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The Shape of the Classification of the Finite Simple Groups

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This is a general survey of the Classification of the Finite Simple Groups with particular emphasis on the current project of Gorenstein, Lyons and Solomon (GLS) directed towards the revision of a substantial segment of the Classification proof.

There are two principal strategies at present directed towards a Classification proof. The one employed in the first successful proof and also, with certain modifications, in the GLS proof, I shall refer to as the Semisimple Approach to the Classification. The other, which has been the object of considerable activity recently, I shall refer to as the Unipotent Approach to the Classification. Each has its advantages and its drawbacks and neither is, at present, completely independent of the other. In unison they provide a complete proof of the Classification Theorem. A question at present is the natural domain for each of these methods. Of course the future may bring entirely new and wonderful approaches to the subject.

The modern history of the Classification began around 1950 when several mathematicians – notably Brauer, Suzuki and Wall – began to investigate simple groups of even order satisfying certain local conditions. This work eventually congealed into the Brauer-Suzuki-Wall Theorem [BSW] characterizing the two-dimensional projective special linear groups over finite fields. Brauer in particular championed the strategy of characterizing finite simple groups of even order by the centralizer of an involution. Suzuki, on the other hand, established the nonexistence of finite simple CA-groups of odd order [S1]. (A group G is a CA-group if the centralizer of every nonidentity element of G is abelian.) This result was the inspiration for the Feit-Thompson Theorem proving the nonexistence of nonabelian finite simple groups of odd order.

Meanwhile Suzuki pursued the classification of transitive permutation groups of odd degree in which the stabilizer of a point has a regular

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normal subgroup and a cyclic complement of odd order [S2]. This formed the foundation for the later classification by Bender of finite groups G with a strongly 2-embedded subgroup M. (M is a strongly p-embedded subgroup of G if M is a proper subgroup of G of order divisible by p such that $M \cap M^g$ has order prime to p for all $g \in G - M$.)

We remark that the Odd Order Theorem [FT] of Feit and Thompson can be regarded as a strong embedding result as well. Indeed the Feit-Thompson Theorem together with the Suzuki-Bender Theorem [B2] establish the following result.

Theorem. *Let G be a finite simple group and let p be the smallest prime divisor of $|G|$. If G has a strongly p-embedded subgroup, then $p = 2$ and G is isomorphic to $SL(2, 2^n)$, $Sz(2^{2n-1})$ or $PSU(3, 2^n)$ for some $n \geq 2$.*

Clearly this is a corollary of the Feit-Thompson and Suzuki-Bender Theorems. As remarked in [So], the Feit-Thompson Theorem is an easy consequence of the above theorem, although this observation does not seem to afford a route to a new proof of the Feit-Thompson Theorem.

The Feit-Thompson and Suzuki-Bender Theorems form the two principal Background Results underlying the GLS proof of the Classification Theorem. (Also in the background is the theory of linear algebraic groups, the determination of the Schur multipliers of the finite simple groups, and the existence, uniqueness and local structure of the sporadic simple groups. And in the "foreground", i.e. essential to the complete proof but not included in the GLS series, is the forthcoming proof of the Quasithin Theorem by Aschbacher and Smith.)

Chapter I. Semisimple Approach

When Brauer did specific characterizations of finite simple groups by the centralizer of an involution, the groups were almost always classical groups defined over fields of odd characteristic. (Of course M_{11} also arose, having an isomorphic involution centralizer to $PSL(3, 3)$.) This focus continued in the work of Brauer's students, Fong, Wong and Harris, who (along with Phan) systematically pursued the characterization of the finite simple groups of Lie type over fields of odd characteristic via the centralizer of an involution during the 1960's.

Of course, when the characteristic is odd, an involution is a semisimple (indeed split semisimple) element of the Lie type group G. Thus it is reasonable to expect (and indeed is true) that the characterization theorems established in particular by Wong and Phan can be generalized to characterizations of finite simple groups of Lie type in any characteristic via the centralizer of a semisimple element of prime order, or more precisely via the centralizer of a suitable element x of prime order contained either in the split maximal torus of G or in a "half-split" maximal torus, i.e. a torus which splits in a quadratic extension of the field of definition of G. Such a characterization over fields of characteristic 2 was accomplished by Gilman and Griess in [GG].

In order to convert this fact into a strategy for the classification, it is useful first to give a definition of a semisimple element for an abstract group, not simply for a group with a preferred linear representation. As our attention will focus on centralizers of such elements, it is natural that the definition should reflect a fundamental property of their centralizers. In the context of a semisimple linear algebraic group G, it is well-known that the centralizer C of a semisimple element is a reductive group, i.e. the product of a semisimple group and a torus (which is central in C if G is simply connected). Extending the work of Fitting, Bender in 1970 [B1] defined the appropriate subgroups of a finite group needed to formulate the analogous structural hypotheses. We recall some definitions.

