Advanced Studies in Pure Mathematics 28, 2000
Combinatorial Methods in Representation Theory
pp. 113–132

Finite Crystals and Paths

Goro Hatayama, Yoshiyuki Koga, Atsuo Kuniba, Masato Okado and Taichiro Takagi

Dedicated to Professor Tetsuji Miwa on his fiftieth birthday

Abstract.

We consider a category of finite crystals of a quantum affine algebra whose objects are not necessarily perfect, and set of paths, semi-infinite tensor product of an object of this category with a certain boundary condition. It is shown that the set of paths is isomorphic to a direct sum of infinitely many, in general, crystals of integrable highest weight modules. We present examples from $C_n^{(1)}$ and $A_{n-1}^{(1)}$, in which the direct sum becomes a tensor product as suggested from the Bethe Ansatz.

§1. Introduction

The main object of this note is to define a set of paths from a *finite* crystal B, which is not necessarily perfect, and investigate its crystal structure. The set of paths $\mathcal{P}(\boldsymbol{p}, B)$ is, roughly speaking, a subset of the semi-infinite tensor product $\cdots \otimes B \otimes \cdots \otimes B \otimes B$ with a certain boundary condition related to $\boldsymbol{p}$. If B is perfect, it is known [KMN1] that as crystals, $\mathcal{P}(\boldsymbol{p}, B)$ is isomorphic to the crystal base $B(\lambda)$ of an integrable highest weight module with highest weight λ of the quantum affine algebra $U_q(\mathfrak{g})$. While trying to generalize this notion, we had two examples in mind: (a) $\mathfrak{g} = C_n^{(1)}, B = B^{1,l}\,(l : \text{odd})$; (b) $\mathfrak{g} = A_{n-1}^{(1)}, B = B^{1,l} \otimes B^{1,m}\,(l \geq m)$. For this parametrization of finite crystals, we refer to [HKOTY]. $B^{1,l}$ stands for the crystal base of an irreducible finite-dimensional $U_q'(\mathfrak{g})$-module. In case (a) (resp. (b)) this finite-dimensional module is isomorphic to $V_{l\overline{\Lambda}_1} \oplus V_{(l-2)\overline{\Lambda}_1} \oplus \cdots \oplus V_{\overline{\Lambda}_1}$ (resp. $V_{l\overline{\Lambda}_1}$) as $U_q(\overline{\mathfrak{g}})$-module, where V_λ is the irreducible finite-dimensional module with highest weight λ. In both cases B is not perfect except when $l = m$ in (b). For precise treatment see section 4.1 for (a) and 4.2 for (b).

Received January 14, 1999.

Let us consider case (a) first. When $l = 1$ it has already been known [DJKMO] that the formal character of $\mathcal{P}(\boldsymbol{p}, B^{1,1})$ for suitable $\boldsymbol{p}$ agrees with that of the irreducible highest weight $A^{(1)}_{2n-1}$-module with fundamental highest weight Λ_i regarded as $C^{(1)}_n$-module via the natural embedding $C^{(1)}_n \hookrightarrow A^{(1)}_{2n-1}$. On the other hand, the Bethe Ansatz suggests [Ku] that $\mathcal{P}(\boldsymbol{p}, B^{1,l})$ is equal to $B(\lambda)\otimes\mathcal{P}(\boldsymbol{p}^\dagger, B^{1,1})$ for suitable $\boldsymbol{p}, \boldsymbol{p}^\dagger$ and a level $\frac{l-1}{2}$ dominant integral weight λ at the level of the Virasoro central charge.

Let us turn to case (b). In [HKMW] the $U'_q(\widehat{sl}_2)$-invariant integrable vertex model with alternating spins is considered. To translate the physical states and operators of this model into the language of representation theory of the quantum affine algebra $U_q(\widehat{sl}_2)$, they considered a set of paths with alternating spins and showed that it is isomorphic to the tensor product of crystals with highest weights. Another appearance of example (b) can be found in [HKKOTY]. They considered the inductive limit of $(B^{1,l})^{\otimes L_1} \otimes (B^{1,m})^{\otimes L_2}$ when $L_1, L_2 \to \infty, L_1 \equiv r_1, L_1 + L_2 \equiv r_2$ (mod n), and showed that there is a weight preserving bijection between the limit and $B((l-m)\Lambda_{r_1}) \otimes B(m\Lambda_{r_2})$. Since there is a natural isomorphism $B^{1,l} \otimes B^{1,m} \simeq B^{1,m} \otimes B^{1,l}$, the above result claims that $\mathcal{P}(\boldsymbol{p}, B^{1,l}\otimes B^{1,m})$ for suitable $\boldsymbol{p}$ is bijective to $B((l-m)\Lambda_{r_1})\otimes B(m\Lambda_{r_2})$ with weight preserved. These results are consistent with the earlier Bethe ansatz calculations on "mixed spin" models [AM, DMN].

