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Representation Theory in Characteristic p

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Let k be an algebraically closed field of characteristic p. (Thus, p is either 0 or a prime number.) Let G be a group which is at the same time an affine algebraic variety over k (that is, an algebraic group over k). A *representation* of G is a homomorphism $\rho : G \longrightarrow GL(V)$ of G into the group of automorphisms of a finite dimensional k-vector space V which is at the same time a morphism of algebraic varieties. We also say that V is a *G-module.* We say that V is *irreducible* if $V \neq 0$ and there is no subspace V' of V (other than 0 or V) such that $\rho(g)V' \subset V'$ for all $g \in G$.

Let $\mathfrak{g}$ be the Lie algebra of G. A *representation* of $\mathfrak{g}$ is a k-linear map $\tau : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ where V is a finite dimensional k-vector space such that

$$\tau([\xi, \xi']) = \tau(\xi)\tau(\xi') - \tau(\xi')\tau(\xi)$$

for all $\xi, \xi' \in \mathfrak{g}$. We also say that V is a *$\mathfrak{g}$-module.* The notion of irreducibility of a $\mathfrak{g}$-module is defined in the same way as in the group case.

We will assume that G is connected, almost simple (that is, G has finite centre and G modulo its centre is a simple group) and simply connected (in a suitable sense). Chevalley [C] proved the remarkable result that the classification of such G is the same as the classification of simple complex Lie algebras (achieved by É. Cartan and Killing). Thus, G must be a special linear group, a symplectic group, a spin group or one of five exceptional groups.

The problem that we will discuss in this paper is that of classifying the irreducible G-modules and $\mathfrak{g}$-modules and that of understanding as much as possible the structure of those irreducible modules. Work on these problems have occupied mathematicians throughout much of this century. We will review some of this work. Towards the end of the paper we will engage in speculation about possible future directions.

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§1. Algebraic groups

1.1. Characteristic 0

In this subsection we assume that $p = 0$. In this case, taking the differential defines an equivalence between the category of G-modules and the category of $\mathfrak{g}$-modules. É. Cartan [Ca] showed that the isomorphism classes of irreducible $\mathfrak{g}$-modules (hence also those of irreducible G-modules) are naturally indexed by a set X given by the intersection of an open cone $E^{>}$ in an euclidean vector space E with a lattice in E. Let V_x be the irreducible G-module corresponding to $x \in X$. H. Weyl [W] gave an explicit character formula and a dimension formula for V_x. An algebraic geometric construction for V_x was given by Borel and Weil (see [S]).

For example, for $G = SL_n(k)$, we can take

$$E = \mathbf{R}^n/\mathbf{R}(1, \dots, 1),$$
$$X = \{(a_1, \dots, a_n) \in \mathbf{Z}^n/\mathbf{Z}(1, \dots, 1) \mid a_1 > a_2 > \cdots > a_n\}.$$

If $G = SL_2(k)$ and $(a, 0) \in X$, we can take $V_{a,0}$ to be the space of homogeneous polynomials in two variables t_1, t_2 with coefficients in k of degree $a - 1$. (This is naturally a G-module.)

1.2. Late 1950's to early 1970's

In the rest of this paper we assume that p is > 0 and sufficiently large, unless we specify otherwise.

Taking the differential defines a functor from the category of G-modules to the category of $\mathfrak{g}$-modules but this is by no means an equivalence. A very small part of the characteristic 0 theory survives: the definition of V_x (for each $x \in X$) given by the Borel-Weil construction still makes sense. It provides a G-module which has the same structure (in particular the same dimension) as in characteristic 0, although it is no longer irreducible in general. Chevalley [C] has shown that V_x contains a unique irreducible G-submodule, denoted by L_x and that $\{L_x \mid x \in X\}$ is a set of representatives for the isomorphism classes of irreducible G-modules.