Definition. A finite group K is *quasisimple* if $K = [K, K]$ and $K/Z(K)$ is a nonabelian simple group. A finite group E is *semisimple* if E is the commuting product of certain quasisimple subgroups, called its *components*.

Definition. Let H be a finite group. The join of all normal nilpotent subgroups of H is called the *Fitting subgroup* of H, $F(H)$. It is the unique maximal normal nilpotent subgroup of H. Similarly the join of all normal semisimple subgroups of H is denoted $E(H)$. It is the unique maximal normal semisimple subgroup of H. Moreover $E(H)$ and $F(H)$ commute with each other. Their commuting product is called the *generalized Fitting subgroup* of H, $F^*(H)$.

We can now identify a characteristic property of the centralizers of many semisimple elements in linear groups and make this a definition in an arbitrary finite group.

Definition. Let G be a finite group. We call an element x of G *semisimple* if $E(C_G(x)) \neq 1$.

We remark that if G is a classical linear group, then for every unipotent element y of G, $E(C_G(y)) = 1$, as a corollary of the Borel-Tits Theorem. On the other hand typically many of the semisimple (in the linear group sense) elements of G will also be semisimple in the above sense. However not all will be because for certain linear semisimple elements x, $C_G(x)$ will be a torus. In an extreme case like $SL(2,q)$, no noncentral linear semisimple element will be semisimple in the sense of the above definition. This reflects a fundamental limitation on the semisimple approach: it does not work for "very small" simple groups. (A good replacement for the term "very small" is quasithin, as we shall see below.)

The first goal of the semisimple approach to the Classification is to search for a semisimple element x such that $E(C_G(x))$ has a component K of maximum possible order. This corresponds in the context of linear groups to the search for a semisimple element with an eigenspace of maximum possible dimension. When chosen judiciously this component K will be a slightly smaller version of the target group G. Moreover by making a similar choice of a semisimple element y inside $K - Z(K)$, one can find a second large component, L, of $E(C_G(y))$ such that K and L generate G. Indeed it is possible not only to find generators for G but also to infer sufficient relations to characterize G via theorems of Coxeter, Steinberg or Curtis and Tits. This strategy was implemented for semisimple involutions by Aschbacher in his Classical Involution Paper [A2] and for semisimple elements of odd prime order by Gilman and Griess [GG].

However there is an important reason to modify this strategy slightly. It is extremely important to control the embedding of such subgroups as $C_K(y)$ in $C_G(y)$. More specifically it is desirable to know that

$$E(C_K(y)) \leq E(C_G(y)).$$

It is not however possible to achieve an a priori proof of this fact because of the following type of example:

Let $H = SL(V)$ with V a 6-dimensional vector space over the finite field F of odd order q. Let $G = VH$ be the semidirect product with the natural action of H on V. Let x be an involution in H with a 4-dimensional -1-eigenspace. Then $E(C_G(x)) = K \cong SL(4,q)$. Next let y be an involution in K with a 2-dimensional 1-eigenspace on V, contained in the -1-eigenspace for x. Then $E(C_K(y)) = L_1 * L_2$ with $L_i \cong SL_2(q)$ acting on the $(-1)^i$-eigenspace for y on V. We can easily compute that $L_1 \leq E(C_G(y))$ but $L_2 \not\leq E(C_G(y))$.

Of course in the example G is far from being a finite simple group. However it is precisely the problem of detecting from *local* information that such a G is not simple which constitutes one of the major chapters of the Classification proof. (A *local subgroup* of a group G is the normalizer of a non-identity p-subgroup of G. Local information is information about the structure of the local subgroups of G.)

Gorenstein and Walter [GW] discovered an important "gravitational principle", called *L-Balance* concerning a subgroup closely related to $E(H)$.

Definition. Let p be a prime and H a p-local subgroup of G. The *p-layer* of H, $L_{p'}(H)$ is the smallest normal subgroup of H covering $E(H/O_{p'}(H))$, where $O_{p'}(H)$ denotes the largest normal subgroup of H of order relatively prime to p.

