Advanced Studies in Pure Mathematics 32, 2001
Groups and Combinatorics — in memory of Michio Suzuki
pp. 279–288

Rationally Determined Group Modules

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Abstract.

Green's correspondence of group modules finds its simplest expression when a finite multiplicative group G has a trivial intersection Sylow p-subgroup P, for some prime p. Then it is between all isomorphism classes of projective-free $\mathbf{R}G$-lattices $\mathbf{L}$ and all isomorphism classes of projective-free $\mathbf{R}N$-lattices $\mathbf{K}$, where $\mathbf{R}$ is a suitable valuation ring and N is the normalizer of P in G. In that case we show in Theorem 3.2 below that the $\mathbf{R}G$-lattice $\mathbf{L}$ is determined by its associated lattices over the residue field and field of fractions of $\mathbf{R}$ if and only if $\mathbf{K}$ has this same property. By Theorem 3.7 some important $\mathbf{R}G$-lattices $\mathbf{L}$ have this property of being "rationally determined." So it would be worthwhile to see if the $\mathbf{R}N$-lattices with this property (and perhaps with other properties preserved by this Green correspondence) could be classified.

§1. Projective-Free Lattices

Let $\mathbf{S}$ be any principal ideal domain. As usual, an $\mathbf{S}$-*order* $\mathbf{O}$ is just an associative $\mathbf{S}$-algebra with identity element $1 = 1_{\mathbf{O}}$ such that $\mathbf{O}$ is free of finite rank when considered as an $\mathbf{S}$-module. When we speak of an $\mathbf{O}$-*lattice* $\mathbf{L}$ we mean a unitary right $\mathbf{O}$-module such that $\mathbf{L}$ is also free of finite rank as an $\mathbf{S}$-module. Of course, a *homomorphism* $\phi\colon \mathbf{L} \to \mathbf{K}$ of $\mathbf{O}$-lattices is just a homomorphism between $\mathbf{O}$-modules $\mathbf{L}$ and $\mathbf{K}$ which are $\mathbf{O}$-lattices. We write any such ϕ on the left, so that it sends any $l \in \mathbf{L}$ to $\phi(l) \in \mathbf{K}$.

In the special case where the principal ideal domain $\mathbf{S}$ is a field, an $\mathbf{S}$-order is just a finite-dimensional associative $\mathbf{S}$-algebra $\mathbf{O}$ with identity element. Furthermore, an $\mathbf{O}$-lattice is just a unitary right $\mathbf{O}$-module $\mathbf{L}$ which is finite-dimensional as a vector space over $\mathbf{S}$.

This research was supported by grant number DMS 96-00106 and by grant number DMS 99-70030, both from the National Science Foundation.

Received May 29, 1999.

Revised May 18, 2000.

Throughout this note we fix a finite group G and a prime p. We also fix $\mathbf{R}$, $\mathbf{p}$, $\mathbf{F}$ and $\overline{\mathbf{F}}$ satisfying

(1.1) $\mathbf{R}$ *is a local principal ideal domain* (*i.e., a real discrete valuation ring*) *with unique maximal ideal* $\mathbf{p}$*, such that the field of fractions* $\mathbf{F}$ *of* $\mathbf{R}$ *is a splitting field of characteristic zero for every subgroup of* G*, and the residue class field* $\overline{\mathbf{F}} = \mathbf{R}/\mathbf{p}$ *of* $\mathbf{R}$ *has characteristic* p.

Notice that each of $\mathbf{R}$, $\mathbf{F}$ and $\overline{\mathbf{F}}$ is a principal ideal domain $\mathbf{S}$, to which all the above definitions apply. Furthermore, the group algebra $\mathbf{S}H$ over $\mathbf{S}$ of any subgroup H of G is an $\mathbf{S}$-order. The following result says that $\mathbf{S}H$-lattices have the Krull-Schmidt property.

Proposition 1.2. *Suppose that* $\mathbf{S}$ *is either* $\mathbf{F}$, $\overline{\mathbf{F}}$ *or* $\mathbf{R}$*, and that* H *is any subgroup of* G*. Then any* $\mathbf{S}H$*-lattice* $\mathbf{L}$ *is isomorphic to a finite direct sum* $\mathbf{L}_1 \oplus \cdots \oplus \mathbf{L}_l$ *of indecomposable* $\mathbf{S}H$*-lattices* $\mathbf{L}_i$*. Furthermore, this direct sum is uniquely determined to within order and isomorphisms by the* $\mathbf{S}H$*-lattice* $\mathbf{L}$*, i.e., if* $\mathbf{L}$ *is also isomorphic to a finite direct sum* $\mathbf{K}_1 \oplus \cdots \oplus \mathbf{K}_k$ *of indecomposable* $\mathbf{S}H$*-lattices* $\mathbf{K}_i$*, then* $k = l$ *and there is some permutation* π *of* $1, 2, \ldots, k$ *such that* $\mathbf{K}_i$ *is* $\mathbf{S}H$*-isomorphic to* $\mathbf{L}_{\pi(i)}$ *for* $i = 1, 2, \ldots, k$.

Proof. When $\mathbf{S}$ is a field $\mathbf{F}$ or $\overline{\mathbf{F}}$, this is the usual Krull-Schmidt Theorem for the finite-dimensional $\mathbf{S}$-algebra $\mathbf{S}H$. When $\mathbf{S}$ is $\mathbf{R}$, its field of fractions $\mathbf{F}$ is a splitting field of characteristic zero for the finite group H by (1.1). So $\mathbf{F}H$ is a split, semi-simple algebra of finite dimension over $\mathbf{F}$. Since $\mathbf{R}H$ is an $\mathbf{R}$-order spanning $\mathbf{F}H$ over $\mathbf{F}$, the basic hypotheses [1, 4.1] and [1, 4.2] of [1, §4] are satisfied by $\mathbf{D} = \mathbf{R}H$. The proposition for $\mathbf{S} = \mathbf{R}$ now holds by [1, 4.7]. Q.E.D.

