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On the vertices of modules in the Auslander–Reiten quiver III

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§0. Introduction

Let kG be the group algebra of a finite group G over a field k of characteristic p, where p is a prime. We denote the stable Auslander–Reiten quiver (AR quiver for short) of kG by $\Gamma_s(kG)$. For the definition of an AR quiver, see [B]. It is known that each connected component Γ of $\Gamma_s(kG)$ has the uniquely determined tree class $\mathcal{T}$. The AR component Γ is isomorphic as graphs to $\mathbf{Z}\mathcal{T}/\pi$, where $\mathbf{Z}\mathcal{T}$ is the graph obtained in a standard way from countably many copies of the tree $\mathcal{T}$ and π is a certain subgroup of $Aut(\mathbf{Z}\mathcal{T})$. Since the important paper by Webb [W] was published, many results concerning the tree classes have been obtained. (See [Be], [E3], [E4], [ES] and [O1].) In the present paper, assuming that k is a perfect field, we determine all the tree classes, not the possibilities of them, completely. The following should be the final result in this nature.

Theorem A. *Let k be a perfect field. Then the tree class of a connected component of $\Gamma_s(kG)$ is one of the following: A_n, $\tilde{A}_{1,2}$, A_∞, $\tilde{B}_3$, B_∞, D_∞, or A_∞^∞. Moreover, each of the above in fact occurs. Furthermore, the following hold. Here D is a defect group of the block to which the modules in Γ belong.*

(i) *B_∞ occurs only when D is dihedral.*
(ii) *D_∞ occurs only when D is semidihedral.* ([E3], [E4])
(iii) *A_∞^∞ occurs only when D is dihedral or semidihedral.* ([E3], [E4])
(iv) *$\tilde{A}_{1,2}$ or $\tilde{B}_3$ occurs only when D is a four group.* ([Be], [ES])

For the notation of the tree classes, we follow 2.30 of [B]. In particular,

$$\tilde{A}_{1,2} : \cdot \xrightarrow{(2,2)} \cdot, \quad \tilde{B}_3 : \cdot \xrightarrow{(1,2)} \cdot \longrightarrow \cdot \xrightarrow{(2,1)} \cdot, \quad B_\infty : \cdot \xrightarrow{(1,2)} \cdot \longrightarrow \cdot \longrightarrow \cdots .$$

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Remark 1. Each of the possibilities in Theorem A occurs in the following case.

(i) A_n occurs if and only if D is cyclic. See [B].

(ii) $\tilde{A}_{1,2}$ occurs for a four group with $p = 2$. See p. 180 of [B].

(iii) $\tilde{B}_3$ occurs for the alternating group on 4 letters with $p = 2$ if k does not contain a cube root of unity. See p. 195 of [B].

(iv) An example of a B_∞-component is given in Section 4.

(v) A_∞^∞ occurs for dihedral groups of order greater than 4 with $p = 2$.

(vi) D_∞ occurs for semidihedral groups with $p = 2$.

Furthermore, the following are known.

(vii) If the tree class is A_∞^∞, then we have $\Gamma \cong \mathbf{Z}A_\infty^\infty$ unless D is a four group. (See [ES].)

(viii) If k is algebraically closed, then one of A_n, $\tilde{A}_{1,2}$, A_∞, D_∞, or A_∞^∞ must occur. (See p.160 of [B].)

(ix) If the modules in Γ are periodic, then its tree class is A_∞. (2.31.11 of [B])

Since B_∞ appears when we have a certain involutive automorphism of an A_∞^∞ component, the block is tame in this case, too. However, it seems that no example of a B_∞-component has been known so far, and this is the reason why we give an example here. From the results known so far, A_n is the only finite Dynkin tree class and $\tilde{A}_{1,2}$ and $\tilde{B}_3$ are only Euclidean tree classes. The rest are infinite Dynkin tree classes, and only A_∞, D_∞, A_∞^∞, B_∞ and C_∞ are possible. (See [B].) Hence, in order to prove Theorem A, it suffices to give an example of a B_∞-component and prove that C_∞ does not occur. In fact, we prove the following.

Theorem 1. *Let k be a perfect field. Then the following hold.*

(i) *As a tree class of a component of $\Gamma_s(kG)$, C_∞ does not occur.*

(ii) *If B_∞ occurs, then a defect group D of the block to which the modules in Γ belong is dihedral of order at least* 8.

On the vertices of modules, beginning with the result for p-groups in [E2], there are several developments [U2], [OU2] which were obtained by using the generalization of Green correspondence due to Kawata [K1] and the results on vertices of modules in the Auslander-Reiten sequences [U1], [OU1]. In this paper, we have the following, which would be also the final result for non-periodic components.

