For the same reasons, one may interchange the sum in j and the limit in r, to get

$$\lim_{r\to\infty} \nabla_\theta^\ell \hat{\phi}(r,\theta) = \sum_{k=0}^{\infty}\sum_{j=1}^{q(k)} A_{k,j}\nabla_\theta^\ell Y_{k,j}(\theta) = \nabla_\theta^\ell A.$$

□

It is also useful to estimate the rate of convergence in Corollary 2.7. In this direction, there are constants B_ℓ, so that

Proposition 2.8. *For $r > r_1$, $|\nabla_\theta^\ell A - \nabla_\theta^\ell \hat{\phi}(r,\theta)| \le B_\ell e^{-2r}$.*

Proof. Set $\hat{\phi}_{k,j} = \overline{\phi}_{k,j}\overline{h}_0^{-1}$, the coefficient of $Y_{k,j}(\theta)$ in the spherical harmonic expansion for $\hat{\phi}$. Then $\hat{\phi}'_{k,j} = [\overline{\phi}'_{k,j}\overline{h}_0 - \overline{\phi}_{k,j}\overline{h}'_0]\overline{h}_0^{-2}$. Define $w_{k,j} = \overline{h}_0^{2}\hat{\phi}'_{k,j} = \overline{\phi}'_{k,j}\overline{h}_0 - \overline{\phi}_{k,j}\overline{h}'_0$. By equation (2.1), we deduce that $w'_{k,j} = \lambda_k g^{-2}\overline{\phi}_{k,j}\overline{h}_0$. □

Now the functions $\overline{\phi}_{k,j}$ and $\overline{h}_0$ are both of order $e^{-\sqrt{E}r}$, according to Lemma 2.2. Moreover, $g^{-2} = 0(e^{-2r})$. So we may integrate up to infinity, yielding

$$w_{k,j}(x) = -\int_x^\infty \lambda_k g^{-2}(y)\overline{\phi}_{k,j}(y)\overline{h}_0(y)dy.$$

Note that $\lim_{x\to\infty} w_{k,j}(x) = 0$, by Lemma 2.2.

Recalling the definition of $w_{k,j}$ gives

$$\hat{\phi}'_{k,j}(x) = -\overline{h}_0^{-2}(x) \int_x^\infty \lambda_k g^{-2}(y)\overline{\phi}_{k,j}(y)\overline{h}_0(y)dy.$$

By Lemma 2.5, $|\overline{\phi}_{k,j}(y)| = |\overline{\phi}_{k,j}(r_1)\overline{h}_k(y)| \leq \overline{\phi}_{k,j}(r_1)\overline{h}_0(y)$. So

$$|\hat{\phi}'_{k,j}(x)| \leq \overline{h}_0^{-2}(x) \int_x^\infty \lambda_k g^{-2}(y)\overline{h}_0^2(y)dy|\overline{\phi}_{k,j}(r_1)|.$$

Using Lemma 2.2, $\overline{h}_0(y) \sim \overline{b}_0 e^{-\sqrt{E}r}$. Thus one has

$$|\hat{\phi}'_{k,j}(x)| \leq c_4 \int_x^\infty \lambda_k g^{-2}(y)dy|\overline{\phi}_{k,j}(r_1)|.$$

Now from (2.4)

$$|\hat{\phi}'_{k,j}(x)| \leq c_5 \int_x^\infty g^{-2}(y)dy\; e^{-c_6\sqrt{\lambda_k}}.$$

By definition $g(y) = \sinh y$. Therefore

$$|\hat{\phi}'_{k,j}(x)| \leq c_7 e^{-c_6\sqrt{\lambda_k}} e^{-2x}. \tag{2.9}$$

The estimate (2.9) is quite appropriate for our present purpose. In fact

$$\nabla_\theta^\ell A(\theta) - \nabla_\theta^\ell \hat{\phi}(r,\theta) = \sum_{k=0}^\infty \sum_{j=1}^{q(k)} (A_{k,j} - \hat{\phi}_{k,j}(r))\nabla_\theta^\ell Y_{k,j}(\theta). \tag{2.10}$$

