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The History of the Theorem of Shafarevich in the Theory of Class Formations

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The Theorem of Shafarevich or, as it is mostly called, the Theorem of Shafarevich-Weil always seemed to me to be the coronation of the cohomological approach to class field theory showing that the notion of the canonical class is much more than an auxiliary tool for proving the main theorems of class field theory. The history of the Theorem of Shafarevich and the development of the notion of the canonical class is quite interesting and this is the subject of my talk.

I begin with some remarks about the periodization of class field theory. Thereafter follows the formulation of the Theorem of Shafarevich and its background as it presents itself to the mathematician of today. Then I will speak about the local case which is the content of Shafarevich's paper in Doklady 53 (1946). In the next section we consider the paper of Weil of 1951 "Sur la théorie du corps de classes" which had a great influence on the further development of class field theory. Then follows the development of the notion of the global canonical class and the formulation of the global theorem by Hochschild and Nakayama.

I will conclude the talk with the consideration of the role played by Hasse and his school.

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§1. Some remarks about the periodization of class field theory

The concepts of class field theory grew out of the work of Kronecker [Kr1853, 1882] on cyclotomic fields and complex multiplication and were formulated by Weber [We1891, 1897] and Hilbert [Hi1898].

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Ph. Furtwängler [Fu1907] in the unramified case and Takagi [Ta1920] in general proved the conjectures of Weber and Hilbert except for the Principal Ideal Theorem. Then Artin [Ar1924, 1927] added his reciprocity map to the theory and on its foundation the Principal Ideal Theorem was proved by Furtwängler in 1930.

With this result of Furtwängler the classical theory of class fields was established. The proofs were complete but looked very mysterious.

In the same year a new period of the theory began with three papers of Hasse [Ha1930a-c] creating local class field theory out of classical class field theory.

This was a period of reformulation and simplification of the theory which was completed by Tate [Tt1952] with the full establishment of the cohomological approach. Besides Hasse and Tate the main actors of this period were Chevalley, Herbrand, Nakayama, Hochschild and Weil, with Emmy Noether and her modern algebra in the background. It is this period 1930-1952 in which the Theorem of Shafarevich is placed. But before we are going into details I would like to give a brief presentation of the cohohomological approach to class field theory as it presents itself to the mathematician of today.

§2. The Theorem of Shafarevich in the Theory of Class Formations

2.1. In our present understanding of class field theory, the Theorem of Shafarevich is a theorem about class formations. We have therefore to begin with a short consideration of this notion which was introduced by Artin-Tate [ArTt1952]. We will use a less abstract definition as is to be found in [Ko1992] which is adequate for our purpose.

Let F be a field and Ω/F a finite or infinite Galois extension. For our purpose F will be a local or global field and Ω the separable algebraic closure of F. We denote by $\mathfrak{K}_F$ the category of finite extensions of F contained in Ω. The morphisms of this category are the field homomorphisms which fix F elementwise.

A *field formation* is a functor A from $\mathfrak{K}_F$ into the category $\mathfrak{A}$ of abelian groups such that the following properties are fulfilled.

(Ia) *For any morphism φ of $\mathfrak{K}_F$ the image $A(\varphi)$ is injective.*
If K, L are fields in $\mathfrak{K}_F$ with $K \subseteq L$, then we identify $A(K)$ with its image in $A(L)$.

(Ib) *If L/K is a normal extension in $\mathfrak{K}_F$ then*

$$A(L)^{G(L/K)} := \{a \in A(L) \mid ga = a \text{ for } g \in G(L/K)\} = A(K).$$

In the following we write $A_L := A(L)$.

The only interesting field formations for our purpose are $A_K = K^\times$, the multiplicative group of K, if F is a local field, i.e. a complete field with discrete valuation and finite residue class field, called the local formation, and $A_K = \mathfrak{C}_K$, the idele class group of K, if F is a global field, i.e. $F = \mathbb{Q}$ the field of rational numbers or $F = \mathbb{F}_q(x)$, the rational function field over the field $\mathbb{F}_q$ with q elements.

2.2. Now we are going to define the notion of a class formation. We use the modified cohomological groups $\hat{H}(G, M)$ of Tate [Tt1952] and for any normal extension L/K in $\mathfrak{K}_F$ we write for short

$$H^n(L/K) := H^n(G(L/K), A(L)), n \geq 0,$$
$$\hat{H}^n(L/K) := \hat{H}^n(G(L/K), A(L)), n \in \mathbb{Z}.$$

These cohomology groups have functorial properties. In particular we need the following:

(IIa) *Let L/K be a finite normal extension in $\mathfrak{K}_F$ and let M be an intermediate field of L/K which is normal over K. Then one has the inflation map*

$$\mathrm{Inf}_{M\to L} : H^n(M/K) \to H^n(L/K)$$

for all $n \geq 0$.