If we forget about the degree of the null root δ from weight, this phenomenon is explained using the theory of crystals with core [KK]. (See also [HKMW] section 3.2.) Let $\{B_k\}_{k\geq 1}$ be a coherent family of perfect crystals and B'_m be a perfect crystal of level m. Fix l such that $l \geq m$ and take dominant integral weights λ and μ of level $l-m$ and m. Then there exists an isomorphism of crystals:

$$\begin{aligned} B(\lambda)\otimes B(\mu) &\simeq B(\sigma\lambda)\otimes B_{l-m}\otimes B(\sigma'\mu)\otimes B'_m \\ &\simeq B(\sigma\lambda)\otimes B(\sigma\sigma'\mu)\otimes(B_l\otimes B'_m), \end{aligned}$$

where σ and σ' are automorphisms on the weight lattice P related to $\{B_k\}_{k\geq 1}$ and B'_m. Iterating this isomorphism infinitely many times, we can expect

$$\mathcal{P}(\boldsymbol{p}^{(\lambda,\mu)}, B_l\otimes B'_m) \simeq B(\lambda)\otimes B(\mu)$$

as $P/\mathbf{Z}\delta$-weighted crystals with suitable $\boldsymbol{p}^{(\lambda,\mu)}$.

In both cases (a),(b) we have illustrated above, what we expect is an isomorphism of P-weighted crystals of the following type:

$$\mathcal{P}(\boldsymbol{p}, B) \simeq B(\lambda)\otimes\mathcal{P}(\boldsymbol{p}^\dagger, B^\dagger) \tag{1.1}$$

and we shall prove it in this paper. First we examine the crystal structure of $\mathcal{P}(\boldsymbol{p}, B)$ and show it is isomorphic to a direct sum of $B(\lambda)$'s. Therefore, the structure of $\mathcal{P}(\boldsymbol{p}, B)$ is completely determined by the set of highest weight elements. In the LHS of (1.1), such set $\mathcal{P}(\boldsymbol{p}, B)_0$ is easy to describe, and in the RHS, this set turns out to be the set of restricted paths $\mathcal{P}^{(\lambda)}(\boldsymbol{p}^\dagger, B^\dagger)$, which is familiar to the people in solvable lattice models. Thus establishing a weight preserving bijection between $\mathcal{P}(\boldsymbol{p}, B)_0$ and $\mathcal{P}^{(\lambda)}(\boldsymbol{p}^\dagger, B^\dagger)$ directly, we can show (1.1).

§2. Crystals

2.1. Notation

Let $\mathfrak{g}$ be an affine Lie algebra. We denote by I the index set of its Dynkin diagram. Note that 0 is included in I. Let α_i, h_i, Λ_i $(i \in I)$ be the simple roots, simple coroots, fundamental weights for $\mathfrak{g}$. Let $\delta = \sum_{i\in I} a_i\alpha_i$ denote the standard null root, and $c = \sum_{i\in I} a_i^\vee h_i$ the canonical central element, where $a_i, a_i^\vee$ are positive integers as in [Kac]. We assume $a_0 = 1$. Let $P = \bigoplus_{i\in I} \mathbf{Z}\Lambda_i \oplus \mathbf{Z}\delta$ be the weight lattice, and set $P^+ = \sum_{i\in I} \mathbf{Z}_{\geq 0}\Lambda_i \oplus \mathbf{Z}\delta$.

Let $U_q(\mathfrak{g})$ be the quantum affine algebra associated to $\mathfrak{g}$. For the definition of $U_q(\mathfrak{g})$ and its Hopf algebra structure, see e.g. section 2.1 of [KMN1]. For $J \subset I$ we denote by $U_q(\mathfrak{g}_J)$ the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, t_i $(i \in J)$. In particular, $U_q(\mathfrak{g}_{I\setminus\{0\}})$ is identified with the quantized enveloping algebra for the simple Lie algebra whose Dynkin diagram is obtained by deleting the 0 vertex from that of $\mathfrak{g}$. We also consider the quantum affine algebra without derivation $U_q'(\mathfrak{g})$. As its weight lattice, the classical weight lattice $P_{cl} = P/\mathbf{Z}\delta$ is needed. We canonically identify P_{cl} with $\bigoplus_{i\in I} \mathbf{Z}\Lambda_i \subset P$. For the precise treatment, see section 3.1 of [KMN1]. We further define the following subsets of P_{cl}: $P_{cl}^0 = \{\lambda \in P_{cl} \mid \langle\lambda, c\rangle = 0\}$, $P_{cl}^+ = \{\lambda \in P_{cl} \mid \langle\lambda, h_i\rangle \geq 0 \text{ for any } i\}$, $(P_{cl}^+)_l = \{\lambda \in P_{cl}^+ \mid \langle\lambda, c\rangle = l\}$. For $\lambda, \mu \in P_{cl}$, we write $\lambda \geq \mu$ to mean $\lambda - \mu \in P_{cl}^+$.

2.2. Crystals and crystal bases

We summarize necessary facts in crystal theory. Our basic references are [K1], [KMN1] and [AK].

A crystal B is a set B with the maps

$$\tilde{e}_i, \tilde{f}_i : B \sqcup \{0\} \longrightarrow B \sqcup \{0\}$$

satisfying the following properties:

$\tilde{e}_i 0 = \tilde{f}_i 0 = 0,$

for any b and i, there exists $n > 0$ such that $\tilde{e}_i^n b = \tilde{f}_i^n b = 0$,
for $b, b' \in B$ and $i \in I$, $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$.

