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Non-Abelian Representations of Geometries

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Abstract.

Let $\mathcal{G}$ be a geometry in which the elements of one type are called *points* and the elements of some other type are called *lines*. Suppose that every line is incident to exactly $p+1$ points where p is a prime number. A (non-abelian) representation of $\mathcal{G}$ is a pair (R, ψ), where R is a group and ψ is a mapping of the set of points of $\mathcal{G}$ into the set of subgroups of order p in R such that R is generated by the image of ψ and whenever $\{x_\infty, x_0, ..., x_{p-1}\}$ is the set of points incident to a line, the subgroups $\psi(x_\infty), \psi(x_0), ..., \psi(x_{p-1})$ are pairwise different and generate in R a subgroup of order p^2. In this article we discuss representations of some classical and sporadic geometries and their applications to certain problems in algebraic combinatorics and group theory.

§1. Abelian representations

Our terminology concerning diagram geometries is mostly standard [Pas94], [Iv99a]. The types of elements on a diagram increase rightward from 1 to the rank of geometry. The elements of type 1, 2 and 3 are called *points*, *lines* and *planes*, respectively. Many important geometries are naturally defined as collections of subspaces in a finite dimensional vector space V so that the type of a subspace equals to its dimension and two subspaces are incident if one of them contains the other one (in this case we say that the incidence is via inclusion). For the projective geometry of V we take all the proper subspaces and for a polar space we take the subspaces which are totally singular with respect to a fixed non-degenerate symplectic, orthogonal or unitary form f on V. These constructions can be generalized as follows (we consider vector spaces over prime fields since they are sufficient to cover our main examples).

Construction A. Let V be an n-dimensional $GF(p)$-space, where p is a prime, let G be a subgroup of $GL(V) \cong GL_n(p)$ and U be a

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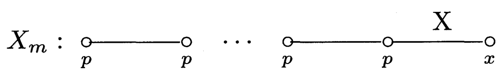
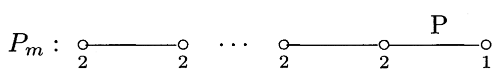
subspace of dimension m in V such that the stabilizer of U in G induces on U an action which contains $SL(U) \cong SL_m(p)$. Let $0 < U_1 < .... < U_{m-1} < U_m = U$ be a maximal flag in U. Define $\mathcal{G}_A(V,G)$ to be an incidence system whose elements of type i are the images of U_i under G, $1 \leq i \leq m$; the incidence is via inclusion.

Under some non-degeneracy assumptions $\mathcal{G}_A(V,G)$ is a geometry which belongs to the diagram

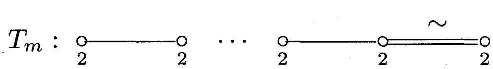


where the rightmost edge indicates the rank 2 geometry formed by the images of U_{m-1} and $U_m = U$ under that stabilizer of U_{m-2} in G. Furthermore, G induces on $\mathcal{G}_A(V,G)$ a flag-transitive action. Notice that the structure of $\mathcal{G}_A(V,G)$ depends on U but not on the maximal flag in U.

In these terms the projective geometry of V can be obtained as $\mathcal{G}_A(V, GL(V))$ for a hyperplane U in V while the polar space associated with a form f as $\mathcal{G}_A(V,G)$ where G is the subgroup in $GL(V)$ which preserves f up to scalar multiplication and U is a maximal totally isotropic subspace in V with respect to f. Some sporadic geometries can also be obtained by Construction A. Recall [Iv99a] that Petersen geometries of rank m have the following diagram (m nodes)



where the rightmost edge indicates the geometry of edges and vertices of the Petersen graph with the natural incidence relation; and tilde geometries of rank m have the following diagram (m nodes)



where the rightmost edge indicates the triple cover of the generalized quadrangle of order $(2,2)$ associated with the non-split extension $3 \cdot Sp_4(2) \cong 3 \cdot Sym_6$.