For example, if $G = SL_2(k)$, then $V_{a,0}$ is defined as in 1.1 in terms of our field k and $L_{a,0}$ is the subspace of $V_{a,0}$ spanned by $(a - 1)$-th powers of linear polynomials. Note that $L_{a,0}$ is not necessarily equal to $V_{a,0}$. The following table gives (for $p = 3$) the value $\dim V_{a,0} = a$ for $a = 1, 2, \dots, 20$ (in the first row) and the corresponding value of $\dim L_{a,0}$

(in the second row):

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	3	2	4	6	3	6	9	2	4	6	4	8	12	6	12	18	3	6

Note that $\dim L_{a,0}$ is linear in a, for a in a fixed interval $(lp, (l+1)p)$ (even in $(lp, (l+1)p]$). The ends of these intervals (the points lp) make sense for $SL_n(k)$ as well; they form the family $\mathcal{F}$ of affine hyperplanes in E:

$$x_i - x_j = lp \quad \text{for some } i \neq j \text{ in } [1, n] \text{ and some integer } l.$$

$\mathcal{F}$ is similarly defined for any G (in terms of roots). For each $H \in \mathcal{F}$ let $\sigma_H : E \to E$ be the map which takes any point of E to its mirror image with respect to H. Let W be the group of affine transformations of E generated by the reflections σ_H. We now remove from E the points that lie in at least one hyperplane of $\mathcal{F}$. The resulting set is a disjoint union of open simplices called *alcoves* (the analogues of the open intervals $(lp, (l+1)p)$ for $SL_2(k)$). Let $\mathcal{A}$ be the set of alcoves; let $\mathcal{A}^>$ be the set of alcoves contained in $E^>$.

For any $x \in X$ which does not lie in any hyperplane of $\mathcal{F}$ and any $A \in \mathcal{A}^>$ we denote by x_A the unique point of $X \cap A$ which is in the W-orbit of x.

Let R_G be the free abelian group with basis $\{L_x \mid x \in X\}$. Any G-module M gives rise to an element

$$\sum_x (L_x : M) L_x \in R_G$$

where $(L_x : M)$ is the number of times that L_x appears in a composition series of M. We sometimes write M instead of $\sum_x (L_x : M) L_x \in R_G$. Thus the elements $V_x \in R_G$ are well defined for $x \in X$. They again form a basis of R_G.

The following result has been conjectured by Verma [V] and proved by Jantzen [J1].

There exists a function $\mathcal{A}^> \times \mathcal{A}^> \to \mathbf{Z}$ denoted by $A, B \mapsto (A, B)$ such that for any $B \in \mathcal{A}^>$ and any $x \in X \cap B$ we have

$$L_x = \sum_{A \in \mathcal{A}^>} (A : B) V_{x_A} \in R_G.$$

(In particular, for fixed B we have $(A, B) = 0$ for all but finitely many A.) Since the V_{x_A} can be considered as known, this shows that the dimension of L_x (and other information about L_x) will be known provided the quantities (A, B) are known. Jantzen also showed that even the L_x with $x \in X$ on some hyperplane in $\mathcal{F}$ are explicitly determined by the (A, B).

We define $A^+ \in \mathcal{A}^>$ by the condition that 0 is in the closure of A^+. Let h be the Coxeter number of G. (Thus, $h = n$ for $SL_n(k)$). If the (A, B) are known whenever $A, B \in \mathcal{A}^>$ are contained in $(p-h)A^+$ then, by a tensor product theorem of Steinberg [St], the (A, B) will be determined for general A, B.

Thus, the problem of understanding L_x is reduced to the determination of the unknown quantities (A, B) with A, B contained in $(p-h)A^+$.

1.3. Late 1970's

W together with the reflections σ_H (where $H \in \mathcal{F}$ contains a codimension 1 face of $A^- = -A^+$) is a *Coxeter group* of affine type. Hence by a general definition which applies to any Coxeter group, to any two elements $y, w \in W$ one can attach a polynomial $P_{y,w} \in \mathbf{Z}[v]$ (v an indeterminate) as in [KL1].

For each $w \in W$ we set $A_w = w(A^-)$. Then $w \mapsto A_w$ is a bijection $W \xrightarrow{\sim} \mathcal{A}$. The following was conjectured in [L1].

(a) *Assume that $A, B \in \mathcal{A}$ are contained in $(p-h)A^+$. Define $y, w \in W$ by $A = A_y$, $B = A_w$. Let $m_{A,B}$ be the number of hyperplanes in $\mathcal{F}$ that separate A from B. Then*

$$(A, B) = (-1)^{m_{A,B}} P_{y,w}(1).$$

(See [KL1] for a precursor of this conjecture, which involves only the $P_{y,w}$ where y, w preserve 0.)

