Advanced Studies in Pure Mathematics 31, 2001
Taniguchi Conference on Mathematics Nara '98
pp. 127–145

The Topology of Real and Complex Algebraic Varieties

János Kollár

Contents

§1. Introduction

Definition 1.1. Let $f_i(x_1, \dots, x_n)$ be polynomials whose coefficients are real or complex numbers. An *affine algebraic variety* is the common zero set of finitely many such polynomials

$$X = X(f_1, \dots, f_k) := \{\mathbf{x} \mid f_i(\mathbf{x}) = 0, \forall i\}.$$

To be precise, I also have to specify where the variables x_i are. If the f_i have complex coefficients then the only sensible thing is to let the x_i be complex. The resulting topological space is

$$X(\mathbb{C}) := \{\mathbf{x} \in \mathbb{C}^n \mid f_i(\mathbf{x}) = 0, \forall i\},$$

which we always view with its Euclidean topology. If the f_i have real coefficients then we can let the x_i be real or complex. Thus we obtain two "incarnations" of a variety

$$X(\mathbb{R}) := \{\mathbf{x} \in \mathbb{R}^n \mid f_i(\mathbf{x}) = 0, \forall i\}, \quad \text{and}$$
$$X(\mathbb{C}) := \{\mathbf{x} \in \mathbb{C}^n \mid f_i(\mathbf{x}) = 0, \forall i\}.$$

Received April 21, 1999.
Partial financial support was provided by the Taniguch Foundation and by the NSF under grant number DMS-9622394.

Again, both of these are topological spaces where the topology is induced by the Euclidean topology on $\mathbb{R}^n$ or $\mathbb{C}^n$.

It is frequently inconvenient that $X(\mathbb{C})$ is essentially never compact. To remedy this, we introduce projective varieties which are closed subsets of the projective n-space $\mathbb{P}^n$. Since the coordinates of a point in $\mathbb{P}^n$ are defined only up to a scalar multiple, the zero set makes sense only for homogeneous polynomials. Given any number of homogeneous polynomials $F_i(x_0, \dots, x_n)$ we obtain the corresponding *projective variety*

$$X = X(F_1, \dots, F_k) := \{\mathbf{x} \in \mathbb{P}^n \mid F_i(\mathbf{x}) = 0, \forall i\}.$$

As before, we can look at the set of real or complex points

$$X(\mathbb{R}) := \{\mathbf{x} \in \mathbb{RP}^n \mid F_i(\mathbf{x}) = 0, \forall i\}, \quad \text{and}$$
$$X(\mathbb{C}) := \{\mathbf{x} \in \mathbb{CP}^n \mid F_i(\mathbf{x}) = 0, \forall i\}.$$

Basic Question 1.2. My main interest is to establish connections between the algebraic properties of X and the topological properties of $X(\mathbb{C})$ and $X(\mathbb{R})$. There are two main direction that one can follow.

Determining which topological spaces can be obtained as $X(\mathbb{C})$ or $X(\mathbb{R})$ is called the *realization problem.* One may also ask for realizations where there is a strong connection between various algebraic and topological properties.

We may also want to know which algebraic properties of X are determined by topological properties of $X(\mathbb{C})$ or $X(\mathbb{R})$. This is the *recognition problem.* Ideally we would like to have a way of computing algebraic invariants from topology.

One can also say that the recognition problem is about obstructions to the realization problem. A recognition result leaves us with less freedom in the realization problem.

The following example illustrates the general features of the recognition problem, which is the main focus of these notes.

Example 1.3. Let $F(x_0, \dots, x_n)$ be a real, homogeneous polynomial of degree d and set $X_F := (F = 0) \subset \mathbb{P}^n$. The basic algebraic invariant of F is its degree $\deg F$. The simplest form of the recognition problem asks if $\deg F$ is determined by $X_F(\mathbb{C})$ or $X_F(\mathbb{R})$.

In this generality the answer is no. Indeed, F and F^2 have the same zero sets but different degrees. Thus it is sensible to assume to start with that F is irreducible. With this assumption the degree is easy to read off from topological data.

$H_{2n-2}(\mathbb{CP}^n, \mathbb{Z}) \cong \mathbb{Z}$ where the generator is given by the hyperplane class $[H]$. It is easy to see that $X_F(\mathbb{C}) \subset \mathbb{CP}^n$ has a homology class $[X_F(\mathbb{C})] \in H_{2n-2}(\mathbb{CP}^n, \mathbb{Z})$ and $[X_F(\mathbb{C})] = \deg F \cdot [H]$.

It is somewhat harder to obtain $\deg F$ from $X_F(\mathbb{C})$ alone. First of all, there are some exceptions. For instance, if $G = x_0^{d-1}x_2 - x_1^d$ then $X_G(\mathbb{C})$ is homeomorphic to S^2. This is caused by the fact that $X_G(\mathbb{C})$ is not a submanifold of $\mathbb{CP}^2$ near the point $(0:0:1)$.

If we restrict our attention to the case when $X_F(\mathbb{C})$ is a submanifold of $\mathbb{CP}^n$ then we are in a good situation. For instance, it is easy to write down a formula for the Chern classes of $X_F(\mathbb{C})$ in terms of $\deg F$ (cf. [Hirzebruch66, §22]). From this we see that $X_F(\mathbb{C})$ determines $\deg F$ with the sole exception $X_F(\mathbb{C}) \sim S^2$ where $\deg F$ can be 1 or 2.