Theorem B. *Let k be a perfect field, and let Γ be a connected component of $\Gamma_s(kG)$. Suppose that it is not a tube. Then one of the following holds.*

(i) *All the modules in Γ have vertices in common.*

(ii) *We can take* $T : X_1 - X_2 - X_3 - \cdots - X_n - \ldots$ *in* Γ *with* $\Gamma \cong \mathbf{Z}T$ *and* $vx(X_1) < vx(X_2) = vx(X_3) = vx(X_4) = \cdots = vx(X_n) = \ldots.$

(iii) $p = 2$, $\Gamma = \mathbf{Z}A_\infty^\infty$, *and only two distinct vertices* P *and* Q *occur, with* $|P : Q| = 2$. *Moreover, one of the following holds.*

(iiia) Q *is a dihedral group of order greater than* 4, *and the modules with vertex* Q *lie in a subquiver* Γ_Q *such that both* Γ_Q *and* $\Gamma \setminus \Gamma_Q$ *are isomorphic to* $\mathbf{Z}A_\infty$ *as graphs.*

(iiib) Q *is a Kleinian four group and* P *is a dihedral group of order 8, and the modules with vertex* Q *lie in two or four adjacent* τ*-orbits.*

Moreover, each of the above possibilities in fact occurs.

Remark 2. The above (i) and (ii) occur in many cases, (iiia) occurs for a dihedral 2-group. See (3.3) of [E1]. (iiib) occurs for a dihedral group D_8 of order 8 and the symmetric group S_4 on 4 letters. The group algebra kD_8 has an AR component satisfying (iiib) above with two adjacent τ-orbits of modules having four group as vertex, and kS_4 has an AR component satisfying (iiib) above with four adjacent τ-orbits of modules having four group as vertex. See also [E1] and V.3 of [E2].

Most parts of Theorem B have been proved in [OU2]. More precisely, it has been shown there that there are only three possibilities (i), (ii) and (iii), of which (i) and (iii) are exactly the same as in Theorem B above. However, the part (ii) of the main theorem in [OU2] asserts that there are three possibilities, namely,

(iia) $vx(X_1) < vx(X_2) = vx(X_3) = vx(X_4) = \cdots = vx(X_n) = \ldots,$
(iib) $vx(X_1) < vx(X_2) = vx(X_3) < vx(X_4) = \cdots = vx(X_n) = \ldots,$
(iic) $vx(X_1) = vx(X_2) < vx(X_3) = vx(X_4) = \cdots = vx(X_n) = \ldots.$

Thus, in order to prove Theorem B, it suffices to show that (iib) and (iic) above do not occur. More precisely, it suffices to prove the following.

Theorem 2. *In the situation of Theorem B, suppose that* $\Gamma \cong \mathbf{Z}A_\infty$. *Then* (i) *or* (ii) *of Theorem B holds.*

The purpose of this paper is of course to prove Theorems 1 and 2. For the both theorems, semidihedral groups play an important role. Thus, after giving some preliminary results in Section 1, we consider modules over dihedral and semidihedral groups in Section 2. The theorems are proved in Section 3. Notation is standard. See [F] and [NT]. The Auslander-Reiten translate is denoted by τ. For symmetric algebras, τ is the composite Ω^2 of two Heller translates. For a non-projective indecomposable module M, the AR sequence terminating at M is denoted by $\mathcal{A}(M)$.

§1. Preliminaries

In this section, we first consider automorphisms of an AR component Γ of $\Gamma_s(kG)$. The following is well known.

Lemma 1.1. *Let σ be an automorphism of the graph Γ which commutes with τ. Suppose that σ has finite order.*

(i) *If $\Gamma \cong \mathbf{Z}A_\infty$, then σ is trivial.*

(ii) *If $\Gamma \cong \mathbf{Z}D_\infty$, then σ is trivial or interchanges the two modules in the end with the same predecessor.*

(iii) *If $\Gamma \cong \mathbf{Z}A_\infty^\infty$, then σ is trivial or a reflection with respect to a certain τ-orbit.*

Let k' be a finite Galois extension of k. Assume that every indecomposable direct summand of $M \otimes_k k'$ for $M \in \Gamma$ is absolutely indecomposable. The proof of the following can be found in 2.33.3 of [B].

Lemma 1.2. *In the situation above, direct summands of $M \otimes_k k'$ for $M \in \Gamma$ belong to a finite set of connected components $\Gamma_1, \cdots, \Gamma_m$ of $\Gamma_s(k'G)$ and $Gal(k'/k)$ acts transitively among the Γ_i's. In particular, Γ_i's are isomorphic to each other.*

Assume that Γ has tree class B_∞ or C_∞. In view of Remark 1 (viii), we have another tree class for components of $\Gamma_s(k'G)$. When tensoring Γ with k', we get the following tree classes.