If we want to emphasize I, B is called an I-crystal. A crystal can be regarded as a colored oriented graph by defining

$$b \xrightarrow{i} b' \quad \Longleftrightarrow \quad \tilde{f}_i b = b'.$$

For an element b of B we set

$$\varepsilon_i(b) = \max\{n \in \mathbf{Z}_{\geq 0} \mid \tilde{e}_i^n b \neq 0\}, \quad \varphi_i(b) = \max\{n \in \mathbf{Z}_{\geq 0} \mid \tilde{f}_i^n b \neq 0\}.$$

We also define a P-weighted crystal. It is a crystal with the weight decomposition $B = \sqcup_{\lambda \in P} B_\lambda$ such that

$$\text{(2.1)} \qquad \tilde{e}_i B_\lambda \subset B_{\lambda+\alpha_i} \sqcup \{0\}, \quad \tilde{f}_i B_\lambda \subset B_{\lambda-\alpha_i} \sqcup \{0\},$$

$$\text{(2.2)} \qquad \langle h_i, wt\, b\rangle = \varphi_i(b) - \varepsilon_i(b).$$

Set

$$\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b)\Lambda_i, \quad \varphi(b) = \sum_{i \in I} \varphi_i(b)\Lambda_i.$$

Then (2.2) is equivalent to $\varphi(b) - \varepsilon(b) = wt\, b$. P_{cl}-weighted crystal is defined similarly.

For two weighted crystals B_1 and B_2, the tensor product $B_1 \otimes B_2$ is defined.

$$B_1 \otimes B_2 = \{b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2\}.$$

The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are defined by

$$\text{(2.3)} \qquad \tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\text{(2.4)} \qquad \tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

Here $0 \otimes b$ and $b \otimes 0$ are understood to be 0. ε_i, φ_i and wt are given by

$$\text{(2.5)} \qquad \varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_1) + \varepsilon_i(b_2) - \varphi_i(b_1)),$$

$$\text{(2.6)} \qquad \varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \varphi_i(b_2) - \varepsilon_i(b_2)),$$

$$\text{(2.7)} \qquad wt\,(b_1 \otimes b_2) = wt\, b_1 + wt\, b_2.$$

Definition 2.1 ([AK]). *We say a P (or P_{cl})-weighted crystal is regular, if for any $i, j \in I$ $(i \neq j)$, B regarded as $\{i,j\}$-crystal is a disjoint union of crystals of integrable highest weight modules over $U_q(\mathfrak{g}_{\{i,j\}})$.*

Crystal is a notion obtained by abstracting the properties of crystal bases [K1]. Let $V(\lambda)$ be the integrable highest weight $U_q(\mathfrak{g})$-module with highest weight $\lambda \in P^+$ and highest weight vector u_λ. It is shown in [K1] that $V(\lambda)$ has a crystal base $(L(\lambda), B(\lambda))$. We regard u_λ as an element of $B(\lambda)$ as well. $B(\lambda)$ is a regular P-weighted crystal. A finite-dimensional integrable $U_q'(\mathfrak{g})$-module V does not necessarily have a crystal base. If V has a crystal base (L, B), then B is a regular P_{cl}^0-weighted crystal with finitely many elements.

Let W be the affine Weyl group associated to $\mathfrak{g}$, and s_i be the simple reflection corresponding to α_i. W acts on any regular crystal B [K2]. The action is given by

$$S_{s_i} b = \begin{cases} \tilde{f}_i^{\langle h_i, wt\, b\rangle} b & \text{if } \langle h_i, wt\, b\rangle \geq 0 \\ \tilde{e}_i^{-\langle h_i, wt\, b\rangle} b & \text{if } \langle h_i, wt\, b\rangle \leq 0. \end{cases}$$

An element b of B is called *i-extremal* if $\tilde{e}_i b = 0$ or $\tilde{f}_i b = 0$. b is called *extremal* if $S_w b$ is i-extremal for any $w \in W$ and $i \in I$.

Definition 2.2 ([AK] Definition 1.7). *Let B be a regular P_{cl}^0-weighted crystal with finitely many elements. We say B is simple if it satisfies*

(1) *There exists $\lambda \in P_{cl}^0$ such that the weights of B are in the convex hull of $W\lambda$.*
(2) $\sharp B_\lambda = 1$.
(3) *The weight of any extremal element is in $W\lambda$.*

Remark 2.3. *Let B be a regular P_{cl}^0-weighted crystal with finitely many elements. We have the following criterion for simplicity. Let $B(\lambda)$ denote the crystal base of the irreducible highest weight $U_q(\mathfrak{g}_{I\setminus\{0\}})$-module with highest weight λ. If B decomposes into $B \simeq \bigoplus_{j=0}^m B(\lambda_j)$ as $U_q(\mathfrak{g}_{I\setminus\{0\}})$-crystal and λ_j satisfies*

(1) *$\lambda_j \in \lambda_0 + \sum_{i\neq 0} \mathbf{Z}_{\leq 0}\alpha_i$ and $\lambda_j \neq \lambda_0$ for any $j \neq 0$,*
(2) *The highest weight element of $B(\lambda_j)$ is not 0-extremal for any $j \neq 0$,*

then B is simple.