1.4. Late 1980's

Around 1985, quantum groups appeared on the scene, due to the work of Drinfeld and Jimbo. These were some strange deformations of complex algebraic groups depending on a parameter $v \in \mathbf{C}^*$. In the original definition v had to be generic, but it turned out that a good definition can be given for arbitrary $v \in \mathbf{C}^*$. The case where v is root of 1 was particularly interesting and in [L3] I found that in this case the representation theory of the quantum group is very similar to that of G in characteristic > 0. For example, in the case of SL_2, in the table in 1.2 one can add a new row (in between the two rows of the original table) giving the value of the dimension of the quantum analogue of $L_{a,0}$ in the case where $v^3 = 1$, $v \neq 1$. One obtains

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	3	2	4	6	3	6	9	4	8	12	5	10	15	6	12	18	7	14
1	2	3	2	4	6	3	6	9	2	4	6	4	8	12	6	12	18	3	6

Note that the new row (quantum case) has values which are sandwiched between those in the first and third row. This suggests that the quantum group is some kind of stepping stone between the algebraic groups in characteristic 0 and p. This led me in 1988 to formulate four steps in a possible proof of Conjecture 1.3 (a) (at least in types A, D, E).

(I) Show that the process of "reduction modulo p" from representations of the quantum group at a p-th root of 1 to representations of G in characteristic p, takes irreducible modules to irreducible modules (in a suitable range).

(II) Show that the representations of the quantum group at a p-th root of 1 are closely related to certain representations of the corresponding affine Lie algebra with central charge $-p-h$.

(III) Show that the characters of the irreducible highest weight representations of the affine Lie algebra with central charge $-p-h$ can be related to the intersection cohomology of Schubert varieties in an affine flag manifold.

(IV) Show that the intersection cohomology in (III) is computed by the polynomials $P_{y,w}$ of [KL1].

1.5. Early 1990's

Steps (I), (II), (III) were attacked by three teams on three continents. (Step (IV) was already known from [KL2]. A simpler version of step (III) dealing with finite dimensional Lie algebras was also known since the early 1980's.) Thus, (I) was solved (for p in the complement of an unknown finite set) by Andersen, Jantzen and Soergel [AJS]; (II) was solved (for p in the complement of a known finite set) in [KL3]; (III) was solved by Kashiwara and Tanisaki [KT1].

For G of type B, C, F_4, G_2 the four steps had to be modified (see [L4]). In the modified framework, steps (I) and (II) were covered by the existing works [AJS], [KL3]. But the solution of step (II) required a new work of Kashiwara and Tanisaki [KT2]; even step (IV) presented a new problem (solved in [L4]).

The combination of these works provides a solution of the Conjecture 1.3 (a) hence of the problem of describing the structure of the irreducible G-modules L_x (for p in the complement of an unknown finite set).

The last restriction on p is very unsatisfactory. It would be very desirable if somebody will remove the restriction on p from [AJS] and also the (milder) restriction on p from [KL3, IV].

§2. Lie algebras

2.1.

The representation theory of $\mathfrak{g}$ is in a much poorer state than that of G. Here not even a classification of the irreducible modules is available. But a conjectural picture is beginning to emerge, promising a very rich theory.

2.2. Early 1970's

There is a canonical map $\xi \mapsto \xi^{[p]}$ of $\mathfrak{g}$ into itself. For example, if $G = SL_n(k)$ then $\mathfrak{g} = \mathrm{End}(k^n)$ and $\xi^{[p]}$ is the p-th power of ξ as an endomorphism of k^n.

Let $\epsilon : \mathfrak{g} \longrightarrow k$ be a linear form. Following Weisfeiler and Kac [WK], we consider the quotient U_ϵ of the enveloping algebra U of $\mathfrak{g}$ by the two-sided ideal generated by the central elements $\xi^p - \xi^{[p]} - \epsilon(\xi)^p 1$ for various $\xi \in \mathfrak{g}$. Then U_ϵ is a finite dimensional algebra and any simple $\mathfrak{g}$-module can be regarded as a module over U_ϵ for a unique ϵ as above [WK]. Hence the problem of understanding all $\mathfrak{g}$-modules is reduced to the problem of understanding all U_ϵ-modules for any linear function $\epsilon : \mathfrak{g} \rightarrow k$. This last problem for general ϵ can be reduced (see [WK]) to the special case where ϵ is nilpotent. (We identify a linear form on $\mathfrak{g}$ with an element of $\mathfrak{g}$ by the Killing form.)