Lemma 1.3. *In the situation of Lemma 1.2, the following hold.*

(i) *If $\Gamma \cong \mathbf{Z}B_\infty$, then $\Gamma_i \cong \mathbf{Z}A_\infty^\infty$ for each i, and some element in $Gal(k'/k)$ stabilizes Γ_i and gives a reflection with respect to a certain τ-orbit.*

(ii) *If $\Gamma \cong \mathbf{Z}C_\infty$, then $\Gamma_i \cong \mathbf{Z}D_\infty$ for each i, and some element in $Gal(k'/k)$ stabilizes Γ_i and interchanges its two ends.*

In [U2] the relationship between the tree classes of components of $\Gamma_s(kG)$ and $\Gamma_s(kN)$ for a normal subgroup N of G is investigated. There it is assumed that k is an algebraically closed field. However, those assertions hold in more general situation. One of the important and crucial points in the argument is to introduce two indices $a(M)$ and $b(M)$ for an indecomposable N-projective kG-module M. They are defined by $a(M) = \dim_k eE_G(V^G)/eJ(E_G(V^G))$ and $b(M) = \dim_k eE_G(V^G)/eL_G(V^G)$, where V is an indecomposable N-source of M, $E_G(V^G) = \mathrm{End}_{kG}(V^G)$, $L_G(V^G) = J(E_N(V))E_G(V^G)$, and e is the idempotent of $E_G(V^G)$ with $eV^G = M$. However, we use only the fact that the multiplicities of direct summands can be described in terms of them. Thus, if $E_N(V)/J(E_N(V)) \cong k$, then the same conclusions hold. On the other hand, if k is a perfect field, then a kG-module M is absolutely indecomposable if and only if $\mathrm{End}_{kG}(M)/J(\mathrm{End}_{kG}(M)) \cong k$ by VII.6.9 of [HB]. Thus modifying the results in sections 2, 3 and 4 of [U2] in such a way, we can summarize them as follows.

Lemma 1.4. *Let N be a normal subgroup of G and Λ a connected component of $\Gamma_s(kN)$. Suppose that k is a perfect field, all the modules in Λ are G-invariant absolutely indecomposable, and that all the arrows in Λ are multiplicity free. Let V be in Λ and M an indecomposable direct summand of V^G. Let Γ be the connected component of $\Gamma_s(kG)$ containing M. Then one of the following holds.*

(i) *All the modules in Γ are N-projective and $\Gamma \cong \Lambda$.*

(ii) *Γ is isomorphic to $\mathbf{Z}A_\infty$ or a tube, that is $\mathbf{Z}A_\infty/\langle\tau^n\rangle$.*

Proof. As remarked above, the arguments in sections 2, 3 and 4 in [U2] can be still applied. In particular, if the modules in $\mathcal{A}(M)$ are N-projective, then the conclusions of 3.5, 3.7, 3.8 and 3.9 of [U2] yield (i). If some direct summand of modules in $\mathcal{A}(M)$ is not N-projective, then the arguments in 4.1 and 4.2 of [U2] almost give (ii). Here we say "almost" because in the proof of 4.2 of [U2], only the D_∞ case is excluded in order to conclude that the tree class of Γ is A_∞. This works since we assume there that k is algebraically closed. However, in the present situation, we have to exclude also the case of B_∞, since this is the only remaining case where $\mathcal{A}(M)$ has an indecomposable (modulo projectives) middle term. Assume that M lies at the end of an AR component with tree class B_∞. Then we have AR sequences

$$\begin{aligned}\mathcal{A}(M) &: 0 \to \tau M \to X \oplus F \to M \to 0, \quad \text{and}\\ \mathcal{A}(X) &: 0 \to \tau X \to Y \oplus 2\tau M \oplus F' \to X \to 0,\end{aligned}$$

where X and Y are non-projective indecomposable kG-modules and F and F' are projective or zero. Note that we are considering the case where X is not N-projective. Hence $\mathcal{A}(X)_N$ splits and we have $Y_N \oplus 2b(M)\tau V \cong X_N \oplus \tau X_N$ modulo projectives. On the other hand, considering $\mathcal{A}(M)_N$, 2.6 of [U2] implies that $X_N \cong a(M)\mathcal{M}(V) \oplus (b(M) - a(M))(V \oplus \tau V)$ modulo projectives, where $\mathcal{M}(V)$ is the middle term of $\mathcal{A}(V)$. Since $\mathcal{M}(V)$ and $\mathcal{M}(\tau V)$ do not have τV as a direct summand and since modules in a B_∞-component are not periodic, we have $2b(M) \leq 2(b(M) - a(M))$. But this gives $a(M) \leq 0$, a contradiction.

Q.E.D.

§2. Modules over dihedral and semidihedral group algebras

We first consider a semidihedral group G of order 2^n. Here $n \geq 4$. For a filed k of characteristic 2, the group algebra kG is tame and $\Gamma_s(kG)$ has non-periodic components of type only of A_∞^∞ and D_∞. (See [E3].) Let A be a k-algebra generated by two elements a and b with the relations

$$a^3 = b^2 = a^2 - b(ab)^{2^{n-2}-1} = 0.$$

In [BD], Bondarenko and Drozd claim the following. Since we can not find a literature which describes an explicit isomorphism, we give it here.

