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The calculation of the character of Moonshine VOA

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§1. Introduction

In Miyamoto [M3], [M6] and Dong-Griess-Höhn [DGH], they described the structure of the Moonshine VOA $\mathbf{V}^\natural$ by using two binary codes $D^\natural, S^\natural$ and Ising models $L(\frac{1}{2},0),\ L(\frac{1}{2},\frac{1}{2}),\ L(\frac{1}{2},\frac{1}{16})$.

The purpose of this note is to calculate the character of $\mathbf{V}^\natural$ and the Thompson series of two involutions of Monster $\mathrm{Aut}(\mathbf{V}^\natural)$ ($2A, 2B$-involutions of Monster) explicitly by following the descriptions of $\mathbf{V}^\natural$ in [M3],[M6] and [DGH]. As is well known (cf.[CN]), these are equal to

$$j(z) - 744,\ \left(\frac{\eta(z)}{\eta(2z)}\right)^{24} + 2^{12}\left(\frac{\eta(2z)}{\eta(z)}\right)^{24} + 24,\ \left(\frac{\eta(z)}{\eta(2z)}\right)^{24} + 24$$

respectively, where $j(z)$ is the well known elliptic modular function and $\eta(z)$ is Dedekind's η-function. Also see a remark at the end of §4 for the calculations of Thompson series for some other elements. Finally, in §5, we will mention a little bit about VOA of "Reed Müller type".

§2. Ising models

2.1. Virasoro Algebra

An infinite dimensional Lie algebra **Vir** having a basis $\{L(m)\ (m \in \mathbf{Z}),\ \mathbf{c}\}$ is called Virasoro algebra if they satisfies

$$[L(m), \mathbf{c}] = 0,\ [L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}\delta_{m+n,0}\mathbf{c}.$$

Let $L(c,h)$ be an irreducible module of **Vir** with central charge $c\ (\in \mathbf{C})$ and highest weight $h\ (\in \mathbf{C})$. Namely, there exists a vector $v \in$

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$L(c,h)$ such that $L(n)v = 0$ $(n > 0)$, $L(0)v = hv$, $\mathbf{c}v = cv$ and $L(c,h)$ is spanned by $L(-n_1)L(-n_2)\cdots L(-n_r)v$ $(n_1 \geq n_2 \geq \cdots \geq n_r > 0$).

As is easily seen from the commutator relations between the $L(n)$, $L(-n_1)L(-n_2)\cdots L(-n_r)v$ is an eigen vector of $L(0)$ with an eigen value $h+n_1+n_2+\cdots+n_r$ and so $L(c,h)$ is a direct sum of eigenspaces $\mathbf{V}_{h+n}$ of $L(0)$ with eigen value $h+n$ $(0 \leq n \in \mathbf{Z})$ $:L(c,h) = \oplus_{n\geq 0}\mathbf{V}_{h+n}$.
Now define a q-series

$$ch(L(c,h)) = \sum_{n\geq 0}(dim\ \mathbf{V}_{h+n})q^{h+n}.$$

This series is called the character of $L(c,h)$. More generally, for a graded space $\mathbf{U} = \oplus_{n\in\mathbf{Q}}U_n$, a q-series $ch(\mathbf{U}) = \sum_{n\in\mathbf{Q}}(dim\ U_n)q^n$ is called the character of a graded space $\mathbf{U}$.

An important thing is that, if $h = 0$, $L(c,0)$ has a structure of **VOA** . Such VOA is called Virasoro VOA and is the most fundamental example of VOA.

In the following, we will consider the case $c = \frac{1}{2}$.

2.2. Ising models

2.2.1. *Irreducible modules of* $L(\frac{1}{2},0)$. As for modules of **VOA** $L(\frac{1}{2},0)$, the following is known:

(2,1) *Any modules of* **VOA** $L(\frac{1}{2},0)$ *is completely reducible and* **VOA** $L(\frac{1}{2},0)$ *has just three irreducible modules* $L(\frac{1}{2},0)$, $L(\frac{1}{2},\frac{1}{2})$, $L(\frac{1}{2},\frac{1}{16})$. (cf. [DMZ])

Let $\mathbf{T}_n$ be the tensor product $L(\frac{1}{2},0)\otimes\cdots\otimes L(\frac{1}{2},0)$ of n copies of $L(\frac{1}{2},0)$. Then, by a general theory of **VOA**,