2.3. Late 1970's to 1997

From now on we assume that $\epsilon \in \mathfrak{g}$ is nilpotent. Let C' be a maximal torus of the centralizer of ϵ in G and let $\bar{C}'$ be the image of C' in the adjoint group of G. Following an idea of Jantzen [J2] (see also [FP], [J4]) one can consider the category $\mathcal{C}_\epsilon$ of U_ϵ-modules which are also $\bar{C}'$-modules, the two module structures being compatible in a natural way; then one studies the simple objects of $\mathcal{C}_\epsilon$ instead of the simple U_ϵ-modules. In the case where ϵ is regular nilpotent inside a Levi subalgebra of some parabolic algebra, the classification of the simple objects of $\mathcal{C}_\epsilon$ has been obtained by Friedlander and Parshall [FP]. A conjecture which would describe much of the structure of these simple objects was given in [L5]. Some examples computed by Jantzen [J3] give support to the conjecture.

2.4. Speculation

Returning to a general nilpotent $\epsilon \in \mathfrak{g}$, we note that $\mathcal{C}_\epsilon$ is a direct sum of indecomposable categories, or blocks. Let us fix a generic block of $\mathcal{C}_\epsilon$. Let $\mathbf{I}$ be an indexing set for the simple objects in this block. For

$\mathbf{i} \in \mathbf{I}$, let $L_{\mathbf{i}}$ be the corresponding simple object of $\mathcal{C}_\epsilon$ and let $Q_{\mathbf{i}}$ be the corresponding indecomposable projective object. For $\mathbf{i} \in \mathbf{I}$ we have

$$Q_{\mathbf{i}} = \sum_{\mathbf{i}' \in \mathbf{I}} n_{\mathbf{i},\mathbf{i}'} L_{\mathbf{i}'}$$

in the appropriate Grothendieck group, where $n_{\mathbf{i},\mathbf{i}'} \in \mathbf{N}$ are zero for all but finitely many $\mathbf{i}'$.

For simplicity we assume that the centralizer of ϵ in the adjoint group of G is connected. In [L6, 14.4, 14.5] I stated a conjecture which gives a parametrization of $\mathbf{I}$ and an interpretation of the matrix $(n_{\mathbf{i},\mathbf{i}'})$ in terms of some apparently totally unrelated K-theoretic objects. (I thank Jantzen for his criticism of that conjecture.)

Here I will state a revised form of the original conjecture. To do this I will review some K-theoretic constructions from [L6].

Let $\mathbf{G}$ be an algebraic group of the same type as G, but over $\mathbf{C}$ instead of k. Let $\mathbf{g}$ be the Lie algebra of $\mathbf{G}$. Let r be the rank of $\mathbf{g}$. Let $\mathcal{B}$ be the variety of all Borel subalgebras of $\mathbf{g}$. Let $e \in \mathbf{g}$ be a nilpotent element of $\mathbf{g}$ of the same type as $\epsilon \in \mathfrak{g}$. We can complete e to an $\mathfrak{sl}_2$-triple (e,h,f) in $\mathbf{g}$. Let $\mathcal{B}_e = \{\mathfrak{b} \in \mathcal{B} \mid e \in \mathfrak{b}\}$.

Following Slodowy, we define Λ_e to be the variety consisting of all pairs $(y, \mathfrak{b})$ where $\mathfrak{b} \in \mathcal{B}$ and y is a nilpotent element of $\mathfrak{b}$ such that $[y - e, f] = 0$. Then $\mathcal{B}_e$ is naturally imbedded in Λ_e by $j : \mathfrak{b} \mapsto (e, \mathfrak{b})$. Moreover, Λ_e is a smooth irreducible variety. Let C be a maximal torus of the simultaneous centralizer of e, h, f in $\mathbf{G}$. Let $H = C \times \mathbf{C}^*$. We regard H as a subgroup of $\mathbf{G} \times \mathbf{C}^*$ as in [L6, 11.1].