Proposition 2.4 ([AK] Lemma 1.9 & 1.10). *Simple crystals have the following properties.*

(1) *A simple crystal is connected.*
(2) *The tensor product of simple crystals is also simple.*

2.3. Category $\mathcal{C}^{fin}$

Let B be a regular P_{cl}^0-weighted crystal with finitely many elements. For B we introduce the *level* of B by

$$lev\, B = \min\{\langle c, \varepsilon(b)\rangle \mid b \in B\} \in \mathbf{Z}_{\geq 0}.$$

Note that $\langle c, \varepsilon(b)\rangle = \langle c, \varphi(b)\rangle$ for any $b \in B$. We also set $B_{\min} = \{b \in B \mid \langle c, \varepsilon(b)\rangle = lev\, B\}$ and call an element of $B_{\min}$ *minimal*.

Definition 2.5. *We denote by $\mathcal{C}^{fin}(\mathfrak{g})$ (or simply $\mathcal{C}^{fin}$) the category of crystal B satisfying the following conditions:*

(1) *B is a crystal base of a finite-dimensional $U_q'(\mathfrak{g})$-module.*
(2) *B is simple.*
(3) *For any $\lambda \in P_{cl}^+$ such that $\langle c, \lambda\rangle \geq lev\, B$, there exists $b \in B$ satisfying $\varepsilon(b) \leq \lambda$. It is also true for φ.*

We call an object of $\mathcal{C}^{fin}(\mathfrak{g})$ *finite crystal*.

Remark 2.6. (i) *Condition (1) implies B is a regular P_{cl}^0-weighted crystal with finitely many elements.*

(ii) *Set $l = lev\, B$. Condition (3) implies that the maps ε and φ from $B_{\min}$ to $(P_{cl}^+)_l$ are surjective. (cf. (4.6.5) in* [KMN1]*)*
(iii) *Practically, one has to check condition (3) only for $\lambda \in P_{cl}^+$ such that there is no $i \in I$ satisfying $\lambda - \Lambda_i \geq 0$ and $\langle c, \lambda - \Lambda_i\rangle \geq lev\, B$. In particular, if $a_i^\vee = 1$ for any $i \in I$ ($\mathfrak{g} = A_n^{(1)}, C_n^{(1)}$), the surjectivity of ε and φ assures (3).*
(iv) *The authors do not know a crystal satisfying (1) and (2), but not satisfying (3).*

Let B_1 and B_2 be two finite crystals. Definition 2.5 (1) and the existence of the universal R-matrix assures that we have a natural isomorphism of crystals.

$$B_1 \otimes B_2 \simeq B_2 \otimes B_1. \tag{2.8}$$

The following lemma is immediate.

Lemma 2.7. *Let B_1, B_2 be finite crystals.*

(1) *$lev(B_1 \otimes B_2) = \max(lev\, B_1, lev\, B_2)$.*
(2) *If $lev\, B_1 \geq lev\, B_2$, then $(B_1 \otimes B_2)_{\min} = \{b_1 \otimes b_2 \mid b_1 \in (B_1)_{\min}, \varphi_i(b_1) \geq \varepsilon_i(b_2)$ for any $i\}$.*
(3) *If $lev\, B_1 \leq lev\, B_2$, then $(B_1 \otimes B_2)_{\min} = \{b_1 \otimes b_2 \mid b_2 \in (B_2)_{\min}, \varphi_i(b_1) \leq \varepsilon_i(b_2)$ for any $i\}$.*

$\mathcal{C}^{fin}(\mathfrak{g})$ forms a tensor category.

Proposition 2.8. *If B_1 and B_2 are objects of $\mathcal{C}^{fin}(\mathfrak{g})$, then $B_1 \otimes B_2$ is also an object of $\mathcal{C}^{fin}(\mathfrak{g})$.*

Proof. We need to check the conditions in Definition 2.5 for $B_1 \otimes B_2$. (1) is obvious and (2) follows from Proposition 2.4 (2).

Let us prove condition (3) for ε. Set $l_1 = lev\, B_1, l_2 = lev\, B_2$. Using (2.8) if necessary, we can assume $l_1 \geq l_2$. Thus we have $lev\, B_1 \otimes B_2 = l_1$. For any $\lambda \in P_{cl}^+$ such that $\langle c, \lambda \rangle \geq l_1$, one can take $b_1 \in B_1$ satisfying $\varepsilon(b_1) \leq \lambda$. Since $\langle c, \varphi(b_1)\rangle \geq l_1 \geq l_2$, one can take $b_2 \in B_2$ satisfying $\varepsilon(b_2) \leq \varphi(b_1)$. In view of (2.5) one has $\varepsilon(b_1 \otimes b_2) = \varepsilon(b_1) \leq \lambda$.

For the proof of φ, repeat a similar exercise for $B_2 \otimes B_1 (\simeq B_1 \otimes B_2)$ using (2.6). ∎

2.4. Category $\mathcal{C}^h$

If an element b of a crystal B satisfies $\tilde{e}_i b = 0$ for any i, we call it a *highest weight* element.