Since $A_{k,j} = \lim_{r\to\infty} \hat{\phi}_{k,j}(r)$, we have

$$A_{k,j} - \hat{\phi}_{k,j}(r) = \int_r^\infty \hat{\phi}'_{k,j}(x)dx.$$

The estimate (2.9) guarantees that the integral converges and also yields

$$|A_{k,j} - \hat{\phi}_{k,j}(r)| \leq c_8 e^{-c_6\sqrt{\lambda_k}} e^{-2r}.$$

Returning to (2.10), one finds that

$$|\nabla_\theta^\ell A - \nabla_\theta^\ell \hat{\phi}(r,\theta)| \leq \sum_{k=0}^\infty \sum_{j=1}^{q(k)} c_8 e^{-c_6\sqrt{\lambda_k}} |\nabla_\theta^\ell Y_{k,j}(\theta)| e^{-2r}.$$

Proposition 2.8 follows easily.

§3. Asymptotic estimate

We proceed to obtain more detailed information concerning the convergence of $\hat{\phi}(r,\theta)$ to $A(\theta)$ as $r \to \infty$. The first step is to derive a basic integral equation satisfied by $\hat{\phi}(r,\theta)$. This leads to an iterative scheme for developing $\hat{\phi}(r,\theta)$ in terms of $A(\theta)$. Near the zeroes of A, $\hat{\phi}$ may be approximated by certain polynomials. The order of these polynomials coincides with the order of vanishing of A.

Recall that $\overline{\phi}(r,\theta) = \phi(r,\theta)g^{(n-1)/2}(r)$ and ϕ is an eigenfunction of the hyperbolic Laplacian with eigenvalue λ. It follows that $\overline{\phi}$ satisfies the partial differential equation

$$-\frac{\partial^2\overline{\phi}}{\partial r^2} + (\gamma(r) - \lambda)\overline{\phi} + g^{-2}\Delta_\theta\overline{\phi} = 0. \tag{3.1}$$

Here Δ_θ is the Laplacian on S^{n-1}. In fact, (3.1) follows by summing the equations (2.1). Alternatively, one derives (3.1) directly from the local coordinate formula for the Laplacian Δ.

The basic idea is to convert the partial differential equation for $\overline{\phi}$ into an integral equation for $\hat{\phi}(r,\theta) = \overline{\phi}(r,\theta)/\overline{h}_0(r)$. We may write

Proposition 3.2. *If $s > r > r_0$, then*

$$\hat{\phi}(s,\theta) = \hat{\phi}(r,\theta) - \int_r^s \overline{h}_0^{-2}(x)\int_x^\infty \overline{h}_0^2(y)g^{-2}(y)\Delta_\theta\hat{\phi}(y,\theta)dydx.$$

Proof. One has $\partial\hat{\phi}/\partial r = \partial/\partial r(\overline{\phi}\,\overline{h}_0^{-1}) = (\partial\overline{\phi}/\partial r)\overline{h}_0 - \overline{\phi}(\partial\overline{h}_0/\partial r)\overline{h}_0^{-2}$. Set $H = \overline{h}_0^2\partial\hat{\phi}/\partial r = (\partial\overline{\phi}/\partial r)\overline{h}_0 - \overline{\phi}(\partial\overline{h}_0/\partial r)$. Then, using equations (2.1) and (3.1), we find $\partial H/\partial r = \overline{h}_0\partial^2\overline{\phi}/\partial r^2 - \overline{\phi}\partial^2\overline{h}_0/\partial r^2 = g^{-2}\Delta_\theta\overline{\phi}\,\overline{h}_0$. So $\partial H/\partial r = g^{-2}\overline{h}_0^2\Delta_\theta\hat{\phi}$. □

By Proposition 2.8, $|\Delta_\theta\hat{\phi}|$ is uniformly bounded in both r and θ. Also, $g^{-2} = 0(e^{-2r})$ and $\overline{h}_0 = 0(e^{-\sqrt{E}r})$, from Lemma 2.2. So we may integrate up to infinity, yielding

$$H(x) = -\int_x^\infty \overline{h}_0^2(y)g^{-2}(y)\Delta_\theta\hat{\phi}(y,\theta)dy.$$

Note that $H = (\partial\overline{\phi}/\partial r)\overline{h}_0 - \overline{\phi}(\partial\overline{h}_0/\partial r)$ approaches zero as $r \to \infty$. In fact, both $\overline{\phi}$ and $\overline{h}_0$ are of order $e^{-\sqrt{E}r}$, by Lemma 2.2 and Theorem 2.6. Equations (2.1) and (3.1) may be integrated to verify that $\partial\overline{\phi}/\partial r$ and $\partial\overline{h}_0/\partial r$ are bounded.