Now $\mathbf{G} \times \mathbf{C}^*$ acts on $\mathcal{B}$ by $(g, \lambda) : \mathfrak{b} \mapsto Ad(g)\mathfrak{b}$ and on $\mathbf{g}$ by $(g, \lambda) : y \mapsto \lambda^{-2} Ad(g) y$.

This restricts to an H-action on $\mathcal{B}$ and one on $\mathbf{g}$. The product H-action on $\mathbf{g} \times \mathcal{B}$ leaves the subvariety Λ_e stable and leaves the subvariety $\mathcal{B}_e$ of Λ_e stable. Note also that the H-action on $\mathbf{g}$ restricts to an H-action on the centralizer $\mathfrak{z}$ of f in $\mathbf{g}$.

Hence the equivariant K-groups $K_H(\mathcal{B}_e)$, $K_H(\Lambda_e)$ (based on H-equivariant coherent sheaves on $\mathcal{B}_e$, Λ_e) are well defined (these are naturally R_H-modules where R_H is the representation ring of H.) Moreover, we can form

$$\mathbf{d} = \sum_{t \geq 0} (-1)^t \mathfrak{z}^{(t)} \in R_H$$

where $\mathfrak{z}^{(t)}$ is the t-th exterior power of $\mathfrak{z}$. One can show that

(a) *the R_H-linear map $j_* : K_H(\mathcal{B}_e) \longrightarrow K_H(\Lambda_e)$ induced by the closed imbedding $j : \mathcal{B}_e \longrightarrow \Lambda_e$ induces an isomorphism after tensoring by $R_H[\mathbf{d}^{-1}]$.*

Let $(\ |\ )_{\Lambda_e}$ be the pairing on $K_H(\Lambda_e)$ defined in [L6, 12.16]. By definition, this pairing takes values in the quotient field of $\mathfrak{R}$; but, due to (a), its values are actually in $R_H[\mathbf{d}^{-1}]$. Let

$$^{-}: K_K(\Lambda_e) \to K_H(\Lambda_e)$$

be as in [L6, 12.16]. (The definition involves Serre-Grothendieck duality on Λ_e.)

(The definition of $(\ |\ )_{\Lambda_e}$ and that of $^{-}$ involve an integer $d(e)$. In [L6, 12.9] one should replace $d(e) = (1/2)\dim Ad(L)e$ by $d(e) = (1/2)\dim Ad(G)e$.)

Note that $R_H[\mathbf{d}^{-1}]$ is naturally imbedded in the ring $\mathfrak{U}_H$ of power series in an indeterminate v^{-1} with coefficients in the ring R_C. (Here v is identified with the standard generator of $R_{\mathbf{C}^*}$.) Indeed, $R_H = R_C[v, v^{-1}] \subset \mathfrak{U}_H$ and $\mathbf{d}$ is a product of factors of form $1 - v^c\alpha$ with $c < 0$ and α a character of C; hence $\mathbf{d}^{-1} \in R_C[[v^{-1}]]$. Let

$$\delta : \mathfrak{U}_H \to \mathbf{Z}((v^{-1}))$$

be the group homomorphism defined by $\sum_{n\in\mathbf{Z}} p_n v^n \mapsto \sum_{n\in\mathbf{Z}} \tilde{p}_n v^n$ where $p_n \in R_C$ and $p \mapsto \tilde{p}$ is the group homomorphism which sends a non-trivial representation of C to 0 and sends the unit representation of C to 1.

Let $\bar{C}$ be the image of C into the adjoint group of $\mathbf{G}$ and let $\bar{H} = \bar{C} \times \mathbf{C}^*$.

Following [L6, 12.18] we define $\mathbf{B}^{\pm}_{\Lambda_e}$ to be the set of all elements $\xi \in K_H(\Lambda_e)$ such that

$$\bar{\xi} = \xi \quad \text{and} \quad \delta(\xi\,|\,\xi)_{\Lambda_e} \in 1 + v^{-1}\mathbf{Z}[[v^{-1}]].$$

Following [L6, 12.22] we define $\mathbf{B}^{\pm}_{\Lambda_e,\mathrm{ad}}$ as the intersection of $\mathbf{B}^{\pm}_{\Lambda_e}$ with

$$\mathrm{Im}(K_{\bar{H}}(\Lambda_e) \to K_H(\Lambda_e)) = K_{\bar{H}}(\Lambda_e).$$