The real case is trickier. $H_{n-1}(\mathbb{RP}^n, \mathbb{Z}_2) \cong \mathbb{Z}_2$ and the generator is given by the hyperplane class $[H]$. $X(\mathbb{R}) \subset \mathbb{RP}^n$ has a homology class $[X(\mathbb{R})] \in H_{n-1}(\mathbb{RP}^n, \mathbb{Z}_2)$ and $[X(\mathbb{R})] = \deg F \cdot [H]$. Thus the topology determines $\deg F$ mod 2. In fact one can not do better than this. It is not hard to see that if X_F is smooth and F^* is a small homogeneous perturbation of $(x_0^2 + \cdots + x_n^2) \cdot F$ then the pairs $(\mathbb{RP}^n, X_F(\mathbb{R}))$ and $(\mathbb{RP}^n, X_{F^*}(\mathbb{R}))$ are diffeomorphic. This shows that $X(\mathbb{R})$ does not provide an upper bound for the degree.

On the other hand, $X(\mathbb{R})$ does provide a lower bound for $\deg F$. Indeed it is a priori clear that only finitely many topological types can be realized by hypersurfaces of bounded degeree. Thus if $X(\mathbb{R})$ is complicated then $\deg F$ has to be large. [Milnor64] is an explicit result in this direction. I do not even have a conjecture about the precise answer.

Thus we can summarize our results as follows:

Conclusion. *Let $X \subset \mathbb{P}^n$ be a smooth hypersurface. Then*

1. $X(\mathbb{C})$ *determines* $\deg F$.
2. $X(\mathbb{R})$ *determines* $\deg F$ mod 2.
3. *If* $X(\mathbb{R})$ *is complicated then* $\deg F$ *is large.*

The aim of these notes is to collect a series of results and conjectures concerning the recognition problem of real and complex algebraic varieties. As in the hypersurface case, the main idea can be summarized as follows.

Principle 1.4.

1. $X(\mathbb{C})$ *determines the important algebraic invariants of* X.
2. *If* $X(\mathbb{R})$ *is complicated then* X *is also complicated.*

§2. Homological Methods

As an intermediate step of our answers, we can study the relationship between $X(\mathbb{C})$ and $X(\mathbb{R})$. The simplest case is when X is the zero set of a real polynomial in one variable. Then $X(\mathbb{C})$ is the set of complex roots and $X(\mathbb{R})$ the set of real roots. We have the following two basic relationships:

1. #(real roots) $\leq$ #(complex roots), and
2. #(real roots) $\equiv$ #(complex roots) mod 2.

It is quite amazing that these elementary assertions can be generalized to arbitrary dimensions.

Theorem 2.1 [Floyd52], [Thom65]. *Let X be a projective variety over $\mathbb{R}$. Then*

$$\sum_i h^i(X(\mathbb{R}), \mathbb{Z}_2) \leq \sum_i h^i(X(\mathbb{C}), \mathbb{Z}_2),$$

where $h^i(X(\mathbb{R}),\mathbb{Z}_2)$ is the dimension of the $\mathbb{Z}_2$-vector space $H^i(X(\mathbb{R}),\mathbb{Z}_2)$.

Theorem 2.2 [Sullivan71]. *Let X be a projective variety over $\mathbb{R}$. Then*

$$\chi(X(\mathbb{R})) \equiv \chi(X(\mathbb{C})) \mod 2,$$

where χ denotes the Euler characteristic. (The choice of the coefficient field does not matter.)

It is slightly disappointing from the algebraic point of view that both of these results are essentially topological. Complex conjugation gives an involution $\tau : X(\mathbb{C}) \to X(\mathbb{C})$ whose fixed point set is precisely $X(\mathbb{R})$. 2.1 and 2.2 hold for the fixed point set of any involution.

Equality frequently holds in 2.1, even in the algebraic case. A generalization is given in [Krasnov83], but the main part is again purely topological. Thus it is possible that the homological aspect of comparing $X(\mathbb{C})$ and $X(\mathbb{R})$ has very little to do with algebraic geometry.

The homological methods give sharp results which are especially useful in dimensions 1 and 2. The reason is that a topological surface is determined by its homology groups. By contrast, the homology groups of a 3-manifold carry very little information. For instance, there are many 3-manifolds M, called *homology spheres* such that

$$H_0(M,\mathbb{Z}) \cong \mathbb{Z}, \quad H_1(M,\mathbb{Z}) \cong H_2(M,\mathbb{Z}) \cong 0, \quad H_3(M,\mathbb{Z}) \cong \mathbb{Z}.$$

For 3-manifolds the crucial information is carried by the fundamental group. In higher dimensions one needs the other homotopy groups as well. One of the main challenges of the theory is to connect the homotopy theoretic properties of $X(\mathbb{C})$ and $X(\mathbb{R})$ with the algebraic nature of X.

§3. The Realization Problem

Over the real numbers, the realization problem has a very nice complete solution. The first results of this type were proved by Seifert. The main contribution to the subject is [Nash52] which was sharpened by [Tognoli73].

Theorem. *For every compact differentiable manifold M there is a real, smooth, projective variety X such that M is diffeomorphic to $X(\mathbb{R})$.*

The case of singular varieties is still not completely solved. For some recent results see [Akbulut-King92].

The realization problem behaves very differently over the complex numbers. There are very few manifolds M which can be written as $X(\mathbb{C})$ for a smooth projective variety X.

First of all, the dimension of M has to be even. The Hodge structure on the cohomology groups of M gives further restrictions. The deepest results in this direction are in [DGMS75].

There has been a lot of recent interest in the fundamental group of complex algebraic varieties. The conclusion is that most finitely presented groups can not be the fundamental group of a smooth projective variety. The simplest such examples are free Abelian groups of odd rank. There is a quite extensive theory of those groups which occur as the fundamental group of a smooth projective or Kähler variety, see for instance [ABCKT96].