Definition 2.9. *We denote by $\mathcal{C}^h(I, P)$ (or simply $\mathcal{C}^h$) the category of regular P-weighted crystal B satisfying the following condition:*

For any $b \in B$, there exist $l \geq 0, i_1, \cdots, i_l \in I$ such that $b' = \tilde{e}_{i_1} \cdots \tilde{e}_{i_l} b \in B$ is a highest weight element.

Clearly, $\mathcal{C}^h(I, P)$ forms a tensor category.

Proposition 2.10 ([KMN1] Proposition 2.4.4). *An object of $\mathcal{C}^h(I, P)$ is isomorphic to a direct sum (disjoint union) of crystals $B(\lambda)$ ($\lambda \in P^+$) of integrable highest weight $U_q(\mathfrak{g})$-modules.*

Let O be an object of $\mathcal{C}^h(I, P)$. By O_0 we mean the set of highest weight elements in O. Suppose that $O_0 = \{b_j \mid j \in J\}$ and $wt\, b_j = \lambda_j \in P^+$, then from the above proposition we have an isomorphism

$$O \simeq \bigoplus_{j \in J} B(\lambda_j) \quad \text{as } P\text{-weighted crystals.}$$

J can be an infinite set.

The following lemma is standard.

Lemma 2.11. *Let B_1, B_2 be weighted crystals. Then $b_1 \otimes b_2 \in B_1 \otimes B_2$ is a highest weight element, if and only if b_1 is a highest weight element and $\tilde{e}_i^{\langle h_i, wt\, b_1\rangle + 1} b_2 = 0$ for any i.*

Let O be an object of $\mathcal{C}^h(I,P)$. From this lemma we have the following bijection.

$$\begin{array}{ccc} (B(\lambda)\otimes O)_0 & \longrightarrow & O^{\leq\lambda} := \{b \in O \mid \tilde{e}_i^{\langle h_i,\lambda\rangle+1} b = 0 \text{ for any } i\} \\ u_\lambda \otimes b & \mapsto & b. \end{array}$$

Note that $O^{\leq 0} = O_0$.

§3. Paths

In this section we construct a set of paths from a finite crystal and consider its structure.

3.1. Energy function

Let us recall the energy function used in [NY] to identify the Kostka-Foulkes polynomial with a generating function over classically restricted paths.

Let B_1 and B_2 be two finite crystals. Suppose $b_1 \otimes b_2 \in B_1 \otimes B_2$ is mapped to $\tilde{b}_2 \otimes \tilde{b}_1 \in B_2 \otimes B_1$ under the isomorphism (2.8). A **Z**-valued function H on $B_1 \otimes B_2$ is called an *energy function* if for any i and $b_1 \otimes b_2 \in B_1 \otimes B_2$ such that $\tilde{e}_i(b_1 \otimes b_2) \neq 0$, it satisfies

$$\begin{aligned} H(\tilde{e}_i(b_1\otimes b_2)) &= H(b_1\otimes b_2)+1 && \text{if } i=0, \varphi_0(b_1)\geq\varepsilon_0(b_2), \\ &&& \quad \varphi_0(\tilde{b}_2)\geq\varepsilon_0(\tilde{b}_1), \\ &= H(b_1\otimes b_2)-1 && \text{if } i=0, \varphi_0(b_1)<\varepsilon_0(b_2), \\ &&& \quad \varphi_0(\tilde{b}_2)<\varepsilon_0(\tilde{b}_1), \\ &= H(b_1\otimes b_2) && \text{otherwise.} \end{aligned} \tag{3.1}$$

When we want to emphasize $B_1 \otimes B_2$, we write $H_{B_1B_2}$ for H. The existence of such function can be shown in a similar manner to section 4 of [KMN1] based on the existence of *combinatorial R-matrix*. The energy function is unique up to additive constant, since $B_1 \otimes B_2$ is connected. By definition, $H_{B_1B_2}(b_1 \otimes b_2) = H_{B_2B_1}(\tilde{b}_2 \otimes \tilde{b}_1)$.

If the tensor product $B_1 \otimes B_2$ is homogeneous, i.e., $B_1 = B_2$, we have $\tilde{b}_2 = b_1, \tilde{b}_1 = b_2$. Thus (3.1) is rewritten as

$$\begin{aligned} H(\tilde{e}_i(b_1\otimes b_2)) &= H(b_1\otimes b_2)+1 && \text{if } i=0, \varphi_0(b_1)\geq\varepsilon_0(b_2), \\ &= H(b_1\otimes b_2)-1 && \text{if } i=0, \varphi_0(b_1)<\varepsilon_0(b_2), \\ &= H(b_1\otimes b_2) && \text{if } i\neq 0. \end{aligned} \tag{3.2}$$

The following proposition, which is shown by case-by-case checking, reduces the energy function of a tensor product to that of each component.