The definition of H now yields

$$\frac{\partial\hat{\phi}}{\partial r}(x,\theta) = -\overline{h}_0^{-2}(x)\int_x^\infty \overline{h}_0^2(y)g^{-2}(y)\Delta_\theta\hat{\phi}(y,\theta)dy.$$

Proposition 3.2 follows by integrating this equation between r and s.

We now let $s \to \infty$ in Proposition 3.2. Note that $\overline{h}_0(r) \sim \overline{b}_0 e^{-\sqrt{E}r}$ from Lemma 2.2. Moreover, the function $g^{-2}(r) = 0(e^{-2r})$ and $|\Delta_\theta\hat{\phi}|$ is bounded by Proposition 2.8. By Theorem 2.6 and the dominated convergence theorem,

$$A(\theta) = \hat{\phi}(r,\theta) - \int_r^\infty \overline{h}_0^{-2}(x)\int_x^\infty \overline{h}_0^2(y)g^{-2}(y)\Delta_\theta\hat{\phi}(y,\theta)dydx.$$

Let T denote the integral-differential operator defined by

$$Tf(r,\theta) = \int_r^\infty \overline{h}_0^{-2}(x)\int_x^\infty \overline{h}_0^2(y)g^{-2}(y)\Delta_\theta f(y,\theta)dydx.$$

The domain of T consists of functions where $|\Delta_\theta f|$ is bounded, uniformly in y and θ.

We may write

$$\hat{\phi}(r,\theta) = A(\theta) + T\hat{\phi}(r,\theta).$$

Substituting $\hat{\phi} = A + T\hat{\phi}$ in the right hand side gives

$$\hat{\phi} = A + TA + T^2\hat{\phi}.$$

Iterating any finite number of times yields, for any positive integer m,

$$\hat{\phi} = \sum_{j=0}^m T^j A + T^m(\hat{\phi} - A).$$

Proposition 2.8 and the dominated convergence theorem guarantee that we always remain within the domain of T.

An elementary calculation using Proposition 2.8 yields

$$T^m(\hat{\phi} - A) = 0(e^{-2(m+1)r}).$$

For this computation we use the familiar estimates $\overline{h}_0(r) \sim \overline{b}_0 e^{-\sqrt{E}r}$ and $g(r) = 0(e^{-2r})$, as noted repeatedly above.

So, with arbitrary m, we have

$$\hat{\phi}(r,\theta) = \sum_{j=0}^m T^j A(r,\theta) + 0(e^{-2(m+1)r}). \tag{3.3}$$

The function $A(\theta)$ is analytic and therefore has zeroes of finite order. We use (3.3) to investigate the behavior of $\hat{\phi}$ near an m'th order zero of A. Choosing a coordinate system centered at this zero, we have $A(\theta) = A_m(\theta) + 0(|\theta|^{m+1})$, where A_m is a non-zero polynomial of order m. We may denote $A_m(\theta) = \sum_{|L|=m} a_L \theta^L + 0(|\theta|^{m+1})$. Here $L = (\ell_1, \ell_2, \dots, \ell_{n-1})$ is a multi-index of total length $|L| = \ell_1 + \ell_2 + \cdots + \ell_{n-1}$, with each ℓ_i being a non-negative integer, and furthermore $\theta^L = \theta_1^{\ell_1}\theta_2^{\ell_2}\cdots\theta_{n-1}^{\ell_{n-1}}$.