The L-Balance Theorem. *Let G be a finite group all of whose proper simple sections satisfy the (weak) Schreier Conjecture. Let p be a prime and let x and y be commuting elements of G of order p. Let L_x and L_y denote the p-layers of $C_G(x)$ and $C_G(y)$ respectively. Then*

$$L_{p'}(C_{L_x}(y)) \leq L_y.$$

In the vernacular, the L-Balance Theorem asserts that the p-layer of a p-local subgroup of G always sinks into the p-layer of G. Hence it is a kind of gravitational (or non-buoyancy) principle. The proof of the L-Balance Theorem depends on a weak version of the following old conjecture.

Schreier's Conjecture. *Let S be a finite simple group. Then $Aut(S)/S$ is a solvable group.*

Schreier's Conjecture is a fairly easy corollary of the Classification Theorem. No independent proof is known. I shall not bother to state the weak version of the Schreier Conjecture here but I note that it was proved when $p = 2$ by Glauberman as a corollary of his Z^*-Theorem [Gl]. Thus for $p = 2$ the hypothesis on proper simple sections of G may be omitted. In the context of an inductive proof of the Classification Theorem, proper simple sections always satisfy the Schreier Conjecture and so the L-Balance Theorem may be used for all primes p.

Notice that the L-Balance Theorem provides a correct analogue of the wished-for property of Bender's subgroup $E(H)$. Inspired by this, we reformulate our semisimple strategy in the following language:

Definition. Let G be a finite group and p a prime. A p-element x of G is said to be *weakly semisimple* if $L_{p'}(C_G(x)) \neq 1$.

Of course every semisimple element of prime power order is weakly semisimple. The converse statement is false in general as is easily seen, for instance, by modifying the example above slightly. Take $G^* = VH^*$ where $H^* = SL^{\pm}(V)$, the group of linear transformations of determinant

± 1. Then an involution t with a 1-dimensional -1-eigenspace is weakly semisimple but not semisimple. However the converse statement is true (and deep) for finite simple groups. For the prime $p = 2$ it was first formulated by Thompson, who called it the B-Conjecture. Its proof for $p = 2$ forms a major chapter in the proof of the Classification Theorem and weak analogues of it for all primes p also play a pivotal role in the work of Gorenstein and Lyons [GL] on the classification of simple groups of characteristic 2-type (roughly speaking, groups in which no involution is semisimple). As a corollary of the Classification Theorem, we obtain the full B-Theorem.

B-Theorem. *Let G be a finite simple group. For all primes p, every weakly semisimple p-element of G is semisimple.*

We can now formulate a somewhat over-simplified version of the Semisimple Strategy for the Classification of Finite Simple Groups based on the Feit-Thompson and Suzuki-Bender Theorems.

Step 1. *Find a prime p for which G has a weakly semisimple of prime order p. Choose $p = 2$, if possible.*

Step 2. *Establish the B_p-Theorem for G, i.e. that every weakly semisimple p-element of G is semisimple.*

Step 3. *Among all semisimple p-elements of G choose one, x, with some component K of $E(C_G(x))$ as large as possible.*

Now the Component Theorem comes into play. This theorem was established first by Aschbacher [A1], extending an earlier result of Powell and Thwaites [PT]. It was reproved shortly thereafter by Gilman [Gi]. For a minimal counterexample to the Classification Theorem, analogues were established for all primes p by GLS [GLS2].

Component Theorem. *Let G be a finite simple group and x a semisimple element of G of prime order p chosen with some component K of $E(C_G(x))$ as large as possible. Suppose that the p-rank of K is greater than 1. Then K does not commute with any G-conjugate of K. Moreover a Sylow p-subgroup of $C_G(K)$ is either cyclic or of maximal class (with $p = 2$ in the latter case).*

A typical example to imagine is $G = SL(V)$ and x a diagonal element with one eigenspace W of codimension 1 or 2. Then $SL(W)$ will be the unique large component of $E(C_G(x))$ and its centralizer will have cyclic Sylow p-subgroups for odd p and a cyclic or quaternion Sylow 2-subgroup. A slightly different example arises when $G = A_n$ and x is a product of two transpositions. Then the Sylow 2-subgroup of the centralizer of the large component (isomorphic to A_{n-4}) is a Klein 4-group.

The point is that the Component Theorem assures us that the centralizer of x is almost precisely determined, the possibilities for K being afforded by the induction hypothesis. This permits us to proceed to the final step.

Step 4. *Identify G, given the approximate structure of $C_G(x)$, via the methodology developed by Brauer, Fong, Wong, Phan and Harris.*

This constitutes the Semisimple Strategy for the Classification of Finite Simple Groups, modulo one serious problem and one difficult theorem which I have swept under the rug. The difficult theorem is the Strongly p-Embedded Theorem, the analogue for odd primes of the Suzuki-Bender Theorem.