(2,2) $\mathbf{T}_n$ *has VOA-structure and any module of* $\mathbf{T}_n$ *is completely reducible. Also,* $\mathbf{T}_n$ *has just* 3^n *irreducible modules*

$$L(h_1,h_2,\cdots,h_n) = L(\frac{1}{2},h_1)\otimes L(\frac{1}{2},h_2)\otimes\cdots\otimes L(\frac{1}{2},h_n)\ (h_i = 0,\frac{1}{2}\ or\ \frac{1}{16})$$

2.2.2. *Characters of* $L(\frac{1}{2},h)$. As for the characters of $L(\frac{1}{2},h)$ $(h = 0,\frac{1}{2}\ or\ \frac{1}{16})$, the followings are known: Let

$$q_+ = \prod_{n=0}^{\infty}(1+q^{n+\frac{1}{2}}),\ q_- = \prod_{n=0}^{\infty}(1-q^{n+\frac{1}{2}}),\ q_0 = \prod_{n=1}^{\infty}(1+q^n).$$

Then we have
$ch(L(\frac{1}{2},0)) = \frac{1}{2}(q_+ + q_-)$, $ch(L(\frac{1}{2},\frac{1}{2})) = \frac{1}{2}(q_+ - q_-)$, $ch(L(\frac{1}{2},\frac{1}{16})) = q^{\frac{1}{16}}q_0$.

The characters of $L(h_1, h_2, \cdots, h_n) = \otimes^n L(\frac{1}{2}, h_i)$ is

$$ch(L(h_1, h_2, \cdots, h_n)) = \prod_{i=1}^{n} ch(L(\frac{1}{2}, h_i)).$$

Let $\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n)$ $(q = exp(2\pi iz))$ be Dedekind's η-function. Then it is clear that

$$q_0 = q^{-\frac{1}{24}} \frac{\eta(2z)}{\eta(z)}, \quad q_+ q_- = q^{\frac{1}{24}} \frac{\eta(z)}{\eta(2z)}, \quad q_0 q_+ q_- = 1.$$

Also we have

$$16 q^{\frac{1}{2}} q_0^8 = q_+^8 - q_-^8 \text{ (Jacobi).}$$

Furthermore, for the calculations of the character of some VOA, it is convenient to note

$$j(z)^{\frac{1}{3}} = 2^8 \left(\frac{\eta(2z)}{\eta(z)}\right)^{16} + \left(\frac{\eta(z)}{\eta(2z)}\right)^{8},$$

where $j(z)$ is the well known elliptic modular function.

2.2.3. *Fusion rules.* Let Λ_n be the set of all irreducible modules of $\mathbf{T}_n = \otimes^n L(\frac{1}{2}, 0)$:$\Lambda_n = \{L(h_1, h_2, \cdots, h_n) \mid h_i = 0, \frac{1}{2} \text{ or } \frac{1}{16}\}$.
Let a binary word $\delta = (\delta_1, \delta_2, \cdots, \delta_n) \in \mathbf{F}_2^n$ of length n act on Λ_n as follows:For $\mathbf{F}_2^n \ni \delta = (\delta_1, \cdots, \delta_n)$

$$L(h_1, h_2, \cdots, h_n) \longrightarrow L(h_1 + \frac{\delta_1}{2}, h_2 + \frac{\delta_2}{2}, \cdots, h_n + \frac{\delta_n}{2}).$$

Here the sum " $h_i + \frac{\delta_i}{2}$" is defined as follows:

$$\frac{1}{2} + 0 = \frac{1}{2}, \ \frac{1}{2} + \frac{1}{2} = 0, \ \frac{1}{16} + 0 = \frac{1}{16}, \ \frac{1}{16} + \frac{1}{2} = \frac{1}{16}$$

These come from well known fusion rules of Ising models which are the most important in the theory of Framed VOA described in the next section. Note that every orbit of the action of $\mathbf{F}_2^n$ on Λ_n is the set of $L(h_1, h_2, \cdots, h_n)$ which have $h_i = \frac{1}{16}$ in the same position.