To compute the spherical Laplacian Δ_θ, it is convenient to employ normal coordinates for the standard metric on S^{n-1}. A result of Cartan [1] gives $g_{ij} = \delta_{ij} + 0(|\theta|^2)$, and thus

$$\Delta_\theta = -\sum_{i=1}^{n-1} \frac{\partial^2}{\partial\theta_i^2} + \sum_{i,j=1}^{n-1} a_{ij}\frac{\partial^2}{\partial\theta_i\partial\theta_j} + \sum_{i=1}^{n-1} b_i \frac{\partial}{\partial\theta_i},$$

where $a_{ij} = 0(|\theta|^2)$ and $b_i = 0(|\theta|)$.

To isolate the dominant contributions for the asymptotic expansion of $\hat{\phi}$, it is convenient to define $\overline{\Delta}_\theta = -\sum_{i=1}^{n-1} \partial^2/\partial\theta_i^2$. There is a corresponding integral-differential operator $\overline{T}f = \sum_{i=1}^{n-1} \overline{T}_i f$, where

$$\overline{T}_i f = -\int_r^\infty \overline{h}_0^{-2}(x) \int_x^\infty \overline{h}_0^2(y) g^{-2}(y) \frac{\partial^2 f}{\partial\theta_i^2}(y,\theta)\,dy\,dx.$$

Our asymptotic estimate of $\hat{\phi}(r,\theta)$ will be valid in the region $H_c = \{(r,\theta) \mid |\theta| < ce^{-r},\ r > r_2\}$, for any given $c > 0$ and $r_2 > r_0$, large enough. Suppose that B is an analytic function of θ satisfying $B(\theta) = 0(|\theta|^p)$ for some p. If $j \geq 0$, then the definition of T and Lemma 2.2 give $T^j B = 0(e^{-pr})$ in H_c. The point is that each application of T involves at most two θ-derivatives, but the double integration in r contributes a factor e^{-2r}. Similarly, one may bound $\overline{T}^j B = 0(e^{-pr})$. Since $\Delta_\theta - \overline{\Delta}_\theta = \Sigma a_{ij}\partial^2/\partial\theta_i\partial\theta_j + \Sigma b_i \partial/\partial\theta_i$, with $a_{ij} = 0(|\theta|^2)$ and $b_i = 0(|\theta|)$, we see that $(T - \overline{T})^j B = 0(e^{-(p+2j)r})$. Combining these observations with formula (3.3), where m is the order of vanishing of A, yields

$$\hat{\phi} = \sum_{j=0}^{m} \overline{T}^j A_m + 0(e^{-(m+1)r}).$$

Since A_m is a polynomial of total order m, in θ, and each $\overline{T}_i$ involves two derivatives in θ_i, we have

$$\begin{aligned}\hat{\phi} &= \sum_{j=0}^{\infty} \overline{T}^j A_m + 0(e^{-(m+1)r}) \\ &= \sum_{j=0}^{\infty} \left(\sum_{i=1}^{n-1} \overline{T}_i\right)^j A_m + 0(e^{-(m+1)r}) \\ &= \sum_{j=0}^{\infty} \sum_{j_1+j_2+\cdots+j_{n-1}=j} \frac{j!}{j_1!j_2!\cdots j_{n-1}!} \\ &\qquad \times \overline{T}_1^{j_1}\overline{T}_2^{j_2}\cdots\overline{T}_{n-1}^{j_{n-1}} A_m + 0(e^{-(m+1)r}).\end{aligned}$$

Thus

$$\begin{aligned}(3.4)\quad \hat{\phi} &= \sum_{|L|=m} a_L \sum_{j=0}^{\infty} \sum_{j_1+j_2+\cdots+j_{n-1}=j} \frac{j!}{j_1!j_2!\cdots j_{n-1}!} \\ &\quad \times(\overline{T}_1^{j_1}\theta_1^{\ell_1})(\overline{T}_2^{j_2}\theta_2^{\ell_2})\cdots(\overline{T}_{n-1}^{j_{n-1}}\theta_{n-1}^{\ell_{n-1}}) + 0(e^{-(m+1)r}).\end{aligned}$$

It remains to evaluate the individual expressions $\overline{T}_i^{j_i}\theta_i^{\ell_i}$, for fixed i. For this purpose, we need the following improvement of Lemma 2.2:

Lemma 3.5. $\overline{h}_0(r) = \overline{b}_0 e^{-\sqrt{E}r} + 0(e^{-(\sqrt{E}+2)r})$.