Lemma 2.1. *Let k be a perfect field of characteristic 2 and G a semidihedral group of order 2^n, where $n \geq 4$. Then we have a k-algebra isomorphism $kG/\mathrm{soc}kG \cong A$.*

Proof. Write $G = \langle x, y | x^{2^{n-1}} = y^2 = 1, yxy = x^{-1+2^{n-2}} \rangle$, and define u in kG by

$$u = x^{2^{n-2}-2} + x^{2^{n-2}-3} + \cdots + x^2 + x + 1 = (x-1)^{2^{n-2}-1} + x^{2^{n-2}-1}.$$

Then, $u^{2^{n-2}} = x^{2^{n-2}}$, $(x-1)u = x^{2^{n-2}-1} - 1 = yxy - 1$, and

$$(1) \quad u - 1 = (x-1)^{2^{n-2}-1} + x^{2^{n-2}-1} + 1 = (x-1)^{2^{n-2}-1} + (x-1)u.$$

We also have

$$\begin{aligned}(2) \qquad (uy-1)(x-1) &= u(yxy-1)y - (x-1) \\ &= u(x-1)uy + (x-1) = (x-1)(u^2y-1).\end{aligned}$$

Let

$$\alpha = (uy-1) + (x-1)^{2^{n-1}-3}(y-1), \quad \beta = y-1, \quad \text{and} \quad \hat{G} = \sum_{g \in G} g.$$

Then, $\beta^2 = 0$. Moreover, $\alpha^2 = (x-1)^{2^{n-1}-2}(y-1)$, since we have

$$(3) \qquad (uy-1)^2 = (x-1)^{2^{n-1}-1},$$

$$(4) \qquad (uy-1)(x-1)^{2^{n-1}-3}(y-1) = \hat{G},$$

$$\begin{aligned}(5) \qquad &(x-1)^{2^{n-1}-3}(y-1)(uy-1) \\ &= \hat{G} + (x-1)^{2^{n-1}-1} + (x-1)^{2^{n-1}-2}(y-1),\end{aligned}$$

$$(6) \qquad ((x-1)^{2^{n-1}-3}(y-1))^2 = 0.$$

(6) is easy to show. For (3), note that $(uy)^2$ is equal to

$$\begin{aligned}
&u(yuy) = u((yxy-1)^{2^{n-2}-1} + x) = u((x-1)^{2^{n-2}-1}u^{2^{n-2}-1} + x) \\
&= (x-1)^{2^{n-2}-1}u^{2^{n-2}} + ux \\
&= (x-1)^{2^{n-2}-1}x^{2^{n-2}} + (x-1)^{2^{n-2}-1}x + x^{2^{n-2}} \\
&= (x-1)^{2^{n-1}-1} + 1.
\end{aligned}$$

The left hand side of (4) is equal to $(x-1)^{2^{n-1}-4}(uy-1)(x-1)(y-1)$ as $(x-1)^{2^{n-1}-4}$ is central in kG. Then (4) can be seen by using (2). (5) is proved by using (1) and the following. (We use also (3) above.)

$$\begin{aligned}
(y-1)(uy-1) &= (uy-1)(y-1) + (yuy-u) \\
&= (u-1)(y-1) + (1-u^2)u^{-1} + (x-1)^{2^{n-1}}.
\end{aligned}$$

From $\alpha^2 = (x-1)^{2^{n-1}-2}(y-1)$, we also obtain $\alpha^3 = \hat{G}$.

Finally, we claim that $\beta(\alpha\beta)^{2^{n-2}-1} = \alpha^2 + \hat{G}$. Note first that $\alpha\beta$ equals to $(uy-1)(y-1) = (u-1)(y-1)$. Thus,

$$\begin{aligned}
\beta(\alpha\beta) &= (y-1)(u-1)(y-1) = (yuy-u)(y-1) \\
&= (u^{-1}-u)(y-1) + \hat{G} = (u^2-1)u^{-1}(y-1) + \hat{G},
\end{aligned}$$

and by using induction, we obtain

$$\beta(\alpha\beta)^{2^{n-2}-1} = (u^2-1)^{2^{n-2}-1}u^{-2^{n-2}+1}(y-1).$$

Now by (1) we have

$$\begin{aligned}
\beta(\alpha\beta)^{2^{n-2}-1} &= (x-1)^{2^{n-1}-2}u^{-2^{n-2}+1}(y-1) = (x-1)^{2^{n-1}-2}x(y-1) \\
&= (x-1)^{2^{n-1}-2}(y-1) + \hat{G} = \alpha^2 + \hat{G}.
\end{aligned}$$

Since $\alpha \equiv uy - 1 \equiv (u-1) + (y-1) \equiv (x-1) + (y-1)$ modulo $(J(kG))^2$, the two elements α and β generate kG. Note also that sockG is generated by $\hat{G}$ over k. Define a map $\varphi : A \to kG/\mathrm{soc}kG$ by $\varphi(a) = \alpha + \mathrm{soc}kG$ and $\varphi(b) = \beta + \mathrm{soc}kG$. Then, the above computations show that φ is a well defined k-algebra isomorphism. Q.E.D.