Strongly p-Embedded Theorem. *Let G be a finite simple group and p a prime such that G has p-rank at least 3. If M is a strongly p-embedded subgroup of G, then G is a finite simple group of Lie type of Lie rank 1 and M is a Borel subgroup of G.*

For odd primes p, this theorem is proved only as a corollary of the Classification Theorem. However a weak version of this theorem is required for the Classification proof, in particular for the proof of the B-Theorem. A sufficient theorem was established by Aschbacher [A3] and a slightly more general variant has been established recently by Stroth. Both proofs are quite long and difficult, and even the statements of the theorems established are long and obscure.

Let's move on from the difficult theorem to the serious problem:

What if G does not contain any weakly semisimple elements of prime order?

The answer to this question is: G is quasithin.

Definition. Let G be a finite simple group. We say that G is *quasithin* if either G has 2-rank at most 2 or every 2-local subgroup of G has p-rank at most 2 for every odd prime p.

In the usual definition of quasithin, G is assumed to be of characteristic 2-type (or even type) and to have 2-rank at least 3. We use the extended definition here for expository purposes.

Klinger-Mason Theorem. *Let G be a finite simple group with no weakly semisimple elements. Then G is quasithin.*

The proof relies on an easier version of some of the signalizer functor analysis used in the proof the the B-Theorem. The principal new ingredient is a lovely and elementary argument of John Thompson in [T2], which was later elaborated slightly in [KM] and has henceforth been known as the Klinger-Mason Method. Thompson's argument establishes easily under the given hypotheses that either G is quasithin or G contains an involution x such that $F^*(C_G(x))$ is a 2-group of symplectic type (indeed extraspecial) and $C_G(x)$ has p-rank at most 2 for all primes greater than 3. Indeed with the extra help of the Thompson-Bender Signalizer Lemma, it is possible to rule out the extraspecial case as well. (See [GLS1; 23.3; §24].)

The occurrence of extraspecial 2-groups in the Klinger-Mason argument reflects the proximity of many of the larger sporadic simple groups such as the sporadic Suzuki group, the Conway groups, the Fischer groups, the Harada group, the Thompson group, the Baby Monster and the Monster, as well as certain small classical linear groups. Although these groups do not satisfy the hypotheses of the Klinger-Mason Theorem, they are quite close. Indeed with slightly weakened hypotheses, Gorenstein and Lyons proved an analogous result whose conclusion is roughly that either G is quasithin or G is one of the large sporadic groups mentioned above or a small classical linear group. The full classification of simple groups containing an involution x such that $F^*(C_G(x))$ is a 2-group of symplectic type was accomplished in the mid 1970's largely through the efforts of Timmesfeld [Ti] and was rightly recognized by many as bringing down the final curtain on the search for sporadic simple groups.

I find the resulting Trichotomy Theorem, implicit in the work of Gorenstein and Lyons, to be one of the more elegant justifications for the Semisimple Approach to the Classification.

Trichotomy Theorem. *Let G be a finite simple group. Then one of the following holds:*

(1) *There is a prime p such that G has p-rank at least 3 and G contains generic weakly semisimple elements of order p; or*
(2) *G is quasithin; or*
(3) *G is on a (short) finite list including A_{12}, eleven sporadic simple groups and several small classical groups defined over $\mathbf{F}_2$ or $\mathbf{F}_3$.*

In the statement above, the term generic reflects a restriction on the allowable components in the centralizers of the semisimple elements of order p. In particular the centralizer of a generic semisimple element of order p must have a component which is not a group of Lie type in characteristic p. For $p = 2$ or 3, most sporadic components are likewise not allowed.

Chapter II. Unipotent Approach

The Semisimple Approach to the Classification runs aground on the rocks of the Quasithin Problem. The Classification proof is rescued at this juncture by the Unipotent Approach, which evolved from the methods developed by Thompson in his classification of simple groups all of whose local subgroups are solvable [T1]. (Clearly such a group has no non-identity weakly semisimple element.) In brief the Unipotent Approach, instead of studying semisimple elements, seeks to identify the *characteristic* of the finite simple group G by finding a prime p for which G has a rich supply of p-local subgroups of "parabolic type".

Definition. Let G be a finite group and p a prime. We say that a p-local subgroup H of G is of *parabolic type* if H contains a Sylow p-subgroup of G and $F^*(H)$ is a p-group.

Definition. Let G be a finite group and p a prime. We say that G is of *characteristic p-type* if every p-local overgroup of a Sylow p-subgroup of G is of parabolic type. G is of connected characteristic p-type if G is of characteristic p-type and G is generated by the overgroups of a fixed Sylow p-subgroup P of G.

