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Representations of Degenerate Affine Hecke Algebra and $\mathfrak{gl}_n$

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Abstract.

We study the representation theory of the degenerate affine Hecke algebra H_ℓ of GL_ℓ using functors that connect the representation theory of H_ℓ and that of the Lie algebra $\mathfrak{gl}_n$. In particular, a new algebraic approach to the classification theorem of simple H_ℓ-modules is given.

Introduction

Let H_ℓ denote the degenerate (or graded) affine Hecke algebra of GL_ℓ introduced by Drinfeld [Dr] as a certain limit of the affine Hecke algebra. Lusztig [Lu1, Lu2] introduced the degenerate affine Hecke algebra associated to a general reductive group, and proved that the representation theory of the degenerate affine Hecke algebra and that of the corresponding affine Hecke algebra are very close, and one can be essentially recovered from the other.

The representation theory of the (degenerate) affine Hecke algebra has been developed by some methods. Zelevinsky [Ze1] classified simple admissible modules over $GL_\ell(F)$, where F is a p-adic field. This gives a classification of simple modules over the affine Hecke algebra of GL_ℓ through a theorem due to Bernstein, Borel and Matsumoto. In Zelevinsky's classification, the simple modules are constructed as unique simple quotient modules (resp. unique simple submodules) of certain induced modules called standard modules (resp. co-standard modules). In [Ze2, Ze3], Zelevinsky conjectured that the multiplicities of simple modules in the composition series of an induced module are described by Kazhdan-Lusztig polynomials of the symmetric group.

This conjecture was proved by Ginzburg [Gi1] (see also [CG]) through geometric methods. (In fact, Ginzburg gave the multiplicity formulas for general affine Hecke algebras in terms of intersection cohomologies. For

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degenerate affine Hecke algebras, the corresponding formulas were given by Lusztig [Lu3].)

As shown in [BB1, BK], the Kazhdan-Lusztig polynomials also occur in the multiplicity formulas for highest weight modules over semisimple Lie algebras. Consequently, the multiplicity formulas for H_ℓ-modules and those for $\mathfrak{gl}_\ell$-modules are both described by the Kazhdan-Lusztig polynomials of the symmetric group.

This observation led us to the study of a family of functors from the category $\mathcal{O}$ of $\mathfrak{gl}_n$-modules to the category of finite-dimensional H_ℓ-modules in [AS, Su]. It turned out that these functors, which arose from conformal field theory [AST], transform the composition series of a Verma module to the composition series of a standard module under certain conditions, and they connect multiplicity formulas in two categories directly. They give a new approach to the representation theory of H_ℓ. For example, some results for H_ℓ-modules can be deduced from the corresponding results for $\mathfrak{gl}_n$-modules through the functors.

The purpose of this paper is to survey the theory of the functors and to see how it is applied to the study of the representation theory of H_ℓ

After some preliminaries in §1 and §2, we define the functors in §3. It turns out that the functors map a Verma module over $\mathfrak{gl}_n$ to an induced module over H_ℓ, which we introduce in §4. One of the most important statement concerning induced modules is Theorem 5.3, which states that an induced module has a unique simple quotient under certain conditions. Using Theorem 5.3, we prove that a simple module over $\mathfrak{gl}_n$ is mapped to a simple module over H_ℓ (or zero) in §5. Theorem 5.3 also plays an essential role in §6, where we give a new proof for the classification of simple H_ℓ-modules. The functors reduce a part of the problem to the classification of simple modules in the category $\mathcal{O}$. In §7, we apply the functors to get some explicit consequences concerning a special class of simple modules parameterized by skew Young diagrams. §8 is on Kazhdan-Lusztig multiplicity formulas. We see that the multiplicity formulas for $\mathfrak{gl}_n$ (given in [BB1, BK]) imply those for H_ℓ (given in [Gi1, Lu3]) via the functors. We also obtain a refinement of the multiplicity formulas concerning the Jantzen filtration on the induced modules (Rogawski's conjecture).

We treat the degenerate affine Hecke algebra in this paper but it is not hard to extend the story to the non-degenerate case, where the degenerate affine Hecke algebra is replaced by the affine Hecke algebra, and $\mathfrak{gl}_n$ is replaced by its quantum enveloping algebra. In Appendix B, we give an action of the affine Hecke algebra on the tensor product of modules over the quantized enveloping algebra. A q-analogue of the functors is constructed from this action.

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§1. Root system and Lie algebra $\mathfrak{gl}_n$

Let $n \in \mathbb{Z}_{\geq 2}$. Let $\mathfrak{gl}_n$ denote the Lie algebra consisting of all $n \times n$ matrices with entries in $\mathbb{C}$. An inner product is defined on $\mathfrak{gl}_n$ by

$$(1.1) \qquad (x|y)_n = \mathrm{tr}(xy)$$

for $x, y \in \mathfrak{gl}_n$. Let $\mathfrak{t}_n$ be the Cartan subalgebra of $\mathfrak{gl}_n$ consisting of all diagonal matrices, and let $\mathfrak{t}_n^*$ be its dual space. The natural pairing is denoted by $\langle\ ,\ \rangle_n : \mathfrak{t}_n^* \times \mathfrak{t}_n \to \mathbb{C}$. Let $E_{i,j}$ $(1 \leq i, j \leq n)$ denote the matrix with only nonzero entries 1 at the (i,j)-th component. Define a basis $\{\epsilon_i\}_{i=1,\dots,n}$ of $\mathfrak{t}_n^*$ by $\epsilon_i(E_{j,j}) = \delta_{i,j}$, and define the roots by $\alpha_{ij} = \epsilon_i - \epsilon_j$ and the simple roots by $\alpha_i = \epsilon_i - \epsilon_{i+1}$.

Put

$$(1.2) \qquad R_n = \{\alpha_{ij} \mid 1 \leq i \neq j \leq n\},$$

$$(1.3) \qquad R_n^+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}, \quad R_n^- = R_n \setminus R_n^+,$$

$$(1.4) \qquad \Pi_n = \{\alpha_i \mid i = 1, \dots, n-1\}.$$

Then $R_n \subseteq \mathfrak{t}_n^*$ is a root system of type A_{n-1}. Since the restriction of $(\ |\ )_n$ to $\mathfrak{t}_n$ is non-degenerate, we have an isomorphism $\mathfrak{t}_n^* \xrightarrow{\sim} \mathfrak{t}_n$, whose image of $\xi \in \mathfrak{t}_n^*$ is denoted by $\xi^\vee$. In particular we have $\epsilon_i^\vee = E_{i,i}$ and $\alpha_i^\vee = E_{i,i} - E_{i+1,i+1}$. We often identify $\mathfrak{t}_n^*$ with $\mathbb{C}^n$ by $\sum_{i=1}^n \lambda_i \epsilon_i \leftrightarrow (\lambda_1, \dots, \lambda_n)$.

Define

$$(1.5) \qquad Q_n = \bigoplus_{i=1}^{n-1} \mathbb{Z}\alpha_i,$$

$$(1.6) \qquad P_n = \bigoplus_{i=1}^{n} \mathbb{Z}\epsilon_i, \quad P_n^+ = \{\lambda \in P_n \mid \langle \lambda, \alpha^\vee \rangle_n \geq 0 \text{ for all } \alpha \in R_n^+\}.$$

An element of P_n (resp. P_n^+) is called a *integral* (resp. *dominant integral*) weight.

Putting $\mathfrak{n}_n^+ = \oplus_{i<j} \mathbb{C}E_{i,j}$, $\mathfrak{n}_n^- = \oplus_{i>j} \mathbb{C}E_{i,j}$, we have a triangular decomposition $\mathfrak{gl}_n = \mathfrak{n}_n^+ \oplus \mathfrak{t}_n \oplus \mathfrak{n}_n^-$. We put $\mathfrak{b}_n^\pm = \mathfrak{n}_n^\pm \oplus \mathfrak{t}_n$.

The Weyl group W_n associated to the root system (R_n, Π_n) is, by definition, a subgroup of $GL(\mathfrak{t}_n^*)$ generated by the reflections s_α $(\alpha \in R_n)$ defined by

(1.7) $$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle_n \alpha \quad (\lambda \in \mathfrak{t}_n^*).$$

We often use another action of W_n on $\mathfrak{t}_n^*$, which is given by

(1.8) $$w \circ \lambda = w(\lambda + \rho) - \rho \quad (w \in W_n,\ \lambda \in \mathfrak{t}_n^*),$$

where $\rho = (n-1, n-2 \dots, 0) \in \mathfrak{t}_n^*$.

For a $\mathfrak{t}_n$-module X and $\lambda \in \mathfrak{t}_n^*$, put

(1.9) $$X_\lambda = \{v \in X \mid hv = \langle \lambda, h \rangle_n v \text{ for all } h \in \mathfrak{t}_n\},$$

(1.10) $$X_\lambda^{\text{gen}} = \{v \in X \mid (h - \langle \lambda, h \rangle_n)^k v = 0 \text{ for all } h \in \mathfrak{t}_n, \text{ some } k \in \mathbb{Z}_{>0}\},$$

(1.11) $$P(X) = \{\lambda \in \mathfrak{t}_n^* \mid X_\lambda \neq 0\}.$$

The space X_λ (resp X_λ^{gen}) is called the *weight space* (resp. *generalized weight space*) of weight λ with respect to $\mathfrak{t}_n$, and an element of $P(X)$ is called a weight of X.

Let $U(\mathfrak{gl}_n)$ denote the universal enveloping algebra of $\mathfrak{gl}_n$. There is a unique anti-involution σ of $U(\mathfrak{gl}_n)$ such that $\sigma(E_{ij}) = E_{ji}$. For a $\mathfrak{gl}_n$-module X, a bilinear form $(\ |\ ) : X \times X \to \mathbb{C}$ is called a $\mathfrak{gl}_n$-contravariant form if $(u|xv) = (\sigma(x)u|v)$ for all $u, v \in X$ and $x \in \mathfrak{gl}_n$.

For $\lambda \in \mathfrak{t}_n^*$, let $M(\lambda) = U(\mathfrak{gl}_n) \otimes_{U(\mathfrak{b}_n^+)} \mathbb{C}v_\lambda$ denote the Verma module with highest weight λ, where v_λ denotes the highest weight vector. There is a unique $\mathfrak{gl}_n$-contravariant form on $M(\lambda)$ such that $(v_\lambda|v_\lambda) = 1$. It follows that the radical of $(\ |\ )$ is the unique maximal submodule of $M(\lambda)$. (See e.g. [Ja] for the proofs.) The unique simple quotient module of $M(\lambda)$ is denoted by $L(\lambda)$.

Let $\mathcal{O} = \mathcal{O}(\mathfrak{gl}_n)$ denote the category of $\mathfrak{gl}_n$-modules which are finitely generated over $U(\mathfrak{gl}_n)$, $\mathfrak{n}_n^+$-locally finite and $\mathfrak{t}_n$-semisimple (see [BGG]). The modules $M(\lambda)$ and $L(\lambda)$ are objects of $\mathcal{O}$. Let $\chi_\lambda : Z(U(\mathfrak{gl}_n)) \to \mathbb{C}$ denote the infinitesimal character of $M(\lambda)$ (i.e. $zv = \chi_\lambda(z)v$ for all $z \in Z(U(\mathfrak{gl}_n))$, $v \in M(\lambda)$). We introduce an equivalence relation in $\mathfrak{t}_n^*$ by

(1.12) $$\lambda \sim \mu \Leftrightarrow \lambda = w \circ \mu \text{ for some } w \in \mathfrak{S}_n.$$

Then it follows that $\chi_\lambda = \chi_\mu$ if and only if $\lambda \sim \mu$. Define the full subcategory $\mathcal{O}^{\chi_\lambda}$ of $\mathcal{O}$ by

(1.13) $$ob\,\mathcal{O}^{\chi_\lambda} = \{X \in ob\,\mathcal{O} \mid (\mathrm{Ker}\chi_\lambda)^k X = 0 \text{ for some } k\}.$$

Then any $X \in ob\, \mathcal{O}$ admits a decomposition

$$X = \bigoplus_{\lambda} X^{\chi_\lambda} \tag{1.14}$$

such that $X^{\chi_\lambda} \in ob\, \mathcal{O}^{\chi_\lambda}$, where λ runs over all representatives of $\mathfrak{t}_n^*/\sim$. The correspondence $X \mapsto X^{\chi_\lambda}$ gives an exact functor on $\mathcal{O}$.