(IIb) *Let L/K be a finite normal extension in Ω/F and let M be an arbitrary intermediate field of L/K. Then one has the restriction and the corestriction map*

$$\mathrm{Res}_{K\to M} : \hat{H}^n(L/K) \to \hat{H}^n(L/M),$$
$$\mathrm{Cor}_{M\to K} : \hat{H}^n(L/M) \to \hat{H}^n(L/K), n \in \mathbb{Z}.$$

(IIc) *Let L/K be a finite normal extension and $s \in G(\Omega/F)$. Then the compatible maps $A(L) \to A(sL)$, $G(sL/sK) \to G(L/K)$ given by $a \to sa$ for $a \in A(L)$ and $t \to sts^{-1}$ for $t \in G(sL/sK)$ induce the map*

$$s^* : \hat{H}^n(L/K) \to \hat{H}^n(sL/sK).$$

A field formation $K \to A(K)$ is called a class formation if for any finite normal extension L/K in Ω/F the following axioms are fulfilled:

(IIIa) *There is a canonical isomorphism*

$$H^2(L/K) \to \mathbb{Z}/[L:K]\mathbb{Z}.$$

The preimage of $1 + [L : K]\mathbb{Z}$ is called the canonical class and will be denoted by $u_{L/K}$.

(IIIb) $H^1(L/K) = \{0\}$.
(IIIc) *For any finite extension M/K let $\iota_{K\to M}$ be the map*

$$\iota_{K\to M} : H^2(L/K) \to H^2(LM/M)$$

induced by the compatible maps $A_L \to A_{LM}$, $G(LM/M) \to G(L/K)$. Then $\iota_{L\to M} u_{L/K} = [M : K] u_{LM/M}$.

From these axioms one derives the functorial properties of the canonical class: For L/K a normal extension and M an intermediate field of L/K one has

(IVa) $\mathrm{Inf}_{M\to L} u_{M/K} = [L : M] u_{L/K}$ *if M/K is normal.*
(IVb) $\mathrm{Res}_{K\to M} u_{L/K} = u_{L/M}$, $\mathrm{Cor}_{M\to K} u_{L/M} = [M : K] u_{L/K}$.
(IVc) $u_{sL/sK} = s^* u_{L/K}$ *with s defined as above.*

2.3. In the cohomological setting the inverse of the Artin map, $G \to A_K/N_{L/K}A_L$, for an abelian extension L/K with Galois group G is given by

$$g \to \prod_{x\in G} f_{L/K}(x, g) N_{L/K} A_L \tag{2.1}$$

where $f_{L/K}(h, g)$ is a cocycle belonging to the canonical class $u_{L/K}$. The map (2.1) was defined by Nakayama [Na1936] and is called the *Nakayama map.*

Tate [Tt1952] interpreted (2.1) as the map from $\hat{H}^{-2}(G(L/K), \mathbb{Z}) \cong G(L/K)$ to $\hat{H}^0(L/K) = A_K/N_{L/K}A_L$ given by cup multiplication with the canonical class. More generally he proved that for any $n \in \mathbb{Z}$ and any normal extension L/K in Ω/F the cup multiplication of $\hat{H}^n(G(L/K), \mathbb{Z})$ with the canonical class gives an isomorphism of $\hat{H}^n(G(L/K), \mathbb{Z})$ onto $\hat{H}^{n+2}(L/K)$. (The case $n = 0$ is axiom (IIIa) and the case $n = -1$ is axiom (IIIb)). For $n = -2$ this gives us an isomorphism of $G/[G, G]$ onto $A_K/N_{L/K}A_L$. One deduces the functorial properties of the Artin map from the functorial properties (IVa-c) of the canonical class. For a finite group G we denote by G^{ab} the quotient group $G/[G, G] = \hat{H}^{-2}(G, \mathbb{Z})$. We keep the notation of (IVa-c).

The following diagrams are commutative:

(Va)
$$\begin{array}{ccc} A_K & \xrightarrow{=} & A_K \\ \downarrow & & \downarrow \\ G(L/K)^{ab} & \longrightarrow & G(M/K)^{ab} \end{array}$$

if M/K normal,

(Vb)
$$\begin{array}{ccc} A_K & \xrightarrow{incl} & A_M \\ \downarrow & & \downarrow \\ G(L/K)^{ab} & \longrightarrow & G(L/M)^{ab}, \\ A_M & \xrightarrow{N_{M/K}} & A_K \\ \downarrow & & \downarrow \\ G(L/M)^{ab} & \longrightarrow & G(L/K)^{ab} \end{array}$$

(Vc)
$$\begin{array}{ccc} A_K & \xrightarrow{s} & A_{sK} \\ \downarrow & & \downarrow \\ G(L/K)^{ab} & \longrightarrow & G(sL/sK)^{ab}. \end{array}$$

The bottom maps in (Va) and (Vc) are induced by projection and conjugation while the bottom maps in (Vb) are restriction and corestriction of $\hat{H}^{-2}(\ ,\mathbb{Z})$. The restriction map is called transfer or Verlagerung. It was introduced by Artin in connection with the Principal Ideal Theorem [Ar1929]. But it already appears in Schur's paper [Su1902] as a nameless tool in the proofs. The corestriction map is induced by restriction to the subgroup.