If $\bar{\mathcal{C}}_{11}$ is the irreducible (11-dimensional) Todd module for the Mathieu group Mat_{24} then there is a 3-dimensional subspace U in $\bar{\mathcal{C}}_{11}$ such that $\mathcal{G}(Mat_{24}) := \mathcal{G}_A(\bar{\mathcal{C}}_{11}, Mat_{24})$ is a tilde geometry of rank 3. Consider the Mathieu group Mat_{22} as a subgroup in Mat_{24}. Then under a suitable choice of U the geometry $\mathcal{G}(Mat_{22}) := \mathcal{G}_A(\bar{\mathcal{C}}_{11}, Mat_{22})$ is a

Petersen geometry of rank 3 and by the construction it is a subgeometry in $\mathcal{G}(Mat_{24})$. Let V_{12} be the natural module for $SU_6(2)$ (considered as a 12-dimensional $GF(2)$-module). Then the non-split extension $3 \cdot Mat_{22}$ is embedded into $SU_6(2)$ and hence V_{12} is a module for this extension. There is a 3-dimensional subspace U in V_{12} such that $\mathcal{G}(3 \cdot Mat_{22}) := \mathcal{G}_A(V_{12}, 3 \cdot Mat_{22})$ is a Petersen geometry of rank 3 which is the universal (triple) cover of $\mathcal{G}(Mat_{22})$.

If $\bar{\Lambda}_{24}$ is the Leech lattice taken modulo 2 (a 24-dimensional $GF(2)$-space) then there is a 4-dimensional subspace U in $\bar{\Lambda}_{24}$ such that $\mathcal{G}(Co_1) := \mathcal{G}_A(\bar{\Lambda}_{24}, Co_1)$ is a tilde geometry of rank 4, containing $\mathcal{G}(Mat_{24})$ as a residue. The second Conway group Co_2 is the stabilizer in Co_1 of a vector $\bar{\lambda}$ in $\bar{\Lambda}_{24}$. Let $\bar{\Lambda}_{23}$ be the orthogonal complement of $\bar{\lambda}$ in $\bar{\Lambda}_{24}$ with respect to the unique non-zero orthogonal form preserved by Co_1. Then under a suitable choice of U and $\bar{\lambda}$ the geometry $\mathcal{G}(Co_2) := \mathcal{G}_A(\bar{\Lambda}_{23}, Co_2)$ is a Petersen geometry of rank 4 containing $\mathcal{G}(Mat_{22})$ as a residue. By the construction $\mathcal{G}(Co_2)$ is a subgeometry in $\mathcal{G}(Co_1)$.

It is natural to ask which geometries with diagrams X_m can be obtained by Construction A. This question leads the following.

Definition 1.1. Let $\mathcal{G}$ be a geometry in which the elements of one type are called *points* and the elements of some other type are called *lines*. Suppose that every line is incident to exactly $p+1$ points where p is a prime number. An *abelian representation* of $\mathcal{G}$ is a pair (V, ψ), where V is a vector space over $GF(p)$ and ψ is a mapping of the set of points of $\mathcal{G}$ into the set of 1-dimensional subspaces in V, such that V is generated by the image of ψ and whenever $\{x_\infty, x_0, ..., x_{p-1}\}$ is the set of points incident to a line l, the subspaces $\psi(x_\infty), \psi(x_0), ..., \psi(x_{p-1})$ are pairwise different and generate in V a 2-dimensional subspace denoted by $\psi(l)$.

An abelian representation (V, ψ) is said to be *faithful* if ψ is injective. If H is an automorphism group of $\mathcal{G}$ then the representation (V, ψ) as above is *H-admissible* if there is a subgroup G of $GL(V)$ and a homomorphism χ of G onto H such that $\psi(u)^g = \psi(u^{\chi(g)})$ for every point u and every $g \in G$. The following result is quite obvious.

Lemma 1.2. *Let $\mathcal{G}$ be a rank m geometry with diagram X_m and let H be a flag-transitive automorphism group of $\mathcal{G}$. Suppose that there is an isomorphism φ of $\mathcal{G}$ onto $\mathcal{G}_A(V, G)$ which commutes with the action of G and that the action of G induced on $\mathcal{G}$ via the isomorphism φ coincides with H. Let ψ be the restriction of φ to the point-set of $\mathcal{G}$. Then (V, ψ) is a faithful H-admissible abelian representation of $\mathcal{G}$ and for an element $u \in \mathcal{G}$ the subspace $\varphi(u)$ is generated by the 1-spaces $\varphi(x)$ taken for all the points x incident to u.*

Thus a possible way to decide whether or not a geometry $\mathcal{G}$ can be obtained by Construction A is to study the abelian representations of $\mathcal{G}$. Abelian representations of various geometries were intensively studied for a long time [Ti74], [RSm89], [Yos92] and [Sh93] (sometimes under different names like *embeddings*). Let us mention just one of the numerous applications of such representations.