In the situation of the preceding proposition we follow Green [2] in saying that an $\mathbf{S}H$-lattice $\mathbf{K}$ *divides* an $\mathbf{S}H$-lattice $\mathbf{L}$ if $\mathbf{L}$ is isomorphic to the direct sum $\mathbf{K} \oplus \mathbf{M}$ of $\mathbf{K}$ and some $\mathbf{S}H$-lattice $\mathbf{M}$. We say that $\mathbf{L}$ is *projective-free* if the only projective $\mathbf{S}H$-lattice $\mathbf{P}$ dividing $\mathbf{L}$ is $\mathbf{P} = 0$. The Krull-Schmidt property implies that any $\mathbf{S}H$-lattice $\mathbf{L}$ is isomorphic to a direct sum $\mathbf{L}_{\text{pf}} \oplus \mathbf{L}_{\text{pr}}$ of a projective-free $\mathbf{S}H$-lattice $\mathbf{L}_{\text{pf}}$ and a projective $\mathbf{S}H$-lattice $\mathbf{L}_{\text{pr}}$, either or both of which could be zero. Furthermore, these conditions determine both $\mathbf{L}_{\text{pf}}$ and $\mathbf{L}_{\text{pr}}$ to within $\mathbf{S}H$-isomorphisms. We call $\mathbf{L}_{\text{pf}}$ and $\mathbf{L}_{\text{pr}}$ the *projective-free part* and the *projective part*, respectively, of $\mathbf{L}$.

If $\mathbf{L}$ is an $\mathbf{R}H$-lattice, then we denote by $\overline{\mathbf{L}}$ its *residual* $\overline{\mathbf{F}}H$-lattice

$$\overline{\mathbf{L}} = \mathbf{L}/(\mathbf{p}\mathbf{L}).$$

We write $\eta_{\mathbf{L}}$ for the natural epimorphism of $\mathbf{L}$ onto its factor $\mathbf{R}H$-module $\overline{\mathbf{L}}$. When $\mathbf{L}$ is the regular $\mathbf{R}H$-lattice $\mathbf{R}H$, its residual $\overline{\mathbf{F}}H$-lattice $\overline{\mathbf{L}}$ can be identified with $\overline{\mathbf{F}}H$. In that case $\eta_{\mathbf{L}}$ is the natural epimorphism $\eta_{\mathbf{R}H}$ of $\mathbf{R}H$ onto $\overline{\mathbf{F}}H$ as $\mathbf{R}$-algebras.

Our hypotheses (1.1) allow us to lift projective lattices.

Lemma 1.3. *If* $\mathbf{Q}$ *is a projective* $\overline{\mathbf{F}}H$*-lattice, for some subgroup* H *of* G*, then there is some projective* $\mathbf{R}H$*-lattice* $\mathbf{P}$ *whose residual* $\overline{\mathbf{F}}H$*-lattice* $\overline{\mathbf{P}}$ *is isomorphic to* $\mathbf{Q}$.

Proof. The completion $\mathbf{R}^*$ of $\mathbf{R}$ is a local principal ideal domain with unique maximal ideal $\mathbf{p}^* = \mathbf{p}\mathbf{R}^*$. Since $\mathbf{F}$ is a splitting field of characteristic zero for H (see (1.1)), Heller's Theorem [4, 2.5] tells us that the map sending any $\mathbf{R}H$-lattice $\mathbf{L}$ to its completion $\mathbf{L}^*$ induces a bijection of the isomorphism classes of $\mathbf{R}H$-lattices onto those of $\mathbf{R}^*H$-lattices. Clearly any free $\mathbf{R}^*H$-lattice is the completion of a free $\mathbf{R}H$-lattice. Because completion preserves direct sums, we conclude that any projective $\mathbf{R}^*H$-lattice (i.e., any direct summand of a free $\mathbf{R}^*H$-lattice) is the completion of some projective $\mathbf{R}H$-lattice.

We may identify $\overline{\mathbf{F}} = \mathbf{R}/\mathbf{p}$ with the residue class field $\mathbf{R}^*/\mathbf{p}^*$ of $\mathbf{R}^*$. Since $\mathbf{R}^*$ is complete, there is some projective $\mathbf{R}^*H$-lattice $\mathbf{P}^*$ such that $\mathbf{P}^*/\mathbf{p}^*\mathbf{P}^*$ is isomorphic to the projective $\overline{\mathbf{F}}H$-lattice $\mathbf{Q}$. As we saw above, $\mathbf{P}^*$ is isomorphic to the completion of some projective $\mathbf{R}H$-lattice $\mathbf{P}$. Then $\overline{\mathbf{P}} = \mathbf{P}/\mathbf{p}\mathbf{P}$ is isomorphic to both $\mathbf{P}^*/\mathbf{p}^*\mathbf{P}^*$ and $\mathbf{Q}$ as an $\overline{\mathbf{F}}H$-lattice. Q.E.D.

Once we can lift projective $\overline{\mathbf{F}}H$-lattices to projective $\mathbf{R}H$-lattices, all the standard results about $\mathbf{p}$-adic lattices become available. As an example we have the following lemma from [5].