For the proof of the fact that $A(K) = K^\times$ for local fields K and $A(K) = \mathfrak{C}_K$ for global fields are class formations one uses apparently weaker axioms and proves the axioms above by the mechanism of group cohomology (see e.g. [Ko1992]).

2.4. The Theorem of Shafarevich gives an answer to the following question: Let L/K be a normal extension in $\mathfrak{K}_F$ and let M/L be an abelian extension given by the corresponding subgroup U of $A(L)$. It follows from (Vc) that the extension M/K is normal if and only if $sU = U$ for all $s \in G(L/K)$ and the conjugation of $G(M/L)$ with an extension s' of s to $G(M/K)$ corresponds to the action of s on $A(L)/U$.

Now assume that M/K is normal such that we have a group extension

$$G(M/L) \to G(M/K) \to G(L/K). \tag{2.2}$$

We put $G := G(M/K),\ H := G(M/L)$. Then if H is given as a G/H-module, where G/H acts on H by conjugation, the group extension (2.2) is determined by an element of $H^2(G/H, H)$, which is defined by means of a 2-cocycle $f(\sigma,\tau)$ of G/H with values in H as follows. Take a set of representatives $\overline{\tau}$ in G for the elements τ of G/H. Then

$$f(\sigma,\tau) = \overline{\sigma}\,\overline{\tau}\,\overline{\sigma\tau}^{-1}.$$

The question is the following: Which element of $H^2(G(L/K), G(M/L))$ corresponds to this group extension. The answer is given by the following

Theorem 1 (Theorem of Shafarevich). *The class in* $H^2(G(L/K), G(M/L))$ *corresponding to* (2.2) *is the image of the canonical class with respect to the map from* $H^2(G(L/K), A_L)$ *to* $H^2(G(L/K), G(M/L))$ *induced by the Artin map.*

This theorem seems to me to be an example of pre-stabilized harmony in the architecture of mathematics, independent of the human mind: The canonical class defined in the local case as an auxiliary notion in the theory of simple algebras and afterwards developed as a tool in class field theory, appears as the class being associated via the Artin map to any extension of Galois groups over the normal extension to which the canonical class belongs. Historically this picture is true for the local class field theory. But in the global case history is more complicated.

§3. The Local Case

3.1. Hasse [Ha1930a-c] created local class field theory by means of global class field theory. Some assumptions he had to make were proved by F.K. Schmidt [Sc1930] to be always fulfilled. The corresponding local Artin map was called *norm residue symbol*, it is now sometimes called *Hasse map.*

Following almost immediately Hasse, Chevalley [Ch1930] and F.K. Schmidt [unpublished] built the theory on the theory of simple algebras, avoiding global class field theory. But a self contained definition of the local Artin map was only given later in the cyclic case by Hasse [Ha1933] and in general by Chevalley [Ch1933].

3.2. At the origin of the local canonical class is the notion of invariant of a simple central algebra over a local field as defined by Hasse [Ha1931]. In this paper Hasse shows that a central simple algebra $\mathcal{A}$ of dimension n^2 over a local field K has a splitting field L of degree n which is unramified, hence L/K is cyclic.

Already Dickson [Di1914] considered simple algebras $\mathcal{A}$ defined by cyclic field extensions L/K in the form

$$\mathcal{A} = \left\{ \sum_{i=0}^{n-1} \xi_i u^i \mid \xi_i \in L \right\}$$

with $u^n = \alpha$ for a fixed $\alpha \in K^\times$ and $u\xi = s(\xi)u$ for a generator s of $G(L/K)$ and $\xi \in L$. Such algebras are called *cyclic* or of *Dickson type.*

In the case of Hasse we can take for s the Frobenius automorphism of L/K and since α is determined by $\mathcal{A}$ only up to multiplication of α by norms of L/K, we can assume $\alpha = \pi^{\nu}$, $\nu = 0, 1, \dots, n-1$, where π is a prime element of K. Hence $\mathcal{A}$ is determined by its dimension n^2 and by its *invariant* $\frac{\nu}{n}$ mod $\mathbb{Z}$.