For $X \in ob\, \mathcal{O}$, put

$$H^0(\mathfrak{n}_n^+, X) = \{v \in X \mid \mathfrak{n}_n^+ v = 0\}, \tag{1.15}$$

$$H_0(\mathfrak{n}_n^-, X) = X/\mathfrak{n}_n^- X. \tag{1.16}$$

Then these are finite-dimensional $\mathfrak{t}_n$-modules. By the universality of the Verma module and (1.14), we have $H^0(\mathfrak{n}_n^+, X)_\lambda \cong \mathrm{Hom}_{\mathfrak{gl}_n}(M(\lambda), X) = \mathrm{Hom}_{\mathfrak{gl}_n}(M(\lambda), X^{\chi_\lambda})) \cong H^0(\mathfrak{n}_n^+, X^{\chi_\lambda})_\lambda$. It also holds that $H_0(\mathfrak{n}_n^-, X)_\lambda \cong H_0(\mathfrak{n}_n^-, X^{\chi_\lambda})_\lambda$. Hence we have a natural injective (resp. surjective) map $H^0(\mathfrak{n}_n^+, X)_\lambda \to (X^{\chi_\lambda})_\lambda$, (resp. $(X^{\chi_\lambda})_\lambda \to H_0(\mathfrak{n}_n^-, X)_\lambda$). Set

$$D_n = \{\lambda \in \mathfrak{t}_n^* \mid \langle \lambda + \rho, \alpha^\vee \rangle_n \notin \mathbb{Z}_{<0} \text{ for all } \alpha \in R_n^+\}. \tag{1.17}$$

Lemma 1.1 ([AS]). *Let $\lambda \in D_n$. Then the maps defined above are both bijective: $H^0(\mathfrak{n}_n^+, X)_\lambda \cong (X^{\chi_\lambda})_\lambda \cong H_0(\mathfrak{n}_n^-, X)_\lambda$.*

§2. Symmetric group and degenerate affine Hecke algebra

Let $\ell \in \mathbb{Z}_{\geq 2}$. Let $\mathfrak{S}_\ell$ denote the symmetric group. Let s_i denote the simple reflection $(i, i+1)$. Then $\mathfrak{S}_\ell$ is generated by $s_1, \ldots, s_{\ell-1}$, and the correspondence $s_i \mapsto s_{\alpha_i}$ gives an isomorphism from $\mathfrak{S}_\ell$ to the Weyl group W_ℓ of the root system (R_ℓ, Π_ℓ).

The length function $l : \mathfrak{S}_\ell \to \mathbb{Z}_{\geq 0}$ is defined by $l(w) = \sharp R_\ell(w)$ for $w \in \mathfrak{S}_\ell$, where

$$R_\ell(w) = R_\ell^+ \cap w^{-1}(R_\ell^-). \tag{2.1}$$

We write $w \to y$ if $y = s_\alpha w$ for some $\alpha \in R_\ell$ and $l(w) < l(y)$. Define $w < y$ if there is a sequence $w \to w_1 \to w_2 \to \cdots \to y$. The resulting relation $\leq$ in $\mathfrak{S}_\ell$ defines a partial order called the *Bruhat order*. Put

$$P_n(\ell) = \{\lambda \in P_n \mid \lambda_i \geq 0 \ (i = 1, \ldots, n) \text{ and } \sum_{i=1}^n \lambda_i = \ell\},$$

$$P_n^+(\ell) = P_n(\ell) \cap P_n^+.$$

An element of $P_n(\ell)$ is called a partition of ℓ with n components. The set $P_n^+(\ell)$ is in one to one correspondence with the set of Young diagrams with ℓ boxes consisting of at most n rows.

Define a surjective map $P_n(\ell) \to P_n^+(\ell)$ by the correspondence $\lambda \mapsto \lambda^+$, where λ^+ denotes the unique element in $P_n^+(\ell) \cap \{w(\lambda) \mid w \in \mathfrak{S}_\ell\}$.

Let us recall that simple $\mathfrak{S}_\ell$-modules are parameterized by the set $P_\ell^+(\ell)$. Let S_λ denote the simple module corresponding to $\lambda \in P_\ell^+(\ell)$.

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in P_n(\ell)$, consider the parabolic subgroup $\mathfrak{S}_\lambda := \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_n}$ of $\mathfrak{S}_\ell$ and set

$$(2.2) \quad \mathfrak{S}_\lambda^\perp = \{w \in \mathfrak{S}_\ell \mid l(ws) > l(w) \text{ for all } s \in \mathfrak{S}_\lambda \cap \{s_1, \ldots, s_{\ell-1}\}\}.$$

Then an element w of $\mathfrak{S}_\lambda^\perp$ is the unique shortest element in the coset $w\mathfrak{S}_\lambda$.

The group $\mathfrak{S}_\ell$ acts on the set $\mathfrak{S}_\lambda^\perp \cong \mathfrak{S}_\ell/\mathfrak{S}_\lambda$ and thus the space $\mathbb{C}[\mathfrak{S}_\lambda^\perp]$ spanned by the elements in $\mathfrak{S}_\lambda^\perp$ is regarded as a $\mathbb{C}[\mathfrak{S}_\ell]$-module. The $\mathfrak{S}_\ell$-module structure of $\mathbb{C}[\mathfrak{S}_\lambda^\perp]$ depends only on the image $\lambda^+ \in P_n(\ell)$.

Let $\lambda \in P_n^+(\ell)$. It is known that the $\mathfrak{S}_\ell$-module $\mathbb{C}[\mathfrak{S}_\lambda^\perp]$ decomposes into

$$(2.3) \qquad \mathbb{C}[\mathfrak{S}_\lambda^\perp] \cong S_\lambda \oplus \bigoplus_{\nu \in P_n^+(\ell),\ \nu \rhd \lambda} S_\nu^{\oplus K_{\nu,\lambda}},$$

where $\rhd$ denotes the dominance order in the set of partitions, and $K_{\nu,\lambda}$ denotes some non-negative integer called Kostka number (see e.g. [Mac, Sa]).

Let $S(\mathfrak{t}_\ell)$ denote the symmetric algebra of $\mathfrak{t}_\ell$, which is isomorphic to the polynomial ring $\mathbb{C}[\epsilon_1^\vee, \ldots, \epsilon_\ell^\vee]$.

Definition 2.1. The *degenerate* (*or graded*) *affine Hecke algebra* H_ℓ of GL_ℓ is the unital associative algebra over $\mathbb{C}$ defined by the following properties:

(i) As a vector space, $H_\ell \cong \mathbb{C}[\mathfrak{S}_\ell] \otimes S(\mathfrak{t}_\ell)$.

(ii) The subspaces $\mathbb{C}[\mathfrak{S}_\ell] \otimes \mathbb{C}$ and $\mathbb{C} \otimes S(\mathfrak{t}_\ell)$ are subalgebras of H_ℓ in a natural fashion (their images will be identified with $\mathbb{C}[\mathfrak{S}_\ell]$ and $S(\mathfrak{t}_\ell)$ respectively).

(iii) The following relations hold in H_ℓ:

$$(2.4) \qquad s_i \cdot \xi - s_i(\xi) \cdot s_i = -\langle \alpha_i, \xi \rangle_\ell \quad (i = 1, \ldots, \ell,\ \xi \in \mathfrak{t}_\ell).$$

Proposition 2.2. [Lu1] *The center of H_ℓ is*

$$S(\mathfrak{t}_\ell)^{\mathfrak{S}_\ell} := \{f \in S(\mathfrak{t}_\ell) \mid w(f) = f \quad \textit{for any } w \in \mathfrak{S}_\ell\}.$$

It is easy to verify that there exists a unique anti-involution ι on H_ℓ such that

$$(2.5) \qquad \iota(s_i) = s_i \ (i = 1, \ldots, \ell - 1), \quad \iota(\epsilon_i^\vee) = \epsilon_i^\vee \ (i = 1, \ldots, \ell).$$

For an H_ℓ-module Y, a bilinear form $(\ |\ ) : Y \times Y \to \mathbb{C}$ is called an H_ℓ-contravariant form if $(u|xv) = (\iota(x)u|v)$ for all $u, v \in Y$ and all $x \in H_\ell$.

Let us introduce intertwining operators, which are useful tools for the investigation of representation theory of H_ℓ. In the rest of this section we refer to e.g. [Lu1, AST] for the proofs of statements.

For each $i \in \{1, \ldots, \ell - 1\}$, we put

$$\phi_i = 1 + s_i \alpha_i^\vee \in H_\ell.$$

Then we have

$$\phi_i \cdot \xi = s_i(\xi) \cdot \phi_i \quad (\xi \in \mathfrak{t}_\ell).$$

Proposition 2.3. *The elements $\{\phi_i\}_i$ defined above satisfy the following relations:*

$$\phi_i \cdot \phi_{i+1} \cdot \phi_i = \phi_{i+1} \cdot \phi_i \cdot \phi_{i+1} \quad (i = 1, \ldots, \ell - 2), \tag{2.6}$$

$$\phi_i \cdot \phi_j = \phi_j \cdot \phi_j \quad (|i - j| \neq 1), \tag{2.7}$$

$$\phi_i^2 = 1 - \alpha_i^{\vee 2} \quad (i = 1, \ldots, \ell - 1)). \tag{2.8}$$

For $w \in \mathfrak{S}_\ell$, let $w = s_{j_1} \cdots s_{j_s} \in \mathfrak{S}_\ell$ be a reduced expression. Put

$$\phi_w = \phi_{j_1} \cdots \phi_{j_s} \in H_\ell.$$

Then the element ϕ_w does not depend on the choice of reduced expressions by Proposition 2.3, and it holds that

$$\phi_{wy} = \phi_w \cdot \phi_y \text{ if } l(wy) = l(w) + l(y). \tag{2.9}$$

By (2.6), we have

$$\phi_w \cdot \xi = w(\xi) \cdot \phi_w \quad (w \in \mathfrak{S}_\ell,\ \xi \in \mathfrak{t}_\ell)\,. \tag{2.10}$$

For an H_ℓ-module Y and $\zeta \in \mathfrak{t}_\ell^*$, we define Y_ζ, Y_ζ^{gen}, and $P(Y)$ by the same formulas as (1.9), (1.10), and (1.11) respectively.

Note that any finite-dimensional H_ℓ-module Y admits the decomposition $Y = \oplus_{\zeta \in \mathfrak{t}_\ell^*} Y_\zeta^{\mathrm{gen}}$.

Proposition 2.4. *Let Y be an H_ℓ-module. Let $\zeta \in \mathfrak{t}_\ell^*$ and $w \in \mathfrak{S}_\ell$. Then $\phi_w(Y_\zeta) \subseteq Y_{w(\zeta)}$ and $\phi_w(Y_\zeta^{\mathrm{gen}}) \subseteq Y_{w(\zeta)}^{\mathrm{gen}}$.*

The element ϕ_w is called the *intertwining operator* (of weight spaces).

Proposition 2.5. *Let $w \in \mathfrak{S}_\ell$. The following relations hold in H_ℓ :*

(i)

$$\phi_w = w \cdot \prod_{\alpha \in R_\ell(w)} \alpha^\vee + \sum_{y<w} y \cdot p_y,$$

for some $p_y \in S(\mathfrak{t}_\ell)$. Here $R_\ell(w) = R_\ell^+ \cap w^{-1}(R_\ell^-)$.

(ii)

$$\phi_{w^{-1}} \cdot \phi_w = \prod_{\alpha \in R_\ell(w)} (1 - \alpha^{\vee 2}).$$

§3. Functors F_λ

Let us recall the definition of the functor

$$F_\lambda : \mathcal{O}(\mathfrak{gl}_n) \to \mathcal{R}(H_\ell)$$

introduced in [AS]. Here $\mathcal{R}(H_\ell)$ denotes the category of finite-dimensional representations of H_ℓ. Let $V_n = \mathbb{C}^n$ denote the vector representation of $\mathfrak{gl}_n$.

Proposition 3.1 ([AS]). *For any $X \in \mathcal{O}(\mathfrak{gl}_n)$, there exists a unique homomorphism*

$$\theta : H_\ell \to \mathrm{End}_{U(\mathfrak{gl}_n)}(X \otimes V_n^{\otimes \ell}) \tag{3.1}$$

such that

$$\theta(s_i) = \Omega_{i\,i+1} \qquad (i = 1, \ldots, \ell - 1), \tag{3.2}$$

$$\theta(\epsilon_i^\vee) = \sum_{0 \le j < i} \Omega_{ji} + n - 1 \quad (i = 1, \ldots, \ell), \tag{3.3}$$

where Ω_{ji} denote the operator given by the element

$$\sum_{1 \le k,m \le n} 1^{\otimes j} \otimes E_{k,m} \otimes 1^{\otimes i-j-1} \otimes E_{m,k} \otimes 1^{\otimes \ell - i} \in \mathfrak{gl}^{\otimes \ell + 1}. \tag{3.4}$$

Remark 3.2. The action of $\mathfrak{S}_\ell$ given by (3.2) is just the natural action of $\mathfrak{S}_\ell$ on $V_n^{\otimes \ell}$.

Let $\lambda \in D_n$ and $X \in ob\,\mathcal{O}(\mathfrak{gl}_n)$. We define

$$F_\lambda(X) = (X \otimes V_n^{\otimes \ell})_\lambda^{\chi_\lambda} \tag{3.5}$$

with an induced H_ℓ-module structure through the homomorphism θ. Obviously F_λ defines an exact functor from $\mathcal{O}(\mathfrak{gl}_n)$ to $\mathcal{R}(H_\ell)$.

Let $X, Y \in ob\,\mathcal{O}(\mathfrak{gl}_n)$ with $\mathfrak{gl}_n$-contravariant forms $(\ |\ )_X$, $(\ |\ )_Y$. Then the tensor product $X \otimes Y$ is equipped with a $\mathfrak{gl}_n$-contravariant bilinear form $(\ |\ )_X \times (\ |\ )_Y$.

The following Proposition immediately follows from the definition of the action θ.