Proof. Lemma 2.2 gives $\overline{h}_0(r) \sim \overline{b}_0 e^{-\sqrt{E}r}$. Consider the ratio $u(r) = \overline{h}_0(r)/e^{-\sqrt{E}r} = \overline{h}_0(r)e^{\sqrt{E}r}$. Then $u' = (\overline{h}_0' + \sqrt{E}\,\overline{h}_0)e^{\sqrt{E}r}$. Set $w(r) = u'(r)e^{-2\sqrt{E}r} = e^{-\sqrt{E}r}(\overline{h}_0' + \sqrt{E}\,\overline{h}_0)$. Differentiating this gives the formula $w' = (\overline{h}_0'' - E\overline{h}_0)e^{-\sqrt{E}r} = (-E+\gamma-\lambda)\overline{h}_0 e^{-\sqrt{E}r} = (\gamma - (n-1)^2/4)\overline{h}_0 e^{-\sqrt{E}r}$. Here we used equation (2.1), with $k = 0$, and the definition $E = (n-1)^2/4 - \lambda$. Integrating up to infinity yields $w(x) = -\int_x^\infty (\gamma(y) - (n-1)^2/4)\overline{h}_0(y)e^{-\sqrt{E}y}dy$. Note that $w \to 0$ as $r \to \infty$, since $w(r) = (\overline{h}_0' + \sqrt{E}\,\overline{h}_0)e^{-\sqrt{E}r} = 0(e^{-2\sqrt{E}r})$, by Lemma 2.2. Moreover, the definition of w gives $u'(x) = w(x)e^{2\sqrt{E}x} = -e^{2\sqrt{E}x}\int_x^\infty (\gamma(y) - (n-1)^2/4)\overline{h}_0(y)e^{-\sqrt{E}y}dy$. Integrating up to infinity yields

$$u(r) - \overline{b}_0 = \int_r^\infty e^{2\sqrt{E}x}\int_x^\infty \left(\gamma(y) - \frac{(n-1)^2}{4}\right)\overline{h}_0(y)e^{-\sqrt{E}y}dydx.$$

Now $\gamma(y) - (n-1)^2/4 = 0(e^{-2y})$. The integral then converges by Lemma 2.2. A calculation gives $u(r) - \overline{b}_0 = 0(e^{-2r})$. Lemma 3.5 follows after multiplying by $e^{-\sqrt{E}r}$. $\square$

Suppose $\ell = \ell_i \geq 2$ is any integer. Clearly

$$\overline{T}_i \theta_i^\ell = \left(- \int_r^\infty \overline{h}_0^{-2}(x) \int_x^\infty \overline{h}_0^2(y) g^{-2}(y) dy dx \right) \ell(\ell - 1) \theta_i^{\ell-2}.$$

Using Lemma 3.5 and the elementary estimate $g(r) = e^r(1 + 0(e^{-2r}))/2$ implies

$$\overline{T}_i \theta_i^\ell(r, \theta) = -\ell(\ell - 1)(\sqrt{E} + 1)^{-1} \theta_i^{\ell-2} e^{-2r}(1 + 0(e^{-2r})).$$

An argument by mathematical induction gives

Lemma 3.6. *For $k \leq [\ell/2]$, the greatest integer in $\ell/2$,*

$$\overline{T}_i^k \theta_i^\ell(r, \theta) = \frac{(-1)^k}{k!} \prod_{s=1}^{k} (\sqrt{E} + s)^{-1} \frac{\ell!}{(\ell - 2k)!} \theta_i^{\ell-2k} e^{-2kr}(1 + 0(e^{-2r})).$$

If $k > [\ell/2]$, then $\overline{T}_i^k \theta_i^\ell = 0$.