Note that $\xi \mapsto -\xi$ is an involution of $\mathbf{B}^{\pm}_{\Lambda_e}$ and of $\mathbf{B}^{\pm}_{\Lambda_e,\mathrm{ad}}$. Let $\mathbf{B}_{\Lambda_e}$, $\mathbf{B}_{\Lambda_e,\mathrm{ad}}$ be the corresponding sets of orbits of this involution. One can show that

(b) $\mu_{\mathbf{b},\mathbf{b}'} := (1-v^2)^{-r}\delta(\mathbf{db}\,|\,\mathbf{b}')_{\Lambda_e} = (1-v^2)^{-r}\delta(\mathbf{b}\,|\,\mathbf{db}')_{\Lambda_e} \in \mathbf{Z}[v, v^{-1}]$

for all $\mathbf{b}, \mathbf{b}' \in \mathbf{B}^{\pm}_{\Lambda_e}$. We conjecture that

(c) *there exists a canonical map* $\zeta : \mathbf{B}^{\pm}_{\Lambda_e,\mathrm{ad}} \to \mathbf{I}$ *which induces a bijection* $\mathbf{B}_{\Lambda_e,\mathrm{ad}} \xrightarrow{\sim} \mathbf{I}$ *such that*

$$\pm\mu_{\mathbf{b},\mathbf{b}'}(1) = n_{\zeta(\mathbf{b}),\zeta(\mathbf{b}')}$$

for all $\mathbf{b}, \mathbf{b}' \in \mathbf{B}^{\pm}_{\Lambda_e,\mathrm{ad}}$.
The sign is taken so that $\pm\mu_{\mathbf{b},\mathbf{b}'}(1) \geq 0$. It is likely that in (b) we have either $\mu_{\mathbf{b},\mathbf{b}'} \in \mathbf{N}[v^{-1}]$ or $-\mu_{\mathbf{b},\mathbf{b}'} \in \mathbf{N}[v^{-1}]$.

Note that both $\mathbf{I}$ and $\mathbf{B}_{\Lambda_e,\mathrm{ad}}$ have natural actions of a free abelian group of rank $\dim C$ and the bijection in (c) should be compatible with these actions.

This conjecture is actually true if $e = 0$ (for those p for which 1.3 (a) holds); this follows from results in [L6].

2.5.

Assume that e is regular (nilpotent). In this case $H = \mathbf{C}^*$ and

$$\mathbf{d} = \prod_{i\in[1,r]} (1 - v^{-2e_i-2}),$$

where $e_1, \ldots, e_r$ are the exponents of $\mathbf{G}$. Also, $\mathbf{B}^{\pm}_{\Lambda_e} = \mathbf{B}^{\pm}_{\Lambda_e,\mathrm{ad}}$ consists of $\pm\mathbf{b}$ where $\mathbf{b} = \mathcal{O}_{\Lambda_e}$ is the structure sheaf of the point Λ_e. Hence $\mathbf{B}_{\Lambda_e,\mathrm{ad}}$ consists of a single element. We have $\mu_{\mathbf{b},\mathbf{b}} = \mathcal{P}$ where

(a) $$\mathcal{P} = \prod_{i\in[1,r]} \frac{1 - v^{-2e_i-2}}{1 - v^{-2}} \in \mathbf{N}[v^{-1}].$$

Hence

$$\mu_{\mathbf{b},\mathbf{b}}(1) = |\bar{W}|$$

where $\bar{W}$ is the Weyl group of $\mathbf{G}$. Thus 2.4 (c) holds in this case. (Compare [J4, 10.10].)

2.6.