Lemma 3.3 ([Su]). *Let X be a $\mathfrak{gl}_n$-module with a $\mathfrak{gl}_n$-contravariant form. The $\mathfrak{gl}_n$-contravariant form on $X \otimes V_n^{\otimes \ell}$ is also H_ℓ-contravariant, and it induces an H_ℓ-contravariant form on $(X \otimes V_n^{\otimes \ell})_\lambda^{\chi_\lambda} = F_\lambda(X)$.*

§4. Induced modules

Let $\lambda, \mu \in \mathfrak{t}_n^*$ be such that $\lambda - \mu \in P_n(\ell)$, and put

(4.1) $$\ell_i = \lambda_i - \mu_i, \quad (i = 1, \ldots, n).$$

Put $H_{\lambda-\mu} := H_{\ell_1} \otimes \cdots \otimes H_{\ell_n} = \mathbb{C}[\mathfrak{S}_{\lambda-\mu}] \otimes S(\mathfrak{t}_\ell)$ and regard it as a subalgebra of H_ℓ. There exists a one-dimensional representation $\mathbb{C}_{\lambda,\mu} = \mathbb{C}\mathbf{1}_{\lambda,\mu}$ of $H_{\lambda,\mu}$ such that

(4.2) $$w\mathbf{1}_{\lambda,\mu} = \mathbf{1}_{\lambda,\mu} \quad (w \in \mathfrak{S}_{\lambda-\mu}),$$

(4.3) $$\xi\mathbf{1}_{\lambda,\mu} = \langle \zeta_{\lambda,\mu}, \xi \rangle_\ell \mathbf{1}_{\lambda,\mu} \quad (\xi \in \mathfrak{t}_\ell),$$

where $\zeta_{\lambda,\mu} \in \mathfrak{t}_\ell^*$ is given by

(4.4)
$$\langle \zeta_{\lambda,\mu}, \epsilon_j^\vee \rangle_\ell = \mu_i + n - i + j - \sum_{k=1}^{i-1} \ell_k - 1 \quad \text{for} \quad \sum_{k=1}^{i-1} \ell_k < j \leq \sum_{k=1}^{i} \ell_k.$$

Note, in particular, that if we put $a_i = \sum_{k=1}^{i-1} \ell_k + 1$ and $b_i = \sum_{k=1}^{i} \ell_k$, then

(4.5) $$\langle \zeta_{\lambda,\mu}, \epsilon_{a_i}^\vee \rangle_\ell = \langle \mu + \rho, \epsilon_i^\vee \rangle_n, \quad \langle \zeta_{\lambda,\mu}, \epsilon_{b_i}^\vee \rangle_\ell = \langle \lambda + \rho, \epsilon_i^\vee \rangle_n - 1,$$

(4.6) $$\langle \zeta_{\lambda,\mu}, \alpha_i^\vee \rangle_\ell = -1 \text{ for } i \notin \{b_1, b_2, \ldots, b_n\}.$$

Define an H_ℓ-module $\mathcal{M}(\lambda, \mu)$ by

(4.7) $$\mathcal{M}(\lambda, \mu) = H_\ell \underset{H_{\lambda-\mu}}{\otimes} \mathbb{C}_{\lambda,\mu}.$$

It is obvious that $\mathcal{M}(\lambda, \mu) \downarrow_{\mathbb{C}[\mathfrak{S}_\ell]} \cong \mathbb{C}[\mathfrak{S}_{\lambda-\mu}^\perp]$ and thus its dimension is given by

$$\dim \mathcal{M}(\lambda, \mu) = \frac{\ell!}{\ell_1! \cdots \ell_n!}.$$

For $\zeta \in \mathfrak{t}_\ell^*$, let $\mathfrak{S}_\ell[\zeta]$ denote the stabilizer of ζ:

$$\mathfrak{S}_\ell[\zeta] = \{s \in \mathfrak{S}_\ell \mid w(\zeta) = \zeta\}. \tag{4.8}$$

Lemma 4.1. *For $\lambda, \mu \in \mathfrak{t}_n^*$ such that $\lambda - \mu \in P_n(\ell)$, we have*

(i) $P(\mathcal{M}(\lambda, \mu)) = \{w(\zeta_{\lambda,\mu}) \mid w \in \mathfrak{S}_{\lambda-\mu}^\perp\}$.

(ii) *For $\eta \in P(\mathcal{M}(\lambda, \mu))$, we have*

$$\dim \mathcal{M}(\lambda, \mu)_\eta^{\text{gen}} = \#\{w \in \mathfrak{S}_{\lambda-\mu}^\perp \mid w(\zeta_{\lambda,\mu}) = \eta\}.$$

In particular, $\dim \mathcal{M}(\lambda, \mu)_{\zeta_{\lambda,\mu}}^{\text{gen}} = \# \left(\mathfrak{S}_{\lambda-\mu}^\perp \cap \mathfrak{S}_\ell[\zeta_{\lambda,\mu}]\right)$.

Proof. First, note that $\{w\mathbf{1}_{\lambda,\mu} \mid w \in \mathfrak{S}_{\lambda-\mu}^\perp\}$ gives a basis of $\mathcal{M}(\lambda, \mu)$. For $\xi \in \mathfrak{t}_\ell$ and $w \in \mathfrak{S}_{\lambda-\mu}^\perp$, it follows from the relation (2.4) that

$$\begin{aligned} \xi \cdot w\mathbf{1}_{\lambda,\mu} &= w \cdot w^{-1}(\xi)\mathbf{1}_{\lambda,\mu} + \sum_{y<w} a_y y\mathbf{1}_{\lambda,\mu} \\ &= \langle w(\zeta_{\lambda,\mu}), \xi\rangle_\ell w\mathbf{1}_{\lambda,\mu} + \sum_{y<w} a_y y\mathbf{1}_{\lambda,\mu}, \end{aligned} \tag{4.9}$$

for some numbers a_y, where y runs over those elements of $\mathfrak{S}_{\lambda-\mu}^\perp$ such that $y < w$. Hence we have (i). Now (ii) is obvious. Q.E.D.

We extend the definition of $\mathcal{M}(\lambda, \mu)$ for any $\lambda, \mu \in \mathfrak{t}_n^*$ by

$$\mathcal{M}(\lambda, \mu) = 0 \text{ for } \lambda, \mu \in \mathfrak{t}_n^* \text{ such that } \lambda - \mu \notin P_n(\ell). \tag{4.10}$$

Theorem 4.2 ([AS]). *Let $\lambda \in D_n$ and $\mu \in \mathfrak{t}_n^*$. Then there is an isomorphism of H_ℓ-modules*

$$F_\lambda(M(\mu)) \cong \mathcal{M}(\lambda, \mu).$$

For $w \in \mathfrak{S}_n$, let w_μ^λ denote the unique longest element in the coset $\mathfrak{S}_n[\lambda + \rho]w\mathfrak{S}_n[\mu + \rho]$.

Lemma 4.3. *Let $\lambda, \mu \in D_n$ and $w \in \mathfrak{S}_n$ be such that $\lambda - w \circ \mu \in P_n(\ell)$. Then $\mathcal{M}(\lambda, w \circ \mu) \cong \mathcal{M}(\lambda, w_\mu^\lambda \circ \mu)$.*

Proof. We prove the statement using the fact known in the representation theory of $\mathfrak{gl}_n$; there exists an injective homomorphism $M(w_\mu^\lambda \circ \mu) \to M(w \circ \mu)$. By applying the exact functor F_λ, we have an injective homomorphism $\mathcal{M}(\lambda, w_\mu^\lambda \circ \mu) \to \mathcal{M}(\lambda, w \circ \mu)$. It is easy to see that $(\lambda - w \circ \mu)^+ = (\lambda - w_\mu^\lambda \circ \mu)^+$. This implies $\dim \mathcal{M}(\lambda, w_\mu^\lambda \circ \mu) = \dim \mathcal{M}(\lambda, w \circ \mu)$ and thus $\mathcal{M}(\lambda, w \circ \mu) \cong \mathcal{M}(\lambda, w_\mu^\lambda \circ \mu)$. Q.E.D.

§5. Simple quotient

We give a sufficient condition for an induced module to have a unique simple quotient (Theorem 5.3), which is an essential step to the classification of simple representations of H_ℓ. Theorem 5.3 has been obtained by Zelevinsky [Ze1]. We give another proof and the key Lemma 5.2 seems to be new.

Lemma 5.1. *Let $\lambda, \mu \in \mathfrak{t}_n^*$ be such that $\lambda - \mu \in P_n(\ell)$, and suppose that $\langle \lambda + \rho, \alpha_i^\vee \rangle_n = 0$ or $\langle \lambda + \rho, \alpha_i^\vee \rangle_n \notin \mathbb{Z}$. Then $\mathcal{M}(\lambda, \mu) \cong \mathcal{M}(s_i \circ \lambda, s_i \circ \mu)$.*

Proof. If $\langle \lambda + \rho, \alpha_i^\vee \rangle_n = 0$, then the statement follows from Lemma 4.3.

Suppose $\langle \lambda + \rho, \alpha_i^\vee \rangle_n \notin \mathbb{Z}$. Put $\ell_j = \lambda_j - \mu_j$ $(j = 1, \ldots, n)$ and let w be the element of $\mathfrak{S}_{\ell_i + \ell_{i+1}}$ corresponding to the permutation $(1, 2, \ldots, \ell_i + \ell_{i+1}) \mapsto (\ell_i + 1, \ell_i + 2, \ldots, \ell_i + \ell_{i+1}, 1, 2, \ldots, \ell_i)$. Regard $\mathfrak{S}_{\ell_i + \ell_{i+1}}$ as a subgroup of $\mathfrak{S}_\ell$ via $\{1\} \times \mathfrak{S}_{\ell_i + \ell_{i+1}} \times \{1\} \subseteq \mathfrak{S}_{\ell_1 + \cdots + \ell_{i-1}} \times \mathfrak{S}_{\ell_i + \ell_{i+1}} \times \mathfrak{S}_{\ell_{i+2} + \cdots + \ell_n} \subseteq \mathfrak{S}_\ell$. Then $\zeta_{s_i \circ \lambda, s_i \circ \mu} = w(\zeta_{\lambda, \mu})$ and there exists an H_ℓ-homomorphism $\mathcal{M}(s_i \circ \lambda, s_i \circ \mu) \to \mathcal{M}(\lambda, \mu)$ such that $\mathbf{1}_{s_i \circ \lambda, s_i \circ \mu} \mapsto \phi_w \mathbf{1}_{\lambda, \mu}$. It follows from Proposition 2.5-(ii) that $\phi_{w^{-1}} \phi_w \mathbf{1}_{\lambda, \mu}$ is nonzero and thus ϕ_w is invertible. Hence it gives an isomorphism. Q.E.D.

For $\eta \in \mathfrak{t}_n^*$, put $R_n[\eta] = \{\alpha \in R_n \mid \langle \eta, \alpha^\vee \rangle_n = 0\}$. It is not difficult to see that $R_n[\eta]$ is a root system and its Weyl group is the stabilizer $\mathfrak{S}_n[\eta]$ of η, i.e. $\mathfrak{S}_n[\eta] = \langle s_\alpha \mid \alpha \in R_n[\eta] \rangle$.

Put

$$(5.1) \qquad P_\eta^+ = \{\mu \in \mathfrak{t}_n^* \mid \langle \mu, \alpha^\vee \rangle_n \in \mathbb{Z}_{\geq 0} \quad \text{for any } \alpha \in R_n^+ \cap R_n[\eta]\},$$

$$(5.2) \qquad P_\eta^- = \{\mu \in \mathfrak{t}_n^* \mid \langle \mu, \alpha^\vee \rangle_n \in \mathbb{Z}_{\leq 0} \quad \text{for any } \alpha \in R_n^+ \cap R_n[\eta]\}.$$

The proof of the following important lemma is given in Appendix A.

Lemma 5.2. *Let $\lambda, \mu \in \mathfrak{t}_n^*$ be such that $\lambda - \mu \in P_n(\ell)$. Suppose the following conditions:*

(a) $\lambda \in D_n$. (b) $\mu + \rho \in P_{\lambda+\rho}^+$.

(c) *There exists numbers $1 = m_0 < m_1 < \cdots < m_k = \ell$ for which we have*

$$(5.3) \qquad \lambda_i - \lambda_j \in \mathbb{Z} \Leftrightarrow m_{r-1} < i, j \leq m_r \text{ for some } r \in \{1, \ldots, k\}.$$

Then, we have $\mathcal{M}(\lambda, \mu)_{\zeta_{\lambda,\mu}} = \mathbb{C}\mathbf{1}_{\lambda,\mu}$.

Theorem 5.3. *Let $\lambda, \mu \in \mathfrak{t}_n^*$ be such that $\lambda - \mu \in P_n(\ell)$. If $\lambda \in D_n$, then $\mathcal{M}(\lambda, \mu)$ has a unique simple quotient module, which is denoted by $\mathcal{L}(\lambda, \mu)$.*

Proof. By Lemma 5.1, it is enough to prove the statement assuming that λ satisfies the conditions in Lemma 5.2. Let N be a submodule of $\mathcal{M}(\lambda,\mu)$. If $N^{\text{gen}}_{\zeta_{\lambda,\mu}} \neq 0$, then $N_{\zeta_{\lambda,\mu}} \neq 0$. By Lemma 5.2, this implies $\mathbf{1}_{\lambda,\mu} \in N$ and thus $N = \mathcal{M}(\lambda,\mu)$. Hence a proper submodule N must satisfy $N \subseteq \oplus_{\eta\neq\zeta_{\lambda,\mu}} \mathcal{M}(\lambda,\mu)^{\text{gen}}_{\zeta_{\lambda,\mu}}$. The sum of all the proper submodules also satisfies this property and it is a unique maximal proper submodule. Q.E.D.