Let $\mathcal{G}$ be a geometry with diagram X_m, $m \geq 3$. Then the points and lines incident to a plane z form a projective plane Π_z of order p. Let G be a flag-transitive automorphism group of $\mathcal{G}$ and suppose that the stabilizer of a plane z in G induces on the residual projective plane Π_z an action containing $L_3(p)$ and that the stabilizer of a line l in G induces on the set of points incident to l an action containing $L_2(p)$. Let x be a point and $\mathcal{G}_x$ be the residue of x in $\mathcal{G}$ whose points and lines are the lines and planes in $\mathcal{G}$ incident to x.

Lemma 1.3. *In the above terms let $G(x)$ be the stabilizer of x in G, L be the kernel of the action of $G(x)$ on the set of lines incident to x and K be the kernel of the action of L on the set of points collinear to x. Suppose that $O_p(L) \neq K$. Then $O_p(L)/K$ is an elementary abelian p-group and (V^*, ψ) is a abelian faithful $G(x)/L$-admissible representation of $\mathcal{G}_x$ where V^* is the module dual to $O_p(L)/K$ and if y is a line incident to x then $\psi(y)$ is the action induced by $O_p(L)$ on the set of points incident to y.*

Proof. Let l be a line incident to x. Then $G(l) \cap G(x)$ induces on the set of points incident to l a Frobenius group F of order $p(p-1)/\varepsilon$ where ε is 1 or 2. Then $O_p(F)$ is of order p and it is contained in every proper normal subgroup of F. Since $O_p(L) \neq K$ and $G(x)$ acts transitively on the set of points in $\mathcal{G}_x$, we conclude that $O_p(L)$ induces on the set of points incident to l the group $O_p(F)$ of order p. Hence $O_p(L)/K$ is an elementary abelian p-group and l corresponds to its pointwise stabilizer which is of index p in $O_p(L)/K$. Let z be a plane incident to x and M be the action induced on Π_z by $G(x) \cap G(z)$. Then $O_p(M)$ is of order p^2 and $M/O_p(M)$ acts on $O_p(M)$ irreducibly. By the above $O_p(L)$ is normal in $G(x) \cap G(z)$, this action coincides with $O_p(M)$ and the result follows. Q.E.D.

The above lemma can be used to decide whether or not a given geometry with diagram X_{m-1} can appear as a point residue in a flag-transitive geometry with diagram X_m.

§2. Non-abelian representations

Petersen and tilde geometries of "large" sporadic simple groups, J_4, BM and M do not possess abelian representations [ISh90], [ISh94a]. Using this fact and Lemma 1.3 it was shown that these geometries do not appear as residues in Petersen and tilde geometries of higher ranks. The latter result was of a crucial importance for completing the classification of the flag-transitive Petersen and tilde geometries [ISh94b]. By Lemma 1.2 the geometries $\mathcal{G}(J_4)$, $\mathcal{G}(BM)$ and $\mathcal{G}(M)$ can not be obtained by Construction A, but in fact they can be obtained by a similar construction [ISh89].

Construction B. Let R be a group, G be a subgroup in the automorphism group of R and U be an elementary abelian subgroup of order p^m in R where p is a prime number, such that the stabilizer of U in G induces on U an action which contains $SL_m(p)$. Let $0 < U_1 < ... < U_{m-1} < U_m = U$ be a maximal flag in U. Define $\mathcal{G}_B(R, G)$ to be an incidence system of rank m whose elements of type i are the images of U_i under G; the incidence is via inclusion.

Again under some non-degeneracy assumptions $\mathcal{G}_B(R, G)$ is a geometry with diagram X_m. For a suitable choice of subgroups U of order 2^4, 2^5 and 2^5, respectively, we have $\mathcal{G}(J_4) = \mathcal{G}_B(J_4, J_4)$, $\mathcal{G}(BM) = \mathcal{G}_B(2 \cdot BM, BM)$, $\mathcal{G}(M) = \mathcal{G}_B(M, M)$. Furthermore, $2 \cdot BM$ can be identified with a subgroup in the Monster M so that in the last two cases the subgroup U can be taken to be the same, which shows that $\mathcal{G}(BM)$ is a subgeometry in $\mathcal{G}(M)$. Construction B leads to the following

Definition 2.1. In terms of Definition 1.1 a pair (R, ψ) is a *representation* of $\mathcal{G}$ if R is a group and ψ is a mapping of the set of points of $\mathcal{G}$ into the set of subgroups of order p in R such that R is generated by the image of ψ and whenever $\{x_\infty, x_0, ..., x_{p-1}\}$ is the set of points incident to a line l, the subgroups $\psi(x_\infty), \psi(x_0), ..., \psi(x_{p-1})$ are pairwise different and generate in R a subgroup of order p^2 denoted by $\psi(l)$.