Proof. Suppose the required formula has been established for a given value $k - 1$ with $k \leq [\ell/2]$. Then

$$\begin{aligned}
\overline{T}_i^k \theta_i^\ell(r, \theta) &= \overline{T}_i(\overline{T}_i^{k-1} \theta_i^\ell)(r, \theta) = \frac{(-1)^{k-1}}{(k - 1)!} \prod_{s=1}^{k-1} (\sqrt{E} + s)^{-1} \\
&\times \frac{\ell!}{(\ell - 2k + 2)!} \left(\int_r^\infty -\overline{h}_0^{-2}(x) \int_x^\infty \overline{h}_0^2(y) g^{-2}(y) e^{-2(k-1)y} \right. \\
&\times \left. (1 + 0(e^{-2y})) dy dx \right) (\ell - 2k + 2)(\ell - 2k + 1) \theta_i^{\ell-2k}.
\end{aligned}$$

Using Lemma 3.5 and $g(r) = (1/2)e^r(1 + 0(e^{-2r}))$ yields

$$\begin{aligned}
\overline{T}_i^k \theta_i^\ell(r, \theta) &= \frac{(-1)^k}{(k - 1)!} \prod_{s=1}^{k-1} \left(\sqrt{E} + s \right)^{-1} \frac{\ell!}{(\ell - 2k)!} \theta_i^{\ell-2k} \\
&\quad \times \int_r^\infty e^{2\sqrt{E}x}(1 + 0(e^{-2x})) \int_x^\infty e^{-2\sqrt{E}y} \\
&\quad \times 4e^{-2y} e^{-2(k-1)y}(1 + 0(e^{-2y})) dy dx.
\end{aligned}$$

The integral is easily calculated, which completes the induction. $\square$

Define polynomials of $x = (x_1, x_2, \ldots, x_{n-1})$ by

$$\begin{aligned} P_L(x) &= \sum_{0 \le j \le (1/2)|L|} \sum_{\substack{j_1+j_2+\cdots+j_{n-1}=j \\ 2j_i \le \ell_i}} \frac{(-1)^j j!}{(\ell_1 - 2j_1)!(\ell_2 - 2j_2)! \cdots (\ell_{n-1} - 2j_{n-1})!} \\ &\times \frac{\ell_1! \ell_2! \ldots \ell_{n-1}!}{((j_1)!(j_2)! \cdots (j_{n-1})!)^2} \prod_{s_1=1}^{j_1} (\sqrt{E} + s_1)^{-1} \\ &\cdots \prod_{s_{n-1}=1}^{j_{n-1}} (\sqrt{E} + s_{n-1})^{-1} \prod_{i=1}^{n-1} (x_i)^{\ell_i - 2j_i}. \end{aligned}$$

Combining (3.4) and Lemma 3.6 gives our main result.

Theorem 3.7. *In the region $H_c = \{(r, \theta) \mid |\theta| < ce^{-r}, r > r_2\}$, for any given $c > 0$, one has*

$$\hat{\phi}(r, \theta) = e^{-mr} \sum_{|L|=m} a_L P_L(e^r \theta) + 0(e^{-(m+1)r}).$$

In the case of the hyperbolic plane, $n = 2$, Theorem 3.7 was proved in [3]. There one had a single polynomial $P_m(x)$, with m distinct real zeroes. This led to a detailed analysis of the nodal structure of $\hat{\phi}$, near $\theta = 0$, and as $r \to \infty$. For $n > 2$, a similar discussion does not appear feasible. The difficulty lies in the complicated structure of the zero set of polynomials of several variables and the related instability of the zero set under perturbation. Analogous problems arose in the earlier investigations [6] of Schrödinger operators in R^n, $n > 2$.

§4. Appendix — Analyticity and expansion in spherical harmonics

Let S^m denote the standard round m-dimensional sphere. The spherical harmonics $Y_{k,j}(\theta)$, for $k \ge 0$ and $1 \le j \le q(k)$, form a complete orthonormal basis for $L^2(S^m)$. Each $Y_{k,j}(\theta)$ is obtained by restriction, to $S^m \subset R^{m+1}$, of a homogeneous harmonic polynomial of degree k. The dimension of the space of degree k harmonic polynomials is $q(k) = 0(k^{m-1})$. Moreover, the spherical harmonics $Y_{k,j}(\theta)$ are eigenfunctions of the spherical Laplacian, with corresponding eigenvalue $\lambda_k = 0(k^2)$. The reader may consult [1] for detailed proofs of these elementary results.