For $\lambda \in D_n$, we call $\mathcal{M}(\lambda,\mu)$ a *standard module.* The following lemma is also a consequence of Lemma 5.2.

Lemma 5.4. *Let $\lambda \in D_n$ and $\mu \in \lambda - P_n(\ell)$. Let $(\ |\ )$ be a non-zero H_ℓ-contravariant form on $\mathcal{M}(\lambda,\mu)$ and let N be a unique maximal submodule of $\mathcal{M}(\lambda,\mu)$. Then $N = \mathrm{rad}(\ |\ )$.*

Proof. It is obvious that $\mathrm{rad}(\ |\ ) \subseteq N$. To prove the opposite inclusion, first note that $\mathcal{M}(\lambda,\mu)^{\text{gen}}_\eta \perp \mathcal{M}(\lambda,\mu)^{\text{gen}}_\zeta$ with respect to $(\ |\ )$ unless $\eta = \zeta$. For any $u \in N$ and $x \in H_\ell$, we have $(u|x\mathbf{1}_{\lambda,\mu}) = (\iota(x)u|\mathbf{1}_{\lambda,\mu}) = 0$ because $\iota(x)u \in N \subseteq \oplus_{\eta\neq\zeta_{\lambda,\mu}} \mathcal{M}(\lambda,\mu)^{\text{gen}}_\eta$ and $\mathbf{1}_{\lambda,\mu} \in \mathcal{M}(\lambda,\mu)^{\text{gen}}_{\zeta_{\lambda,\mu}}$. This implies $N \subseteq \mathrm{rad}(\ |\ )$. Q.E.D.

By Lemma 3.3, the $\mathfrak{gl}_n$-contravariant form on $L(\mu)$ induces an H_ℓ-contravariant form on $\mathcal{L}(\lambda,\mu) = F_\lambda(L(\mu))$, and it turns out to be non-degenerate. Now, Lemma 5.4, implies that the H_ℓ-module $F_\lambda(L(\mu))$ is simple unless it is zero. More precisely, we have

Theorem 5.5 ([AS, Su]). *Let $\lambda \in D_n$ and $\mu \in \lambda - P_n(\ell)$.*

(i) *If $\mu + \rho \in P^-_{\lambda+\rho}$ then we have $F_\lambda(L(\mu)) \cong \mathcal{L}(\lambda,\mu)$.*

(ii) *If $\mu + \rho \notin P^-_{\lambda+\rho}$ then we have $F_\lambda(L(\mu)) = 0$.*

Remark 5.6. (i) One can express μ in Theorem 5.5 as $\mu = w \circ \tilde{\mu}$ with some $w \in \mathfrak{S}_n$ and $\tilde{\mu} \in D_n$. Then the condition $\mu+\rho \in P^-_{\lambda+\rho}$ is equivalent to

$$\mu = w^\lambda \circ \tilde{\mu} \quad \text{or equivalently} \quad \mu = w^\lambda_{\tilde{\mu}} \circ \tilde{\mu}.$$

Here w^λ (resp. $w^\lambda_{\tilde{\mu}}$) denotes the unique longest element in the coset $\mathfrak{S}_n[\lambda+\rho]w$ (resp. $\mathfrak{S}_n[\lambda+\rho]w\mathfrak{S}_n[\tilde{\mu}+\rho]$). (See [Su, Remark 3.2.3] for the proof.)

(ii) In [Su], we give a proof of Theorem 5.5 using the result by Zelevinsky [Zel, Theorem 6.1] that describes when two simple modules are isomorphic. In the following, we give a modified proof of Theorem 5.5 without referring to Zelevinsky's result. (See Theorem 6.5.)

Proof of Theorem 5.5. The statement (ii) follows from Lemma 4.3 easily (see [Su]).

Let us prove (i). It is enough to see that $F_\lambda(L(\mu))$ is nonzero under the condition $\mu+\rho \in P^-_{\lambda+\rho}$, by which we can write μ as $\mu = w^\lambda_{\tilde{\mu}} \circ \tilde{\mu}$, where $\tilde{\mu} \in D_n$ and $w^\lambda_{\tilde{\mu}}$ is the longest element in $\mathfrak{S}_n[\lambda+\rho]w\mathfrak{S}_n[\tilde{\mu}+\rho]$. In the Grothendieck group of $\mathcal{O}(\mathfrak{gl}_n)$, we write

$$M(w^\lambda_{\tilde{\mu}} \circ \tilde{\mu}) = L(w^\lambda_{\tilde{\mu}} \circ \tilde{\mu}) + \sum_{y_{\tilde{\mu}}} a_{y_{\tilde{\mu}}} L(y_{\tilde{\mu}} \circ \tilde{\mu}). \tag{5.4}$$

Here the sum runs over those elements $y_{\tilde{\mu}} \in W_n$ such that $y_{\tilde{\mu}}$ is longest in $y_{\tilde{\mu}} W_n[\tilde{\mu}+\rho]$ and $y_{\tilde{\mu}} > w^\lambda_{\tilde{\mu}}$. Note that this implies

$$y_{\tilde{\mu}} \notin \mathfrak{S}_n[\lambda+\rho]w^\lambda_{\tilde{\mu}}\mathfrak{S}_n[\tilde{\mu}+\rho]. \tag{5.5}$$

Applying F_λ to (5.4) we have

$$\mathcal{M}(\lambda, w^\lambda_{\tilde{\mu}} \circ \tilde{\mu}) = F_\lambda(L(w^\lambda_{\tilde{\mu}} \circ \tilde{\mu})) + \sum_{y_{\tilde{\mu}}} a_{y_{\tilde{\mu}}} F_\lambda(L(y_{\tilde{\mu}} \circ \tilde{\mu})) \tag{5.6}$$

in the Grothendieck group of $\mathcal{R}(H_\ell)$. Note that

$$F_\lambda(L(y_{\tilde{\mu}} \circ \tilde{\mu}))\downarrow_{\mathbb{C}[\mathfrak{S}_\ell]} \subseteq \mathcal{M}(\lambda, y_{\tilde{\mu}} \circ \tilde{\mu})\downarrow_{\mathbb{C}[\mathfrak{S}_\ell]} = \oplus_{\nu \trianglerighteq (\lambda - y_{\tilde{\mu}} \circ \tilde{\mu})^+} S_\nu^{\oplus a_\nu}$$

with some $a_\nu \in \mathbb{Z}_{\geq 0}$. By Lemma 5.7 below, it follows from (5.5) that $(\lambda - y_{\tilde{\mu}} \circ \tilde{\mu})^+ \triangleright (\lambda - w^\lambda_{\tilde{\mu}} \circ \tilde{\mu})$, and thus $F_\lambda(L(y_{\tilde{\mu}} \circ \tilde{\mu}))$ does not contain $S_{(\lambda - w^\lambda_{\tilde{\mu}} \circ \tilde{\mu})^+}$, which must be contained in $\mathcal{M}(\lambda, w^\lambda_{\tilde{\mu}} \circ \tilde{\mu})$. Therefore $F_\lambda(L(w^\lambda_{\tilde{\mu}} \circ \tilde{\mu}))$ cannot be zero. Q.E.D.

Lemma 5.7. *Let $\lambda, \mu \in D_n$ and $w, y \in \mathfrak{S}_n$ be such that $\lambda - w \circ \mu, \lambda - y \circ \mu \in P_n(\ell)$. If $y \notin \mathfrak{S}_n[\lambda+\rho]w\mathfrak{S}_n[\mu+\rho]$ and $y > w$, then $(\lambda - y \circ \mu)^+ \triangleright (\lambda - w \circ \mu)^+$*

Proof. First suppose that $y = s_\alpha w$ for $\alpha \in R_n^+$. Then $l(y) > l(w)$ implies $w^{-1}(\alpha) \in R_n^+$, and it follows that $\langle \lambda+\rho, \alpha^\vee \rangle_n \geq 0$ and $\langle w(\mu+\rho), \alpha^\vee \rangle_n = \langle \mu+\rho, w^{-1}(\alpha^\vee) \rangle \geq 0$. Hence we have

$$\begin{aligned} |\langle \lambda - y \circ \mu, \alpha^\vee \rangle_n| =& |\langle \lambda+\rho, \alpha^\vee \rangle_n + \langle w(\mu+\rho), \alpha^\vee \rangle_n| \geq \\ & |\langle \lambda+\rho, \alpha^\vee \rangle_n - \langle w(\mu+\rho), \alpha^\vee \rangle_n| = |\langle \lambda - w \circ \mu, \alpha^\vee \rangle_n|. \end{aligned}$$

This implies $(\lambda - y \circ \mu)^+ \trianglerighteq (\lambda - w \circ \mu)^+$. The equality holds only when $\langle \lambda+\rho, \alpha^\vee \rangle_n = 0$ or $\langle w(\mu+\rho), \alpha^\vee \rangle_n = 0$, that is, only when $y = s_\alpha w \in \mathfrak{S}_n[\lambda+\rho]w\mathfrak{S}_n[\mu+\rho]$.

Now let us consider the general case. Since $y > w$, there is a sequence $\alpha^{(1)}, \ldots, \alpha^{(m)}$ in R_n^+ such that $y = s_{\alpha^{(m)}} \cdots s_{\alpha^{(1)}} w$ and $l(w^{(k+1)}) > l(w^{(k)})$ $(k \geq 0)$, where $w^{(k)} = s_{\alpha^{(k)}} \cdots s_{\alpha^{(1)}} w$. Now the statement follows by the induction on m. Q.E.D.

In the proof of Theorem 5.5, we have also proved the following

Corollary 5.8. *Let $\lambda \in D_n$ and $\mu \in \lambda - P_n(\ell)$. Then*

$$\mathcal{L}(\lambda, \mu)\downarrow_{\mathbb{C}[\mathfrak{S}_\ell]} \cong S_{(\lambda-\mu)^+} \oplus \bigoplus_{\nu \rhd (\lambda-\mu)^+} S_\nu^{\oplus N_{\mu,\nu}^\lambda}$$

for some non-negative integers $N_{\mu,\nu}^\lambda$.

§6. Classification of simple modules

Let us consider the particular case where $\ell = n$. For $\zeta \in \mathfrak{t}_\ell^*$, we put

$$\mathcal{I}(\zeta) := \mathcal{M}(\zeta - \rho + \epsilon, \zeta - \rho), \quad \mathbf{1}_\zeta := \mathbf{1}_{\zeta-\rho+\epsilon, \zeta-\rho},$$

where $\epsilon = (1, \ldots, 1) \in \mathfrak{t}_\ell^*$. The H_ℓ-module $\mathcal{I}(\zeta)$ is called the principal series representation associated with ζ. As a $\mathbb{C}[\mathfrak{S}_\ell]$-module, $\mathcal{I}(\zeta)$ is isomorphic to the regular representation. Note also that $\xi \mathbf{1}_\zeta = \langle \zeta, \xi \rangle_\ell \mathbf{1}$ for $\xi \in \mathfrak{t}_\ell$, and that

$$\mathrm{Hom}_{H_\ell}(\mathcal{I}(\zeta), Y) = Y_\zeta$$

for any H_ℓ-module Y.

Lemma 6.1 ([Ro]). *Let $\zeta \in \mathfrak{t}_\ell^*$ and $w \in \mathfrak{S}_\ell$. Then $\mathcal{I}(\zeta)$ and $\mathcal{I}(w(\zeta))$ have the same composition factors.*

Proof. It is enough to prove the statement when w is the simple reflection, say s_i. The intertwining operator $\phi_i = 1 + s_i \alpha_i^\vee$ defines H_ℓ-homomorphisms $\Phi_i : \mathcal{I}(\zeta) \to \mathcal{I}(s_i(\zeta))$ and $\Phi^i : \mathcal{I}(s_i(\zeta)) \to \mathcal{I}(\zeta)$ such that $\mathbf{1}_\zeta \mapsto \phi_i \mathbf{1}_{s_i(\zeta)}$ and $\mathbf{1}_{s_i(\zeta)} \mapsto \phi_i \mathbf{1}_\zeta$ respectively. If $\langle \zeta, \alpha_i^\vee \rangle_\ell \neq \pm 1$, then by (2.8), Φ_i is an isomorphism. Now, it is enough to prove the statement in the case $\langle \zeta, \alpha_i^\vee \rangle_\ell = 1$. Through $\mathcal{I}(\zeta)\downarrow_{\mathbb{C}[\mathfrak{S}_\ell]} \cong \mathcal{I}(s_i(\zeta))\downarrow_{\mathbb{C}[\mathfrak{S}_\ell]} \cong \mathbb{C}[\mathfrak{S}_\ell]$, the Φ_i and Φ^i are regarded as the maps between $\mathbb{C}[\mathfrak{S}_\ell]$ given by $v \mapsto v(1 - s_i)$ and $v \mapsto v(1 + s_i)$ $(v \in \mathbb{C}[\mathfrak{S}_\ell])$ respectively. Therefore the sequence $\mathcal{I}(\zeta) \xrightarrow{\Phi_i} \mathcal{I}(s_i(\zeta)) \xrightarrow{\Phi^i} \mathcal{I}(\zeta)$ is exact. Hence, in the Grothendieck group of $\mathcal{R}(H_\ell)$, we have

$$\mathcal{I}(s_i(\zeta)) = (\mathcal{I}(\zeta)/\mathrm{Ker}(\Phi_i)) \oplus \mathrm{Im}(\Phi^i) = (\mathcal{I}(\zeta)/\mathrm{Ker}(\Phi_i)) \oplus \mathrm{Ker}(\Phi_i) = \mathcal{I}(\zeta),$$

as required. Q.E.D.