[Kapovich-Millson97] found examples of groups which can not be the fundamental group of a smooth quasi-projective variety.

It is quite remarkable that the topological spaces $X(\mathbb{C})$ are special even among compact complex manifolds. As observed by Carlson and Kotschick, the main theorem of [Taubes92] implies that every finitely presented group is the fundamental group of a compact complex 3-manifold.

§4. The Recognition Problem over $\mathbb{C}$

Let X_t be a family of smooth projective varieties depending continuously on a parameter t. Then $X_t(\mathbb{C})$ is a continuously varying family of smooth manifolds, hence locally constant. Thus the topology of $X(\mathbb{C})$ is not able to distinguish the individual varieties X_t from each other. The best we can hope for is that the topology tells us in which family we are.

For a smooth projective curve C the only discrete invariant is the genus $g(C)$. By definition, this is the dimension of the space of holomorphic 1-forms, denoted by $H^0(C, \Omega_C)$. By Serre duality this is dual to the first cohomology group of the structure sheaf $H^1(C, \mathcal{O}_C)$.

The set of complex points $C(\mathbb{C})$ is a compact topological surface. It is also orientable, so it can be obtained from S^2 by attaching handles. The basic invariant is the number of handles.

Theorem 4.1. *Let C be a smooth, projective, algebraic curve. Then*

1. (*Riemann,* 1857) $h^0(C, \Omega_C) =$ *number of handles of* $C(\mathbb{C})$.
2. (*Hurwitz,* 1891) *All curves with the same genus form a connected family.*

Ideally one would like to get similar results in higher dimensions.

Definition 4.2 (Kodaira dimension). Let X be a smooth projective variety of dimension n. In analogy with the curve case it is natural to consider the space of holomorphic n-forms. Unfortunately this is not enough and we have to look at multivalued holomorphic n-forms as well. It is technically easier to work with sections of powers of the line bundle of holomorphic n-forms $H^0(X, (\Omega_X^n)^{\otimes m}) = H^0(X, \mathcal{O}(mK_X))$. The Kodaira dimension, denoted by $\kappa(X)$, essentially measures the growth of these vector spaces.

To be precise, if these groups are always zero then we set $\kappa(X) = -\infty$. Otherwise it turns out that there is a unique integer $0 \leq \kappa(X) \leq \dim X$ and constants $0 < c_1, c_2$ such that

$$c_1 \cdot m^{\kappa(X)} \leq h^0(X, \mathcal{O}(mK_X)) \leq c_2 \cdot m^{\kappa(X)}$$

holds whenever $h^0(X, \mathcal{O}(mK_X)) \neq 0$.

(It is conjectured that there is an integer N and finitely many polynomials $P_i(m)$ such that $h^0(X, \mathcal{O}(mK_X)) = P_i(m)$ if $m \equiv i \bmod N$ and $m \gg 1$, but this is proved only for $\dim X \leq 3$.)

In analogy with 4.1.1 one can ask if the Kodaira dimension of a surface is determined by $X(\mathbb{C})$. It was noticed that the answer is no if we consider $X(\mathbb{C})$ as a topological manifold [Dolgachev66].

As Donaldson theory started to discover the difference between diffeomorphism and homeomorphism in real dimension 4, the hope emerged that this may hold for diffeomorphism. This has been one of the motivating questions of the differential topology of algebraic surfaces. After many contributions, the final step was accomplished by [Pidstrigach95], [Friedman-Qin95]. With the methods of Seiberg-Witten theory, the proof is quite short [Okonek-Teleman95]:

Theorem 4.3. *Let S be a smooth, projective algebraic surface over $\mathbb{C}$. Then $\kappa(S)$ is determined by the differentiable manifold $S(\mathbb{C})$.*

As in 4.1.2 we may also look at the family of all algebraic structures on a given 4-manifold. It is known that they form finitely many connected families. Recent examples of [Manetti98] show that in general there are several families.

Starting with dimension 3, the differentiable stucture of $X(\mathbb{C})$ does not determine the Kodaira dimension of X, as it was first obeserved in [Friedman-Morgan88]. It becomes necessary to find additional topological data. A natural candidate is the symplectic structure of $X(\mathbb{C})$. Hopefully, this provides the right setting in all dimensions.

Definition 4.4 (Symplectic manifolds). A *symplectic* manifold is a pair (M^{2n}, ω) where M is a differentiable manifold of dimension $2n$ and ω is a 2-form $\omega \in \Gamma(M, \wedge^2 T^*)$ which is d-closed and nondegenerate. That is, $d\omega = 0$ and ω^n is nowhere zero.

For a smooth projective variety X the following construction gives a symplectic structure on $X(\mathbb{C})$. On $\mathbb{C}^{n+1}$ consider the *Fubini-Study* 2-form

$$\omega' := \frac{\sqrt{-1}}{2\pi}\left[\frac{\sum dz_i \wedge d\bar{z}_i}{\sum |z_i|^2} - \frac{(\sum \bar{z}_i\, dz_i) \wedge (\sum z_i\, d\bar{z}_i)}{(\sum |z_i|^2)^2}\right].$$

It is closed, nondegenerate on $\mathbb{C}^{n+1} \setminus \{0\}$ and invariant under scalar multiplication. Thus ω' descends to a symplectic 2-form ω on $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$.

If $X \subset \mathbb{CP}^n$ is any smooth variety, then the restriction $\omega|_X$ makes $X(\mathbb{C})$ into a symplectic manifold.