E. Noether [No1929] generalized the construction of Dickson for arbitrary Galois extensions L/K. Now we get a simple algebra $\mathcal{A}$ in the form

$$\mathcal{A} = \left\{ \sum_{g \in G(L/K)} \xi_g u_g \mid \xi_g \in L \right\}$$

with $u_g \xi = g(\xi) u_g$ for $\xi \in L$ and

$$u_g u_h = a(g,h) u_{gh},$$

where $a(g,h)$ is a 2-cocycle of $G(L/K)$ with values in $L^\times$. This construction is determined by $\mathcal{A}$ and L/K only up to multiplication by a coboundary such that in fact $\mathcal{A}$ is determined by an element of $H^2(G(L/K), L^\times)$. These considerations show that the Brauer group of algebra classes which are central over K and split by L is isomorphic to $H^2(G(L/K), L^\times)$. If in particular K is a local field, then the algebra class is determined by its invariant and the *canonical class* of $H^2(G(L/K), L^\times)$ corresponds to the algebra class with invariant $\frac{1}{[L:K]}$ mod $\mathbb{Z}$.

3.3. One of the most important steps towards the cohomological foundation of class field theory was a new interpretation of the local Artin map by Nakayama [Na1936]: He proved, using local class field theory, that the inverse map

$$G(L/K) \to K^\times / N_{L/K}(L^\times)$$

for a local abelian extension L/K is given by (2.1).

More generally let L/K be a Galois extension with Galois group G and let G' be the commutator subgroup of G. Then (2.1) induces by definition a homomorphism of G/G' into $K^\times / N_{L/K}(L^\times)$. Y. Akizuki [Ak1936] showed that this homomorphism is injective for arbitrary base fields K, if we use for the definition of (2.1) instead of $u_{L/K}$ an arbitrary element in $H^2(G, L^\times)$ of order $|G|$. This implies $N_{L/K}(L^\times) = N_{M/K}(M^\times)$ for the maximal abelian subextension M/K of L/K if K is a local field.

3.4. The next important step in direction of the cohomological foundation of class field theory came in 1950, when Hochschild eliminated the theory of simple algebras, working only with the corresponding factor systems [Ho1950]. This was in fact only a reformulation, but soon afterwards it became clear that the same procedure is possible in the global case.

3.5. In his paper [Sh1946] Shafarevich proved Theorem 1 in the local case. This paper of less than two pages is perhaps the shortest paper among the essential papers written about a mathematical subject. Besides some functorial properties of the canonical class which at that time appeared as functorial properties of the invariant of simple algebras he uses the Nakayama map (2.1) and a relation for factor systems by Witt [Wi1935]. In the latter paper, which consists only of one and a half pages, Witt proved two rules about classes of factor systems. Shafarevich used the second rule, which we formulate more generally as a property of 2-cocycles $a(g,h)$ of a finite group G with values in a G-module A. Let H be a normal subgroup of G and let $\{\tilde{g}|g \in G\}$ be a system of representatives of the classes of G/H in G.

Then

$$f(g,h) := \prod_{x \in H} x a(\tilde{g},\tilde{h}) a(x, \tilde{g}\tilde{h}) a^{-1}(x, \widetilde{gh}), \tag{3.1}$$

depends only on the classes of g and h in G/H and $f(g,h)$ is a cocycle of G/H with values in A^H. Furthermore, the class of $f(g,h)$ in $H^2(G,A)$ is equal to the class of $[H:\{1\}]a(g,h)$.

In the case of simple algebras we have $A = L^\times$. Witt shows that the algebra corresponding to $[H:\{1\}]a(g,h)$ with $g,h \in G$, which a priori splits already over the fixed field L^H of H, is similar to the algebra corresponding to $f(g,h)$ considered as factor system G/H with values in $(L^H)^\times$. Witt's motivation was to find an explicit expression for $f(g,h)$ in terms of $a(g,h)$.

Shafarevich's proof is so short that we reproduce it here: He formulates the theorem and the proof in terms of simple algebras over local fields and his theorem formulated in 1946 concerns only the local case. But if we pass from the simple central algebra with invariant $\frac{1}{[L:K]} + \mathbb{Z}$ to the corresponding canonical class, as we will do, then his proof goes through for class formations.

Proof of Theorem 1. We put $G := G(M/K)$, $H := G(M/L)$, hence $G(L/K) = G/H$. Let $a(\sigma,\tau)$ be a cocycle of G/H belonging to the canonical class $u_{L/K}$, let $b(g,h)$ be a cocycle of G belonging to $u_{M/K}$, and let $c(g,h)$ be a cocycle of H belonging to $u_{M/L}$. Furthermore, let $\alpha(\sigma,\tau)$ be a cocycle of G/H belonging to the class of (2.2). Fix a representative $\overline{\sigma} \in G$ for $\sigma \in G/H$. Then $\overline{\sigma}\,\overline{\tau} = \alpha(\sigma,\tau)\overline{\sigma\tau}$. By means of the Nakayama map (2.1) for M/L we can write

$$\alpha(\sigma,\tau) = \alpha(\prod_{x\in H} c(x,\alpha(\sigma,\tau))) \text{ with } \sigma,\tau \in G/H, \tag{3.2}$$

where α denotes the Artin map $L^\times \to H$.