§3. Framed *VOA*

We will consider a simple **VOA** $\mathbf{V} = \oplus_{n=0}^{\infty} \mathbf{V}_n$ satisfying the following conditions:

(3.1) *$dim \mathbf{V}_0 = 1$, i.e. $\mathbf{V}_0 =< \mathbf{1} >$ where $\mathbf{1}$ is the vacuum of $\mathbf{V}$,*

(3.2) $\mathbf{V}$ *contains* $\mathbf{T}_n = \otimes^n L(\frac{1}{2}, 0)$ *as a subVOA which has Virasoro element in common.*
Recently VOA of this type is called Framed VOA. Viewing $\mathbf{V}$ as a $\mathbf{T}_n$-module, the complete reducibility (2,2) of $\mathbf{T}_n$ yields the decomposition

$$\mathbf{V} \simeq \oplus_{(h_1,h_2,\cdots,h_n)} a_{(h_1,h_2,\cdots,h_n)} L(h_1, h_2, \cdots, h_n) \text{ (as } \mathbf{T}_n\text{-module)}$$

where the $a_{(h_1,h_2,\cdots,h_n)}$ express multiplicity. This decomposition yields an isomorphism as graded space by the condition (3.2) and so we have

$$ch(\mathbf{V}) = \sum_{n=0}^{\infty} (dim \mathbf{V}_n) q^n = \sum_{(h_1,h_2,\cdots,h_n)} a_{(h_1,h_2,\cdots,h_n)} ch(L(h_1, h_2, \cdots, h_n)).$$

Thus , if we know the multiplicities $a_{(h_1,h_2,\cdots,h_n)}$, the character of $\mathbf{V}$ can be written down immediately by using the characters of Ising models. Note that $h_1+h_2+\cdots+h_n$ is a nonnegative integer, because the weights of VOA are integers.

Now Miyamoto [M3], [M6] and Dong-Griess-Höhn [DGH] showed that the above decomposition of $\mathbf{V}$ has a "2-structure " described in terms of two binary even codes S and D which will be explained in the following. However we will mention just the results and the proofs of the statements will be omitted. For the proofs, we refer the readers to [M1], [M4], [M6] (or [DGH]) together with [DM].

3.1. Code S

For $\mathbf{h} = (h_1, h_2, \cdots, h_n)$ ($h_i = 0, \frac{1}{2}$ or $\frac{1}{16}$), we assign a binary word $\tilde{\mathbf{h}} = (h'_1, h'_2, \cdots, h'_n) \in \mathbf{F}_2^n$ as follows:

$$h'_i = \begin{cases} 1 & \text{if } h_i = \frac{1}{16} \\ 0 & \text{if } h_i = 0 \text{ or } \frac{1}{2}. \end{cases}$$

Thus a word $\tilde{\mathbf{h}}$ shows positions in which the $h_i = \frac{1}{16}$ appear. Let

$$S = \{\tilde{\mathbf{h}} \mid a_{\mathbf{h}} = a_{(h_1,h_2,\cdots,h_n)} \neq 0\}.$$

Then we have

(3,1,1) S *is a linear code.*

(3,1,2) $S \ni \alpha \implies 8|wt(\alpha)$, i.e.*the weight of every word of S is divisible by* 8.

(3,1,3) $\tilde{\mathbf{h}} = \tilde{\mathbf{h}'} \implies a_{\mathbf{h}} = a_{\mathbf{h}'}$, i.e. *the multiplicities $a(h_1, h_2, \cdots, h_n)$ of two $L(h_1, h_2, \cdots, h_n)$ coincide if $\frac{1}{16}$ appear in the same positions.*
Therefore, gathering the $L(h_1, h_2, \cdots, h_n)$ with $\frac{1}{16}$ in the same positions, we get the decomposition.

(3,1,4) $\mathbf{V}^\alpha = a_\alpha(\oplus_{\tilde{\mathbf{h}}=\alpha} L(h_1, h_2, \cdots, h_n))$, $\mathbf{V} = \oplus_{\alpha \in S} \mathbf{V}^\alpha$.
We will call this decomposition $\mathbf{T}_n$*-decomposition of* $\mathbf{V}$.

3.2. Code D

In this subsection, we will consider the $L(h_1, h_2, \cdots, h_n)$ with $\tilde{\mathbf{h}} = 0$. Thus we have $h_i = 0$ *or* $\frac{1}{2}$. For $\mathbf{h} = (h_1, h_2, \cdots, h_n)$ with $\tilde{\mathbf{h}} = 0$, assign a binary word $(2h_1, 2h_2, \cdots, 2h_n) \in \mathbf{F}_2^n$ and set

$$D = \{(2h_1, 2h_2, \cdots, 2h_n) \mid \tilde{\mathbf{h}} = 0 \text{ and } a_{\mathbf{h}} \neq 0\}.$$

Then we have

(3,2,1) *D is an even linear code and $D^{\perp} \supset S$* ,

(3,2,2) $\tilde{\mathbf{h}} = 0 \Longrightarrow a_{\mathbf{h}} = 1$.