Lemma 6.1 implies the following

Proposition 6.2. *Any finite-dimensional irreducible H_ℓ-module is a composition factor of $\mathcal{I}(\zeta)$ for some $\zeta \in \rho + D_\ell$.*

Theorem 6.3. (*cf.* [Zel, Theorem 6.1] [Ch1])
Any finite-dimensional simple module over H_ℓ is isomorphic to $\mathcal{L}(\lambda, w \circ (\lambda - \epsilon))$ for some $\lambda \in D_\ell$ and $w \in \mathfrak{S}_\ell$ such that $\lambda - w \circ (\lambda - \epsilon) \in P_\ell(\ell)$.

Proof. Let L be a finite-dimensional simple H_ℓ-module. By Proposition 6.2, we can suppose that L is a composition factor of $\mathcal{I}(\zeta)$ for some $\zeta \in \rho + D_\ell$. Put $\lambda = \zeta - \rho + \epsilon \in D_\ell$. By Theorem 4.2 and Theorem 5.5, the functor F_λ transforms the composition series of $M(\lambda - \epsilon)$ to the composition series of $\mathcal{I}(\lambda + \rho - \epsilon) = \mathcal{I}(\zeta)$. Therefore L is of the form $\mathcal{L}(\lambda, w \circ (\lambda - \epsilon)) = F_\lambda(L(w^\lambda_{\lambda-\epsilon} \circ (\lambda - \epsilon))$ for some $w \in \mathfrak{S}_\ell$. Q.E.D.

We say that an H_ℓ-module Y is of *level* n if $Y\downarrow_{\mathbb{C}[\mathfrak{S}_\ell]} \cong \oplus_{\nu \in P_n^+(\ell)} S_\nu^{\oplus a_\nu}$ for some $a_\nu \in \mathbb{Z}_{\geq 0}$. The induced module $\mathcal{M}(\lambda, \mu)$ $(\lambda, \mu \in \mathfrak{t}_n^*)$ is of level n. Any finite-dimensional H_ℓ-module is of level ℓ.

Corollary 6.4. *Any simple H_ℓ- module of level n is isomorphic to $\mathcal{L}(\lambda, \mu)$ for some $\lambda \in D_n$ and $\mu \in \lambda - P_n(\ell)$.*

Theorem 6.5. (*cf.* [Zel, Theorem 6.1]) *Suppose that $\lambda, \mu \in D_n$ and $w, y \in \mathfrak{S}_n$ satisfy $\lambda - w \circ \mu, \lambda - y \circ \mu \in P_n(\ell)$. Then the following are equivalent:*

(a) $y \in \mathfrak{S}_n[\lambda + \rho] w \mathfrak{S}_n[\mu + \rho]$,

(b) $\mathcal{M}(\lambda, w \circ \mu) \cong \mathcal{M}(\lambda, y \circ \mu)$,

(c) $\mathcal{L}(\lambda, w \circ \mu) \cong \mathcal{L}(\lambda, y \circ \mu)$.

Proof. (a)$\Rightarrow$(b) follows from Lemma 4.3. (b)$\Rightarrow$(c) is obvious.

(c)$\Rightarrow$(b): Suppose (c), then there is a weight vector $v \in \mathcal{M}(\lambda, y \circ \mu)$ whose weight is $\zeta_{\lambda, w\circ\mu}$. Let $s_i \in \mathfrak{S}_{\lambda - w\circ\mu}$. Then $\phi_i v$ is a weight vector of weight $s_i(\zeta_{\lambda, w\circ\mu})$. But $s_i(\zeta_{\lambda, w\circ\mu})$ does not belong to $P(\mathcal{L}(\lambda, y \circ \mu)) = P(\mathcal{L}(\lambda, w \circ \mu))$ because it does not belong to $P(\mathcal{M}(\lambda, w \circ \mu)) = \{x(\zeta_{\lambda, w\circ\mu}) \mid x \in \mathfrak{S}^\perp_{\lambda - w\circ\mu}\}$ (Lemma 4.1-(i)). Hence $\phi_i v = (1 - s_i)v = 0$ for any $s_i \in \mathfrak{S}_{\lambda - w\circ\mu}$. Therefore there exists an H_ℓ-homomorphism $f : \mathcal{M}(\lambda, w \circ \mu) \to \mathcal{M}(\lambda, y \circ \mu)$ such that $f(\mathbf{1}_{\lambda, w\circ\mu}) = v$. By Corollary 5.8, the image $f(\mathcal{M}(\lambda, w \circ \mu))$ contains $S_{(\lambda - w\circ\mu)^+} = S_{(\lambda - y\circ\mu)^+}$ as a $\mathbb{C}[\mathfrak{S}_\ell]$-submodule. Since $S_{(\lambda - y\circ\mu)^+}$ generates $\mathcal{M}(\lambda, y \circ \mu)$ over H_ℓ, the homomorphism f is surjective and thus bijective.

(b)$\Rightarrow$(a): We prove the statement only for the case $\lambda \in P_\ell$. The general case is reduced to this case. Suppose (b). It is enough to prove $w = y$ assuming that w (resp. y) is the shortest element in $\mathfrak{S}_n[\lambda + \rho] w \mathfrak{S}_n[\mu + \rho]$ (resp. $\mathfrak{S}_n[\lambda + \rho] y \mathfrak{S}_n[\mu + \rho]$). Note that this assumption implies $w \circ \mu + \rho \in P^+_{\lambda+\rho}$ and $y \circ \mu + \rho \in P^+_{\lambda+\rho}$ (see [Su, Remark 3.2.3] for the proof). First we prove $\zeta_{\lambda, w\circ\mu} = \zeta_{\lambda, y\circ\mu}$. For this purpose, we introduce a total order in P_ℓ by

$$\zeta > \eta \Leftrightarrow {}^\exists k \in \{1, \dots, \ell\} \text{ such that } \zeta_k > \eta_k \text{ and } \zeta_i = \eta_i \text{ for } i \geq k+1.$$

Through some combinatorial argument, it follows from the assumption $w \circ \mu + \rho \in P^+_{\lambda+\rho}$ (resp. $y \circ \mu + \rho \in P^+_{\lambda+\rho}$) that $\zeta_{\lambda, w\circ\mu}$ (resp. $\zeta_{\lambda, y\circ\mu}$) is the minimal element in $P(\mathcal{M}(\lambda, w \circ \mu)) = \{x(\zeta_{\lambda,\mu}) \mid x \in \mathfrak{S}^\perp_{\lambda - w\circ\mu}\}$ (resp. in $P(\mathcal{M}(\lambda, y \circ \mu))$. Therefore (b) implies $\zeta_{\lambda, w\circ\mu} = \zeta_{\lambda, y\circ\mu}$.

Next, let us prove $\mathfrak{S}_{\lambda - w\circ\mu} = \mathfrak{S}_{\lambda - y\circ\mu}$. Let $s_i \in \mathfrak{S}_{\lambda - w\circ\mu}$. Then by the same argument we used in the proof of the implication (c)$\Rightarrow$(b), we have $s_i \mathbf{1}_{\lambda, y\circ\mu} = \mathbf{1}_{\lambda, y\circ\mu}$ for any $s_i \in \mathfrak{S}_{\lambda - w\circ\mu}$. This implies $\mathfrak{S}_{\lambda - w\circ\mu} \subseteq \mathfrak{S}_{\lambda - y\circ\mu}$. Similarly we have $\mathfrak{S}_{\lambda - y\circ\mu} \subseteq \mathfrak{S}_{\lambda - w\circ\mu}$, and thus $\mathfrak{S}_{\lambda - w\circ\mu} = \mathfrak{S}_{\lambda - y\circ\mu}$.

Finally, let us see $w \circ \mu = y \circ \mu$, that is equivalent to $w = y$. Put $p_i = \langle \lambda - w \circ \mu, \epsilon_i^\vee \rangle_n$ and $q_i = \langle \lambda - y \circ \mu, \epsilon_i^\vee \rangle_n$. Suppose $w \circ \mu \neq y \circ \mu$ and let $k \in \{1, \dots, n\}$ be the largest number such that $p_k \neq q_k$. We may assume that $p_k \neq 0$. Then $\mathfrak{S}_{\lambda - w\circ\mu} = \mathfrak{S}_{\lambda - y\circ\mu}$ implies that there exists $j < k$ such that $q_i = 0$ for $i = j+1, j+2, \dots, k$ and $p_k = q_j$. Put $m = \sum_{i=1}^k p_i = \sum_{i=1}^j q_i$. Now $\zeta_{\lambda, w\circ\mu} = \zeta_{\lambda, y\circ\mu}$ implies $\langle \lambda + \rho, \epsilon_k^\vee \rangle_n = \langle \zeta_{\lambda, w\circ\mu}, \epsilon_m^\vee \rangle_\ell + 1 = \langle \zeta_{\lambda, y\circ\mu}, \epsilon_m^\vee \rangle_\ell + 1 = \langle \lambda + \rho, \epsilon_j^\vee \rangle_n$, and thus $\alpha_{jk} \in R_n^+ \cap R_n[\lambda + \rho]$. But $\langle y \circ \mu, \alpha_{jk}^\vee \rangle_n = -q_j < 0$. This contradicts the assumption $y \circ \mu + \rho \in P^+_{\lambda+\rho}$. Hence $w \circ \mu = y \circ \mu$. Q.E.D.

§7. Skew shape representations

As remarked in [AS], the construction of the functors gives a generalization of the Frobenius-Schur-Weyl reciprocity. Let us recall the classical Frobenius-Schur-Weyl reciprocity between $\mathfrak{S}_\ell$ and $\mathfrak{gl}_n$. Let $\mathfrak{gl}_n$ and $\mathfrak{S}_\ell$ act on the space $V_n^{\otimes \ell}$ from the left naturally. Then each of the images of $U(\mathfrak{gl}_n)$ and $\mathbb{C}[\mathfrak{S}_\ell]$ in $\mathrm{End}_{\mathbb{C}}(V_n^{\otimes \ell})$ is the commutant of the other. This gives the following decomposition law:

$$V_n^{\otimes \ell} = \bigoplus_{\lambda \in P_n^+(\ell)} L(\lambda) \otimes S_\lambda, \tag{7.1}$$

as a $U(\mathfrak{gl}_n) \times \mathbb{C}[\mathfrak{S}_\ell]$-module.

Proposition 7.1. *Let* $\mu \in P_n^+$.

(i) *As a* $U(\mathfrak{gl}_n) \times H_\ell$*-module,*

$$L(\mu) \otimes V_n^{\otimes \ell} = \bigoplus_{\lambda \in P_n^+, \lambda - \mu \in P_n(\ell)} L(\lambda) \otimes \mathcal{L}(\lambda, \mu), \tag{7.2}$$

(ii) *Each of the images of* $U(\mathfrak{gl}_n)$ *and* H_ℓ *on* $\mathrm{End}_{\mathbb{C}}(L(\mu) \otimes V_n^{\otimes \ell})$ *is the commutant of the other.*

Proof. (i) Note that, for $\lambda \in P_n^+$ and a finite-dimensional $\mathfrak{gl}_n$-module X, we have

$$\mathrm{Hom}_{U(\mathfrak{gl}_n)}(L(\lambda), X) = \mathrm{Hom}_{U(\mathfrak{gl}_n)}(M(\lambda), X). \tag{7.3}$$

The right hand side is isomorphic to $H^0(\mathfrak{n}_n^+, X)_\lambda$. Hence, by Lemma 1.1, we have

$$\begin{aligned} L(\mu) \otimes V_n^{\otimes \ell} &= \bigoplus_{\lambda \in P_n^+} L(\lambda) \otimes \mathrm{Hom}_{U(\mathfrak{gl}_n)}(L(\lambda), L(\mu) \otimes V_n^{\otimes \ell}) \\ &= \bigoplus_{\lambda \in P_n^+} L(\lambda) \otimes F_\lambda(L(\mu)). \end{aligned}$$

Now, Theorem 5.5 implies the statement. (ii) follows from (i). Q.E.D.

Suppose $\lambda, \mu \in P_n^+$ and $\lambda - \mu \in P_n(\ell)$. Then λ/μ gives a skew Young diagram (skew shape) with ℓ boxes. The corresponding simple module $\mathcal{L}(\lambda, \mu)$ is called *a skew shape representation*, which has been studied e.g. in [Ch1, Ch2, Ch3, Ra]. We will recover some results on them as consequences of the applications of the functors.