If $|k| \geq 3$, Crawley-Boevey gives a description of indecomposable A-modules in [CB]. Thus it gives a classification of indecomposable modules over semidihedral group algebras. We now give the following remark.

Lemma 2.2. *Let k be a perfect field of characteristic 2 and G a semidihedral group of order 2^n, where $n \geq 4$. Then all the non-periodic indecomposable kG-modules are absolutely indecomposable.*

Proof. Since k is perfect, by VII.6.9 of [HB], it suffices to prove that $M \otimes_k k'$ is indecomposable for any indecomposable kG-module M and any extension k' of k. In the classification of indecomposable kG-modules in [CB], it is required that k has at least 3 elements. However, if this is the case, then the classification is exactly the same in all the cases. Hence, if $|k| \geq 3$, the assertion holds. Now suppose that $k = GF(2)$. Let $k_1 = GF(2^2)$, $k_2 = GF(2^3)$ and $k_3 = GF(2^6)$. Let M be an indecomposable kG-module. Let

$$M \otimes_k k_1 = M_1 \oplus \cdots \oplus M_r \quad \text{and} \quad M \otimes_k k_2 = M_1' \oplus \cdots \oplus M_s'$$

be decompositions of $M \otimes_k k_1$ and $M \otimes_k k_2$ into direct sums of indecomposable k_1G-modules and k_2G-modules, respectively. Then $M_1, \ldots, M_r$ are $Gal(k_1/k)$-conjugates and $M_1', \ldots, M_s'$ are $Gal(k_2/k)$-conjugates. Here r is 1 or 2 and s is 1 or 3, since $Gal(k_1/k)$ and $Gal(k_2/k)$ are cyclic of order 2 and 3, respectively. However, we know that M_i's and M_j''s are absolutely indecomposable, and thus

$$(M_1 \otimes_{k_1} k_3) \oplus \cdots \oplus (M_r \otimes_{k_1} k_3) \quad \text{and} \quad (M_1' \otimes_{k_2} k_3) \oplus \cdots \oplus (M_s' \otimes_{k_2} k_3)$$

are both indecomposable direct sum decompositions of $M \otimes_k k_3$. Hence, we have $r = s = 1$. Therefore, $M \otimes_k k_1$ is indecomposable, and it yields that M is absolutely indecomposable. Q.E.D.

Remark 2.3. The assertions in Lemma 2.2 can be proved also in the case where G is a dihedral 2-group, by using the classification of indecomposable kG-modules.

The following is a key result in this paper.

Proposition 2.4. *Let k be a perfect field, G a dihedral or semidihedral 2-group, and σ an automorphism of G sending each involution in G into its G-conjugate. Then every non-periodic indecomposable kG-module is σ-invariant. In particular, every non-periodic indecomposable module over a semidihedral group is invariant under any automorphism of the group.*

Proof. By Lemma 2.2 and Remark 2.3, we may assume that k is algebraically closed. Let M be a non-periodic indecomposable kG-module. If G is a four group, the result holds clearly. We assume that $|G| > 4$. Thus M lies in a component isomorphic to $\mathbf{Z}A_\infty^\infty$ or $\mathbf{Z}D_\infty$.

Consider first the case where M has at least two predecessors in the AR component. Take an indecomposable kG-module X and an irreducible map $f : X \to M$. Suppose that f is surjective. If this is not the case, we take its dual. Let U be the kernel of f. We use the argument in 3.2 of [E4]. There exists a shifted subgroup H of order 2 such that U_H is not projective. Let $V = k_H^G$, the module induced from the trivial module of H. It is concluded that $\dim V \leq |G|/2 \leq \dim U'$, where U' is U or ΩU. Moreover, it is shown that there is no monomorphism from U' to V or $U' \cong V$. Furthermore, in Case 1 (ℓ.-5 on p.155 of [E4]) a contradiction is derived when it is assumed that there is no monomorphism from U' to V. Consequently, $U \cong V$ holds and we have $|G|/2 = \dim V = \dim U$.

Notice that kH is a subalgebra of kN for some elementary abelian subgroup N of order 4 in G. Any such an N is generated by the central involution and a non-central involution of G. Thus from the assumption on σ, there exists $g \in G$ such that $H^\sigma = H^g$. Hence V is σ-invariant. Let $h_1 : V \to V^\sigma$ be an isomorphism. Consider the following commutative diagram. Here, by 1.1 of [E4], either h_1 or h_1^{-1} lifts to a map between X and X^σ, and we may assume that h_1 does.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V & \xrightarrow{g} & X & \xrightarrow{f} & M & \longrightarrow & 0 \\
 & & h_1\downarrow & & h_2\downarrow & & h_3\downarrow & & \\
0 & \longrightarrow & V^\sigma & \xrightarrow{g^\sigma} & X^\sigma & \xrightarrow{f^\sigma} & M^\sigma & \longrightarrow & 0
\end{array}$$

Since σ has finite order and since X and M are indecomposable, h_3 must be an isomorphism by Fitting's lemma.