Let

$$V^0 = \oplus_{\tilde{\mathbf{h}}=0} L(h_1, h_2, \cdots, h_n) = \oplus_{\delta \in D} L(\frac{\delta_1}{2}, \frac{\delta_2}{2}, \cdots, \frac{\delta_n}{2})$$

where $\delta = (\delta_1.\delta_2, \cdots, \delta_n)$.

Then we have

(3,2,3) $\mathbf{V}^0$ is a subVOA of $\mathbf{V}$ and the $\mathbf{V}^{\alpha}$ ($\alpha \in S$) are irreducible modules of $\mathbf{V}^0$.

3.3. The structure of $\mathbf{V}^{\alpha}$

$\mathbf{V}^0$ defined above is what is called Code VOA in a series of Miyamoto's papers [M2], [M4], [M5] and [M6]. In view of Miyamoto [M4], the multiplicities a_{α} ($\alpha \in S$) are described as follows by using D, S, α:

For $\alpha \in S$, set $D_{\alpha} = \{\delta \in D \mid supp(\delta) \subset supp(\alpha)\}$.

(3,3,1) *Let H_{α} be a maximal selforthogonal subcode of D_{α}. Then* $a_{\alpha} = [D_{\alpha} : H_{\alpha}]$.

Now consider the decomposition $\mathbf{V}^{\alpha} = a_{\alpha}(\oplus_{\tilde{\mathbf{h}}=\alpha} L(h_1, h_2, \cdots, h_n))$.

Then *what is the set of the $L(h_1, h_2, \cdots, h_n)$ which appears in the righthandside?*

In order to examine this set, recall the action of $\mathbf{F}_2^n$ on Λ_n defined in §2.2.3. For $\alpha \in S$, we put

$$\Lambda_n(\alpha) = \{L(h_1, h_2, \cdots, h_n) \mid \tilde{\mathbf{h}} = \alpha\}$$

and consider the action of D on $\Lambda_n(\alpha)$. Then we have that

the set of $\{L(h_1, h_2, \cdots, h_n)\}$ which appear in the above decomposition of $\mathbf{V}^{\alpha}$ is equal to an orbit (with integral weight) of the action of D on $\Lambda_n(\alpha)$.

Note that D_{α} is one-point stabilizer of this action of D on $\Lambda_n(\alpha)$. Therefore,

$$\textit{the number of orbits of the action of } D \textit{ on } \Lambda_n(\alpha) = \frac{2^{n-wt(\alpha)}}{[D : D_{\alpha}]}.$$

Thus we see that the decomposition of a Framed VOA as $\mathbf{T}_n$-module

$$\mathbf{V} \simeq \oplus_{(h_1,h_2,\cdots,h_n)} a_{(h_1,h_2,\cdots,h_n)} L(h_1,h_2,\cdots,h_n) \text{ (as } \mathbf{T}_n\text{-module)}$$

can be completely described by two binary codes D, S and the choice of an orbit of the action of D on $\Lambda_n(\alpha)$ for each $\alpha \in S$.

Remark. Now we are naturally led to a problem:

When two binary codes D, S satisfying (3,1,2) *and* (3,2,1) *are given, can we construct a VOA by choosing suitably an orbit* (*with integral weight*) *of the action of D on $\Lambda_n(\alpha)$ for each $\alpha \in S$?*

In [M5], [M6], Miyamoto showed that, under suitable conditions for D, S, Framed VOA can be constructed (for some such examples, see §5 of this note) and, in particular, starting from two special binary codes $D^\natural, S^\natural$ which are described in the next section, Moonshine VOA can be reconstructed.

§4. Moonshine VOA

Dong-Mason-Zhu [DMZ] showed that Moonshine VOA $\mathbf{V}^\natural$ constructed by Frenkel-Lepowsky-Meurman [FLM] satisfies the conditions (3.1), (3.2) in the beginning of the previous section for $n = 48$, and then Miyamoto [M3] and Dong-Greiss-Höhn [DGH] determined two codes D, S. In this section, these codes $D^\natural, S^\natural$ for $\mathbf{V}^\natural$ will be described and the character of $\mathbf{V}^\natural$ will be calculated by using $D^\natural, S^\natural$. Also Thompson series for two involutions of $\mathrm{Aut}(\mathbf{V}^\natural)$ will be calculated.