Lemma 1.4. *Suppose that* H *is a subgroup of* G*, that* $\mathbf{L}$ *is an* $\mathbf{R}H$*-lattice, and that* $\mathbf{Q}$ *is a projective* $\overline{\mathbf{F}}H$*-lattice dividing* $\overline{\mathbf{L}}$*. Then there is some projective* $\mathbf{R}H$*-lattice* $\mathbf{P}$ *such that* $\overline{\mathbf{P}}$ *is* $\overline{\mathbf{F}}H$*-isomorphic to* $\mathbf{Q}$*. Furthermore, any such* $\mathbf{P}$ *divides* $\mathbf{L}$.

Proof. Lemma 1.3 gives us some projective $\mathbf{R}H$-lattice $\mathbf{P}$ whose residual $\overline{\mathbf{F}}H$-lattice $\overline{\mathbf{P}}$ is isomorphic to $\mathbf{Q}$. Once we know that such a $\mathbf{P}$ exists, the rest of the proof of [5, Lemma 1] can be followed almost word for word to prove the rest of the present lemma. Q.E.D.

The preceding lemma allows us to characterize both projective and projective-free $\mathbf{R}H$-lattices by their residuals.

Proposition 1.5. *Let* H *be any subgroup of* G*, and* $\mathbf{L}$ *be any* $\mathbf{R}H$*-lattice. Then* $\mathbf{L}$ *is projective or projective-free if and only if its residual* $\overline{\mathbf{F}}H$*-lattice* $\overline{\mathbf{L}}$ *is respectively projective or projective-free.*

Proof. If the finitely-generated $\mathbf{R}H$-module $\mathbf{L}$ is projective, then it divides the direct sum $(\mathbf{R}H)^n$ of n copies of the regular $\mathbf{R}H$-module $\mathbf{R}H$, for some integer $n > 0$. It follows that $\overline{\mathbf{L}}$ divides the direct sum $(\overline{\mathbf{F}}H)^n$ of n copies of $\overline{\mathbf{F}}H$. So $\overline{\mathbf{L}}$ is a projective $\overline{\mathbf{F}}H$-lattice.

Conversely, if $\overline{\mathbf{L}}$ is $\overline{\mathbf{F}}H$-projective, then Lemma 1.4 with $\mathbf{Q} = \overline{\mathbf{L}}$ gives us some projective $\mathbf{R}H$-lattice $\mathbf{P}$ dividing $\mathbf{L}$ such that $\overline{\mathbf{P}}$ is $\overline{\mathbf{F}}H$-isomorphic to $\overline{\mathbf{L}}$. This can only happen when $\mathbf{L} \simeq \mathbf{P}$ is projective. Thus $\mathbf{L}$ is projective if and only if $\overline{\mathbf{L}}$ is projective.

If some non-zero projective $\mathbf{R}H$-lattice $\mathbf{P}$ divides $\mathbf{L}$, then its residual $\overline{\mathbf{F}}H$-lattice $\overline{\mathbf{P}}$ is non-zero and divides $\overline{\mathbf{L}}$. We saw above that $\overline{\mathbf{P}}$ is projective. Hence $\overline{\mathbf{L}}$ is not projective-free whenever $\mathbf{L}$ is not projective-free.

Conversely, suppose that some non-zero projective $\overline{\mathbf{F}}H$-lattice $\mathbf{Q}$ divides $\overline{\mathbf{L}}$. Then Lemma 1.4 gives us some projective $\mathbf{R}H$-lattice $\mathbf{P}$ dividing $\mathbf{L}$ such that $\overline{\mathbf{P}} \simeq \mathbf{Q} \neq 0$. Evidently $\mathbf{P}$ is not zero. Thus $\mathbf{L}$ is not projective-free if and only if $\overline{\mathbf{L}}$ is not projective-free. Q.E.D.

Another consequence of Lemma 1.4 is the standard correspondence between projective $\mathbf{R}H$-lattices and projective $\overline{\mathbf{F}}H$-lattices.

Proposition 1.6. *If H is a subgroup of G, then there is a one to one correspondence between all isomorphism classes of indecomposable projective $\mathbf{R}H$-lattices $\mathbf{P}$ and all isomorphism classes of indecomposable projective $\overline{\mathbf{F}}H$-lattices $\mathbf{Q}$. Here the isomorphism class of $\mathbf{P}$ corresponds to that of $\mathbf{Q}$ if and only if $\overline{\mathbf{P}}$ is $\overline{\mathbf{F}}H$-isomorphic to $\mathbf{Q}$.*

Proof. Any projective $\mathbf{R}H$-lattice $\mathbf{P}$ has a projective residual $\overline{\mathbf{F}}H$-lattice $\overline{\mathbf{P}}$ by Proposition 1.5. Any projective $\overline{\mathbf{F}}H$-lattice $\mathbf{Q}$ is isomorphic to such a residual $\overline{\mathbf{P}}$ by Lemma 1.3. If $\mathbf{P}_0$ is also a projective $\mathbf{R}H$-lattice, then any isomorphism $\mathbf{P} \simeq \mathbf{P}_0$ of $\mathbf{R}H$-lattices induces an isomorphism $\overline{\mathbf{P}} \simeq \overline{\mathbf{P}_0}$ of residual $\overline{\mathbf{F}}H$-lattices. So we only need show that $\mathbf{P}$ is $\mathbf{R}H$-isomorphic to $\mathbf{P}_0$ whenever $\overline{\mathbf{P}}$ is $\overline{\mathbf{F}}H$-isomorphic to $\overline{\mathbf{P}_0}$. But in that case Lemma 1.4, with $\mathbf{P}_0$ and $\overline{\mathbf{P}_0}$ in place of $\mathbf{L}$ and $\mathbf{Q}$, respectively, implies that $\mathbf{P}$ divides $\mathbf{P}_0$. Since $\overline{\mathbf{P}}$ is isomorphic to $\overline{\mathbf{P}_0}$, this can only happen when $\mathbf{P}$ is isomorphic to $\mathbf{P}_0$. Q.E.D.