Next consider the case where M lies at the end of a D_∞-component. There are indecomposable modules X, Y, Z and irreducible maps $f : X \to M$, $f' : X \to Y$ and $f'' : X \to Z$, where M and Y have only one predecessor. We already know that X and Z are σ-invariant by the above. Thus M^σ is either M or Y. We prove that $M^\sigma \cong M$ by showing that $\dim M \neq \dim Y$. By applying the argument in the first paragraph to the map $f'' : X \to Z$, we have $|G|/2 \equiv \dim X - \dim Z \mod |G|$. Moreover, considering $\mathcal{A}(M)$ and $\mathcal{A}(\tau^{-1}X)$, we have $\dim X \equiv 2\dim M \mod |G|$ and $2\dim X \equiv \dim M + \dim Y + \dim Z \mod |G|$. Hence $\dim M - \dim Y \equiv |G|/2 \mod |G|$. Therefore, we have $\dim M \neq \dim Y$ as desired. Q.E.D.

§3. Proof of Theorems

We prove Theorem 2 first.

Proof of Theorem 2. By the very final remark in [OU2], we may assume that $p=2$, and G is a 2-group. Moreover, it suffices to consider the case where $vx(X_1) = vx(X_2) < vx(X_3) = vx(X_4) = \dots$ in the notation of the theorem. If this is the case, then by Theorem B of [E2], G has a normal subgroup H with $|G:H| = 2$ and $\Gamma_s(kH)$ has an AR component Θ isomorphic to $\mathbf{Z}D_\infty$. Furthermore, the two ends in Θ are G-conjugate. Now, by Theorem 4 of [E4], H must be semidihedral. However, this is impossible by Proposition 2.4. Q.E.D.

Proof of Theorem 1. Let $\bar{k}$ be the algebraic closure of k. Suppose that $\Gamma_s(kG)$ has an AR component Γ isomorphic to either $\mathbf{Z}B_\infty$ or $\mathbf{Z}C_\infty$. Let D be a defect group of the block of G to which the modules in Γ belong. By Theorem B, all the modules have the same vertex Q. Let M be in Γ. Let $\Gamma_1, \cdots, \Gamma_r$ be connected components of $\Gamma_s(\bar{k}G)$ containing indecomposable direct summands of $M \otimes_k \bar{k}$. All the modules in Γ_i have also vertex Q by III.4.14 of [F], and they belong to blocks whose defect group is D by III.9.10 of [F]. Now by Lemma 1.2, Γ_i's are $Gal(\bar{k}/k)$ conjugate, and by Lemma 1.3, $\Gamma_i \cong \mathbf{Z}A_\infty^\infty$ if $\Gamma \cong \mathbf{Z}B_\infty$, and $\Gamma_i \cong \mathbf{Z}D_\infty$ if $\Gamma \cong \mathbf{Z}C_\infty$. Hence D is either dihedral or semidihedral. ([E4]) We will show that $\Gamma \cong \mathbf{Z}B_\infty$ and D is dihedral. The proof consists of several lemmas.

Lemma 3.1. *We may assume that Q is normal in G. Moreover, Q is dihedral or semidihedral and we have $|D:Q| \leq 2$.*

Proof. By [K1] there is a quiver monomorphism from Γ to a component of $\Gamma_s(kN_G(Q))$ which preserves vertices. In particular, M is mapped to its Green correspondent. Since $\mathbf{Z}B_\infty$ and $\mathbf{Z}C_\infty$ can not be a proper subquiver of any AR component of the stable AR quiver, the image of the monomorphism must be a connected component. Moreover, the Green correspondent of M lies in a block of $N_G(Q)$ whose defect group is also dihedral or semidihedral. Hence, it follows from the same argument as in the second paragraph of 4.2 in [E4] (p.158) that the $kN_G(Q)$-modules in the image lie in a block whose defect group is D. Thus, we may assume that Q is normal in G. The last statement holds since Q is a non-cyclic normal subgroup of D. Q.E.D.

Let V be a Q-source of M and Θ the AR component of $\Gamma_s(kQ)$ containing V. Let N be the set of elements in G those which induce automorphisms of Q by conjugation sending each involution in Q into its Q-conjugate.

Lemma 3.2. *It follows that Q is a dihedral group of order at least*

8, $\Theta \cong \mathbf{Z}A_\infty^\infty$, Θ is G-invariant, and that any element in $G \setminus N$ induces a reflection on Θ.