4.1. Codes $D^\natural, S^\natural$

Firstly we define two binary codes $D^\#, S^\#$ of length 16. Let $S^\#$ be a binary code generated by the following five words of length 16: (1^{16}), $(1^8 0^8)$, $((1^4 0^4)^2)$, $((1^2 0^2)^4)$, $((1.0)^8)$

In coding theory, $S^\#$ is known to be the 1st order Reed-Müller code $RM(4,1)$ of length 16. Let $D^\# = (S^\#)^\perp$ =(orthogonal complement of $S^\#$). $D^\#$ is known to be the 2nd order Reed-Müller code $RM(4,2)$.

The code $S^\natural$ is defined to be the set of words of length 48 which put three words of $S^\#$ in order as follows:

$$(\sigma,\sigma,\sigma), (\sigma,\sigma,\bar{\sigma}), (\sigma,\bar{\sigma},\sigma), (\bar{\sigma},\sigma,\sigma) \ \sigma \in S^\#, \ \bar{\sigma} = \sigma + (1^{16})$$

$S^\natural$ is a (48, 7, 16)-binary code and its weight enumerator is

$$x^{48} + 3x^{32}y^{16} + 120x^{24}y^{24} + 3x^{16}y^{32} + y^{48}.$$

Finally let $D^\natural = (S^\natural)^\perp$. Then $D^\natural$ is a $(48, 41, 4)$-binary code and when a word of $D^\natural$ is written in the shape like (ρ_1, ρ_2, ρ_3) $(\rho_i \in \mathbf{F}_2^{16})$, we have

$$(4.1.1) \quad \begin{aligned} &D^\natural \ni (\rho_1, \rho_2, \rho_3)(\rho_i \in \mathbf{F}_2^{16}) \\ &\Longleftrightarrow \rho_i \textit{ is an even word and } \rho_1 + \rho_2 + \rho_3 \equiv 0 \textit{ mod } D^\#. \end{aligned}$$

These $D^\natural, S^\natural$ are codes for Moonshine VOA $\mathbf{V}^\natural$.

4.2. $\mathbf{T}_{48}$-decomposition of $\mathbf{V}^\natural$

The following table gives some datas which are necessary for the description of $\mathbf{T}_{48}$-decomposition (3.1.4) of $\mathbf{V}^\natural$:

	$wt(\alpha)$	# of α	$\lvert D^\natural_\alpha\rvert$	# of orbits	multi., a_α
I	0	1	1	2^7	1
II	16	3	$\lvert D^\#\rvert = \lvert H_8\rvert^2 \cdot 2^3$	4	2^3
III	24	120	$\lvert H_8\rvert^3 \cdot 2^6$	2	2^6
IV	32	3	$\lvert D^\#\rvert^2 \cdot 2^4 = \lvert H_8\rvert^4 \cdot 2^{10}$	2	2^{10}
V	48	1	$\lvert D^\natural\rvert = \lvert H_8\rvert^6 \cdot 2^{17}$	1	2^{17}

What the 1st and 2nd column of this table mean is clear. The most important column is the 3rd one which gives the order of $D^\natural_\alpha$ together with the structure of $D^\natural_\alpha$ for each $\alpha \in S^\natural$. For example, $|D^\#| = |H_8|^2 \cdot 2^3$ in the 2nd row means that $(D^\natural)_\alpha \simeq D^\#$ and $(D^\natural)_\alpha$ contains a direct sum of two copies of Hamming code H_8 as a maximal selforthogonal subcode which has the index 2^3 in $(D^\natural)_\alpha$. This can be easily from §4.1, (4.1.1). The 4th column gives the number of orbits of the action of $D^\natural$ on $\Lambda_{48}(\alpha)(= \frac{2^{n-wt(\alpha)}}{[D^\natural:(D^\natural)_\alpha]})$. The 5th column is the multiplicities$(= [(D^\natural)_\alpha : H_\alpha])$ appearing in irreducible module $(\mathbf{V}^\natural)^\alpha$ in $\mathbf{T}_{48}$-decomposition (3.1.4) of $\mathbf{V}^\natural$ (cf. (3.3.1)).