Assume that e is subregular and $\mathbf{G}$ is of type D or E. In this case $H = \mathbf{C}^*$. One can check that

$$\mathbf{d} = \prod_{i\in[1,r+2]} (1 - v^{-2s_i})$$

where $s_1, s_2, \ldots, s_{r+2}$ is:
$2, 3, 4, 5, 6, 6, 8, 9$ (type E_6),
$2, 4, 6, 6, 8, 9, 10, 12, 14$ (type E_7),
$2, 6, 8, 10, 12, 14, 15, 18, 20, 24$ (type E_8),
$2, 2, 4, 6, \ldots, 2n-4, n-2, n-1, n$ (type D_n).
One can also give a closed formula for $\mathbf{d}$:

(a) $$\mathbf{d} = (1 - v^{-2})^r \mathcal{P} \det(\tilde{A}) \det(A)^{-1}$$

where $\mathcal{P}$ is as in 2.5 (a), $\tilde{A}$ is the square matrix indexed by the vertices of the affine Coxeter diagram J with j, j' entry equal to $1+v^{-2}$ if $j=j'$, equal to $-v^{-1}$ if j, j' are joined in that diagram and equal to 0 in the remaining cases; A is the analogous matrix defined in terms of the ordinary Coxeter diagram.

According to [L7], in this case $\mathbf{B}^{\pm}_{\Lambda_e} = \mathbf{B}^{\pm}_{\Lambda_e,\mathrm{ad}}$ consists of elements $\pm\mathbf{b}_j$ ($j \in J$) where $\mathbf{b}_j$ are certain vector bundles on Λ_e and the matrix $((\mathbf{b}_j \mid \mathbf{b}_{j'})_{\Lambda_e})$ is just the inverse of $\tilde{A}$ above. Now $\tilde{A}^{-1}$ can be computed by the method of [LT]. For $j, j' \in J$, we denote by $[j, j']$ be the subset of J consisting of all vertices that lie on the geodesic joining j, j' in the affine Coxeter graph. Let $\tilde{A}_{j,j'}$ be the submatrix of $\tilde{A}$ obtained by removing all rows and columns indexed by some element of $[j, j']$. Let

$$\tilde{\alpha}_{j,j'} = \det(\tilde{A}_{j,j'}) \in \mathbf{Z}[v^{-1}].$$

Then we have

$$(\mathbf{b}_j \mid \mathbf{b}_{j'})_{\Lambda_e} = \tilde{\alpha}_{j,j'} \det(\tilde{A})^{-1}.$$

Using this and (a), we see that

$$\mu_{\mathbf{b}_j,\mathbf{b}_{j'}} = \mathcal{P} \det(A)^{-1} \tilde{\alpha}_{j,j'} \in \mathbf{Z}[v^{-1}].$$

Hence in this case, 2.4 (c) predicts that $\mathbf{I}$ may be identified with J in such a way that the multiplicities $n_{j,j'}$ are given by

$$n_{j,j'} = |\bar{W}| z_{j,j'} z^{-1}_{j_0,j_0}$$

where $z_{j,j'}$ is the order of the centre of the simply connected group with Coxeter graph $J - [j, j']$ (full subgraph of J) and j_0 is the unique element of J that is not a vertex of the ordinary Coxeter graph.

2.7.

Since $\mathbf{B}^{\pm}_{\Lambda_e,\mathrm{ad}}$ is expected to be a signed basis of the $R_{\mathbf{C}^*}$-module $K_{\bar{H}}(\Lambda_e)$ [L6, 12.23(a)], we see that, in the case where $C = \{1\}$, the Conjecture 2.4 (c) predicts that the number of elements in the block $\mathbf{I}$ is the sum of the Betti numbers of $\mathcal{B}_e$.

2.8.

We expect that the set $\mathbf{B}_{\Lambda_e}$ and the quantities $\mu_{\mathbf{b},\mathbf{b}'}$ are intimately related with the combinatorics of the two-sided cell $\mathbf{c}$ in the extended affine Weyl group attached in [L2, 4.8] to the $\mathbf{G}$-orbit of e.

2.9.

The most urgent task would be to test the Conjecture 2.4 (c) in the example in 2.6. Assuming that the conjecture passes this test, one can try to imagine whether a successsion of steps analogous to (I)–(IV) in 1.4 can be used to prove it. At least step (I) makes sense; one should use the version of quantum group at a root of 1 studied by De Concini and Kac in [DK]. One can also expect that there is a corresponding class of representations of the appropriate affine Lie algebra with negative central charge which are connected with the two-sided cell **c** in 2.9.

Remark added 5.22.1999. After this paper was written, Jantzen (*Subregular nilpotent representations of Lie algebras in prime characteristic*, preprint April 1999) has shown that Conjecture 2.4 (c) holds in the example in 2.6.

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