We have $\mathrm{Res}_{K\to L} u_{M/K} = u_{M/L}$ (IVb) and therefore

$$\alpha(\sigma,\tau) = \alpha(\prod_{x\in H} b(x,\alpha(\sigma,\tau))) \text{ with } \sigma,\tau \in G/H. \tag{3.3}$$

On the other hand $\mathrm{Inf}_{L\to M} u_{L/K} = [M:L]u_{M/K}$ (IVa). Hence by the property (3.1) of Witt

$$a(gH,hH) \sim b(g,h)^{[M:L]} \sim \prod_{x\in H} xb(\tilde{g},\tilde{h})b(x,\tilde{g}\tilde{h})b^{-1}(x,\widetilde{gh}),$$

where $\tilde{g} := \overline{gH}$, which we can write in the form

$$a(\sigma,\tau) \sim \prod_{x\in H} xb(\overline{\sigma},\overline{\tau})b(x,\overline{\sigma}\,\overline{\tau})b^{-1}(x,\overline{\sigma\tau}). \tag{3.4}$$

Combining (3.3) and (3.4) we see that our assertion

$$\alpha(\sigma,\tau) \sim \alpha(a(\sigma,\tau))$$

is equivalent to

$$\prod_{x\in H} b(x,\alpha(\sigma,\tau))U \sim \prod_{x\in H} xb(\overline{\sigma},\overline{\tau})b(x,\overline{\sigma}\,\overline{\tau})b^{-1}(x,\overline{\sigma\tau})U. \tag{3.5}$$

Since $U = N_{M/L}A_M$ we have $\prod_{x\in H} xb(\overline{\sigma},\overline{\tau}) \in U$. Furthermore, the cocycle property

$$xb(\alpha(\sigma,\tau),\overline{\sigma\tau})b(x,\alpha(\sigma,\tau)\overline{\sigma\tau}) = b(x\alpha(\sigma,\tau),\overline{\sigma\tau})b(x,\alpha(\sigma,\tau))$$

implies

$$N_{M/L}b(\alpha(\sigma,\tau),\overline{\sigma\tau}) \prod_{x\in H} b(x,\overline{\sigma}\,\overline{\tau}) = \prod_{x\in H} b(x,\overline{\sigma\tau}) \prod_{x\in H} b(x,\alpha(\sigma,\tau))$$

This proves (3.5). □

§4. The Global Case

4.1. The idele group. Chevalley [Ch1936] considered the generalization of global class field theory to infinite abelian extensions. In terms of the old formulation of the theory this meant that one has to pass to ray class groups for bigger and bigger modules. He avoided this problem by passing from ray class groups to a new group which he baptized fundamental group and whose elements he called ideal elements. Later this group was called *idele group* and its elements *ideles*. In his fundamental group he introduced a topology such that the closure of the group of principal ideal elements, i.e. principal ideles, is the kernel of the natural map φ_K from J_K onto G_K^{ab}, where J_K denotes the idele group of the number field K and G_K^{ab} the Galois group of the maximal abelian extension of K.

The map φ_K is given by Hasse's norm residue symbol $\left(\frac{\alpha_\mathfrak{p}}{L_\mathfrak{P}/K_\mathfrak{p}}\right)$, where $\alpha_\mathfrak{p} \in K_\mathfrak{p}$ and $L_\mathfrak{P}/K_\mathfrak{p}$ is an abelian extension of the completion $K_\mathfrak{p}$ of K for the place $\mathfrak{p}$ (compare 3.1). Let $\{K^n | n = 1, 2, \dots\}$ be a sequence of abelian extensions of K such that $L = \bigcup_{n=1}^{\infty} K^n$ is the maximal abelian extension of K. Then the value of φ_K at the idele $\prod_\mathfrak{p} \alpha_\mathfrak{p}$ is given by

$$\varphi_K\Big(\prod_\mathfrak{p} \alpha_\mathfrak{p}\Big) = \lim_{\longleftarrow n} \prod_\mathfrak{p} \left(\frac{\alpha_\mathfrak{p}}{K^n_{\mathfrak{P}_n}/K_\mathfrak{p}}\right), \tag{4.1}$$

where the product runs over all places $\mathfrak{p}$ of K and $\mathfrak{P}_n$ is a fixed place of K^n over $\mathfrak{p}$.

(4.1) shows that the introduction of ideles was prepared by Hasse [Ha1930a-c]. In particular the product formula for the norm residue symbol in [Ha1930a] means that the principal ideles are in the kernel of φ_K. In [Ch1940] class field theory is presented with full proofs for the first time without using tools from complex function theory.

4.2. The further development of the global theory was very much influenced by Weil's paper [Wl1951].

First of all Weil was fully aware of the fact that the fundamental object to be considered is the idele class group $\mathfrak{C}_K := J_K/P_K$, where P_K denotes the group of principal ideles, which one can identify with the multiplicative group $K^\times$ of K. He uses the topology in J_K which is given by the product topology of the group U_K of unit ideles: $U_K = \prod_\mathfrak{p} U_\mathfrak{p}$, where $U_\mathfrak{p}$ denotes the compact group of units in $K_\mathfrak{p}$ if $\mathfrak{p}$ is a prime ideal of K, and $U_\mathfrak{p} = K_\mathfrak{p}^\times$ if $\mathfrak{p}$ is an archimedean valuation of K. Now P_K is a discrete and therefore closed subgroup of J_K.