Proposition 7.2 ([Ch3, Ra]). *Let* $\lambda, \mu \in P_n^+$ *such that* $\lambda - \mu \in P_n(\ell)$. *Then*

$$\mathcal{L}(\lambda, \mu)\downarrow_{\mathfrak{S}_\ell} \cong \bigoplus_{\nu \in P_n^+, \lambda - \nu \in P_n(\ell)} S_\nu^{\oplus c_{\mu\nu}^\lambda}, \tag{7.4}$$

where the coefficient is given by the Littlewood-Richardson number

$$c_{\mu\nu}^\lambda = \dim_{\mathbb{C}} \mathrm{Hom}_{U(\mathfrak{gl}_n)}(L(\lambda), L(\mu) \otimes L(\nu)).$$

Proof. Follows from $\mathcal{L}(\lambda, \mu) = \mathrm{Hom}_{U(\mathfrak{gl}_n)}(L(\lambda), L(\mu) \otimes V_n^{\otimes \ell})$ and (7.1). Q.E.D.

It is well-known that the *characteristic* (see [Mac]) of the $\mathbb{C}[\mathfrak{S}_\ell]$-module S_ν is given by the Schur function. Hence, Proposition 7.2 states that the characteristic of $\mathcal{L}(\lambda, \mu)$ (as a $\mathbb{C}[\mathfrak{S}_\ell]$-module) is given by the skew Schur function ([Mac]).

Proposition 7.3 ([Ch3]). *Let $\lambda, \mu \in P_n^+$ be such that $\lambda - \mu \in P_n(\ell)$. Then there exists an exact sequence*

$$(7.5) \qquad 0 \leftarrow \mathcal{L}(\lambda, \mu) \leftarrow \mathcal{C}_0 \leftarrow \mathcal{C}_1 \leftarrow \cdots \leftarrow \mathcal{C}_{n(n-1)/2} \leftarrow 0$$

of H_ℓ-modules, where

$$\mathcal{C}_i = \bigoplus_{y \in \mathfrak{S}_n,\ l(y)=i} \mathcal{M}(\lambda, y \circ \mu).$$

Proof. Apply F_λ to the BGG resolution ([BGG]) for the finite-dimensional simple $\mathfrak{gl}_n$-module $L(\mu)$. Q.E.D.

Remark 7.4. By considering the characteristics as $\mathbb{C}[\mathfrak{S}_\ell]$-modules, one can see that the Jacobi-Trudi identity for a skew Schur function ([Mac]) follows from the sequence (7.5) (cf. [Ze4, Ak]).

§8. Multiplicity formulas

For a module M and a simple module L, let $[M : L]$ denote the multiplicity of L in the composition series of M.

Let $\mathfrak{S}_n^\mu$ denote the integral Weyl group of $\mu \in \mathfrak{t}_n^*$:

$$(8.1) \qquad \mathfrak{S}_n^\mu = \{w \in \mathfrak{S}_n \mid \mu - w \circ \mu \in Q_n\}.$$

The following formula is a direct consequence of Theorem 4.2 and Theorem 5.5:

Theorem 8.1. *Let $\lambda, \mu \in D_n$ and let $w, y \in \mathfrak{S}_n^\mu$ such that $\lambda - w \circ \mu, \lambda - y \circ \mu \in P_n(\ell)$. Then we have*

$$(8.2) \qquad [\mathcal{M}(\lambda, w \circ \mu) : \mathcal{L}(\lambda, y \circ \mu)] = [M(w \circ \mu) : L(y^\lambda \circ \mu)],$$

where y^λ denotes the longest element in $\mathfrak{S}_n[\lambda + \rho]y$.

Let $\lambda, \mu \in D_n$ and $w, y \in \mathfrak{S}_n^\mu$ be as in Theorem 8.1. The equality (8.2) has been known (at least in the case $\ell = n$) through the following two multiplicity formulas:

$$(8.3) \qquad [M(w \circ \mu) : L(y \circ \mu)] = P_{w, y_\mu}(1),$$

$$(8.4) \qquad [\mathcal{M}(\lambda, w \circ \mu) : \mathcal{L}(\lambda, y \circ \mu)] = P_{w, y_\mu^\lambda}(1).$$

Here $P_{w,y}(q) \in \mathbb{Z}[q, q^{-1}]$ denotes the Kazhdan-Lusztig polynomial [KL] of the Hecke algebra associated to $\mathfrak{S}_n^\mu$ (we put $P_{w,y}(q) = 0$ for $w \not< y$ for convenience), and y_μ (resp. y_μ^λ) denotes the longest element in $y\mathfrak{S}_n[\mu+\rho]$ (resp. $\mathfrak{S}_n[\lambda + \rho]y\mathfrak{S}_n[\mu + \rho]$).

Remark 8.2. It follows from (8.3) and (8.4) that $P_{w,y_\mu}(1) = P_{w_\mu,y_\mu}(1)$ and $P_{w,y_\mu^\lambda}(1) = P_{w_\mu,y_\mu^\lambda}(1) = P_{w_\mu^\lambda,y_\mu^\lambda}(1)$. The last number is expressed in terms of the intersection cohomology concerning nilpotent orbits on the quiver variety [Ze3].

The formula (8.3) was conjectured by Kazhdan-Lusztig [KL] and proved by Beilinson-Bernstein [BB1] and Brylinski-Kashiwara [BK]. The formula (8.4) was conjectured by Zelevinsky [Ze2] (see also [Ze3]) and proved by Ginzburg [Gi1] (see also [CG]) and by Lusztig [Lu3]. The theory of perverse sheaves plays an essential role in these proofs.

Let us see that Theorem 8.1 (proved in a purely algebraic way) implies that the Kazhdan-Lusztig formula (8.3) is equivalent to its degenerate affine Hecke analogue (or its p-adic analogue) (8.4). The implication (8.3) $\Rightarrow$ (8.4) is obvious. The implication (8.4) $\Rightarrow$ (8.3) is proved as follows. Take any $\mu \in D_n$ and $w, y \in \mathfrak{S}_n^\mu$. Then we can find $\ell \in \mathbb{Z}_{\geq 2}$ and $\lambda \in D_n + \rho$ such that

$$\lambda - z \circ \mu \in P_n(\ell) \text{ for all } z \in \mathfrak{S}_n^\mu.$$

In this case $F_\lambda(L(z \circ \mu))$ never vanishes and thus it is isomorphic to $\mathcal{L}(\lambda, z \circ \mu)$. Now (8.4) implies (8.3).

Note that the formula (8.3) has an inverse formula, which expresses the character of $L(w \circ \mu)$ as a combination of the character of Verma modules. By applying the functor, we have the corresponding formula for H_ℓ-modules.

Corollary 8.3. *Let $\lambda, \mu \in D_n$ and let $y \in \mathfrak{S}_n^\mu$ such that $\lambda - y \circ \mu \in P_n(\ell)$. Then, in the Grothendieck group of $\mathcal{R}(H_\ell)$, we have*

$$\mathcal{L}(\lambda, y \circ \mu) = \mathcal{L}(\lambda, y_\mu^\lambda \circ \mu) =$$

$$\sum_{w_\mu^\lambda \in \mathfrak{S}_n^\mu} \left(\sum_{x \in \mathfrak{S}_n[\lambda+\rho] w_\mu^\lambda \mathfrak{S}_n[\mu+\rho]} (-1)^{l_\mu(x)+l_\mu(y_\mu^\lambda)} P_{x\pi, y_\mu^\lambda \pi}(1) \right) \mathcal{M}(\lambda, w_\mu^\lambda \circ \mu).$$

Here l_μ and π denote the length function and the longest element of $\mathfrak{S}_n^\mu$ respectively, and $\sum_{w_\mu^\lambda \in \mathfrak{S}_n^\mu}$ denotes the summation over those elements $w_\mu^\lambda \in \mathfrak{S}_n^\mu$ such that w_μ^λ is longest in $\mathfrak{S}_n[\lambda + \rho] w_\mu^\lambda \mathfrak{S}_n[\mu + \rho]$.

Next we will consider a refinement of the formula (8.4) concerning the Jantzen filtration. We fix a weight $\delta \in \mathfrak{t}_n^*$. Let $A = \mathbb{C}[[t]]$ denote the ring of formal power series in t. We use the notation: $\eta^t = \eta + \delta t \in \mathfrak{t}_n^* \otimes A$ for $\eta \in \mathfrak{t}_n^*$. For $\mu \in \mathfrak{t}_n^*$, let $M(\mu^t)$ be the Verma module of $\mathfrak{gl}_n \otimes A$ with highest weight μ^t:

$$M(\mu^t) = (U(\mathfrak{gl}_n) \otimes A) \underset{U(\mathfrak{b}_n^+) \otimes A}{\otimes} (Av_{\mu^t}).$$

The $\mathfrak{gl}_n$-contravariant bilinear form on $M(\mu)$ can be naturally extended to a $\mathfrak{gl}_n \otimes A$-contravariant form $(\ |\ )_{M(\mu^t)}$ on $M(\mu^t)$.

Define

$$M(\mu^t)_j = \{v \in M(\mu^t) \mid (v \mid u)_{M(\mu^t)} \in t^j A \text{ for all } u \in M(\mu^t)\}.$$

Putting $M(\mu)_j = M(\mu^t)_j \,/ (tM(\mu^t) \cap M(\mu^t)_j)$ we have a filtration

$$M(\mu) = M(\mu)_0 \supseteq M(\mu)_1 \supseteq M(\mu)_2 \supseteq \cdots$$

by $\mathfrak{gl}_n$-modules called the *Jantzen filtration* [Ja].

It is possible to define an analogous filtration (which we call the Jantzen filtration) on $\mathcal{M}(\lambda, \mu)$ associated to δ, although it is not straitforward (see [Ro, Su]). Let $\mathcal{M}(\lambda, \mu) = \mathcal{M}(\lambda, \mu)_0 \supseteq \mathcal{M}(\lambda, \mu)_1 \supseteq \cdots$ be the Jantzen filtration associated to δ. We refer [Su] for the proof of the following theorem.

Theorem 8.4 ([Su]). *Suppose that $\lambda \in D_n$ and $\mu \in \mathfrak{t}_n^*$ satisfy $\lambda - \mu \in P(V_n^{\otimes \ell})$ and $\mu + \rho \in P_{\lambda+\rho}^-$. Then $F_\lambda(M(\mu)_j) = \mathcal{M}(\lambda, \mu)_j$.*

A priori the Jantzen filtrations depend on the choice of the deformation direction $\delta \in \mathfrak{t}_n^*$. It has been known that the Jantzen filtration on $M(\mu)$ does not depend on the choice of δ for which $(\ |\ )_{M(\mu^t)}$ is non-degenerate [Ba]. Now Theorem 8.4 implies

Proposition 8.5. *Let λ and μ be as above. Then the Jantzen filtration on $\mathcal{M}(\lambda, \mu)$ does not depend on the choice of δ such that*

$$(8.5) \qquad \langle \delta, \alpha^\vee \rangle_n \neq 0 \ \text{ for any } \alpha \in R_n^+ \text{ such that } \langle \mu + \rho, \alpha^\vee \rangle_n \in \mathbb{Z}_{>0}.$$

Let $\{M(\mu)_j\}_j$ and $\{\mathcal{M}(\lambda, \mu)_j\}_j$ be the Jantzen filtrations associated to same δ. As a direct consequence of Theorem 5.5 and Theorem 8.4, we have

Theorem 8.6. *Let $\lambda, \mu \in D_n$ and $w, y \in \mathfrak{S}_n^\mu$ be such that $\lambda - w \circ \mu, \lambda - y \circ \mu \in P_n(\ell)$. Then we have*

$$(8.6) \qquad [\mathcal{M}(\lambda, w \circ \mu)_j : \mathcal{L}(\lambda, y \circ \mu)] = [M(w^\lambda \circ \mu)_j : L(y^\lambda \circ \mu)],$$

where w^λ and y^λ denote the longest element in $\mathfrak{S}_n[\lambda+\rho]w$ and $\mathfrak{S}_n[\lambda+\rho]y$ respectively.

Let $\lambda, \mu \in D_n$ and $w, y \in \mathfrak{S}_n^\mu$ be such that $\lambda - w \circ \mu,\ \lambda - y \circ \mu \in P_n(\ell)$. Suppose that w and y are the longest elements in $\mathfrak{S}_n[\lambda+\rho]w\mathfrak{S}_n[\mu+\rho]$ and $\mathfrak{S}_n[\lambda+\rho]y\mathfrak{S}_n[\mu+\rho]$, respectively. Let $\{M(w \circ \mu)_j\}_j$ and $\{\mathcal{M}(\lambda, w \circ \mu)_j\}_j$ be the Jantzen filtration associated to δ satisfying the condition (8.5).

The following formula was conjectured in [GJ2, GM], and proved in [BB2].

$$\text{(8.7)} \qquad \sum_{j \in \mathbb{Z}_{\geq 0}} [\mathrm{gr}_j M(w \circ \mu) : L(y \circ \mu)] q^{(l_\mu(y) - l_\mu(w) - j)/2} = P_{w,y}(q),$$

where $P_{w,y}(q)$ denotes the Kazhdan-Lusztig polynomial of $\mathfrak{S}_n^\mu$, and l_μ denotes the length function on $\mathfrak{S}_n^\mu$. Combining with Theorem 8.6, the improved Kazhdan-Lusztig formula (8.7) implies its degenerate affine Hecke analogue, which was conjectured in [Ro] and proved in [Gi2] (for the non-degenerate affine Hecke algebras).