In order to distinguish the representations in the above definition from the abelian representations we sometimes call the former ones *non-abelian representations.* Since the group R might or might not be abelian the correct term would probably be *non-necessarily abelian representations.* The notions of faithful and H-admissible representations can be defined analogously to the abelian case. It is easy to see that a statement analogous to Lemma 1.2 holds where $\mathcal{G}_A(V, G)$ is changed to $\mathcal{G}_B(R, G)$ and (V, ψ) is changed to (R, ψ).

If (R, ψ) and (R', ψ') are representations of a geometry $\mathcal{G}$ and $\varphi : R' \to R$ is a homomorphism such that $\psi(x) = \varphi(\psi'(x))$ for every point

x, then φ is said to be a *morphism of representations.* For a representation (R,ψ) there is a *universal representation* (R_U,ψ_U) possessing a morphism φ_U onto (R,ψ) such that whenever a representation (R',ψ') possesses a morphism φ onto (R,ψ) there is a morphism φ' of (R_U,ψ_U) onto (R',ψ') such that φ_U is the composition of φ' and φ. The group R_U can be defined in terms of generators and relations as follows. For every point x choose a generator $r(x)$ of the subgroup $\psi(x)$. Then the generators of R_U are elements $u(x)$ of order p taken for all points x. If $\{x_\infty,x_0,...,x_{p-1}\}$ are the points incident to a line l, then for $1\le i\le p-1$ we have $r(x_i)=r(x_\infty)^{a(i)}r(x_0)^{b(i)}$ for some $1\le a(i),b(i)\le p-1$. Then the relations of R_U associated with the line l are

$$[u(x_\infty),u(x_0)]=1,\ u(x_i)=u(x_\infty)^{a(i)}u(x_0)^{b(i)},\ 1\le i\le p-1,$$

where the $a(i)$ and $b(i)$ are as above. The mapping ψ_U sends x onto the subgroup generated by $u(x)$ and $\varphi_U:u(x)\mapsto r(x)$ for every point x. In general the universal representation (R_U,ψ_U) depends on the particular choice of (R,ψ) although in some circumstances the universal representation is unique. This is the case, for instance, when $p=2$ (in this case $a(1)=b(1)=1$ and the relations are uniquely determined). Another uniqueness situation is described in the following.

Lemma 2.2. *Suppose that H is an automorphism group of $\mathcal{G}$ such that for every line l the stabilizer of l in H induces on the set of points incident to l an action containing $L_2(p)$. Then all H-admissible representations of $\mathcal{G}$ have isomorphic universal representations.*

Proof. Let (R,ψ) be an H-admissible representation and G be the corresponding automorphism group of R which possesses a homomorphism onto H. Let $\{x_\infty,x_0,...,x_{p-1}\}$ be the set of points incident to a line l, $W=\psi(l)$ and F be the action induced on W by its stabilizer in G. Then W can be considered as a 2-dimensional $GF(p)$-space and by the hypothesis F contains $SL(W)\cong SL_2(p)$. Let W_0 be a 1-subspace in W, F_0 be its stabilizer in F and W_1 be the $GF(p)$-module of dimension $p+1$ for F induced from the module W_0 of F_0. Then the result follows from the following fact which is well known and easy to check: W_1 has a unique submodule of codimension 2. Q.E.D.

If (R,ψ) is an abelian representation then the corresponding *universal abelian representation* is a pair (V_U,ψ_U^a) where V_U is the quotient of R_U over the commutator subgroup of R_U and ψ_U^a is the composition of ψ_U and the natural homomorphism of R_U onto V_U.

By Lemma 1.2 and the analogous statement for the non-abelian case Constructions A and B produce geometries together with their representations. It was shown in [RSm89], [ISh89], [Sm92], [ISh94a], [IPS96] and

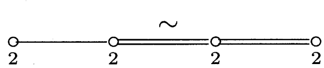
[ISh97] that the representations associated with the above constructions of $\mathcal{G}(Mat_{22})$, $\mathcal{G}(Mat_{24})$, $\mathcal{G}(Co_2)$, $\mathcal{G}(Co_1)$, $\mathcal{G}(J_4)$, $\mathcal{G}(BM)$ and $\mathcal{G}(M)$ are universal (among the non-abelian representations). If $\chi : \tilde{\mathcal{G}} \to \mathcal{G}$ is a covering of geometries and (R, ψ) is a representation of $\mathcal{G}$ then $(R, \psi\chi)$ is a representation of $\tilde{\mathcal{G}}$. In particular $\mathcal{G}(3 \cdot Mat_{22})$ possesses a (non-faithful) representation in $\bar{\mathcal{C}}_{11}$. It can be shown that $\mathcal{G}(3 \cdot Mat_{22})$ possesses a representation in the direct product of $\bar{\mathcal{C}}_{11}$ and the extraspecial group 2_+^{1+12} (a central extension of V_{12}). It is not known whether or not this representation is universal.