Weil states without proof that the Grössencharacters of Hecke [He 1918] can be identified with the continuous characters of $\mathfrak{C}_K$. The proof for this fact goes along the lines of the transition from ray class groups to the idele class groups as explained in [Ch1936]. But since Chevalley uses another topology for the idele group, he could not find this beautiful interpretation of Grössencharacters, which in fact shows that the introduction of infinite components of the idele group as the multiplicative groups $K_\mathfrak{p}^\times$ for archimedean places $\mathfrak{p}$ was the "right" definition, while, if we are only interested in the interpretation of abelian extensions, the complex places, are useless and the real components could be reduced to $\{\pm 1\}$. In contrast to [Ch1936], Weil considers not only number fields but also function fields of one variable over finite fields as base fields.

In his thesis of 1950 at Princeton University, J. Tate introduced Hecke's Grössencharacters in the same way as Weil in 1951. But this thesis was published only in 1967 ([Tt1950]).

The main problem stated and solved in [Wl1951] is related to the Theorem of Shafarevich: Let L/K be a finite Galois extension. Is it possible to find a natural group extension of $G(L/K)$ with $\mathfrak{C}_L$ compatible with the natural action of $G(L/K)$ on $\mathfrak{C}_L$?

Let K first be a function field. Then the Artin map is an isomorphism of $\mathfrak{C}_L$ onto a dense subgroup of the Galois group G_L^{ab} of the maximal abelian extension L^{ab} of L. Hence we have a natural group extension of $G(L/K)$ with $\mathfrak{C}_L$ given by the group extension

$$G_L^{ab} \to G(L^{ab}/K) \to G(L/K).$$

The case of an algebraic number field K is different. There we have a non-trivial connected component $\mathfrak{D}_L$ of the unit element of $\mathfrak{C}_L$ and class field theory gives only a natural group extension of $G(L/K)$ with $\mathfrak{C}_L/\mathfrak{D}_L$. The problem of Weil is the question whether it is possible to lift this group extension of $\mathfrak{C}_L$. He requires functorial properties for this lifting and he shows that there exists one and only one lifting $G_{L,K}$ with these properties. Weil's motivation for the construction of this lifting to the group extension

$$\mathfrak{C}_K \to G_{L,K} \to G(L/K) \tag{4.2}$$

is its application to L-functions. By means of the group $G_{L,K}$ he defines a new kind of L-functions, now called Weil L-functions which combines the notions of Hecke and Artin (non-abelian) L-function. In his commentaries to his collected works Weil writes: Aussi aurais-je pu intituler mon mémoire "le mariage d'Artin et de Hecke".

4.3. The group extension (4.2) defines an element in the group $H^2(G(L/K), \mathfrak{C}_L)$, called later on the canonical class. But with respect to the Theorem of Shafarevich in the global case Weil has nothing to add because he is not looking for an independent description of the class belonging to the group extension

$$G(L^{ab}/L) \to G(L^{ab}/K) \to G(L/K)$$

but he takes this class for his purpose of defining the group extension (4.2).

Nevertheless, the paper [Wl1951] by Weil stimulated Nakayama to give an independent definition of the canonical class and to prove the Theorem of Shafarevich in the global case (together with Hochschild) [HoNa1952].

Weil writes in his commentaries to his collected works that Nakayama got the manuscript of [Wl1951] before it was published and found an essential mistake in it. Weil was able to correct the mistake in time and Nakayama gave his independent definition of the canonical class already in his paper [Na1951] in the same volume of the Journal of the Mathematical Society of Japan in honour to Takagi that contains also Weil's paper.

Nakayama defines the canonical class in complete analogy with the local canonical class using the cohomological treatment of local class field theory given in [Ho1950]. The canonical class is defined by means of "Durchkreuzung" with a cyclic cyclotomic extension of K, a method which goes back to Chebotarev [Ce1926] and was used by Artin [Ar1927] to prove his reciprocity law. Cyclotomic fields play in the global case the same role as unramified extensions in the local case.

In [Na1952] the cohomological construction of class field theory is complete in so far as it is based only on index relations, the theory of cyclotomic fields and Hasse's sum relation for the invariants of Brauer algebra classes. He proves $H^1(G(L/K), \mathfrak{C}_K) = \{0\}$ for arbitrary finite Galois extensions of global fields K, while in [Na1951] this is only proved for cyclic extensions L/K. Furthermore, it is proved that Nakayama's canonical class is the same as Weil's canonical class.