Each $f \in L^2(S^m)$ has a generalized Fourier series, with coefficients $a_{k,j} = \int_{S^m} f(\theta) Y_{k,j}(\theta)$. If $f \in C^\infty(S^m)$, then $a_{k,j} = 0(\lambda_k^{-\ell})$ for any ℓ, according to partial integration. Moreover, one has a uniformly convergent expansion, as a consequence of standard elliptic theory,

$$f(\theta) = \sum_{k=0}^{\infty} \sum_{j=1}^{q(k)} a_{k,j} Y_{k,j}(\theta). \tag{4.1}$$

This expansion may be repeatedly differentiated, term by term, to yield the expansion of any higher order derivative of f.

The purpose of this appendix is to correlate the analyticity of f with the exponential decay of the $a_{k,j}$, in their dependence upon k. This result is implicit in the much more elaborate developments of [6]. However, it seems worthwhile to present a simple elementary proof. We begin with

Proposition 4.2. *If* $|a_{k,j}| \leq c_1 e^{-c_2 k}$*, then the series of* (4.1) *converges to a real analytic function* f*.*

Proof. Since we normalized $\|Y_{k,j}\|_2 = 1$, one has $\|Y_{k,j}\|_\infty \leq c_3 k^{(m-1)/2}$ by standard elliptic theory. The decay hypothesis, about $a_{k,j}$, allows the extension of f to a function

$$u(r, \theta) = \sum_{k=0}^{\infty} r^k \sum_{j=1}^{q(k)} a_{k,j} Y_{k,j}(\theta) \tag{4.3}$$

on some neighborhood of the closure of the unit ball B^{m+1} in R^{m+1}. Since u is the uniform limit of harmonic functions, u is harmonic. The standard Laplacian Δ of R^n is analytic hypoelliptic and thus u is analytic. It follows that the restriction f of u, to S^m, is also analytic. □

For the converse to Proposition 4.2, it is convenient to employ

Lemma 4.4. *Let* u *be a solution of the Dirichlet problem on* $B = B^{m+1}$*, with analytic boundary data* f*. Then* u *extends to a harmonic function on a neighborhood of the closure* $\overline{B}$*.*

Proof. The Cauchy-Kovalevskya theorem provides a harmonic extension h of f to a neighborhood of $S^m \subset R^{m+1}$. Let χ be a smooth cut-off function, equal to one near S^m, and with support contained within the domain of definition of h. Clearly, $u - \chi h = 0$ on S^m, and $\Delta(u - \chi h) = g \in C_0^\infty(B)$. If G is the Greens function of B, one consequently has $u - \chi h = Gg$. The explicit formula for G, obtained by the method of images [2, p. 264], now shows that u extends harmonically past the boundary of B. □

The converse to Proposition 4.2 is

Proposition 4.5. *If f is real analytic, then $|a_{k,j}| \leq c_1 e^{-c_2 k}$ in the expansion* (4.1).

Proof. Let u be the solution of the Dirichlet problem on B^{m+1}, with boundary data f. Since f is C^∞, the coefficients $a_{k,j}$ decay faster than any polynomial in k, as observed above. Thus the series in (4.3) converges for $r \leq 1$. By uniqueness in the Dirichlet problem, we have for $r \leq 1$,

$$u(r,\theta) = \sum_{k=0}^{\infty} r^k \sum_{j=1}^{q(k)} a_{k,j} Y_{k,j}(\theta). \tag{4.6}$$

By Lemma 4.4, we have for some $\delta > 0$ and $r \leq 1 + \delta$, a uniformly convergent expansion

$$u(r,\theta) = \sum_{k=0}^{\infty} \sum_{j=1}^{q(k)} a_{k,j}(r) Y_{k,j}(\theta).$$

Since u is harmonic, separation of variables shows that each $a_{k,j}(r)$ satisfies a second order ordinary differential equation. By the uniqueness theory for ordinary differential equations, (4.6) holds when $r \leq 1 + \delta$. Taking the L^2 norm gives

$$\sum_{k=0}^{\infty} (1+\delta)^{2k} \sum_{j=1}^{q(k)} |a_{k,j}|^2 = \int_{S^m} |u(1+\delta,\theta)|^2.$$

Therefore $|a_{k,j}|$ decays exponentially in k. □

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