Theorem 8.7. (cf. [Gi2, Theorem 2.6.1]) *We have*

$$\text{(8.8)} \qquad \sum_{j \in \mathbb{Z}_{\geq 0}} [\mathrm{gr}_j \mathcal{M}(\lambda, w \circ \mu) : \mathcal{L}(\lambda, y \circ \mu)] q^{(l_\mu(y) - l_\mu(w) - j)/2} = P_{w,y}(q).$$

Remark 8.8. A similar result for affine Hecke algebras has been announced also by I. Grojnowski.

§A. Proof of Lemma 5.2

We proceed by two steps.

Step 1.

In the following we use notations $\mathcal{M}_\ell(\lambda, \mu)$ to denote H_ℓ-module $\mathcal{M}(\lambda, \mu)$, and $\rho^{(n)}$ (resp. $\epsilon^{(n)}$) to denote $\rho = (n-1, \dots, 1, 0) \in \mathfrak{t}_n^*$ (resp. $\epsilon = (1, \dots, 1) \in \mathfrak{t}_n^*$) when we want to clarify the rank. For positive integers ℓ and n such that n divides ℓ, we set

$$\mathcal{M}_{\ell,n} = \mathcal{M}_\ell(-\rho^{(n)} + (\ell/n)\epsilon^{(n)}, -\rho^{(n)}),$$
$$\zeta_{\ell,n} = \zeta_{-\rho^{(n)} + (\ell/n)\epsilon^{(n)}, -\rho^{(n)}} \in \mathfrak{t}_\ell^*, \quad \mathbf{1} = \mathbf{1}_{-\rho^{(n)} + (\ell/n)\epsilon^{(n)}, -\rho^{(n)}} \in \mathcal{M}_{\ell,n}.$$

We will prove

Proposition A.1. *Under the notations given above, we have*

$$(\mathcal{M}_{\ell,n})_{\zeta_{\ell,n}} = \mathbb{C}\mathbf{1}.$$

In the case $\ell = n$, the module $\mathcal{M}_{n,n}$ is nothing but the principal series representation $\mathcal{I}(\mathbf{0}^{(n)})$ (see §6), where $\mathbf{0}^{(n)} = (0, \ldots, 0) \in \mathfrak{t}_n^*$. In this case, Proposition A.1 has been proved by Rogawski (and also by Cherednik), and we will refer to this result later (in the proof of Lemma A.7):

Lemma A.2 ([Ro] [Ch4]). (i) $\dim (\mathcal{I}(\mathbf{0}^{(n)}))_{\mathbf{0}^{(n)}} = 1$.
(ii) $\mathcal{I}(\mathbf{0}^{(n)}) = (\mathcal{I}(\mathbf{0}^{(n)}))^{\mathrm{gen}}_{\mathbf{0}^{(n)}}$.
(iii) $\mathcal{I}(\mathbf{0}^{(n)})$ *is simple.*

Remark A.3. Similar statements hold for $\mathcal{I}(k\epsilon^{(n)})$ $(k \in \mathbb{C})$.

In order to prove Proposition A.1, we need some preparations. For $1 \leq r \leq \ell - 1$ and $1 \leq p \leq \ell - r$, let c_r^p denote the following cyclic permutation

$$c_r^p = s_{r+p-1} \cdots s_{r+1} s_r \in \mathfrak{S}_\ell.$$

Lemma A.4. *Let Y be an H_ℓ-module and suppose that $v \in Y$ is such that*

$$\alpha_k^\vee v = -v \quad (k = r+1, \ldots, r+p-1), \tag{A.1}$$

$$\alpha_r^\vee v = pv, \quad s_{r+p} v = v. \tag{A.2}$$

Then $v \in \mathbb{C}[\mathfrak{S}_\ell] \phi_{c_r^p} v$.

Remark A.5. Since $\alpha_{r,r+p} \in R_n(c_r^p)$ and $\alpha_{r,r+p}^\vee v = v$, it follows from Proposition 2.5-(ii) that $\phi_{(c_r^p)^{-1}} \phi_{c_r^p} v = 0$.

Proof of Lemma A.4. We will construct an element $\psi \in H_\ell$ such that $\psi \phi_{c_r^p} v = v$ explicitly. Note that $\phi_{c_r^p} = \phi_{r+p-1} \phi_{c_r^{p-1}}$ by (2.9). Since $\alpha_{r+p-1}^\vee \phi_{c_r^{p-1}} v = \phi_{c_r^{p-1}} \alpha_{r,r+p}^\vee v = \phi_{c_r^{p-1}} v$, we have

$$\begin{aligned} \phi_{c_r^p} v = \phi_{r+p-1} \phi_{c_r^{p-1}} v &= (1 + s_{r+p-1} \alpha_{r+p-1}^\vee) \phi_{c_r^{p-1}} v \\ &= (1 + s_{r+p-1}) \phi_{c_r^{p-1}} v. \end{aligned} \tag{A.3}$$

It is clear that $s_{r+p} \phi_{c_r^{p-1}} v = \phi_{c_r^{p-1}} s_{r+p} v = \phi_{c_r^{p-1}} v$, from which we have

$$\begin{aligned} &\frac{1}{2}(1 + s_{r+p} - s_{r+p-1} s_{r+p}) \phi_{c_r^p} v \\ =&\frac{1}{2}(1 + s_{r+p} - s_{r+p-1} s_{r+p})(1 + s_{r+p-1}) \phi_{c_r^{p-1}} v = \phi_{c_r^{p-1}} v. \end{aligned}$$

On the other hand, since $R_\ell(c_r^{p-1}) = \{\alpha_{r,k} \mid r+1 \le k \le r+p-1\}$, we have

$$\phi_{(c_r^{p-1})^{-1}}\phi_{c_r^{p-1}}v = \prod_{k=2}^{p}(1-k^2)v \quad \text{(see Proposition 2.5-(ii))}.$$

Therefore we get

$$\text{(A.4)} \qquad \frac{1}{2}\prod_{k=2}^{p}\frac{1}{(1-k^2)}\cdot\phi_{(c_r^{p-1})^{-1}}\cdot(1+s_{r+p}-s_{r+p-1}s_{r+p})\cdot\phi_{c_r^p}v = v$$

as required. Q.E.D.

Assume that n divides ℓ and put $m = \ell/n$. In the set $P(\mathcal{M}_{\ell,n})$ of weights of $\mathcal{M}_{\ell,n}$, there exists a unique anti-dominant element $\zeta^\circ_{\ell,n}$, that is given by

$$\text{(A.5)} \qquad \zeta^\circ_{\ell,n} = (\overbrace{0,\dots,0}^{n},\overbrace{1,\dots,1}^{n},\dots,\overbrace{m-1,\dots,m-1}^{n}).$$

Take an element $\tau \in (\mathfrak{S}_\ell)^{\perp}_{\lambda-\mu}$ such that $\tau(\zeta_{\ell,n}) = \zeta^\circ_{\ell,n}$, which is given by

$$\text{(A.6)} \qquad \tau = \omega^1\cdots\omega^{m-1} \in \mathfrak{S}_\ell.$$

Here

$$\text{(A.7)} \qquad \omega^p = \sigma^p_{n-1}\sigma^p_{n-2}\cdots\sigma^p_1,$$

with

$$\text{(A.8)} \qquad \sigma^p_k = c^p_{k(p+1)-(k-1)}\cdots c^p_{k(p+1)-1}c^p_{k(p+1)}.$$

Note that

$$l(\tau) = \sum_{p=1}^{m-1}\sum_{k=1}^{n-1}\sum_{j=0}^{k-1} l(c^p_{k(p+1)-j}),$$

and thus ϕ_τ is expressed as a product of $\phi_{c_r^p}$'s.

Iterated applications of Lemma A.4 imply the following

Lemma A.6. *The vector $\phi_\tau\mathbf{1}$ is a cyclic vector of $\mathcal{M}_{\ell,n}$:*

$$H_\ell\phi_\tau\mathbf{1} = \mathcal{M}_{\ell,n}.$$

Now we prove the following

Lemma A.7. $(\mathcal{M}_{\ell,n})_{\zeta^\circ_{\ell,n}} = \mathbb{C}\phi_\tau\mathbf{1}.$

Proof. For a subset I of $\{1,\dots,\ell-1\}$, let $\mathfrak{S}_I$ denote the subgroup of $\mathfrak{S}_\ell$ generated by $\{s_i \mid i \in I\}$, and let $\mathfrak{t}_I$ denote the subspace of $\mathfrak{t}_\ell$ spanned by $\{\epsilon_i^\vee \mid i \in I \text{ or } i-1 \in I\}$. Put $H_I = \mathbb{C}[\mathfrak{S}_I] \otimes S(\mathfrak{t}_I)$ and regard it as a subalgebra of H_ℓ.

Put

$$\text{(A.9)} \qquad \begin{aligned} B_i &= \{(i-1)n+1, (i-1)n+2, \dots, in-1\}, \\ B &= B_1 \sqcup \cdots \sqcup B_m. \end{aligned}$$

Consider subalgebras $H_{B_i} \cong H_n$ of H_ℓ corresponding to B_i $(i=1,\dots,m)$, and their modules $K_i := H_{B_i}\phi_\tau \mathbf{1} \subseteq \mathcal{M}_{\ell,n}$. By (A.5) and Lemma A.2, we have

$$\text{(A.10)} \qquad K_i \cong \mathcal{I}((i-1)\epsilon^{(n)}).$$

The subspace

$$H_B\mathbf{1} = (H_{B_1} \otimes \cdots \otimes H_{B_m})\mathbf{1} = K_1 \otimes \cdots \otimes K_m$$

of $\mathcal{M}_{\ell,n}$ is an $S(\mathfrak{t}_\ell)$-submodule. Lemma A.2 implies

$$(H_B\mathbf{1})^{\text{gen}}_{\zeta^\circ_{\ell,n}} = (K_1)^{\text{gen}}_{\mathbf{0}^{(n)}} \otimes (K_2)^{\text{gen}}_{\epsilon^{(n)}} \otimes \cdots \otimes (K_m)^{\text{gen}}_{(m-1)\epsilon^{(n)}},$$

and its dimension is $(n!)^m$. On the other hand, it follows from Lemma 4.1 that

$$\dim (\mathcal{M}_{\ell,n})^{\text{gen}}_{\zeta^\circ_{\ell,n}} = (n!)^m,$$

and thus $(\mathcal{M}_{\ell,n})^{\text{gen}}_{\zeta^\circ_{\ell,n}} = (H_B\mathbf{1})^{\text{gen}}_{\zeta^\circ_{\ell,n}}$. Combining with Lemma A.2-(i), we have

$$(\mathcal{M}_{\ell,n})_{\zeta^\circ_{\ell,n}} = (H_B\mathbf{1})_{\zeta^\circ_{\ell,n}} = (K_1)_{\mathbf{0}^{(n)}} \otimes (K_2)_{\epsilon^{(n)}} \otimes \cdots \otimes (K_m)_{(m-1)\epsilon^{(n)}} = \mathbb{C}\phi_\tau\mathbf{1}.$$

Q.E.D.

Proof of Proposition A.1. Take any $v \in (\mathcal{M}_{\ell,n})_{\zeta_{\ell,n}}$. Lemma A.7 implies that $\phi_\tau v = c \cdot \phi_\tau \mathbf{1}$ for some $c \in \mathbb{C}$. Putting $v_0 = v - c\mathbf{1}$, we have

$$\text{(A.11)} \qquad \phi_\tau v_0 = 0.$$

Let $\alpha_i \in B$. Then we have $\langle \zeta_{\ell,n}, \alpha_i^\vee \rangle_\ell = -1$, and thus $(1-s_i)v_0 = \phi_i v_0 \in (\mathcal{M}_{\ell,n})_{s_i(\zeta_{\ell,n})}$. Since the weight $s_i(\zeta_{\ell,n})$ does not belong to $P(\mathcal{M}_{\ell,n})$, it must be zero. Therefore $s_i v_0 = v_0$.

Hence there exists an H_ℓ-homomorphism

$$\text{(A.12)} \qquad f : \mathcal{M}_{\ell,n} \to \mathcal{M}_{\ell,n}$$

such that $f(\mathbf{1}) = v_0$. By (A.11), we have $\phi_\tau \mathbf{1} \in \mathrm{Ker} f$. This implies $\mathrm{Ker} f = \mathcal{M}_{\ell,n}$ by Lemma A.6. Therefore $v_0 = 0$ and thus $v \in \mathbb{C}\mathbf{1}$.
Q.E.D.

Step 2.