The resulting symplectic manifold $(X(\mathbb{C}), \omega|_X)$ depends on the embedding $X \hookrightarrow \mathbb{CP}^n$, but the dependence is rather easy to understand:

We say that two symplectic manifolds (M, ω_0) and (M, ω_1) are *symplectic deformation equivalent* if there is a continuous family of symplectic manifolds (M, ω_t) starting with (M, ω_0) and ending with (M, ω_1).

To every smooth projective variety the above construction associates a symplectic manifold $(X(\mathbb{C}), \omega|_X)$ which is unique up to symplectic deformation equivalence.

This allows us to formulate the correct generalization of 4.3.

Conjecture 4.5. *Let X be a smooth, projective variety over $\mathbb{C}$. Then $\kappa(X)$ is determined by the symplectic manifold $(X(\mathbb{C}), \omega|_X)$.*

A few special cases of this conjecture are known. It is conjectured that $\kappa(X) = -\infty$ iff X is uniruled. It was proved in [Kollár98a, 4.2.10] that being uniruled is a property of the symplectic manifold $(X(\mathbb{C}), \omega|_X)$. Mirror symmetry suggests that varieties of Kodaira dimension zero can also be recognized from their symplectic structure.

One can also ask if there are only finitely many connected families of varieties with a given symplectic structure (M, ω). If $b_2(M) = 1$ then this is true even for the differentiable structure by [Kollár98a, 4.2.3] but nothing seems to be known in general.

§5. Rational and Uniruled Varieties

The recognition problem is much less understood for real varieties. As shown by 1.3, we can easily get some $\mod 2$ information about X but it is less clear what else to do. The basic works of [Harnack1876] on curves and [Comessatti14] on surfaces were promising, but it is only recently that some meaningful positive results appeared in higher dimensions.

Very little is known about the connection of $X(\mathbb{R})$ with the Kodaira dimension of X. The special case of varieties with Kodaira dimension $-\infty$ is now becoming clearer, so I mainly concentrate on those.

Partly for historical reasons, I focus on rational and uniruled varieties. Rational varieties are very special but the topology of their real part turns out to be quite interesting. It is conjectured that X is uniruled iff $\kappa(X) = -\infty$, so the study of uniruled varieties fits well within the framework of the recognition probem.

Definition 5.1 (Rational and unirational varieties). In Section 1 we have defined varieties as zero sets of functions. There is another way of associating a geometric object to a function by looking at its graph or its image. This leads to the notions of rational and unirational varieties.

Let $\phi_i(t_0, \dots, t_d)$ be homogeneous polynomials of the same degree. They define a map

$$\Phi : \mathbb{P}^d \dashrightarrow \mathbb{P}^N \quad \text{given as} \quad \mathbf{t} \longmapsto (\phi_0(\mathbf{t}) : \cdots : \phi_N(\mathbf{t})).$$

Over $\mathbb{C}$ the image of such a map is automatically a dense subset of an algebraic variety. A variety which can be written this way is called *unirational.* Note that we do not assume that X has dimension d, but it is not hard to see that once X is unirational we can always choose a parametrization $\Psi : \mathbb{P}^{\dim X} \dashrightarrow X$.

One has to be a little more careful over $\mathbb{R}$. First of all, if we have a real variety X then we are interested only in those parametrizations $\Phi : \mathbb{P}^d \dashrightarrow X$ where the coordinate functions of Φ have real coefficients. Second, the image of $\mathbb{RP}^d$ need not be dense in $X(\mathbb{R})$. For instance the image of

$$(t_0 : t_1) \longmapsto (t_0^4 + t_1^4 : 2t_0^2t_1^2 : t_0^4 - t_1^4)$$

is only half of the circle $x_1^2 + x_2^2 = x_0^2$.

There are some examples of X where there is no parametrization $\Psi : \mathbb{P}^d \dashrightarrow X$ such that the image of $\mathbb{RP}^d$ is dense in $X(\mathbb{R})$, but for every $x \in X(\mathbb{R})$ there is a parametrization $\Psi_x : \mathbb{P}^d \dashrightarrow X$ such that $\Psi_x(\mathbb{RP}^d)$ contains an open neighborhood of x.

A parametrization of a variety $\Phi : \mathbb{P}^d \dashrightarrow X \subset \mathbb{P}^N$ is especially useful if Φ has an inverse $X \dashrightarrow \mathbb{P}^d$. Over $\mathbb{C}$ this is equivalent to assuming that Φ is injective an a dense open subset of $\mathbb{P}^d$. It is important to note that this fails over $\mathbb{R}$. For instance $\phi : x \mapsto x^3$ gives an injective map $\mathbb{R} \to \mathbb{R}$ but it is a $3:1$ map if viewed as $\phi : \mathbb{C} \to \mathbb{C}$.

The following is an easy example of the recognition problem.

Lemma 5.2. *Let X be a smooth, real, projective variety which is rational. Then $X(\mathbb{R})$ is connected.*

Rationality and unirationality are very useful and strong properties of an algebraic variety but unfortunately they are exceedingly hard to check in practice. A considerable weakeneing of these notions is given next.

Definition 5.3. A variety X of dimension d is called *uniruled* if there is a variety Y of dimension $d-1$ and a map $\Phi : Y \times \mathbb{P}^1 \dashrightarrow X$ which has dense image over $\mathbb{C}$. If in addition Φ has an inverse then we say that X is *ruled.*

As before, one has to be careful with the real versions.

At first sight this notion seems too general. Since Y can be arbitrary, uniruled can be interpreted to mean that X behaves like a rational variety in one direction only. It would be more convincing to have a notion which requires rational like behaviour in every direction. The concept of *rationally connected varieties* was introduced in [KoMiMo92] with exactly this aim in mind.