When G has no semisimple involutions one is close to knowing that G is of connected characteristic 2-type. The final ingredient is provided by "pushing-up theorems" established in the mid 1970's by Baumann, Glauberman, Niles, Aschbacher and others, which establish that either G is of connected characteristic 2-type or G has a strongly 2-embedded subgroup.

Once G is known to be of connected characteristic p-type, the Unipotent Strategy in brief is to study, in the spirit of Tits, the coset geometry determined by the p-local subgroups of parabolic type and to recognize this geometry as that of a split BN-pair of rank at least 2. (Of course there are exceptions arising from the sporadic simple groups of characteristic p-type.) As noted above, many of the ideas for this approach originate in Thompson's N-group paper, whose main theorem may be paraphrased as:

N-Group Theorem. *Let G be a non-abelian simple group all of whose local subgroups are solvable. Assume that G has 2-rank at least 3 and G does not have a strongly 2-embedded subgroup. Then $G \cong {}^2F_4(2)'$, i.e. G is the Tits group.*

The idea of identifying simple groups of characteristic 2-type as split BN-pairs was initiated in the late 1960's by Suzuki [S3] and his students. This approach was temporarily sidetracked by the Gorenstein Program for the Classification announced in the early 1970's, which featured the Semisimple Approach. It was however pursued in the context of quasithin groups and uniqueness groups by Aschbacher, Gomi [Gm] and others. Later Goldschmidt developed a variant Amalgam Method [Go] aimed at a new proof of the N-Group Theorem, which was eventually obtained by Stellmacher.

In the Semisimple Approach to the Classification, the Unipotent Method is required to treat the Quasithin Problem and the Strongly p-Embedded 2-Local Problem. The latter appears in published work of Aschbacher and the former is currently being completed by Aschbacher and Smith. There is however a program underway, spearheaded by Meierfrankenfeld, Stellmacher and Stroth, to apply the Unipotent Method to all groups of connected characteristic p-type (possibly using a slightly different definition than the one given above).

As a strategy aimed at a complete proof of the Classification Theorem, the Unipotent Strategy collides with obstacles at two ends. One obstacle is the Strongly p-Embedded Subgroup Problem. At present the Unipotent Strategy presupposes that G is generated by the p-local subgroups containing a fixed Sylow p-subgroup P. Except when $p = 2$, there is no known approach to the case when P is contained in a unique maximal subgroup M of G. In particular this problem includes (and can probably be reduced to) the case when M is a strongly p-embedded subgroup of G. This is of course similar to the Strongly p-Embedded 2-Local Problem confronted by the Semisimple Approach and solved in that context by Aschbacher and later by Stroth using unipotent methodology. Conceivably a Unipotent Proof of the Classification could be structured in such a way that a solution of a similar nature would be possible.

A hybrid strategy which assigns to the Unipotent Approach precisely the task of classifying finite simple groups of characteristic 2-type would rely only on the Strongly Embedded Theorem of Suzuki-Bender. From my perspective this hybrid strategy is attractive inasmuch as it bypasses the morass of complicated definitions and difficult theorems related to groups with a strongly (or almost strongly) p-embedded 2-local subgroup. Of course a revolutionary new classification of groups with a strongly p-embedded subgroup would change the landscape for both the Semisimple and Unipotent Strategies, ironically improving the cases for each of them.

It is impossible to conjecture a flowchart for a Unipotent Approach to the entire Classification Theorem without an answer to the following question:

What if the simple group G is not of characteristic p-type for any prime p?

In this case of course G would be full of semisimple elements and the Semisimple Approach would be effective. The problem is that we are missing an analogue of the Klinger-Mason Reduction which would tell us that the residual semisimple problem was "bounded" in some good sense. For example one would like a comparatively short proof of a theorem of the following type.

Theorem. *Let G be a finite simple group of 2-rank at least 3 which is not of characteristic p-type for any prime p. Then for some involution t of G, there is a component K of $E(C_G(t))$ such that $K/Z(K)$ is an alternating group.*

Indeed the only simple groups with no characteristic are alternating groups and J_1. If one could give a proof of this fact of comparable length and elegance to the Klinger-Mason argument which rounds off the Semisimple Analysis, then there would be a strong argument for preferring the Unipotent Approach to the Classification proof. Even without it, there is great value in pursuing the Unipotent Analysis to its logical conclusions. If successful, it will bring to satisfying completion Michio Suzuki's program for the classification of finite simple groups of characteristic 2-type via the unipotent methods he helped to pioneer.

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