Proposition 3.1. *Set $B = B_1 \otimes B_2$, then*

$$\begin{aligned} H_{BB}((b_1 \otimes b_2) \otimes (b'_1 \otimes b'_2)) &= H_{B_1B_2}(b_1 \otimes b_2) + H_{B_1B_1}(\tilde{b}_1 \otimes b'_1) \\ &\quad + H_{B_2B_2}(b_2 \otimes \tilde{b}'_2) + H_{B_1B_2}(b'_1 \otimes b'_2). \end{aligned}$$

Here $\tilde{b}_1, \tilde{b}'_2$ are defined as

$$\begin{aligned} B_1 \otimes B_2 &\simeq B_2 \otimes B_1 \\ b_1 \otimes b_2 &\mapsto \tilde{b}_2 \otimes \tilde{b}_1 \\ b'_1 \otimes b'_2 &\mapsto \tilde{b}'_2 \otimes \tilde{b}'_1. \end{aligned}$$

Remark 3.2. *Decomposition of the energy function is not unique. For instance, the following also gives such decomposition.*

$$\begin{aligned} H_{BB}((b_1 \otimes b_2) \otimes (b'_1 \otimes b'_2)) &= H_{B_2B_1}(b_2 \otimes b'_1) + H_{B_1B_1}(b_1 \otimes \check{b}'_1) \\ &\quad + H_{B_2B_2}(\check{b}_2 \otimes b'_2) + H_{B_1B_2}(\check{b}'_1 \otimes \check{b}_2), \end{aligned}$$

where

$$\begin{aligned} B_2 \otimes B_1 &\simeq B_1 \otimes B_2 \\ b_2 \otimes b'_1 &\mapsto \check{b}'_1 \otimes \check{b}_2. \end{aligned}$$

3.2. Set of paths $\mathcal{P}(\mathbf{p}, B)$

We shall define a set of paths from any finite crystal in $\mathcal{C}^{fin}$ imitating the construction in section 4 of [KMN1] from a perfect crystal.

Definition 3.3. *An element $\mathbf{p} = \cdots \otimes \mathbf{b}_j \otimes \cdots \otimes \mathbf{b}_2 \otimes \mathbf{b}_1$ of the semi-infinite tensor product of B is called a reference path if it satisfies $\mathbf{b}_j \in B_{\min}$ and $\varphi(\mathbf{b}_{j+1}) = \varepsilon(\mathbf{b}_j)$ for any $j \geq 1$.*

Definition 3.4. *Fix a reference path $\mathbf{p} = \cdots \otimes \mathbf{b}_j \otimes \cdots \otimes \mathbf{b}_2 \otimes \mathbf{b}_1$. We define a set of paths $\mathcal{P}(\mathbf{p}, B)$ by*

$$\mathcal{P}(\mathbf{p}, B) = \{p = \cdots \otimes b_j \otimes \cdots \otimes b_2 \otimes b_1 \mid b_j \in B, b_k = \mathbf{b}_k \text{ for } k \gg 1\}.$$

An element of $\mathcal{P}(\mathbf{p}, B)$ is called a *path*. For convenience we denote b_k by $p(k)$ and $\cdots \otimes b_{k+2} \otimes b_{k+1}$ by $p[k]$ for $p = \cdots \otimes b_j \otimes \cdots \otimes b_2 \otimes b_1$.

Definition 3.5. *For a path $p \in \mathcal{P}(\mathbf{p}, B)$, set*

$$\begin{aligned} E(p) &= \sum_{j=1}^{\infty} j(H(p(j+1) \otimes p(j)) - H(\mathbf{p}(j+1) \otimes \mathbf{p}(j))), \\ W(p) &= \varphi(\mathbf{p}(1)) + \sum_{j=1}^{\infty} (\mathrm{wt}\, p(j) - \mathrm{wt}\, \mathbf{p}(j)) - E(p)\delta. \end{aligned}$$

$E(p)$ and $W(p)$ are called the energy and weight of p.

We distinguish $W(p) \in P$ from $wt\, p = \varphi(\mathbf{p}(1)) + \sum_{j=1}^{\infty}(wt\, p(j) - wt\, \mathbf{p}(j)) \in P_{cl}$.

Remark 3.6. (i) *If B is perfect, the set of reference paths is bijective to $(P_{cl}^{+})_l$, where $l = \mathrm{lev}\, B$. For $\lambda \in (P_{cl}^{+})_l$ take a unique $\mathbf{b}_1 \in B_{\min}$ such that $\varphi(\mathbf{b}_1) = \lambda$. The condition $\varphi(\mathbf{b}_{j+1}) = \varepsilon(\mathbf{b}_j)$ fixes $\mathbf{p} = \cdots \otimes \mathbf{b}_j \otimes \cdots \otimes \mathbf{b}_1$ uniquely.*

(ii) *In* [KMN1] *$\mathbf{p}$ is called a ground state path, since $E(p) \geq E(\mathbf{p})$ for any $p \in \mathcal{P}(\mathbf{p}, B)$. But if B is not perfect, it is no longer true in general.*

The following theorem is essential for our consideration below.

Theorem 3.7. *Assume rank $\mathfrak{g} > 2$. Then $\mathcal{P}(\mathbf{p}, B)$ is an object of $\mathcal{C}^h$.*

Proof. Assume $\tilde{e}_i p = \cdots \otimes \tilde{e}_i b_j \otimes \cdots \otimes b_1 \neq 0$. Note that $E(\tilde{e}_i p) = E(p) - \delta_{i0}$ and $wt\, \tilde{e}_i b_j = wt\, b_j + \alpha_i - \delta_{i0}\delta \in P_{cl}$. By Definition 3.5 it is immediate to see $\mathcal{P}(\mathbf{p}, B)$ is a P-weighted crystal. Thus one has to check the following:

(i) If for any $i, j \in I$ $(i \neq j)$, $\mathcal{P}(\mathbf{p}, B)$ regarded as $\{i, j\}$-crystal is a disjoint union of crystals of integrable highest weight modules over $U_q(\mathfrak{g}_{\{i,j\}})$.