For our present purposes uniruled is sufficient. The reason is that the current topological methods are not fine enough to detect different type behaviour in different directions. In some sense rational is analogous to positive curvature. At present we have results that distinguish negative curvature from everyting else but we can not handle the mixed curvature case well.

Example 5.4.

1. Nonempty quadrics are rational as shown by the inverse of the stereographic projection from a point of the quadric.

2. Set $S := (x^2 + y^2 + \prod_{i=1}^m (z - a_i) = 0)$ where the a_i are distinct real numbers. Then S is rational over $\mathbb{R}$ iff $m \leq 2$. Indeed, if $m \leq 2$ then this is a quadric so rational. If $m \geq 3$ then $S(\mathbb{R})$ is disconnected, so it can not be rational by 5.2. On the other hand, a surface of the form $x^2 - y^2 + f(z) = 0$ is rational as shown by the substitution

$$(u, v) \longmapsto \left(\frac{f(v) + u^2}{2u}, \frac{f(v) - u^2}{2u}, v \right)$$

So $x^2 + y^2 = f(z)$ is rational over $\mathbb{C}$ but not over $\mathbb{R}$.

3. [Segre51] The cubic surface $z^2 = x^3 + y^3 + c$ is unirational for any c, as shown by the parametrization

$$x = \frac{u^2}{3}, \quad y = \frac{u^6 + 27c - 27v}{9u(6v + u^3)},$$
$$z = \frac{u^6 + 27c - 27v + 54v^2 + 9u^3 v}{9(u^3 + 6v)}.$$

Uniruled hypersurfaces are easy to characterize.

Theorem 5.5. *Let $X = (f(x_0, \ldots, x_n) = 0)$ be a smooth hypersurface of degree d in $\mathbb{P}^n$. Then X is uniruled iff $d \leq n$.*

The question of (uni)rationality of hypersurfaces is more subtle.

Question 5.6. *Let $X = (f(x_0, \ldots, x_n) = 0)$ be a smooth hypersurface of degree d in $\mathbb{P}^n$. Is it true that if X is rational then $d \leq 3$?*

I do not know any conceptual reason why the answer should be yes. On the other hand, by now we have many ways of constructing cubic hypersurfaces which are rational [Tregub93], [Hassett99] and there are several unirationality constructions for higher degrees as well. In general X is unirational for $d \leq \Phi(n)$ where Φ is a function that goes to infinity very slowly with n. For instance, a smooth cubic of dimension at least 2 or a smooth quartic of dimension at least 6 is unirational (cf. [Kollár96, V.5.18]).

It is also known that if X is rational and "very general" then $d \leq \frac{2}{3}n + 1$ [Kollár95].

§6. The Nash Conjecture for 3-folds

As we saw in 1.3, in the real version of the recognition problem we can only expect results claiming that if $X(\mathbb{R})$ is complicated then so is X. Among algebraic varieties the rational ones are the simplest. Proving the nonrationality of a variety using only its real points may be the simplest version of the real recognition problem.

A following bold conjecture of Nash asserted that this can not be done:

Conjecture 6.1 [Nash52]. *For every compact differentiable manifold M there is a smooth, real, projective variety X such that X is rational over $\mathbb{R}$ and M is diffeomorphic to $X(\mathbb{R})$.*

As far as I know, this has been the shortest lived conjecture in mathematics since it was disproved 38 years before it was posed. (This seemed not to have been realized for quite some time though.)

Theorem 6.2 [Comessatti14]. *Let S be a smooth, real, projective surface. Assume that S is rational and $S(\mathbb{R})$ is orientable. Then $S(\mathbb{R})$ is either a sphere or a torus.*

The examples $(x^2 + y^2 = z^2 \pm u^2) \subset \mathbb{RP}^3$ show that the sphere and the torus both occur. Also, by blowing up points of $\mathbb{RP}^2$ we see that all nonorientable surfaces do occur. Thus the above theorem gives a necessary and sufficient condition for a topological surface to be representable as the set of real points of a smooth, real, projective surface which is birational to $\mathbb{RP}^2$.

While the negative solution of the surface case suggests that the Nash conjecture may fail in higher dimensions, efforts to use the 2-dimensional case to produce higher dimensional counter examples have failed so far. In fact, most of the higher dimensional results were positive.

The first such result is the solution of the *topological Nash conjecture.* It is generally hoped that a birational map between smooth varieties can be factored as a sequence of blow ups and downs of smooth subvarieties[1]. Blowing up and down makes sense in the topological or differentiable setting. Thus it is reasonable to expect that if X is rational then $X(\mathbb{R})$ can be obtained from $\mathbb{RP}^n$ by blow ups and downs.

---

[1]This was recently proved by Włodarczyk and by Abramovich, Karu, Matsuki, Włodarczyk.