Proof. Recall that all the modules in Γ are Q-projective. If Q is a four group, then kQ has two τ-orbits of non-periodic indecomposable modules. Thus Γ has only finitely many τ-orbits, a contradiction. Thus Q is not a four group. By Lemma 3.1, we have $|G : N| \leq 2$, and $G = N$ if Q is semidihedral. Recall that every module in Θ is N-invariant by Proposition 2.4. Thus, if $G = N$, then it would follow from Lemma 1.4 that $\Gamma \cong \Theta$ or Γ has tree class A_∞, a contradiction. Hence $G \neq N$, and in particular, Q must be dihedral of order at least 8. This implies also that $\Theta \cong \mathbf{Z}A_\infty^\infty$. ([E3]) Moreover, if all the modules in Θ are G-invariant, or if $N = I_G(\Theta)$, the inertia group of Θ in G, then Lemma 1.4 and [K2] derive a contradiction similarly. Thus, Θ is G-invariant but some modules in Θ are not G-invariant. This means that every element in $G \setminus N$ induces a reflection on Θ by Lemma 1.1. Q.E.D.

Let $H = QC_G(Q)$. Then H is a normal subgroup of G contained in N. Let X be an indecomposable kH-module such that M is isomorphic to a direct summand of X^G and that the source of X is V, and let Λ be a connected component of $\Gamma_s(kH)$ containing X. Moreover, let b be a block of kH containing X.

Lemma 3.3. *It follows that $\Lambda \cong \Theta \cong \mathbf{Z}A_\infty^\infty$. Moreover, $b \cong kQ \otimes_k A$, where A is the full matrix ring over some finite extension field of k. In particular, Q is a defect group of b. Furthermore, we may assume that b is G-invariant.*

Proof. Again by Lemma 1.4, $\Lambda \cong \Theta \cong \mathbf{Z}A_\infty^\infty$ and all the modules in Λ are Q-projective. The results follow from the argument in the proof of 4.1 of [E4]. The last statement holds by [K2]. Q.E.D.

We fix an isomorphism $b \cong kQ \otimes_k A$ in Lemma 3.3 and identify these two algebras. Let S be the unique (up to isomorphisms) simple A-module. Then, since b is G-invariant, so is S. Moreover, by Lemma 3.3, there is an equivalence between modkQ and mod$b$, by which a kQ-module U corresponds to $U \otimes_k S$. Let $A \otimes_k \overline{k} = \oplus_i A_i$ be the decomposition into a direct sum of simple algebras over $\overline{k}$. Accordingly, we have $S \otimes_k \overline{k} = \oplus_i S_i$ and $b \otimes_k \overline{k} \cong \overline{k}Q \otimes_{\overline{k}} (A \otimes_k \overline{k}) = \oplus_i(\overline{k}Q \otimes_{\overline{k}} A_i)$, where S_i is a simple A_i-module. For each i, let $b_i = \overline{k}Q \otimes_{\overline{k}} A_i$. Then b_i is a block of $\overline{k}H$ and its defect group is Q by III. 9.10 of [F]. Moreover, there is also an equivalence between mod $\overline{k}Q$ and modb_i, by which a $\overline{k}Q$-module W corresponds to $W \otimes_{\overline{k}} S_i$. Let U be a kQ-module and suppose that the b-module $U \otimes_k S$ lies in Λ. Since U is not periodic, $U \otimes_k \overline{k}$ is indecomposable by Remark 2.3. Hence tensoring the modules in Λ with $\overline{k}$, the AR component Λ decomposes into a disjoint union $\cup_i \Lambda_i$. Here Λ_i is an AR component of $\Gamma_s(\overline{k}H)$ and isomorphic to $\mathbf{Z}A_\infty^\infty$ by Lemma 3.3. Write $X \otimes_k \overline{k} = \oplus_i X_i$, where X_i is the direct summand belonging to b_i. Then X_i lies in Λ_i. There is an indecomposable direct summand M_1 of $M \otimes_k \overline{k}$ such that $M_1 \cong X_1^G$. Without loss of generality, we may assume that M_1 lies in Γ_1.

Lemma 3.4. *It follows that $D = Q$, and the conclusions in Theorem 1 hold.*

Proof. Suppose that $D \neq Q$. Considering all the possibilities for Q and D, it follows that $G = DN$ and $D \cap N = Q$. Since G/H is a 2-group, by V.5.15 and V.5.16 of [NT], we may assume that DH is the inertia group $I_G(b_1)$ of b_1 in G. In particular, $DH = I_G(\Lambda_1)$. Without loss of generality, we may assume that V is D-invariant, that is, an element of $DH \setminus H$ induces a reflection on Θ with respect to the τ-orbit of V. Then, X_1 is D-invariant from the above argument. In fact, we have $I_G(X_1) = DH$. Now by 2.5 of [U2], the middle term of $\mathcal{A}(M_1)$ has a direct summand whose vertex is D, a contradiction. Therefore, $Q = D$. Finally, we recall that, if a defect group is dihedral, then D_∞ does not occur. Thus $\Gamma_i \cong \mathbf{Z}A_\infty^\infty$ and we can conclude that $\Gamma \cong \mathbf{Z}B_\infty$ by Lemma 1.3. Q.E.D.