(ii) For any $p \in \mathcal{P}(\mathbf{p}, B)$, there exist $l \geq 0, i_1, \cdots, i_l \in I$ such that $p' = \tilde{e}_{i_1} \cdots \tilde{e}_{i_l} p \in \mathcal{P}(\mathbf{p}, B)$ is a highest weight element.

We prove (i) first. For $p \in \mathcal{P}(\mathbf{p}, B)$ take m, m' such that $p(k) = \mathbf{p}(k)$ for $k > m$ and $m' \gg m$. Note that if $\tilde{f}_{i_N} \cdots \tilde{f}_{i_1} p[m] = p[m'] \otimes b'_{m'} \otimes \cdots \otimes b'_{m+1}$, then $b'_k = \mathbf{p}(k)$ for $k > m + N$. From the assumption, $U_q(\mathfrak{g}_{\{i,j\}})$ is the quantized enveloping algebra associated to a finite-dimensional Lie algebra. Since B is regular, the connected component containing $p[m]$, as $\{i, j\}$-crystal, can be considered to be in $B(\varphi(p[m'])) \otimes B^{\otimes(m'-m)}$. Since $\varepsilon(p[m]) = 0$, we can regard $p[m]$ as highest weight element of some $\{i, j\}$-crystal B_0 which is isomorphic to the crystal of an integrable highest weight $U_q(\mathfrak{g}_{\{i,j\}})$-module. Hence p is contained in a component of the $\{i, j\}$-crystal $B_0 \otimes B^{\otimes m}$, which is a disjoint union of crystals of integrable highest weight $U_q(\mathfrak{g}_{\{i,j\}})$-modules.

To prove (ii) for $p = \cdots \otimes b_k \otimes \cdots \otimes b_1 \in \mathcal{P}(\mathbf{p}, B)$, we take the minimum integer m such that $p' = p[m]$ is a highest weight element. We prove by induction on m.

First let us show that there exist $l \geq 0, i_1, \cdots, i_l \in I$ such that $\tilde{e}_{i_1} \cdots \tilde{e}_{i_l}(p' \otimes b_m)$ is a highest weight element. The proof is essentially the same as a part of that of Theorem 4.4.1 in [KMN1]. Nevertheless we repeat it for the sake of self-containedness. Suppose that there does not exist such $i_1, \cdots, i_l$. Then there exists an infinite sequence $\{i_\nu\}$ in I such that

$$\tilde{e}_{i_k} \cdots \tilde{e}_{i_1}(p' \otimes b_m) \neq 0.$$

Since $\tilde{e}_{i_k} \cdots \tilde{e}_{i_1}(p' \otimes b_m) = p' \otimes \tilde{e}_{i_k} \cdots \tilde{e}_{i_1} b_m$ and B is a finite set, there exists $b^{(1)} \in B$ and $j_1, \cdots, j_l$ such that

$$p' \otimes b^{(1)} = \tilde{e}_{j_l} \cdots \tilde{e}_{j_1}(p' \otimes b^{(1)}).$$

Hence setting $b^{(\nu+1)} = \tilde{e}_{j_\nu} b^{(\nu)}$, we have

$$\tilde{e}_{j_\nu}(p' \otimes b^{(\nu)}) = p' \otimes b^{(\nu+1)} \text{ and } b^{(l+1)} = b^{(1)}.$$

In view of (2.6) we have $\varphi_i(p') \geq \varphi_i(b_{m+1})$ for any i. Thus by (2.3) we have $\varepsilon_{j_\nu}(b^{(\nu)}) > \varphi_{j_\nu}(p') \geq \varphi_{j_\nu}(b')$ for some $b' \in B$. Hence we have

$$\tilde{e}_{j_\nu}(b' \otimes b^{(\nu)}) = b' \otimes b^{(\nu+1)}.$$

Therefore, from (3.2), we have

$$H(b' \otimes b^{(\nu+1)}) = H(b' \otimes b^{(\nu)}) - \delta_{i_\nu 0}.$$

Hence $H(b' \otimes b^{(l+1)}) = H(b' \otimes b^{(1)}) - \sharp\{\nu \mid j_\nu = 0\}$, which implies there is no ν such that $j_\nu = 0$. On the other hand, $\sum_\nu \alpha_{j_\nu} = 0 \bmod \mathbf{Z}\delta$ and hence $\sum_\nu \alpha_{j_\nu}$ is a positive multiple of δ, which contradicts $0 \notin \{j_1, \cdots, j_l\}$.