§2. Green Correspondents

Let $\mathbf{S}$ be either $\mathbf{R}$ or $\overline{\mathbf{F}}$. Then any integer n relatively prime to the characteristic p of $\overline{\mathbf{F}} = \mathbf{R}/\mathbf{p}$ has an image $n1_{\mathbf{S}}$ which is a unit of $\mathbf{S}$. This and the Krull-Schmidt property are enough to imply all of Green's theory in [2] and [3] for $\mathbf{S}H$-lattices.

We're going to apply his theory when G has subgroups P and N satisfying

(2.1) *P is a Sylow p-subgroup of G, and N is its normalizer $\mathrm{N}_G(P)$ in G. Furthermore, the intersection $P \cap P^\sigma$ of P with its conjugate $P^\sigma = \sigma^{-1}P\sigma$ by any $\sigma \in G - N$ is the trivial subgroup* 1 *of G.*

Of course this last condition just says that P is a *trivial intersection subgroup* of G. Green's correspondence in this case simplifies to

Proposition 2.2. *If* (2.1) *holds and $\mathbf{S}$ is either $\mathbf{R}$ or $\overline{\mathbf{F}}$, then there is a one to one correspondence between all isomorphism classes of projective-free $\mathbf{S}G$-lattices $\mathbf{L}$ and all isomorphism classes of projective-free $\mathbf{S}N$-lattices $\mathbf{K}$. Here the isomorphism class of $\mathbf{L}$ corresponds to that of $\mathbf{K}$ if and only if $\mathbf{L}$ is isomorphic to the projective-free part $(\mathbf{K}^G)_{\mathrm{pf}}$ of the $\mathbf{S}G$-lattice $\mathbf{K}^G$ induced by $\mathbf{K}$. This happens if and only if $\mathbf{K}$ is isomorphic to the projective-free part $(\mathbf{L}_N)_{\mathrm{pf}}$ of the $\mathbf{S}N$-lattice $\mathbf{L}_N$ restricted from $\mathbf{L}$.*

Proof. Because $\mathbf{S}H$-lattices have the Krull-Schmidt property, for any subgroup H of G, we may apply all the arguments in [3] to our present situation. Following the notation of that paper as closely as possible, we denote by $a(H)$ the Green ring for the $\mathbf{S}H$-lattices. So $a(H)$ is generated as an additive group by the Green symbols $(\mathbf{U})$, one for each $\mathbf{S}H$-lattice $\mathbf{U}$, subject only to the relations that $(\mathbf{U}) = (\mathbf{U}')$ whenever $\mathbf{U}$ and $\mathbf{U}'$ are isomorphic $\mathbf{S}H$-lattices, and that $(\mathbf{U}) + (\mathbf{U}') = (\mathbf{U} \oplus \mathbf{U}')$ for any $\mathbf{S}G$-lattices $\mathbf{U}$ and $\mathbf{U}'$. (Multiplication in $a(H)$ is irrelevant to our purposes.) The Krull-Schmidt property implies that $a(H)$ is a free additive group with one basis element $(\mathbf{U})$ for each isomorphism class of indecomposable $\mathbf{S}H$-lattices $\mathbf{U}$. Those $(\mathbf{U})$ in this basis for which $\mathbf{U}$ is projective-free form a basis for an additive subgroup $a_{\mathrm{pf}}(H)$ of $a(H)$. Those for which $\mathbf{U}$ is projective form a basis for another additive subgroup $a_{\mathrm{pr}}(H)$. Furthermore, $a(H)$ is the direct sum

$$a(H) = a_{\mathrm{pf}}(H) \oplus a_{\mathrm{pr}}(H) \tag{2.3}$$

of these two subgroups.

As the subgroups D and H of G used in [3] we take the present P and N, respectively. Then $H = N$ contains the normalizer $\mathrm{N}_G(D) = N$ of $D = P$, as required on page 75 of [3]. The index $[G : D]$ of the Sylow p-subgroup $D = P$ is relatively prime to p. Hence its image $[G : D]1_{\mathbf{S}}$ is a unit of $\mathbf{S}$. As in [2, Theorem 2], this implies that any $\mathbf{S}G$-lattice is D-projective. So the additive subgroup $a_D(G)$, generated by the $(\mathbf{L})$ for D-projective $\mathbf{S}G$-lattices $\mathbf{L}$, is all of $a(G)$. Similarly, $a(N)$ is equal to its subgroup $a_D(N)$.

Because $D = P$ is a trivial intersection subgroup of G, the family $\mathbf{X} = \mathbf{X}(D, H)$ of all intersections $D^\sigma \cap D$ with $\sigma \in G - H = G - N$

just consists of the trivial subgroup 1 of G. Hence the additive subgroup $a_{\mathbf{X}}(G) = \sum_{D' \in \mathbf{X}} a_{D'}(G)$ of $a(G)$ is just the additive subgroup $a_1(G)$ generated by the $(\mathbf{P})$, where $\mathbf{P}$ runs over the 1-projective $\mathbf{S}G$-lattices. Since the 1-projective $\mathbf{S}G$-lattices are just the projective ones, we conclude that $a_{\mathbf{X}}(G) = a_{\mathrm{pr}}(G)$. This and (2.3) imply that

$$a_D(G)/a_{\mathbf{X}}(G) = a(G)/a_{\mathrm{pr}}(G) \simeq a_{\mathrm{pf}}(G)$$

as additive groups. Similarly

$$a_D(N)/a_{\mathbf{X}}(N) = a(N)/a_{\mathrm{pr}}(N) \simeq a_{\mathrm{pf}}(N).$$

In view of these natural isomorphisms, [3, Theorem 1] implies the present proposition. Q.E.D.