§4. Examples

The following gives an example of a group G such that $\Gamma_s(kG)$ has a component isomorphic to $\mathbf{Z}B_\infty$. It is due to the first author ([O2]).

Let k be a perfect field of characteristic 2 which does not contain a cube root of unity. Let n be an integer with $n \geq 3$ and G a group generated by x, y , z and t with relations $x^2 = y^2 = z^3 = t^2 = 1$ and

$$(xy)^{2^{n-1}} = 1, xz = zx, yz = zy, tx = yt, ty = xt, tz = z^2t.$$

Then $|G| = 2^{n+1}3$ and G has normal subgroups $D = \langle x, y\rangle$ and $C = \langle z\rangle$ with $D \cap C = \{1\}$. Note that D is a dihedral group of order 2^n and C is a cyclic group of order 3. Let $H = D \times C$. Then G is a semidirect product of H and $\langle t\rangle$. Let σ be a Galois automorphism such that σ interchanges the two cube roots of unity. Since k does not contain a cube root of unity, kC has the unique (up to isomorphisms) simple module T of dimension 2. It is G-invariant, and since G/C is a 2-group, T can be extended to a simple kG-module S. Moreover, it follows that $T \otimes \overline{k} = T_1 \oplus T_2$, where T_1 and T_2 are non-isomorphic simple $\overline{k}C$-modules with $T_1^\sigma = T_2$. However, $S \otimes \overline{k}$ is a simple $\overline{k}G$-module, since $T_1^t = T_2$ and $(S \otimes \overline{k})_C \cong T_1 \oplus T_2$.

Let $X = (x-1)\overline{k}D/s\overline{k}D$ and $Y = (y-1)\overline{k}D/s\overline{k}D$, where s is the sum of all the elements in D. Then X and Y are non-projective indecomposable $\overline{k}D$-modules and we have $X^t = Y$ and $Y^t = X$, but X and Y are invariant under the Galois actions. (See Remark 2.3.) It is known that X and Y lie in the same connected component of $\Gamma_s(\overline{k}D)$ which is isomorphic to $\mathbf{Z}A_\infty^\infty$. (See [W] or [E2].) Now $X \otimes_{\overline{k}} T_1$, $Y \otimes_{\overline{k}} T_1$, $X \otimes_{\overline{k}} T_2$ and $Y \otimes_{\overline{k}} T_2$ are non-isomorphic indecomposable $\overline{k}H$-modules, and we have $(X \otimes_{\overline{k}} T_1)^t = Y \otimes_{\overline{k}} T_2$, $(Y \otimes_{\overline{k}} T_1)^t = X \otimes_{\overline{k}} T_2$, $(X \otimes_{\overline{k}} T_1)^\sigma = X \otimes_{\overline{k}} T_2$ and $(Y \otimes_{\overline{k}} T_1)^\sigma = Y \otimes_{\overline{k}} T_2$. Of course, $X \otimes_{\overline{k}} T_1$ and $Y \otimes_{\overline{k}} T_1$ lie in the same AR component Θ_1, and $X \otimes_{\overline{k}} T_2$ and $Y \otimes_{\overline{k}} T_2$ lie in the same AR component Θ_2. Both Θ_1 and Θ_2 are isomorphic to $\mathbf{Z}A_\infty^\infty$. Let $Z_1 = (X \otimes_{\overline{k}} T_1)^G = (Y \otimes_{\overline{k}} T_2)^G$ and $Z_2 = (X \otimes_{\overline{k}} T_2)^G = (Y \otimes_{\overline{k}} T_1)^G$. Then Z_1 and Z_2 are non-isomorphic indecomposable $\overline{k}G$-modules, and we have $Z_1^\sigma = Z_2$. Moreover, we have $\Theta_1^t = \Theta_2$ and Z_1 and Z_2 lie in the same AR component Γ isomorphic to $\mathbf{Z}A_\infty^\infty$.

Finally, we recall that $\Omega(\overline{k}) \otimes_{\overline{k}} T_i$ lies in Θ_i for $i = 1, 2$. Here $\Omega(\overline{k})$ is the Heller translate of the trivial $\overline{k}D$-module $\overline{k}$, i.e., the kernel of the projective cover of $\overline{k}$. We have $(\Omega(\overline{k}) \otimes_{\overline{k}} T_1)^t = \Omega(\overline{k}) \otimes_{\overline{k}} T_2$ and $(\Omega(\overline{k}) \otimes_{\overline{k}} T_1)^\sigma = \Omega(\overline{k}) \otimes_{\overline{k}} T_2$. Therefore $(\Omega(\overline{k}) \otimes_{\overline{k}} T_1)^G$ lies in Γ and is σ-invariant. Since $Z_1^\sigma = Z_2$ and since Z_1 and Z_2 lie in Γ, from Lemmas 1.1, 1.2 and 1.3, it follows that the tree class of the AR component containing $\Omega(S)$ must be B_∞.

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