Some further examples of geometries and their representation can be obtained by the following

Construction C. Let $\mathcal{G}_A(W, H)$ be a geometry obtained by Construction A via a subgroup U of order p^m. Let R be a group, G be a subgroup in the automorphism group of R and V be a subgroup in R. Suppose that there are isomorphisms $\varphi_1 : W \to V$ and $\varphi_2 : H \to N_G(V)/C_G(V)$ such that $\varphi_1(w^h) = \varphi_1(w)^{\varphi_2(h)}$ for all $w \in W$ and $h \in H$. Suppose also that for $1 \leq i \leq m$ every subgroup contained in V and conjugate to $\varphi_1(U_i)$ in G is conjugate to $\varphi_1(U_i)$ in $N_G(V)$. Define $\mathcal{G}_C(R, V, H)$ to be an incidence system whose elements of type $m+1$ are the images of V under G and for $1 \leq i \leq m$ the elements of type i are the images of $\varphi_1(U_i)$ under G; the incidence is via inclusion.

Again under some non-degeneracy conditions $\mathcal{G}_C(R, V, H)$ is a geometry of rank $m+1$ in which the residue of V is isomorphic to $\mathcal{G}_A(W, H)$.

Let Fi_{24} be the largest Fischer 3-transposition group. The commutator subgroup Fi'_{24} of Fi_{24} contains a subgroup V which is isomorphic to $\bar{\mathcal{C}}_{11}$ as a module for $Mat_{24} \cong N_{Fi_{24}}(V)/C_{Fi_{24}}(V)$. Thus $\mathcal{G}(Mat_{24})$ can be realized in V by Construction A. The geometry $\mathcal{G}_C(Fi'_{24}, 2^{11}, Mat_{24})$ (for $G = Fi_{24}$) is the minimal 2-local parabolic geometry $\mathcal{G}_2(Fi'_{24})$ of Fi'_{24} as in [RSt84] with the diagram

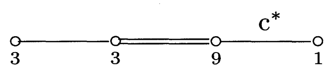


By the construction $\mathcal{G}_2(Fi'_{24})$ possesses a representation in Fi'_{24}. The following obvious result can be used to show that this representation is not universal.

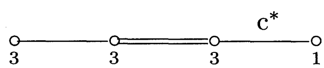
Lemma 2.3. *Let $\mathcal{G}$ be a geometry with $p+1$ points on a line and suppose that (R, ψ) is a representation of $\mathcal{G}$. Let $\tilde{R}$ be a perfect central extension of R whose kernel has order coprime to p. Then $(\tilde{R}, \tilde{\psi})$ is a representation of $\mathcal{G}$, where $\tilde{\psi}(x)$ is the unique Sylow p-subgroup in the preimage of $\psi(x)$ in $\tilde{R}$.*

By the above lemma $\mathcal{G}_2(Fi'_{24})$ possesses a representation in the extension $3 \cdot Fi'_{24}$ of Fi'_{24} by its Schur multiplier. Alternatively we could obtain $\mathcal{G}_2(Fi'_{24})$ by Construction C starting with $R = 3 \cdot Fi'_{24}$. This representation was shown in [Rch99] to be universal.

The Monster group M contains a subgroup V of order 3^8 which is the natural module for $N_M(V)/C_M(V) \cong \Omega_8^-(3).2$ so that the polar space of the latter group can be realized in V by Construction A. The geometry $\mathcal{G}(M, 3^8, \Omega_8^-(3).2)$ is the c-extended dual polar space $\mathcal{G}(M)$ [RSt84] with the diagram