When $\mathbf{S}$ is either $\mathbf{R}$ or $\overline{\mathbf{F}}$, we say that a projective-free $\mathbf{S}G$-lattice $\mathbf{L}$ is an *$\mathbf{S}G$-Green correspondent* of a projective-free $\mathbf{S}N$-lattice $\mathbf{K}$ (or that $\mathbf{K}$ is an *$\mathbf{S}N$-Green correspondent* of $\mathbf{L}$) if the isomorphism classes of $\mathbf{L}$ and $\mathbf{K}$ correspond in the above proposition.

Proposition 2.4. *Let a projective-free $\mathbf{R}N$-lattice $\mathbf{K}$ be an $\mathbf{R}N$-Green correspondent of a projective-free $\mathbf{R}G$-lattice $\mathbf{L}$. Then both the residual $\overline{\mathbf{F}}N$-lattice $\overline{\mathbf{K}}$ of $\mathbf{K}$ and the residual $\overline{\mathbf{F}}G$-lattice $\overline{\mathbf{L}}$ of $\mathbf{L}$ are projective-free. Furthermore, $\overline{\mathbf{K}}$ is an $\overline{\mathbf{F}}N$-Green correspondent of $\overline{\mathbf{L}}$.*

Proof. Proposition 1.5 implies that both $\overline{\mathbf{K}}$ and $\overline{\mathbf{L}}$ are projective-free. The isomorphism $\mathbf{L}_N \simeq (\mathbf{L}_N)_{\mathrm{pf}} \oplus (\mathbf{L}_N)_{\mathrm{pr}}$ of $\mathbf{R}N$-lattices induces an isomorphism

$$\overline{\mathbf{L}_N} \simeq \overline{(\mathbf{L}_N)_{\mathrm{pf}}} \oplus \overline{(\mathbf{L}_N)_{\mathrm{pr}}}$$

of the $\overline{\mathbf{F}}N$-residuals of those lattices. By Proposition 1.5 the $\overline{\mathbf{F}}N$-lattices $\overline{(\mathbf{L}_N)_{\mathrm{pf}}}$ and $\overline{(\mathbf{L}_N)_{\mathrm{pr}}}$ are respectively projective-free and projective. Hence they are respectively isomorphic to the projective free part $(\overline{\mathbf{L}_N})_{\mathrm{pf}}$ and projective part $(\overline{\mathbf{L}_N})_{\mathrm{pr}}$ of $\overline{\mathbf{L}_N}$.

Since $\mathbf{K}$ is an $\mathbf{R}N$-Green correspondent of $\mathbf{L}$, it is $\mathbf{R}N$-isomorphic to $(\mathbf{L}_N)_{\mathrm{pf}}$. So $\overline{\mathbf{K}}$ is $\overline{\mathbf{F}}N$-isomorphic to $\overline{(\mathbf{L}_N)_{\mathrm{pf}}} \simeq (\overline{\mathbf{L}_N})_{\mathrm{pf}}$. But $\overline{\mathbf{L}_N}$ is equal to the restriction $\overline{\mathbf{L}}_N$ of $\overline{\mathbf{L}}$ to an $\overline{\mathbf{F}}N$-lattice. Hence $\overline{\mathbf{K}} \simeq (\overline{\mathbf{L}}_N)_{\mathrm{pf}}$ is an $\overline{\mathbf{F}}N$-Green correspondent of $\overline{\mathbf{L}}$. Q.E.D.

§3. Rationally Determined Lattices

Any $\mathbf{R}H$-lattice $\mathbf{L}$, for any subgroup H of G, extends to an $\mathbf{F}H$-lattice $\mathbf{FL} \simeq \mathbf{F} \otimes_{\mathbf{R}} \mathbf{L}$, determined to within isomorphisms by the fact that any basis for the free module $\mathbf{L}$ over $\mathbf{R}$ is also a basis for the vector space $\mathbf{F}\mathbf{L}$ over $\mathbf{F}$. Thus any $\mathbf{R}H$-lattice $\mathbf{L}$ determines both an $\overline{\mathbf{F}}H$-lattice $\overline{\mathbf{L}} = \mathbf{L}/(\mathbf{pL})$ and an $\mathbf{F}H$-lattice $\mathbf{FL}$. Since $\overline{\mathbf{F}}$ and $\mathbf{F}$ are the two "domains of rationality" associated with $\mathbf{R}$, it is reasonable to make the

Definition 3.1. An $\mathbf{R}H$-lattice $\mathbf{L}$ is *rationally determined* if it is determined to within isomorphisms by its associated $\overline{\mathbf{F}}H$-lattice $\overline{\mathbf{L}}$ and $\mathbf{F}H$-lattice $\mathbf{FL}$, i.e., if $\mathbf{L}$ is $\mathbf{R}H$-isomorphic to any $\mathbf{R}H$-lattice $\mathbf{K}$ such that $\overline{\mathbf{L}}$ is $\overline{\mathbf{F}}H$-isomorphic to $\overline{\mathbf{K}}$ and $\mathbf{FL}$ is $\mathbf{F}H$-isomorphic to $\mathbf{FK}$.