Now, for the description of $\mathbf{T}_{48}$-decomposition of $\mathbf{V}^\natural$, it remains to choose an orbit of the action of $D^\natural$ on $\Lambda_{48}(\alpha)$ for each $\alpha \in S^\natural$. Consider the 2nd row, for example. The number of orbits is 4. As is easily seen, the representatives of each orbit (say, for $\alpha = (1^{16}, 0^{16}, 0^{16})$) are $L((\frac{1}{16})^{16}, 0^{16}, 0^{16})$, $L((\frac{1}{16})^{16}, 0^{16}, \frac{1}{2}0^{15})$, $L((\frac{1}{16})^{16}, \frac{1}{2}0^{15}, 0^{16})$, and $L((\frac{1}{16})^{16}, \frac{1}{2}0^{15}, \frac{1}{2}0^{15})$.
The 2nd and the 3rd one is improper, because they have half-integral weight. The 1st one is also improper, because it has weight 1 but the Moonshine VOA $\mathbf{V}^\natural$ has no vector of weight 1. Thus we must choose the last one as a representative. This orbit is the set of all $L((\frac{1}{16})^{16}, *, *)$ such that $\frac{1}{2}$ appear odd times in each part of two $*$.
For other rows of the above table, the orbit is uniquely determined by

"integral condition" of weight. Thus we have

	$wt(\alpha)$	a representative of orbit
I	0	$L(0^{48})$
II	16	$L((\frac{1}{16})^{16}(\frac{1}{2}0^{15})(\frac{1}{2}0^{15}))$
III	24	$L((\frac{1}{16})^{24}(\frac{1}{2}0^{23}))$
IV	32	$L((\frac{1}{16})^{32}0^{16})$
V	48	$L((\frac{1}{16})^{48})$

4.3. The calculation of the the character

Firstly let us remark about the character of Code VOA.
Let D be a binary even code and M_D be a code VOA for D:

$$M_D = \oplus_{\delta\in D} L(\frac{\delta_1}{2}, \frac{\delta_2}{2}, \cdots, \frac{\delta_n}{2})\ (\delta = (\delta_1, \delta_2, \cdots, \delta_n)).$$

Let $W_D(x, y) = \sum_{\delta\in D} x^{n-wt(\delta)} y^{wt(\delta)}$ (the weight enumerator of D).
Then the character of M_D is expressed as follows:

$$ch(M_D) = W_D(ch(L(\frac{1}{2}, 0)), ch(L(\frac{1}{2}, \frac{1}{2}))$$

Using formulas of $ch(L(\frac{1}{2}, 0))$, and $ch(L(\frac{1}{2}, \frac{1}{2}))$ mentioned in §2.2.2 and MacWilliam's identity in coding theory, we have

$$ch(M_D) = \frac{1}{|D^{\perp}|} W_{D^{\perp}}(q_+, q_-).$$

Let us begin the calculation of the character of $\mathbf{V}^{\natural}$:

$$\mathbf{V}^{\natural} = \oplus_{\alpha\in S^{\natural}} (\mathbf{V}^{\natural})^{\alpha}.$$

For that purpose, let us calculate $ch((\mathbf{V}^{\natural})^{\alpha})$ since we know the $\mathbf{T}_{48}$-decomposition of $(\mathbf{V}^{\natural})^{\alpha}$ in §4.2.

Case I *where* $S^{\natural} \ni \alpha$ *is of Type* I , i.e. $\alpha = (0^{48})$:
In this case, $(\mathbf{V}^{\natural})^{\alpha} \simeq M_{D^{\natural}}$ (code VOA) and so

$$ch((\mathbf{V}^{\natural})^{\alpha}) = \frac{1}{2^7}(q_+^{48} + 3q_+^{32}q_-^{16} + 120q_+^{24}q_-^{24} + 3q_+^{16}q_-^{32} + q_-^{48}).$$

Using a formula $q_0 q_+ q_- = 1$ and Jacobi's formula $16q^{\frac{1}{2}}q_0^8 = q_+^8 - q_-^8$, we get

$$ch((\mathbf{V}^{\natural})^{\alpha}) = 2^{17}q^3 q_0^{48} + 3\cdot 2^{10} q^2 q_0^{24} + (q_+ q_-)^{24} + 24q = Q_I.$$

Here we put the righthandside as Q_I.

Case II *where* $S^\natural \ni \alpha$ *is of Type II* , i.e. $\alpha = (1^{16}0^{16}0^{16}), (0^{16}1^{16}0^{16})$ or $(0^{16}0^{16}1^{16})$.
For simplicity of notations, let

$$X = ch(L(\frac{1}{2}, 0)),\ Y = ch(L(\frac{1}{2}, \frac{1}{2})),\ Z = ch(L(\frac{1}{2}, \frac{1}{16})).$$

Then we have

$$\begin{aligned} ch((\mathbf{V}^\natural)^\alpha) &= 2^3 Z^{16} \left(\sum_{i=1}^{8} \binom{16}{2i-1} X^{16-(2i-1)} Y^{2i-1} \right)^2 \\ &= 2^3 \cdot \frac{1}{4}((X+Y)^{16} - (X-Y)^{16})^2 Z^{16}. \end{aligned}$$