Let μ be a subgroup of order 3 in V which is non-singular with respect to the $N_M(V)$-invariant quadratic form on V. Then $N_M(\mu) \cong 3 \cdot Fi_{24}$, $V = \mu \oplus W$ and W is the natural module for $N_M(W)/V \cong \Omega_7(3).2$. Then $\mathcal{G}_C(3 \cdot Fi'_{24}, 3^7, \Omega_7(3).2)$ is a subgeometry in $\mathcal{G}(M)$ which is the c-extended dual polar space $\mathcal{G}_3(Fi'_{24})$ with the diagram



By the construction $\mathcal{G}(M)$ and $\mathcal{G}_3(Fi'_{24})$ possess representations in M and $3 \cdot Fi'_{24}$, respectively. It was realized in [BIP99] that these representations are not universal and the observation can be generalized as follows

Lemma 2.4. *Let $\mathcal{G}$ be a geometry with $p+1$ points on a line. Let (R, ψ) be a representation of $\mathcal{G}$ and for a point x let $r(x)$ be a generator of $\psi(x)$. Let $\tilde{\psi}$ be the mapping from the point set of $\mathcal{G}$ into the set of subgroups in the direct product $R^{p-1} = \{(r_1, r_2, ..., r_{p-1}) \mid r_i \in R\}$ of $p-1$ copies of R defined by*

$$\tilde{\psi}(x) = \langle (r(x), r(x)^2, ..., r(x)^{p-1}) \rangle$$

and $\tilde{R}$ be the subgroup in R^{p-1} generated by the image of $\tilde{\psi}$. Then $(\tilde{R}, \tilde{\psi})$ is a representation of $\mathcal{G}$.

Proof. If A is an abelian group, then for every positive integer n the mapping defined by $a \mapsto a^n$ for every $a \in A$ is an automorphism of A. This means that whenever X is a set of points such that $[r(x), r(y)] = 1$ for all $x, y \in X$, the subgroup in $\tilde{R}$ generated by $\{\tilde{\psi}(x) \mid x \in X\}$ is isomorphic to the subgroup in R generated by $\{\psi(x) \mid x \in X\}$. Now the result follows by taking X to be the set of points incident to a line. Q.E.D.

Let K be the smallest normal subgroup of R such that for every $2 \leq n \leq p-1$ the mapping $r(x) \mapsto r(x)^n$ for every point x induces an automorphism of R/K. Then one can show that $\tilde{R}$ is isomorphic to the direct product of $p-1$ copies of K extended by R/K. This shows that $\mathcal{G}(M)$ and $\mathcal{G}_3(Fi'_{24})$ possess representations in $M \times M$ and $3 \cdot Fi'_{24} \times 3 \cdot Fi'_{24}$, respectively. We conjecture that these representations are universal.

§3. Machinery

In this section we discuss some available technique for calculating universal representations of geometries.

Let $\mathcal{G}$ be a geometry of rank m with $p+1$ points per a line, (R, ψ) be a representation of $\mathcal{G}$ and $r(x)$ be a generator of $\psi(x)$. Let $\Gamma = \Gamma(\mathcal{G})$ be the collinearity graph of $\mathcal{G}$ which is a graph on the set of points of $\mathcal{G}$, where two points are adjacent if they are incident to a common line. For a point x let $\Gamma_i(x)$ be the set of points at distance i from x in Γ. Let $R_i(x)$ be the subgroup in R generated by the subgroups $\psi(y)$ taken for all points y which are at distance at most i from x in Γ. Let Δ_i be the graph on $\Gamma_i(x)$ in which two points are adjacent if they are incident to a common line, which is also incident to a point in $\Gamma_{i-1}(x)$.

Lemma 3.1. *Suppose that y and z are in the same connected component of Δ_i. Then $R_{i-1}(x)\psi(y) = R_{i-1}(x)\psi(z)$.*

The above lemma is useful for bounding the orders of the factors $R_i(x)/R_{i-1}(x)$ in the case of abelian representations. The first of these factors is of a particular importance (in both abelian and non-abelian cases).

Lemma 3.2. *Suppose that $m \geq 3$, $\mathcal{G}$ belongs to a string diagram and that the points and lines incident to a plane form a projective plane of order p. Let $\tilde{\psi} : l \to \psi(l)/\psi(x)$ where l is a line incident to x. Then $(R_1(x)/\psi(x), \tilde{\psi})$ is a representation of the residue of x in $\mathcal{G}$.*

Lemma 3.3. *Suppose that $p = 2$ and R is abelian. Let $(x_0, x_1, ..., x_k = x_0)$ be a cycle in Γ and for $0 \leq i \leq k-1$ let $\{x_i, x_{i+1}, y_i\}$ be the points incident to a line, then $\Pi_{j=0}^{k-1} r(y_j) = 1$.*

Recall that a *geometric hyperplane* in $\mathcal{G}$ is a proper subset S of points such that for every line l either all the points incident to l are contained in S or l is incident to exactly one point in S. Notice that whenever P is a subgroup of index p in R the set $S = \{x \mid \psi(x) \in P\}$ is a geometric hyperplane.