We will reduce Lemma 5.2 to Proposition A.1. Fix $\lambda \in D_n$ and $\mu \in \lambda - P_n(\ell)$. Put $\ell_i = \lambda_i - \mu_i$ $(i = 1, \dots, n)$ and $a_i = \sum_{k=1}^{i-1} \ell_k + 1$, $b_i = \sum_{k=1}^{i} \ell_k$ $(i = 1, \dots, n)$. Recall that

$$\langle \zeta_{\lambda,\mu}, \alpha^\vee_{a_i,a_j} \rangle_\ell = \langle \mu + \rho, \alpha^\vee_{i,j} \rangle_n, \ \langle \zeta_{\lambda,\mu}, \alpha^\vee_{b_i,b_j} \rangle_\ell = \langle \lambda + \rho, \alpha^\vee_{i,j} \rangle_n.$$

The following lemma is easy to prove.

Lemma A.8. *Let $w \in \mathfrak{S}^\perp_{\lambda-\mu}$ and $k, k' \in \{1, \dots, \ell - 1\}$. If $a_i \le k < k' \le b_i$ for some i, then $w(k) < w(k')$.*

By the conditions in Lemma 5.2, we can find integers

$$0 = n'_0 < n'_1 < n'_2 < \cdots < n'_r = n,$$

$$0 = n_0 < n_1 < n_2 < \cdots < n_s = n$$

such that

$$\{\alpha \in R_n \mid \langle \lambda + \rho, \alpha^\vee \rangle_n = 0\} = R_n \cap \sum_{i \neq n'_0, \dots, n'_r} \mathbb{Z}\alpha_i,$$

$$\{\alpha \in R_n \mid \langle \lambda + \rho, \alpha^\vee \rangle_n = \langle \mu + \rho, \alpha^\vee \rangle_n = 0\} = R_n \cap \sum_{i \neq n_0, \dots, n_s} \mathbb{Z}\alpha_i$$

respectively. Set

$$I'_p = \{a_{n'_{p-1}+1}, a_{n'_{p-1}+1} + 1, \dots, b_{n'_p} - 1\} \ (p = 1, \dots, r), \ I' = I'_1 \sqcup \cdots \sqcup I'_r,$$
$$I_p = \{a_{n_{p-1}+1}, a_{n_{p-1}+1} + 1, \dots, b_{n_p} - 1\} \ (p = 1, \dots, s), \ I = I_1 \sqcup \cdots \sqcup I_s.$$

Note that $\mathfrak{S}_{\lambda-\mu} \subseteq \mathfrak{S}_I \subseteq \mathfrak{S}_{I'}$ and

$$\mathfrak{S}_{I'}/\mathfrak{S}_{\lambda-\mu} \cong \mathfrak{S}^\perp_{\lambda-\mu} \cap \mathfrak{S}_{I'}, \quad \mathfrak{S}_I/\mathfrak{S}_{\lambda-\mu} \cong \mathfrak{S}^\perp_{\lambda-\mu} \cap \mathfrak{S}_I.$$

Lemma A.9. $\mathfrak{S}^\perp_{\lambda-\mu} \cap \mathfrak{S}_\ell[\zeta_{\lambda,\mu}] \subseteq \mathfrak{S}^\perp_{\lambda-\mu} \cap \mathfrak{S}_I.$

Proof. Let $w \in \mathfrak{S}^\perp_{\lambda-\mu} \cap \mathfrak{S}_\ell[\zeta_{\lambda,\mu}]$. First, we will prove $w \in \mathfrak{S}^\perp_{\lambda-\mu} \cap \mathfrak{S}_{I'}$. It is enough to prove that $w(\{1, 2, \dots, b_{n'_k}\}) = \{1, 2, \dots, b_{n'_k}\}$ for any $k = 1, 2, \dots, r$. Suppose that $w(\{1, 2, \dots, b_{n'_k}\}) \neq \{1, 2, \dots, b_{n'_k}\}$ and let c be the largest number such that

(A.13) $\qquad c \notin \{1, 2, \dots, b_{n'_k}\}$ and $w^{-1}(c) \in \{1, 2, \dots, b_{n'_k}\}$.

Since $w \in \mathfrak{S}_{\lambda-\mu}^{\perp}$, it follows from Lemma A.8 that $w^{-1}(c) = b_i$ for some i. Let j be the number such that $a_j \le c \le b_j$. Note that $i \le n'_k < j$ and thus

$$\langle \lambda + \rho, \alpha_{i,j}^{\vee} \rangle_n \ne 0. \tag{A.14}$$

Since $w \in \mathfrak{S}_{\ell}[\zeta_{\lambda,\mu}]$, we have

$$\langle \zeta_{\lambda,\mu}, w^{-1}(\epsilon_c^{\vee}) - \epsilon_c^{\vee} \rangle_{\ell} = \langle w(\zeta_{\lambda,\mu}) - \zeta_{\lambda,\mu}, \epsilon_c^{\vee} \rangle_{\ell} = 0.$$

On the other hand, we have

$$\begin{aligned} &\langle \zeta_{\lambda,\mu}, w^{-1}(\epsilon_c^{\vee}) - \epsilon_c^{\vee} \rangle_{\ell} = \langle \zeta_{\lambda,\mu}, \epsilon_{b_i}^{\vee} - \epsilon_c^{\vee} \rangle_{\ell} \\ =&\langle \zeta_{\lambda,\mu}, \epsilon_{b_i}^{\vee} - \epsilon_{b_j}^{\vee} \rangle_{\ell} + (b_j - c) = \langle \lambda + \rho, \alpha_{i,j}^{\vee} \rangle_n + (b_j - c). \end{aligned} \tag{A.15}$$

Hence we have $c = b_j$ and $\langle \lambda + \rho, \alpha_{i,j}^{\vee} \rangle_n = 0$, that contradicts (A.14). Therefore we proved $w \in \mathfrak{S}_{\lambda-\mu}^{\perp} \cap \mathfrak{S}_{I'}$.

Next, suppose that $w(\{1, 2, \ldots, b_{n_k}\}) \ne \{1, 2, \ldots, b_{n_k}\}$ for some k, and let c be the smallest number such that

$$w(c) \in \{1, 2, \ldots, b_{n_k}\} \text{ and } c \notin \{1, 2, \ldots, b_{n_k}\}. \tag{A.16}$$

Then Lemma A.8 implies $c = a_i$ for some i. Now, similar argument as above deduces a contradiction and thus shows $w \in \mathfrak{S}_{\lambda-\mu}^{\perp} \cap \mathfrak{S}_I$. Q.E.D.

Let $v \in \mathcal{M}(\lambda, \mu)_{\zeta_{\lambda,\mu}}$. For each $p \in \{1, \ldots, s\}$, we can write v as

$$v = \sum_j x_j^{(p)} \cdot z_j^{(p)} \mathbf{1}_{\lambda,\mu}, \tag{A.17}$$

where $\{x_j^{(p)}\}_j$ are linearly independent elements of $\mathbb{C}[\mathfrak{S}_{\lambda-\mu}^{\perp} \cap \mathfrak{S}_{I \setminus I_p}]$, and $z_j^{(p)} \in \mathbb{C}[\mathfrak{S}_{\lambda-\mu}^{\perp} \cap \mathfrak{S}_{I_p}]$.

Lemma A.10. $\xi z_k^{(p)} \mathbf{1}_{\lambda,\mu} = \langle \zeta_{\lambda,\mu}, \xi \rangle z_k^{(p)} \mathbf{1}_{\lambda,\mu}$ *for* $\xi \in \mathfrak{t}_{I_p}$.

Proof. We have

$$0 = (\xi - \langle \zeta_{\lambda,\mu}, \xi \rangle) v = \sum_j x_j^{(p)} \cdot (\xi - \langle \zeta_{\lambda,\mu}, \xi \rangle) \cdot z_j^{(p)} \mathbf{1}_{\lambda,\mu}. \tag{A.18}$$

Since $\mathfrak{S}_{I_p} \subseteq \mathfrak{S}_{\ell}$ is closed with respect to the Bruhat order, we have $\xi z_j^{(p)} \mathbf{1}_{\lambda,\mu} \in \mathbb{C}[\mathfrak{S}_{\lambda-\mu}^{\perp} \cap \mathfrak{S}_{I_p}] \mathbf{1}_{\lambda,\mu}$. Because $\{x_j^{(p)}\}_j$ are linearly independent, each $(\xi - \langle \zeta_{\lambda,\mu}, \xi \rangle) z_j^{(p)} \mathbf{1}_{\lambda,\mu}$ must be zero. Q.E.D.

Let $H_{I_p} = \mathbb{C}[\mathfrak{S}_{I_p}] \otimes S(\mathfrak{t}_{I_p}) \subseteq H_\ell$ be the subalgebra corresponding to $I_p \subseteq \Pi_\ell$. Obviously

$$H_{I_p} \cong H_d,$$

where $d = \#I_p$.

It is clear that H_{I_p}-module $H_{I_p}\mathbf{1}_{\lambda,\mu}$ is isomorphic to $\mathcal{M}_{d,n_p-n_{p-1}}$. Hence Proposition A.1 implies that $z_k^{(p)}\mathbf{1}_{\lambda,\mu} \in \mathbb{C}\mathbf{1}_{\lambda,\mu}$. Thus we have $v \in \mathbb{C}[\mathfrak{S}_{\lambda-\mu}^{\perp} \cap \mathfrak{S}_{I\setminus I_p}]$ for any p. This implies $v \in \mathbb{C}\mathbf{1}_{\lambda,\mu}$ and proves Lemma 5.2.

§B. q-analogue

Let $q \in \mathbb{C}^*$ and suppose that q is not a root of 1.

Definition B.1. The affine Hecke algebra $\mathcal{H}_\ell(q)$ of GL_ℓ is the associative algebra over $\mathbb{C}$ with generators

$$T_i^{\pm 1}\ (i = 1, \dots, \ell - 1), \quad Y_i^{\pm 1}\ (i = 1, \dots, \ell),$$

and relations

$$\begin{gathered} T_i T_i^{-1} = 1 = T_i^{-1} T_i, \quad (T_i + q)(T_i - q^{-1}) = 0, \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \ (\text{if } |i - j| > 1), \\ Y_i Y_i^{-1} = 1 = Y_i^{-1} Y_i, \quad Y_i Y_j = Y_j Y_i, \\ T_i Y_i T_i = Y_{i+1}, \quad T_i Y_j = Y_j T_i \ (\text{if } j \notin \{i, i+1\}). \end{gathered}$$

The subalgebra $\bar{\mathcal{H}}_\ell(q) \subset \mathcal{H}_\ell(q)$ generated by $T_1, \dots, T_{\ell-1}$ is called the Hecke algebra of GL_ℓ.

Let U_q denote the quantized enveloping algebra of $\mathfrak{gl}_n$ with a coproduct $\Delta : U_q \to U_q \otimes U_q$. (We refer to [Ji] for the definition.)

Let X and Y be objects of the BGG category $\mathcal{O}(U_q)$ (see e.g. [Jo]), and suppose that X or Y is finite-dimensional. Let $R_{XY} \in \operatorname{End}_{\mathbb{C}}(X \otimes Y)$ be the *R-matrix* on $X \otimes Y$ in the sense of [Ta]. (Actually, in [Ta], the R-matrix is considered only in the case where X and Y are both finite-dimensional. But it is easy to see that the same construction gives a well-defined operator on $X \otimes Y$ as long as X or Y is finite-dimensional. We also refer to [Ta] for the proof of the properties of the R-matrix below.) The operator R_{XY} is invertible and satisfies

$$\Delta(u)\check{R}_{XY} = \check{R}_{XY}\Delta(u) \quad (u \in U_q), \tag{B.1}$$

where we set $\check{R}_{XY} = p \circ R_{XY}$ with p being the permutation $p(x \otimes y) = y \otimes x$. Let Z be another objects of $\mathcal{O}(U_q)$ such that at least two of $\{X, Y, Z\}$ are finite-dimensional. Then we have the *Yang-Baxter equation* on $X \otimes Y \otimes Z$:

$$(\text{B.2}) \qquad (\check{R}_{YZ} \otimes 1_X)(1_Y \otimes \check{R}_{XZ})(\check{R}_{XY} \otimes 1_Z) = (1_Z \otimes \check{R}_{XY})(\check{R}_{XZ} \otimes 1_Y)(1_X \otimes \check{R}_{YZ}).$$

Regard V_n as the vector representation of U_q. As proved by Jimbo [Ji], the correspondence

$$T_i \mapsto 1^{\otimes i-1} \otimes \check{R}_{V_n V_n} \otimes 1^{\otimes \ell-i-1} \quad (i = 1, \ldots, \ell - 1)$$

gives an action of $\bar{\mathcal{H}}_\ell(q)$ on $V_n^{\otimes \ell}$. The following proposition is easy to prove using (B.1) and (B.2):

Proposition B.2. *There exists a unique homomorphism*

$$\mathcal{H}_\ell(q) \to \mathrm{End}_{U_q}(X \otimes V_n^{\otimes \ell})$$

such that

$$T_i \mapsto \check{R}_i \quad (i = 1, \ldots, \ell - 1),$$
$$Y_i \mapsto \check{R}_{i-1} \cdots \check{R}_1 \left((\check{R}_{V_n X} \check{R}_{X V_n}) \otimes 1^{\otimes \ell-1} \right) \check{R}_1 \cdots \check{R}_{i-1} \quad (i = 1, \ldots, \ell),$$

where

$$\check{R}_i = 1^{\otimes i} \otimes \check{R}_{V_n V_n} \otimes 1^{\otimes \ell-i-1} \quad (i = 1, \ldots, \ell - 1).$$

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