Finally Hochschild-Nakayama [HoNa1952] proved that for any finite Galois extension L of a global field K the group $H^2(G(L/K), \mathfrak{C}_L)$ is cyclic of degree $[L{:}K]$ generated by the canonical class. This paper contains also a full treatment of the functorial properties of the Artin map including the transfer homomorphism (Verlagerung).

Furthermore, the Theorem of Shafarevich is proved in the global case. This proof is identical with Shafarevich's proof (see 3.5) except for the fact that the second property of Witt is not used directly. The authors give a new and less elegant proof for it. They refer to Shafarevich's Theorem and remark that if they apply their procedure to the local case they "obtain a proof for Shafarevich's result". This is of course not surprising, since their procedure is identical to Shafarevich's.

4.4. The ideas of Weil, Hochschild and Nakayama were further developed in the seminar of Artin and Tate, 1951/52, at Princeton University. There is an exposition of class field theory on the basis of group cohomology, the notion of class formation is introduced, and the groups $G_{L,K}$ of 4.2, called *Weil groups*, are defined for local and global fields in the scope of class formations. The Theorem of Shafarevich is treated as a theorem about class formations (exactly as in 2.4) and it is called "Theorem of Shafarevich-Weil". The notes of the seminar were published only in 1967 [ArTt1952].

An essential ingredient of the cohomological treatment of class field theory is still missing in the Artin-Tate notes: The modified cohomological groups which put together homology and cohomology groups for finite groups G and G-modules A to a sequence of groups $\{\hat{H}^n(G, A) | n \in \mathbb{Z}\}$. The modified cohomology groups were introduced by Tate [Tt1952]. With these groups we get the picture of class field theory which we briefly described in 2.1-2.3.

4.5. With Tate's paper [Tt1952] the period of reformulation and simplification of class field theory was completed. The next period in the theory of algebraic number fields was distinguished by the study of infinite extensions: Iwasawa's theory of Γ-extensions and the theory of maximal extensions with restricted ramification (Tate, Serre, Shafarevich). But the description of this development lays outside the scope of this talk.

From the time after 1952, I mention only two results which are related to the Theorem of Shafarevich, both belonging to the local theory.

The first one is the Theorem of Sen and Tate [SnTt1963] which clarifies the connection of the Theorem of Shafarevich with the filtration given by the ramification groups in the upper numbering.

We keep the notation of section 2.4. Let F be a local field and let D be the division algebra with center K corresponding to the splitting field L and the canonical class $u_{L/K}$. If $a(\sigma, \tau)$, $\sigma, \tau \in G(L/K)$, is a cocycle in the class $u_{L/K}$, then D is given as in 3.2 in the form

$$D = \left\{ \sum_{\sigma \in G(L/K)} \xi_\sigma u_\sigma \mid \xi_\sigma \in L \right\}$$

with $u_\sigma \xi = \sigma(\xi) u_\sigma$ for $\xi \in L$ and

$$u_\sigma u_\tau = a(\sigma, \tau) u_{\sigma\tau}.$$

Hence we have an imbedding of the local Weil group $G_{L,K}$ in $D^\times$ given by taking u_σ as the representative of $\sigma \in G(L/K)$ in $G_{L,K}$. Then

$$G_{L,K} = \bigcup_{\sigma \in G(L/K)} L^\times u_\sigma.$$

We introduce a filtration $\{G^v_{L,K} | v \in \mathbb{R}_+\}$ in $G_{L,K}$ by means of

$$G^v_{L,K} = \{g \in G_{L,K} \mid \iota g \in G(L^{ab}/K)^v\}$$

where ι denotes the homomorphism of $G_{L,K}$ in $G(L^{ab}/K)$ given by the Theorem of Shafarevich.

Furthermore, let ν be the exponential valuation of D, normalized such that $\nu(\pi) = 1$ for a prime element π of L, and let $\varphi(x) = \varphi_{L/K}(x)$ be the Herbrand function of L/K. Then one has the following

Theorem of Sen and Tate. *Let G be an element of $G_{L,K}$ and $x \in \mathbb{R}, x > 0$. Then $g \in G^{\varphi(x)}_{L,K}$ if and only if $\nu(g-1) \geq x$.*

The second result I want to mention here, belongs to the theory of Lubin-Tate extensions giving an explicit construction of the fully ramified extensions of a local field [LuTt1965]. If we apply this theory to a normal extension L of K in the notation of 2.4, then the transformation formula for the change of the prime element π of L for the formal multiplication by π can be interpreted as an explicit construction of the group extension 2.2. See [KodS1996] for details, in particular 2.2.