Lemma 3.4. *Suppose that $p = 2$ and that $\mathcal{G}$ contains a geometric hyperplane S such that the subgroups $\psi(x)$ taken for all $x \in S$ generate the whole R. Let T be a group of order 2 generated by an element t. Let $\hat{\psi}$ be a mapping of the set of points into the set of subgroups in the direct product of R and T which sends x to $\langle (r(x), t^{\alpha}) \rangle$ where $\alpha = 0$ if $x \in S$ and $\alpha = 1$ otherwise. Then $(R \times T, \hat{\psi})$ is a representation of $\mathcal{G}$.*

The following result is a slight generalization of Lemma 2.2 in [IPS96].

Lemma 3.5. *In the case $p = 2$ suppose that for every point x there are two subsets $A(x)$ and $B(x)$ of points such that*

(i) *if $y \in A(x)$ then $[r(x), r(y)] = 1$;*
(ii) *the graph on $B(x)$ in which two points are adjacent if there is a line incident to those points as well as to a point in $A(x)$, is connected;*
(iii) *if $z \in B(x)$ then $x \in B(z)$ and the graph on the set of points in which x is adjacent to the points in $B(x)$, is connected.*

Then the subgroup generated by the commutators $[r(x), r(z)]$ taken for every point x and every $z \in B(x)$ is of order at most 2 and contained in the centre of R. In particular, if $A(x) \cup B(x)$ is the whole set of points for every point x, then the commutator subgroup of R has order at most 2.

An important situation covered by Lemma 3.5 is when for every point x the set of points y such that $r(x)$ and $r(y)$ commute, form a geometric hyperplane $A(x)$ and the subgraph in the collinearity graph induced by the complement $B(x)$ of the hyperplane is connected. In a certain sense the next lemma deals with the opposite situation.

Lemma 3.6. *In the case $p = 2$ suppose that $\mathcal{G}$ contains a geometric hyperplane S such that the subgraph in Γ induced by the complement of S has at least two connected components T_1 and T_2. Then the universal representation group R_U of $\mathcal{G}$ is infinite.*

Proof. For a point x let $u(x)$ denote the corresponding generator of R_U. Let $D = \langle a_1, a_2 \mid a_1^2 = a_2^2 = 1 \rangle$ be the infinite dihedral group. Let χ be the mapping which sends $u(x)$ onto a_i if $x \in T_i$, $i = 1$ or 2 and onto the identity element of D otherwise. Then it is clear that χ induces a surjective homomorphism of R_U onto D and the result follows. Q.E.D.

It was checked by D.V. Pasechnik (private communication) that the tilde geometry $\mathcal{G}(3 \cdot Sp_4(2))$ of rank 2 contains a geometric hyperplane with disconnected complement and by the above lemma the universal representation group of $\mathcal{G}(3 \cdot Sp_4(2))$ is infinite.

For a group G containing a Klein four group let $\mathcal{I}(G)$ be a rank 2 geometry, whose points are the involutions in G, whose lines are the Klein four subgroups in G and the incidence is via inclusion. By the construction $\mathcal{I}(G)$ possesses a representation in G. The following result (Proposition 4.5 in [IPS96]), checked by D.V. Pasechnik on a computer, marked a breakthrough in our understanding of the non-abelian representations.

Lemma 3.7. *The universal representation group of $\mathcal{I}(Alt_7)$ is $3 \cdot Alt_7$.*

It follows immediately from Lemma 2.3 that $3 \cdot Alt_7$ is a representation group of $\mathcal{I}(Alt_7)$, but the universality fact is highly non-trivial. It would be interesting to learn more about representations of the geometries $\mathcal{I}(G)$ for other non-abelian simple groups G. The universal representation group of $\mathcal{I}(Mat_{22})$ is $3 \cdot Mat_{22}$ [IPS96] and of $\mathcal{I}(U_4(3))$ is $3^2 \cdot U_4(3)$ [Rch99].