The main observation of this note is

Theorem 3.2. *Suppose that* (1.1) *and* (2.1) *hold, that* $\mathbf{K}$ *is a projective-free* $\mathbf{R}N$*-lattice, and that* $\mathbf{L}$ *is an* $\mathbf{R}G$*-Green correspondent of* $\mathbf{K}$*. Then the projective-free* $\mathbf{R}G$*-lattice* $\mathbf{L}$ *is rationally determined if and only if the* $\mathbf{R}N$*-lattice* $\mathbf{K}$ *is rationally determined.*

Proof. Assume that $\mathbf{L}$ is rationally determined. We must show that $\mathbf{K}$ is rationally determined. In view of Definition 3.1 it suffices to prove that $\mathbf{K}$ is $\mathbf{R}N$-isomorphic to $\mathbf{K}_0$ whenever $\mathbf{K}_0$ is an $\mathbf{R}N$-lattice whose residual $\overline{\mathbf{F}}N$-lattice $\overline{\mathbf{K}_0}$ is isomorphic to $\overline{\mathbf{K}}$, and whose associated $\mathbf{F}N$-lattice $\mathbf{FK}_0$ is isomorphic to $\mathbf{FK}$.

The projective-free $\mathbf{R}N$-lattice $\mathbf{K}$ has a projective-free residual $\overline{\mathbf{F}}N$-lattice $\overline{\mathbf{K}}$ by Proposition 1.5. The isomorphic $\overline{\mathbf{F}}N$-lattice $\overline{\mathbf{K}_0}$ is also projective-free. So Proposition 1.5 implies that $\mathbf{K}_0$ is a projective-free $\mathbf{R}N$-lattice. Hence some projective-free $\mathbf{R}G$-lattice $\mathbf{L}_0$ is a Green correspondent of $\mathbf{K}_0$. Since the Green correspondence is the bijection of isomorphism classes in Proposition 2.2, we can prove that $\mathbf{K}$ is $\mathbf{R}N$-isomorphic to $\mathbf{K}_0$ by showing that $\mathbf{L}$ is $\mathbf{R}G$-isomorphic to $\mathbf{L}_0$. Because $\mathbf{L}$ is rationally determined, it will suffice to show that $\overline{\mathbf{L}}$ is $\overline{\mathbf{F}}G$-isomorphic to $\overline{\mathbf{L}_0}$, and that $\mathbf{FL}$ is $\mathbf{F}G$-isomorphic to $\mathbf{FL}_0$.

The isomorphic $\overline{\mathbf{F}}N$-lattices $\overline{\mathbf{K}} \simeq \overline{\mathbf{K}_0}$ induce isomorphic $\overline{\mathbf{F}}G$-lattices $\overline{\mathbf{K}}^G \simeq \overline{\mathbf{K}_0}^G$. Hence we have $\overline{\mathbf{F}}G$-isomorphisms

$$(\overline{\mathbf{K}}^G)_{\mathrm{pf}} \simeq (\overline{\mathbf{K}_0}^G)_{\mathrm{pf}} \quad \text{and} \quad (\overline{\mathbf{K}}^G)_{\mathrm{pr}} \simeq (\overline{\mathbf{K}_0}^G)_{\mathrm{pr}}. \tag{3.3}$$

By definition $(\overline{\mathbf{K}}^G)_{\mathrm{pf}}$ and $(\overline{\mathbf{K}_0}^G)_{\mathrm{pf}}$ are $\overline{\mathbf{F}}G$-Green correspondents of $\overline{\mathbf{K}}$ and $\overline{\mathbf{K}_0}$, respectively. So Proposition 2.4 tells us that $(\overline{\mathbf{K}}^G)_{\mathrm{pf}}$ is $\overline{\mathbf{F}}G$-isomorphic to the residual $\overline{\mathbf{L}}$ of the Green correspondent $\mathbf{L}$ of $\mathbf{K}$. Similarly $(\overline{\mathbf{K}_0}^G)_{\mathrm{pf}}$ is $\overline{\mathbf{F}}G$-isomorphic to $\overline{\mathbf{L}_0}$. Therefore the first isomorphism in (3.3) implies that $\overline{\mathbf{L}}$ is $\overline{\mathbf{F}}G$-isomorphic to $\overline{\mathbf{L}_0}$.

Evidently $\overline{\mathbf{K}}^G$ is $\overline{\mathbf{F}}G$-isomorphic to the residual $\overline{\mathbf{K}^G}$ of the $\mathbf{R}G$-lattice $\mathbf{K}^G$ induced by $\mathbf{K}$. As in the proof of Proposition 2.4, this implies that $(\overline{\mathbf{K}}^G)_{\mathrm{pr}}$ is $\overline{\mathbf{F}}G$-isomorphic to the residual $\overline{(\mathbf{K}^G)_{\mathrm{pr}}}$ of $(\mathbf{K}^G)_{\mathrm{pr}}$.

Similarly $(\overline{\mathbf{K}_0}^G)_{\mathrm{pr}}$ is $\overline{\mathbf{F}}G$-isomorphic to the residual $\overline{(\mathbf{K}_0^G)_{\mathrm{pr}}}$ of $(\mathbf{K}_0^G)_{\mathrm{pr}}$. So the second isomorphism in (3.3) implies that the projective $\mathbf{R}G$-lattices $(\mathbf{K}^G)_{\mathrm{pr}}$ and $(\mathbf{K}_0^G)_{\mathrm{pr}}$ have isomorphic $\overline{\mathbf{F}}G$-residuals. By Proposition 1.6 this forces $(\mathbf{K}^G)_{\mathrm{pr}}$ to be $\mathbf{R}G$-isomorphic to $(\mathbf{K}_0^G)_{\mathrm{pr}}$. It follows that $\mathbf{F}(\mathbf{K}^G)_{\mathrm{pr}}$ is $\mathbf{F}G$-isomorphic to $\mathbf{F}(\mathbf{K}_0^G)_{\mathrm{pr}}$.