Transforming this in the same way as Case I, we get

$$ch((\mathbf{V}^\natural)^\alpha) = 2^3(2^{14}q^3q_0^{48} + 2^8q^2q_0^{24}) = Q_{II}.$$

Calculating $ch((\mathbf{V}^\natural)^\alpha)$ for α of *Type III, IV, V* similarly, we get

$$\begin{aligned} ch((\mathbf{V}^\natural)^\alpha) &= 2^6(2^{11}q^3q_0^{48} + 3 \cdot 2^3q^2q_0^{24}) &&= Q_{III} \text{ for } \alpha \text{ of } \textit{Type III}, \\ ch((\mathbf{V}^\natural)^\alpha) &= 2^{10}(2^7q^3q_0^{48} + q^2q_0^{24}) &&= Q_{IV} \text{ for } \alpha \text{ of } \textit{Type IV}, \\ ch((\mathbf{V}^\natural)^\alpha) &= 2^{17}q^3q_0^{48} &&= Q_V \text{ for } \alpha \text{ of } \textit{Type V}. \end{aligned}$$

Thus we have

$$\begin{aligned} ch(\mathbf{V}^\natural) &= \sum_{\alpha \in S^\natural} ch((\mathbf{V}^\natural)^\alpha) \\ &= Q_I + 3 \cdot Q_{II} + 120 \cdot Q_{III} + 3 \cdot Q_{IV} + Q_V \\ &= 2^{24}q^3q_0^{48} + 3 \cdot 2^{16}q^2q_0^{24} + (q_+q_-)^{24} + 24q \\ &= q\left(2^{24}\left(\frac{\eta(2z)}{\eta(z)}\right)^{48} + 3 \cdot 2^{16}\left(\frac{\eta(2z)}{\eta(z)}\right)^{24} + \left(\frac{\eta(z)}{\eta(2z)}\right)^{24} + 24\right). \end{aligned}$$

Finally, using $j(z)^{\frac{1}{3}} = 2^8\left(\frac{\eta(2z)}{\eta(z)}\right)^{16} + \left(\frac{\eta(z)}{\eta(2z)}\right)^8$, we get

$$\frac{1}{q}ch(\mathbf{V}^\natural) = j(z) - 744.$$

4.4. Thompson series of some involutions of $Aut(\mathbf{V}^\natural)$

For $\tau \in Aut(\mathbf{V}^\natural)$ (Monster),

$$T_\tau(q) = \frac{1}{q}\sum_{n=0}^{\infty} Tr(\tau|(\mathbf{V}^\natural)_n)q^n$$

is called Thompson series of τ. We will calculate Thompson series of some involutions of $Aut(\mathbf{V}^\natural)$.

For each i ($1 \leq i \leq 48$), we define a linear transformation of $\mathbf{V}^\natural$ as follows:

$$\tau_i|(\mathbf{V}^\natural)^\alpha = \epsilon(i,\alpha) Id_{(\mathbf{V}^\natural)^\alpha}, \ \epsilon(i,\alpha) = \begin{cases} -1 & i \in supp(\alpha) \\ 1 & i \notin supp(\alpha) \end{cases}$$

Then τ_i is an automorphism (as VOA) of $\mathbf{V}^\natural$ (cf. [M1]). In the following, we will calculate Thompson series of $\tau_1 \in Aut(\mathbf{V}^\natural)$ ($2A$-involution) and $\tau_1\tau_2 \in Aut(\mathbf{V}^\natural)$ ($2B$-involution). For each $\alpha \in S^\natural$, let

$$\epsilon_\alpha = \begin{cases} -1 & 1 \in supp(\alpha) \\ 1 & 1 \notin supp(\alpha). \end{cases}$$

Then we have $qT_{\tau_1}(q) = \sum_{\alpha \in S^\natural} \epsilon_\alpha ch((\mathbf{V}^\natural)^\alpha)$ which is equal to

$$Q_I + (-1+1+1)Q_{II} + (1-1-1)Q_{IV} - Q_V + \left(\sum_{\alpha: Type\ III} \epsilon_\alpha \right) Q_{III}.$$

But since the number of α of *Type III* with $1 \in supp(\alpha)$ is equal to the number of α of *Type III* with $1 \notin supp(\alpha)$, the last term is canceled and so we get

$$qT_{\tau_1}(q) = Q_I + Q_{II} - Q_{IV} - Q_V = 2^{12} q^2 q_0^{24} + (q_+ q_-)^{24} + 24q$$

$$= q \left(2^{12} \left(\frac{\eta(2z)}{\eta(z)} \right)^{24} + \left(\frac{\eta(z)}{\eta(2z)} \right)^{24} + 24 \right).$$