The calculation of universal representations can be reduced to studying of covering of certain Cayley graphs. Suppose that (R, ψ) is faithful, let Q be the set of all non-identity elements contained in the subgroups $\psi(x)$ taken for all points x and let Ξ be the Cayley graph of R with respect to the set Q of generators. Let Ξ_U be the similar graph associated with the universal representation (R_U, ψ_U). Since (R_U, ψ_U) must also be faithful, the valency of both Ξ and Ξ_U is $p - 1$ times the number of points in $\mathcal{G}$. This means that the homomorphism of R_U onto R induces a covering $\varphi : \Xi_U \to \Xi$. Furthermore, for the every line l the covering φ induces an isomorphism of the subgraph $\Xi_U(l)$ in Ξ_U induced by the elements in $\psi_U(l)$ onto the analogous subgraph $\Xi(l)$ in Ξ (both $\Xi_U(l)$ and $\Xi(l)$ are complete graphs on p^2 vertices.) This gives the following.

Lemma 3.8. *Suppose that (R, ψ) is faithful. Let C_0 be the set of triangles in Ξ which are contained in the subgraphs induced by the elements in $\psi(l)$ taken for all lines l and let C be the set of images of the triangles in C_0 under R. If the cycles in C generate the fundamental group of Ξ then the representation (R, ψ) is universal.*

For our last statement assume that every element of $\mathcal{G}$ can be identified with the set of points incident to this element so that the incidence is via inclusion. For an element e of $\mathcal{G}$ let $\psi(e)$ denote the subgroup in R generated by the subgroups $\psi(x)$ taken for all points x incident to e. We say that (R, ψ) is *separable* if $\psi(e) \neq \psi(f)$ whenever $e \neq f$. The separability particularly implies that (R, ψ) is faithful. Define $\mathcal{A}(\mathcal{G}, R)$ to be the incidence system of rank $m+1$ whose elements of type 1 are the elements of R (right cosets of the identity subgroup) and for $2 \leq i \leq m+1$

the elements of type i are all the right cosets of the subgroups $\psi(e)$ for all elements e of type $i-1$ in $\mathcal{G}$; the incidence is via inclusion. The following result is a generalization of Lemma 1.1 in [Iv98].

Lemma 3.9. *In the above terms suppose that (R,ψ) is separable. Then*

(i) *$\mathcal{A}(\mathcal{G},R)$ is a geometry in which the residue of an element of type 1 is isomorphic to $\mathcal{G}$ and the elements of type 1 and 2 incident to an element of type 3 form the affine plane of order p;*
(ii) *if G is a flag-transitive automorphism group of $\mathcal{G}$ then the semidirect product $R:G$ acts flag-transitively on $\mathcal{A}(\mathcal{G},R)$;*
(iii) *if (R',ψ') is another representation of $\mathcal{G}$ and $\chi : R' \to R$ is a morphism of representations, then χ induces a 2-covering of $\mathcal{A}(\mathcal{G},R')$ onto $\mathcal{A}(\mathcal{G},R)$.*

Notice that the Cayley graph Ξ introduced in the paragraph before Lemma 3.8 is the collinearity graph of $\mathcal{A}(\mathcal{G},R)$. The case $p=2$ is of a particular interest since the affine plane of order 2 is isomorphic to the c-geometry of 1- and 2-element subsets of a set of size 4. Thus the representations of Petersen and tilde geometries provide c-extensions of these geometries [SW01]. Similarly representations of the dual polar spaces with 3 points on a line (associated with $Sp_{2n}(2)$ and $U_{2n}(2)$) give their c-extensions. Notice that the universal representations of these dual polar spaces are known only for $n=2$ and 3. It was conjectured by A.E. Brouwer that the dimension of the universal abelian representation of the dual polar space associated with $Sp_{2n}(2)$ is

$$1+\left[{n \atop 1}\right]_2+\left[{n \atop 2}\right]_2=(2^n+1)(2^{n-1}+1)/3.$$

Recently this conjecture was proved in [Li00] using some earlier results and methods from [BI97], [McC00] and independently in [BB00] by a different method.

In some cases (compare Theorem 2 (iii) in [Iv98]) one can show that $\mathcal{A}(\mathcal{G},R_U)$ is the universal 2-cover of $\mathcal{A}(\mathcal{G},R)$ and this reduces calculation of the universal representation to the question about 2-simple connectedness. For example the universality of the representation of $\mathcal{G}(M)$ in M established in [IPS96] is equivalent to the 2-simple connectedness of the corresponding c-extension of $\mathcal{G}(M)$. The latter result has been used in [Iv99a] to obtain a new proof identifying Y_{555} with the Bimonster $M \wr 2$.

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