The isomorphism $\mathbf{FK} \simeq \mathbf{FK}_0$ of $\mathbf{F}N$-lattices induces isomorphisms $\mathbf{F}(\mathbf{K}^G) \simeq (\mathbf{FK})^G \simeq (\mathbf{FK}_0)^G \simeq \mathbf{F}(\mathbf{K}_0^G)$ of $\mathbf{F}G$-lattices. Since $\mathbf{K}^G$ and $\mathbf{K}_0^G$ are $\mathbf{R}G$-isomorphic to $(\mathbf{K}^G)_{\mathrm{pf}} \oplus (\mathbf{K}^G)_{\mathrm{pr}}$ and $(\mathbf{K}_0^G)_{\mathrm{pf}} \oplus (\mathbf{K}_0^G)_{\mathrm{pr}}$, respectively, this gives us $\mathbf{F}G$-isomorphisms

$$\mathbf{F}(\mathbf{K}^G)_{\mathrm{pf}} \oplus \mathbf{F}(\mathbf{K}^G)_{\mathrm{pr}} \simeq \mathbf{F}(\mathbf{K}^G) \simeq \mathbf{F}(\mathbf{K}_0^G) \simeq \mathbf{F}(\mathbf{K}_0^G)_{\mathrm{pf}} \oplus \mathbf{F}(\mathbf{K}_0^G)_{\mathrm{pr}}.$$

We saw above that $\mathbf{F}(\mathbf{K}^G)_{\mathrm{pr}} \simeq \mathbf{F}(\mathbf{K}_0^G)_{\mathrm{pr}}$ as $\mathbf{F}G$-lattices. So the Krull-Schmidt property for $\mathbf{F}G$-lattices implies that $\mathbf{FL} \simeq \mathbf{F}(\mathbf{K}^G)_{\mathrm{pf}}$ is $\mathbf{F}G$-isomorphic to $\mathbf{FL}_0 \simeq \mathbf{F}(\mathbf{K}_0^G)_{\mathrm{pf}}$.

We have now shown that $\overline{\mathbf{L}}$ is $\overline{\mathbf{F}}G$-isomorphic to $\overline{\mathbf{L}_0}$, and that $\mathbf{FL}$ is $\mathbf{F}G$-isomorphic to $\mathbf{FL}_0$. As we remarked above, this is enough to imply that $\mathbf{K}$ is rationally determined whenever $\mathbf{L}$ is. A similar argument, using restriction of lattices from G to N instead of induction from N to G, shows that the converse statement also holds. Q.E.D.

Surprisingly enough, for any subgroup H of G there are some important rationally determined $\mathbf{R}H$-lattices. After embedding an arbitrary $\mathbf{R}H$-lattice $\mathbf{L}$ in an $\mathbf{F}H$-lattice $\mathbf{FL}$, we can multiply it by any central idempotent e in $\mathbf{F}H$, obtaining an $\mathbf{R}H$-sublattice $\mathbf{L}e$ spanning the $\mathbf{F}H$-submodule $(\mathbf{FL})e = \mathbf{F}(\mathbf{L}e)$ of $\mathbf{FL}$.

Proposition 3.4. *Suppose that H is a subgroup of G, that $\mathbf{P}$ is a projective $\mathbf{R}H$-lattice, and that e is a central idempotent of $\mathbf{F}H$. Then the $\mathbf{R}H$-lattice $\mathbf{L} = \mathbf{P}e$ is rationally determined.*

Proof. Let $\mathbf{K}$ be any $\mathbf{R}H$-lattice such that $\overline{\mathbf{K}}$ is $\overline{\mathbf{F}}H$-isomorphic to $\overline{\mathbf{L}}$ and $\mathbf{FK}$ is $\mathbf{F}H$-isomorphic to $\mathbf{FL}$. We must prove that $\mathbf{K}$ is $\mathbf{R}H$-isomorphic to $\mathbf{L}$.

Right multiplication by e is an $\mathbf{R}H$-epimorphism ρ of $\mathbf{P}$ onto $\mathbf{L} = \mathbf{P}e$. If we follow ρ by the natural epimorphism $\eta_{\mathbf{L}}$ of $\mathbf{L}$ onto $\overline{\mathbf{L}} = \mathbf{L}/(\mathbf{pL})$, and by some $\overline{\mathbf{F}}H$-isomorphism $\overline{\iota}$ of $\overline{\mathbf{L}}$ onto $\overline{\mathbf{K}}$, we obtain a homomorphism $\overline{\iota} \circ \eta_{\mathbf{L}} \circ \rho \colon \mathbf{P} \to \overline{\mathbf{K}}$ of $\mathbf{R}H$-modules. We also have the natural epimorphism $\eta_{\mathbf{K}}$ of $\mathbf{K}$ onto $\overline{\mathbf{K}} = \mathbf{K}/(\mathbf{pK})$ as $\mathbf{R}H$-modules. Because $\mathbf{P}$ is a projective $\mathbf{R}H$-module, there is some homomorphism $\theta \colon \mathbf{P} \to \mathbf{K}$ of $\mathbf{R}H$-lattices such that

$$\eta_{\mathbf{K}} \circ \theta = \overline{\iota} \circ \eta_{\mathbf{L}} \circ \rho \colon \mathbf{P} \to \overline{\mathbf{K}}. \tag{3.5}$$