Theorem 6.3 [Akbulut-King91], [Mikhalkin97]. *Every compact differentiable manifold is obtainable from $\mathbb{RP}^n$ by a sequence of differentiable blow ups and downs.*

One can try to prove the Nash conjecture by trying to make the above sequence of blow ups and downs algebraic. For blow ups one needs to realize certain submanifolds by algebraic suvarieties. This is relatively easy, though not automatic. The algebraic realization of toplogical blow downs is much harder. Assume for instance that $n = 2$ and we want to realize a blow down $\pi : X(\mathbb{R}) \to S$ algebraically. π contracts a simple closed curve $L \subset X(\mathbb{R})$ to a point. In order to make this algebraic, we have to find a smooth rational curve $C \subset X(\mathbb{C})$ of selfintersection -1 such that $C(\mathbb{R})$ is isotopic to L. An algebraic surface frequently contains only finitely many smooth rational curves of selfintersection -1, so it is usually impossible to find such a C. It is easier to find some curve D such that $D(\mathbb{R})$ is isotopic to L. We can then blow up suitable complex conjugate points to achive that D becomes contractible. Contracting D we obtain a surface X' with a very singular point such that $X'(\mathbb{R})$ is a manifold. (Such examples abound in all dimensions, for istance $Y = (x^a + y^b + z^c = t^{2d+1})$ is homeomorphic to $\mathbb{R}^3$ as shown by $\phi(x,y,z) = (x,y,z, \sqrt[2d+1]{x^a + y^b + z^c})$.)

These ideas can be used to solve another weakening of the Nash conjecture:

Theorem 6.4 [Benedetti-Marin92]. *For every compact 3-manifold M there is a singular real algebraic variety X such that X is rational and M is homeomorphic to $X(\mathbb{R})$.*

It turns out that despite these positive partial results, the Nash conjecture fails in dimension 3 as well. Before stating the precise results we need to review the general features of the topology of 3-manifolds.

6.5 (The topology of 3-manifolds).

As a general reference see [Scott83].

Assume for simplicity that we consider only orientable 3-manifolds. By the results of Kneser and Milnor, any such can be written as a connected sum $M_1 \# \cdots \# M_k$ where $\pi_2(M_i) = 0$ and the summands are uniquely determined. Thus one has to concentrate on those 3-manifolds M such that $\pi_2(M) = 0$.

There are 3 known classes of such 3-manifolds.

Seifert fibered: These are 3-manifolds which admit a differentiable map to a surface $M^3 \to F^2$ such that every fiber is a circle. $M^3 \to F^2$ is a fiber bundle outside finitely many points of F and the behaviour of $M^3 \to F^2$ near the exceptional points is fully understood.

Torus bundles: These are 3-manifolds which can be written as a torus bundle over a circle (or are doubly covered by such).

Hyperbolic: These can be written as the quotient of hyperbolic 3-space by a discrete group of motions $\mathbb{H}^3/\Gamma$ for $\Gamma \subset PO(3,1)$. This is the largest class and it is not sufficiently understood.

The geometrization conjecture of Thurston asserts that every 3-manifold M such that $\pi_2(M) = 0$ can be obtained from the above examples. For this to work one has to consider generalizations of these examples to the case of 3-manifolds with boundary and to allow gluing the pieces along boundary components which are tori.

For us the main consequence to keep in mind is that very few 3-manifolds are Seifert fibered.

We are now ready to formulate the 3-dimensional analog of Comessatti's result:

Theorem 6.6 [Kollár98c]. *Let X be a smooth, real, projective 3-fold. Assume that X is uniruled and $X(\mathbb{R})$ is orientable. Then every component of $X(\mathbb{R})$ is among the following:*

1. *Seifert fibered,*
2. *connected sum of several copies of $S^3/\mathbb{Z}_{m_i}$ (called lens spaces),*
3. *torus bundle over S^1 (or doubly covered by a torus bundle),*
4. *finitely many other possible exceptions, or*
5. *obtained from the above by repeatedly taking connected sum with $\mathbb{RP}^3$ and $S^1 \times S^2$.*

I expect the final answer to be even more precise. Unfortunately the current version of the proof falls short of proving these.

Conjecture 6.7. *Notation and assumptions as above.*

1. *Torus bundles over S^1 do not occur (unless they are also Seifert fibered) and the finitely many exceptions in* 6.6.4 *are also not needed.*
2. *All Seifert fibered* 3*-manifolds and all connected sums of lens spaces do occur, at least with X uniruled.*
3. *The list should be much shorter for X rational.*

6.8. I expect that 6.7.2 can be proved using the constructions of [Kollár99b]. The method of [Kollár99c] fails to exclude torus bundles over S^1, but this may not be very hard to achieve eventually.

It is much less clear to me how to deal with the possible finitely many exceptions. There are two related sources of these.

The first part of the proof of 6.6 given in [Kollár99a] shows that $X(\mathbb{R})$ does not change much if we simplify X using the minimal model program (see, for instance, [Kollár-Mori98]). Thus we are reduced to understanding the topology of $X(\mathbb{R})$ where X is in some kind of "standard form". There are 3 types of standard forms and two of these have been dealt with in [Kollár99b], [Kollár99c]. The third class is the so called Fano 3-folds. These are 3-folds such that minus the canonical class is ample. There is a complete list of smooth Fano 3-folds, but unfortunately the reduction method of [Kollár99a] introduces some singularities. There are only finitely many cases by [Kawamata92], but the explicit list is not known. Even if X is some reasonably well known variety, determining $X(\mathbb{R})$ may be quite hard. One of the simplest concrete open problems is the following.

Question. *Let $X \subset \mathbb{P}^4$ be a smooth hypersurface of degree 4. Can $X(\mathbb{R})$ be hyperbolic?*

Standard effective estimates of real and complex algebraic geometry give the following bound for the number of possible cases.

Proposition. *The number of exceptions in* 6.6.4 *is at most* $10^{10^{49}}$.

Instead of getting bogged down in the minutiae of the precise list, it may be more interesting to consider the following general problem.