Thus Thompson series $T_{\tau_1}(q)$ is equal to a modular function corresponding to 2A-involution of Monster (cf. [CN]). Next, let

$$\epsilon'_\alpha = \begin{cases} 1 & 1,2 \in supp(\alpha) \text{ or } 1,2 \notin supp(\alpha) \\ -1 & \text{otherwise.} \end{cases}$$

Then the Thompson series $qT_{\tau_1\tau_2}(q) = \sum_{\alpha \in S^\natural} \epsilon'_\alpha ch((\mathbf{V}^\natural)^\alpha)$ is equal to

$$Q_I + (1+1+1)Q_{II} + (1+1+1)Q_{IV} + Q_V + \left(\sum_{\alpha: Type III} \epsilon'_\alpha \right) Q_{III}.$$

But since there exist 56 α of *Type III* with $\epsilon'_\alpha = 1$ and 64 α of *Type III* with $\epsilon'_\alpha = -1$, we get

$$\begin{aligned} qT_{\tau_1\tau_2}(q) &= Q_I + 3\cdot Q_{II} + 3\cdot Q_{IV} + Q_V - 8\cdot Q_{III} \\ &= (q_+q_-)^{24} + 24q = q\left(\left(\frac{\eta(z)}{\eta(2z)}\right)^{24} + 24\right). \end{aligned}$$

Thus Thompson series $T_{\tau_1\tau_2}(q)$ is equal to a modular function corresponding to 2B-involution of Monster (cf. [CN]).

Remark. For some $\tau \in Aut(\mathbf{V}^\natural)$ which come from $Aut(D^\natural)$, it is possible to calculate Thompson series $T_\tau(q)$ explicitly. In fact, Miyamoto [M6] has done it for such 3-element of $Aut(\mathbf{V}^\natural)$ (which corresponds to 3C-element of Monster and $T_\tau(q) = j(3z)^{\frac{1}{3}}$.) Also Sakuma [S], one of Miyamoto's graduate students, has written down $T_\tau(z)$ in terms of the characters of Ising models for such 5-element and 7-element which should be $(\frac{\eta(z)}{\eta(5z)})^6 + 6$ and $(\frac{\eta(z)}{\eta(7z)})^4 + 49\cdot(\frac{\eta(7z)}{\eta(z)})^4 + 4$ respectively, although it is a little bit unsatisfactory for these identifications.

§5. VOA of Reed Müller type

For $m \geq 4$, let

$S(m) = RM(m,1)$ (1st order Reed Müller code of length 2^m)

$D(m) = S(m)^\perp = RM(m, m-2)$ ($(m-2)$-th order Reed Müller code of length 2^m)

(Note that $S(4), D(4)$ is nothing but $S^\#, D^\#$ respectively in §4.1). It is easy to see that

(5,1) $D(m), S(m)$ *satisfy the conditions* (3.1.2), (3.2.1)

(5,2) *Orbit of the action of* $D(m)$ *on* $\Lambda_{2^m}(\alpha)$ *for each* $\alpha \in S(m)$ *is uniquely determined under integral condition of weight.*

Furthermore, in view of Miyamoto's theory [M5], [M6], there exists VOA for $D(m), S(m)$. We denote it by $\mathbf{V}(m)$.

Remark. $\mathbf{V}(m)$ can be constructed as VOA over the real number field with a positive definite invariant form and then, if $m \geq 6$, $Aut(\mathbf{V}(m))$ is a finite group (cf. [M6]). For $m = 4, 5$, we see $\mathbf{V}(4) = E_8$-Lattice VOA and $\mathbf{V}(5) = E_{16}$-Lattice VOA. As for the character of $\mathbf{V}(m)$, we can easily see that $q^{-\frac{2^{m-1}}{24}} ch(\mathbf{V}(m))$ is equal to $j(z)^{\frac{1}{3}}$, $j(z)^{\frac{2}{3}}$ and $j(z)^{\frac{1}{3}}(j(z) - 992)$ for $m = 3, 4$ and 5 respectively.
More generally, it seems very likely

$$q^{-\frac{2^{m-1}}{24}} ch(\mathbf{V}(m)) = j(z)^{\frac{\mu}{3}} (\text{a polynomial of } j(z)) \ (\mu = 1 \text{ or } 2).$$

But the author has not yet checked it.

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