The $\mathbf{R}H$-homomorphism $\theta\colon \mathbf{P} \to \mathbf{K}$ extends by $\mathbf{F}$-linearity to an $\mathbf{F}H$-homomorphism $\theta^{\mathbf{F}}\colon \mathbf{FP} \to \mathbf{FK}$. This last homomorphism commutes with multiplication by the central idempotent e of $\mathbf{F}H$. So it restricts to an $\mathbf{R}H$-homomorphism $\iota = (\theta^{\mathbf{F}})_{\mathbf{L}}$ of $\mathbf{L} = \mathbf{P}e$ into $\mathbf{K}e$. But right multiplication by the idempotent e is the identity on both $\mathbf{L} = \mathbf{P}e$ and $\mathbf{FL} = \mathbf{FP}e$. Hence it is the identity on both the $\mathbf{F}H$-lattice $\mathbf{FK}$ isomorphic to $\mathbf{FL}$, and on the $\mathbf{R}H$-sublattice $\mathbf{K}$ of $\mathbf{FK}$. We conclude that ι is an $\mathbf{R}H$-homomorphism of $\mathbf{L}$ into $\mathbf{K} = \mathbf{K}e$. Since the epimorphism ρ in the equation (3.5) is just multiplication by e, that equation implies that

$$\bar{\iota} \circ \eta_{\mathbf{L}} = \eta_{\mathbf{K}} \circ \iota\colon \mathbf{L} \to \overline{\mathbf{K}}.$$

Thus $\iota\colon \mathbf{L} \to \mathbf{K}$ is a homomorphism of $\mathbf{R}H$-lattices inducing the isomorphism $\bar{\iota}\colon \overline{\mathbf{L}} \to \overline{\mathbf{K}}$ of $\overline{\mathbf{F}}H$-lattices. Hence ι is an $\mathbf{R}H$-isomorphism of $\mathbf{L}$ onto $\mathbf{K}$. Q.E.D.

The $\mathbf{R}H$-lattice $\mathbf{P}e$ in the preceding proposition is projective-free in the most important case.

Proposition 3.6. *Suppose that H is a subgroup of G, that $\mathbf{P}$ is an indecomposable projective $\mathbf{R}H$-lattice, and that e is a central idempotent of $\mathbf{F}H$. Then the $\mathbf{R}H$-lattice $\mathbf{P}e$ is either equal to $\mathbf{P}$ or projective-free.*

Proof. Assume that $\mathbf{P}e$ is not projective-free. We must show that it is equal to $\mathbf{P}$, i.e., that right multiplication by e is the identity on $\mathbf{P}$. Since right multiplication by the idempotent e is certainly the identity on $\mathbf{P}e$, it will suffice to show that $\mathbf{P}$ is $\mathbf{R}H$-isomorphic to $\mathbf{P}e$.

Because $\mathbf{P}e$ is not projective-free, it is divisible by some non-zero projective $\mathbf{R}H$-lattice $\mathbf{Q}$. So there is some $\mathbf{R}H$-epimorphism π of $\mathbf{P}e$ onto $\mathbf{Q}$. Right multiplication by e is an $\mathbf{R}H$-epimorphism ρ of $\mathbf{P}$ onto $\mathbf{P}e$. Hence the composite map $\pi \circ \rho\colon \mathbf{P} \to \mathbf{Q}$ is an epimorphism of $\mathbf{R}H$-lattices. Since $\mathbf{Q}$ is $\mathbf{R}H$-projective, there is some $\mathbf{R}H$-monomorphism $\mu\colon \mathbf{Q} \to \mathbf{P}$ such that $\pi \circ \rho \circ \mu$ is the identity map of $\mathbf{Q}$ onto itself. In particular, the non-zero $\mathbf{R}H$-lattice $\mathbf{Q}$ divides the indecomposable $\mathbf{R}H$-lattice $\mathbf{P}$. This can only happen when $\pi \circ \rho$ is an isomorphism of $\mathbf{P}$ onto $\mathbf{Q}$, with μ as its inverse. But then the epimorphism ρ must be an $\mathbf{R}H$-isomorphism of $\mathbf{P}$ onto $\mathbf{P}e$. As we remarked above, this is enough to prove the proposition. Q.E.D.

Putting the preceding results together, we obtain

Theorem 3.7. *Suppose that* (1.1) *and* (2.1) *hold, that $\mathbf{P}$ is an indecomposable projective $\mathbf{R}G$-lattice, and that e is a central idempotent of $\mathbf{F}G$ such that $\mathbf{P}e \neq \mathbf{P}$. Then the $\mathbf{R}G$-lattice $\mathbf{P}e$ is projective-free, and its $\mathbf{R}N$-Green correspondents are rationally determined.*

Proof. The $\mathbf{R}G$-lattice $\mathbf{P}e$ is projective-free by Proposition 3.6, and is rationally determined by Proposition 3.4. So its $\mathbf{R}N$-Green correspondents are rationally determined by Theorem 3.2. Q.E.D.

References

[1] E. C. Dade, Counting characters in blocks, I, Invent. Math., **109** (1992), 187–210.

[2] J. A. Green, On the indecomposable representations of a finite group, Math. Z., **70** (1959), 430–445.

[3] J. A. Green, A transfer theorem for modular representations, J. Algebra, **1** (1964), 73–84.

[4] A. Heller, On group representations over a valuation ring, Proc. Natl. Acad. Sci. USA, **47** (1961), 1194–1197.

[5] J. G. Thompson, Vertices and sources, J. Algebra, **6** (1967), 1–6.

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