Conjecture 6.9. *Let X be a smooth, real, projective, uniruled variety of dimension at least* 3. *Then none of the connected components of $X(\mathbb{R})$ is hyperbolic.* (*Even without assuming orientability.*)

A very substantial step towards proving 6.9 is the following:

Theorem 6.10 [Viterbo98]. *Conjecture* 6.9 *holds if* $H_2(X(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}$ *and X is covered by lines.* (*A line is a morphism* $f : \mathbb{CP}^1 \to X$ *such that* $f_*[\mathbb{CP}^1]$ *generates* $H_2(X(\mathbb{C}), \mathbb{Z})$.)

6.11 (Lagrangian versions). Real algebraic varieties provide the largest known class of Lagrangian and special Lagrangian submanifolds (cf. [McDuff-Salamon95]). It would be quite interesting to know if the above results have their analogs for Lagrangian submanifolds of symplectic varieties. This problem is also related to some of the questions posed by Fukaya during the conference.

§7. The Nonprojective Nash Conjecture

The original conjecture of Nash asked about the existence of a variety X such that $X(\mathbb{R})$ is diffeomorhic to M and X is

1. smooth,
2. projective, and
3. rational.

If we drop the third condition then the asnwer is yes by 3.1. If we drop the smoothness assumption, the answer is again yes by 6.4. Is it possible to drop the projectivity assumption?

Allowing quasi projective varieties instead of projective ones does not help at all. 6.3 suggests to look at compact complex manifolds which can be obtained from $\mathbb{P}^3$ by a sequence of smooth blow ups and downs. The smooth blow up of a projective variety is projective, but it is not clear that the same holds for smooth blow downs, thus we may get something interesting. This class of manifolds was introduced by [Artin68] and [Moishezon67].

Definition–Theorem 7.1. A compact complex manifold Y is called a *Moishezon manifold* or an *Artin algebraic space* if the following equivalent conditions are satisfied:

1. Y is bimeromorphic to a projective variety.
2. Y can be made projective by a sequence of smooth blow ups.

It is not at all clear that there are nonprojective Moishezon manifolds. By a result of Chow and Kodaira, if a smooth compact complex surface is bimeromorphic to a projective variety then it is projective (cf. [BPV84, IV.5]). The first nonalgebraic examples in dimension 3 were found by Hironaka (see [Hartshorne77, App. B.3]).

We of course want to keep the notion of a real structure, thus we look at pairs (Y, τ) where Y is a compact complex manifold and $\tau : Y \to Y$ an antiholomorphic involution. Then $Y(\mathbb{R})$ denotes the fixed point set of τ. Considering such pairs is very natural. The main problem of their theory is that all reasonable names have already been taken. "Real analytic space" is used for something else and "real complex manifold" sounds goofy.

Moishezon manifolds seem quite close to projective varieties. In general, if a property of projective varieties does not obviously involve the existence of an ample line bundle, then it also holds for Moishezon manifolds. It was therefore quite a surpise to me that for the Nash conjecture the nonprojective cases behave very differently.

Theorem 7.2 [Kollár99d]. *Let M be a compact, connected 3-manifold. Then there is a sequence of smooth, real blow ups and downs*

$$\mathbb{P}^3 = Y_0 \dashrightarrow Y_1 \dashrightarrow \cdots \dashrightarrow Y_n = X_M$$

such that $X_M(\mathbb{R})$ is diffeomorphic to M.

Acknowledgements. I thank P. Orlik and D. Toledo for many helpful comments.

References

[ABCKT96] J. Amorós, M. Burger, K. Corlette, D. Kotschick and D. Toledo, "Fundamental groups of compact Kähler manifolds", Amer. Math. Soc., 1996.

[Akbulut-King91] S. Akbulut and H. King, Rational structures on 3-manifolds, Pacific J. Math., **150** (1991), 201–204.

[Akbulut-King92] S. Akbulut and H. King, "Topology of Real Algebraic Sets", MSRI Publ. vol. **25**, Springer, 1992.

[Artin68] M. Artin, The implicit function theorem in algebraic geometry, in "Algebraic geometry", Bombay, Oxford Univ. Press, 1968, pp. 13–34.

[BPV84] W. Barth, C. Peters and A. Van de Ven, "Compact Complex Surfaces", Springer, 1984.

[Benedetti-Marin92] R. Benedetti and A. Marin, Déchirures de variétés de dimension trois, Comm. Math. Helv., **67** (1992), 514–545.

[BCR98] J. Bochnak, M. Coste and M-F. Roy, "Real algebraic geometry", Springer, 1998.

[Comessatti14] A. Comessatti, Sulla connessione delle superfizie razionali reali, Annali di Math., **23**(3) (1914), 215–283.

[DGMS75] P. Deligne, Ph. Griffiths, J. Morgan and D. Sullivan, Real homotopy theory of Kähler manifolds, Invent. Math., **29** (1975), 245–274.

[Dolgachev66] I. Dolgachev, On Severi's conjecture concerning simply connected algebraic surfaces, Soviet Math. Dokl., **7** (1966), 1169–1171.

[Floyd52] E. Floyd, On periodic maps and the Euler characteristics of associated spaces, Trans. Amer. Math. Soc., **72** (1952), 138–147.

[Friedman-Morgan88] R. Friedman and J. Morgan, Bull. Amer. Math. Soc., **18** (1988), 1–19.

[Friedman-Qin95] R. Friedman and Z. Qin, On complex surfaces diffeomorphic to rational surfaces, Inv. Math., **120** (1995), 81–117.

[Harnack1876] A. Harnack, Über die Vieltheiligkeit der ebenen algebraischen Kurven, Math. Ann., **10** (1876), 189–198.

[Hartshorne77] R. Hartshorne, "Algebraic Geometry", Springer, 1977.

[Hassett99] B. Hassett, Some rational cubic fourfolds, J. Algebraic Geom., **8** (1999), 103–114.