§5. The Contribution of Hasse and Jehne in the Time After the War

5.1. Hasse worked from 1946 to 1950 in Berlin, where he attracted a large group of talented students. In 1950 he went to Hamburg followed by almost all his students. His main research project during that time was the theory of field embeddings: Given a Galois extension L/K and a group extension

$$A \to G \to G(L/K)$$

with abelian kernel A. Then one can look for a Galois extension F/K containing L/K such that there is an isomorphism φ of $G(F/K)$ onto G such that the diagram

$$\begin{array}{ccc} G(F/K) & \xrightarrow{=} & G(L/K) \\ \downarrow & & \| \\ G & \longrightarrow & G(L/K) \end{array}$$

is commutative. In [Ha1947] the situation that F/L is given by class field theory is considered but the main interest of Hasse was concentrated on the case where F/L is given by Kummer theory.

The class field theoretic situation is considered in the language of ray class groups and the question of the corresponding two-classes is solved in some simple cases. In general this is called "Widerspiegelungsproblem" and Hasse writes at the end of the paper that one needs an essentially new idea to solve the problem.

5.2. One of Hasse's students, Wolfram Jehne, solved the Widerspiegelungsproblem in his diploma in the spirit of the theory of simple algebras. In [Je1952] Jehne defined for this purpose the notion of an idele of algebras consisting of local algebras $A_{\mathfrak{p}}^*$ with center $K_{\mathfrak{p}}$ for all places $\mathfrak{p}$ of K such that $A_{\mathfrak{p}}^*$ is similar to $K_{\mathfrak{p}}$ for almost all $\mathfrak{p}$. An idele of algebras is called embedding idele with respect to L/K of a Galois algebra A/Ω in the sense of Teichmüller [Te1940] if the components $A_{\mathfrak{p}}^*$ are embeddings of $A_{\mathfrak{p}} = A \otimes_K K_{\mathfrak{p}}$ (i.e. one can embed $A_{\mathfrak{p}}$ in $A_{\mathfrak{p}}^*$ such that $A_{\mathfrak{p}}$ is the centralizer of $K_{\mathfrak{p}}$ in $A_{\mathfrak{p}}^*$). To such an embedding idele Jehne associated an element of $H^2(G(L/K), \mathfrak{C}_L)$ and defined its invariant as the sum of the invariants of the local algebras $A_{\mathfrak{p}}^*$. Finally the canonical class is the uniquely determined element in $H^2(G(L/K), \mathfrak{C}_L)$ with invariant $\frac{1}{[L:K]} + \mathbb{Z}$. One finds a similar procedure in cohomological terms in [ArTt1952]. With this construction Jehne gave a proof of Artin's reciprocity law using only Hasse's sum relation for the invariants of algebra classes.

At the end of his paper Jehne proves the Theorem of Shafarevich in the global case in the same way as Shafarevich proved it in the local case by using the Nakayama map and Witt's second rule.

§6. Concluding Remarks

6.1. Shafarevich proved his Theorem in 1945 in the local case. Though his paper [Sh1946] was simultaneously published in English and Russian it became known rather slowly in the West and in Japan. But it is reviewed in Math. Rev. 8 (1947), p. 250, by G. Whaples, and it is mentioned in the article of Chevalley [Ch1951]. It could have been known therefore also to other mathematicians who worked on class field theory around 1951. However, because of the isolation of the mathematicians in Germany at that time, it was unknown to Hasse's group in Berlin and Hamburg. Probably Weil did not know it when he wrote his paper [Wl1951], otherwise he would have mentioned it.

6.2. We have seen that the Theorem of Shafarevich in the global case was proved independently by Hochschild-Nakayama [HoNa1952] and Jehne [Je1952] at the same time. A correct name for the Theorem would be therefore Theorem of Shafarevich-Hochschild-Nakayama-Jehne. Since this is too long and the proof of the Theorem in the global case is almost the same as in the local case after one has established the notion of a canonical class in the global case, it seems to me that "Theorem of Shafarevich" is a justified name. In this I follow Serre [Se1962] who called (p.172) the Theorem in this way.

6.3. Hasse and Nakayama both could have proved the Theorem of Shafarevich in the local case in 1936 but it seems the time was not ripe for asking the question. Hasse finally published the problem in 1947 but only for algebraic number fields and in a, for that time, old fashioned form as a question about ray class groups. Nevertheless, his student Jehne solved the problem in terms of idele class groups. But Hasse failed to understand or accept the solution of Jehne, such that this result remained rather unknown.

6.4. Hasse [Ha1932] was the first to use factor system classes, i.e. elements of the group $H^2(G(L/K), K^\times)$, in the context of algebraic number theory in connection with the theory of simple algebras. So he should be considered as one of the creators of the homological method in algebraic number theory. But he did not like it as a method of cohomology groups independent of the theory of algebras as can be seen from his talk [Ha1967] about the history of class field theory. Maybe, the theory of simple algebras was too important and dear to him since he was one of its creators in the thirties. He did not want to eliminate the simple algebras in the proofs of class field theory as was done in the local case by Hochschild [Ho1950]. This elimination paved the way for the cohomological approach to global class field theory. We see the cohomological method fully developed in the paper of Hochschild-Nakayama [HoNa1952].

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