[Hirzebruch66] F. Hirzebruch, "Topological methods in algebraic geometry", third enlarged ed., Springer, 1966.

[Iskovskikh80] V. A. Iskovskikh, Anticanonical models of three-dimensional algebraic varieties, J. Soviet Math., **13** (1980), 745–814.

[Kapovich-Millson97] M. Kapovich and J. Millson, Artin groups, projective arrangements, and fundamental groups of smooth complex algebraic varieties, C. R. Acad. Sci. Paris Seir. I Math., **325** (1997), 871–876.

[Kawamata92] Y. Kawamata, Boundedness of $\mathbb{Q}$-Fano threefolds, Proc. Int. Conf. Algebra, Contemp. Math., **131** (1992), 439–445.

[Kharlamov76] V. Kharlamov, The topological type of non-singular surfaces in RP^3 of degree four, Funct. Anal. Appl., **10** (1976), 295–305.

[Kollár93] J. Kollár, Effective Base Point Freeness, Math. Ann., **296** (1993), 595–605.

[Kollár95] J. Kollár, Nonrational hypersurfaces, J. Amer. Math. Soc., **8** (1995), 241–249.

[Kollár96] J. Kollár, "Rational Curves on Algebraic Varieties", Springer Verlag, Ergebnisse der Math., vol. **32**, 1996.

[Kollár98a] J. Kollár, Low degree polynomial equations, in "European Congres of Math", Birkhäuser, 1998, pp. 255–288.

[Kollár98b] J. Kollár, Real Algebraic Threefolds I. Terminal Singularities, Collectanea Math. FERRAN SERRANO, 1957-1997, **49** (1998), 335–360.

[Kollár98c] J. Kollár, The Nash conjecture for threefolds, ERA of Amer. Math. Soc., **4** (1998), 63–73.

[Kollár99a] J. Kollár, Real Algebraic Threefolds II. Minimal Model Program, Jour. Amer. Math. Soc., **12** (1999), 33–83.

[Kollár99b] J. Kollár, Real Algebraic Threefolds III. Conic Bundles, to appear.

[Kollár99c] J. Kollár, Real Algebraic Threefolds IV. Del Pezzo Fibrations, in "Complex analysis and algebraic geometry", de Gruyter, 2000, pp. 317–346.

[Kollár99d] J. Kollár, The nonprojective Nash conjecture for threefolds, in preparation.

[KoMiMo92] J. Kollár, Y. Miyaoka and S. Mori, Rationally Connected Varieties, J. Alg. Geom., **1** (1992), 429–448.

[Kollár-Mori98] J. Kollár and S. Mori, "Birational geometry of algebraic varieties", Cambridge Univ. Press, 1998.

[Krasnov83] V. A. Krasnov, Harnack–Thom inequalities for mappings of real algebraic varieties (Russian), Izv. Akad. Nauk SSSR Ser. Mat., **47** (1983), 268–297.

[Manetti98] M. Manetti, On the moduli space of diffeomorphic algebraic surfaces, preprint.

[McDuff-Salamon95] D. McDuff and D. Salamon, "Introduction to symplectic topology", Clarendon, 1995.

[Mikhalkin97] G. Mikhalkin, Blow up equivalence of smooth closed manifolds, Topology, **36** (1997), 287–299.

[Milnor64] J. Milnor, On the Betti numbers of real varieties, Proc. Amer. Math. Soc., **15** (1964), 275–280.

[Moishezon67] B. Moishezon, On n-dimensional compact varieties with n algebraically independent meromorphic functions, Amer. Math. Soc. Transl., **63** (1967), 51–177.

[Nash52] J. Nash, Real algebraic manifolds, Ann. Math., **56** (1952), 405–421.

[Okonek-Teleman95] C. Okonek and A. Teleman, Les invariants de Seiberg-Witten et la conjecture de Van de Ven, C. R. Acad. Sci. Paris, **321** (1995), 457–461.

[Pidstrigach95] V. Pidstrigach, Patching formulas for spin polynomials and a proof of the Van de Ven conjecture, Izvestiya Russ. A.S., **45** (1995), 529–544.

[Scott83] P. Scott, The geometries of 3-manifolds, Bull. London Math. Soc., **15** (1983), 401–487.

[Segre51] B. Segre, The rational solutions of homogeneous cubic equations in four variables, Notae Univ. Rosario, **2** (1951), 1–68.

[Shafarevich72] R. I. Shafarevich, "Basic Algebraic Geometry", (in Russian), Nauka, 1972. English translation: Springer, 1977, 2nd edition, 1994.

[Sullivan71] D. Sullivan, Combinatorial invariants of analytic spaces, Springer Lecture Notes, vol. **192** (1971), 165–168.

[Taubes92] C. Taubes, The existence of anti-self-dual conformal structures, J. Differential Geom., **36** (1992), 163–253.

[Thom65] R. Thom, Sur l'homologie des variétés algébriques réelles, in "Differential and combinatorial topology", Princeton Univ. Press, 1965, pp. 255–265.

[Tognoli73] A. Tognoli, Su una congettura di Nash, Ann. Sci. Norm. Sup. Pisa, **27** (1973), 167–185.

[Tregub93] S. Tregub, Two remarks on four dimensional cubics, Russ. Math. Surv., **48:2** (1993), 206–208.

[Viterbo98] C. Viterbo, Symplectic real algebraic geometry, to appear.

University of Utah
Salt Lake City UT 84112
`kollar@math.utah.edu`

Current address:
Princeton University
Princeton
NJ 08544-1000
U.S.A.
`kollar@math.princeton.edu`