Now set $p'' = p' \otimes b_m (= p[m-1]), b'' = b_{m-1} \otimes \cdots \otimes b_1$. Notice that for any $i \in I$ satisfying $\tilde{e}_i p'' \neq 0$, there exists $k \geq 1$ such that

$$\tilde{e}_i^k (p'' \otimes b'') = \tilde{e}_i p'' \otimes \tilde{e}_i^{k-1} b''.$$

Therefore there exist $l \geq 0, (i_1, k_1), \cdots, (i_l, k_l) \in I \times \mathbf{Z}_{>0}$ such that

$$\tilde{e}_{i_1}^{k_1} \cdots \tilde{e}_{i_l}^{k_l} p = \tilde{e}_{i_1} \cdots \tilde{e}_{i_l} p'' \otimes \tilde{e}_{i_1}^{k_1-1} \cdots \tilde{e}_{i_l}^{k_l-1} b''$$

and $\tilde{e}_{i_1} \cdots \tilde{e}_{i_l} p''$ is a highest weight element. Now we can use the induction assumption and complete the proof. ∎

Remark 3.8. *As seen in the proof, the theorem does not require the condition* $\mathbf{b}_j \in B_{\min}$ *for the reference path* $\mathbf{p} = \cdots \otimes \mathbf{b}_j \otimes \cdots \otimes \mathbf{b}_1$.

The following proposition describes the set of highest weight elements in $\mathcal{P}(\boldsymbol{p}, B)$.

Proposition 3.9.

$$\mathcal{P}(\mathbf{p}, B)_0 = \{p \in \mathcal{P}(\mathbf{p}, B) \mid p(j) \in B_{\min}, \varphi(p(j+1)) = \varepsilon(p(j)) \text{ for } \forall j\}.$$

Proof. Assume $p = \cdots \otimes b_j \otimes \cdots \otimes b_1$ is a highest weight element. We prove the following by induction on m in decreasing order.

(i) $b_m \in B_{\min}, \varphi(b_{m+1}) = \varepsilon(b_m)$
(ii) $\varphi(p[m-1]) = \varphi(b_m)$

These conditions are satisfied for sufficiently large m. From (ii) for $m+1$ we have $\varphi(p[m]) = \varphi(b_{m+1})$. From Lemma 2.11 we see that $p[m]$ is a highest weight element and $\varepsilon(b_m) \le wt\, p[m] = \varphi(p[m]) = \varphi(b_{m+1})$. Combining this with (i) for $m+1$, we can conclude (i) for m. For (ii) use (2.6). ∎

As seen in the proof, we obtain

Corollary 3.10. *If $p \in \mathcal{P}(\mathbf{p}, B)_0$, then $wt\, p[j] = \varphi(p(j+1))$.*

3.3. Restricted paths

When B is perfect the set of *restricted* paths was defined in [DJO] and shown to be bijective to $(B(\lambda) \otimes B(\mu))_0$ for some $\lambda, \mu \in P_{cl}^+$. Here we shall consider restricted paths for any finite crystal B.

For $\lambda \in P_{cl}^+$ and $p \in \mathcal{P}(\mathbf{p}, B)$, we introduce a sequence of weights $\{\lambda_j(p)\}_{j \ge 0}$ by

$$\lambda_j(p) = \lambda + \varphi(p(j+1)) \text{ for } j \gg 1,$$
$$\lambda_{j-1}(p) = \lambda_j(p) + wt\, p(j).$$

Notice that this definition is well-defined by virtue of the property of the reference path. In fact, $\lambda_j(p) = \lambda + wt\, p[j]$.

Definition 3.11. *For $\lambda \in P_{cl}^+$ we define a subset $\mathcal{P}^{(\lambda)}(\mathbf{p}, B)$ of $\mathcal{P}(\mathbf{p}, B)$ by*

$$\mathcal{P}^{(\lambda)}(\mathbf{p}, B) = \{p \in \mathcal{P}(\mathbf{p}, B) \mid \tilde{e}_i^{\langle h_i, \lambda_j(p)\rangle + 1} p(j) = 0 \text{ for } \forall i, j\}.$$

An element of $\mathcal{P}^{(\lambda)}(\mathbf{p}, B)$ is called a *restricted* path.

Proposition 3.12. *For $\lambda \in P_{cl}^+$ we have*

$$\mathcal{P}(\mathbf{p}, B)^{\le \lambda} = \mathcal{P}^{(\lambda)}(\mathbf{p}, B).$$

Proof. Assume $p = \cdots \otimes b_j \otimes \cdots \otimes b_1 \in \mathcal{P}(\mathbf{p}, B)^{\le \lambda}$, which is equivalent to saying $u_\lambda \otimes p$ is a highest weight element. So is $u_\lambda \otimes p[j] \otimes b_j$ by Lemma 2.11. Using this lemma again we get $\varepsilon(b_j) \le wt\,(u_\lambda \otimes p[j]) = \lambda_j(p)$.

To show the inverse inclusion, assume $p = \cdots \otimes b_j \otimes \cdots \otimes b_1 \in \mathcal{P}^{(\lambda)}(\mathbf{p}, B)$. We prove $\varepsilon(p[j]) \le \lambda$ by induction on j in decreasing order. We know $\varepsilon(p[j]) = 0$ for sufficiently large j. Supposing $\varepsilon(p[j]) \le \lambda$ we immediately obtain $\varepsilon(p[j] \otimes b_j) \le \lambda$ from (2.5) and the condition $\varepsilon(b_j) \le \lambda_j(p)$. ∎

As seen in the proof we have $\lambda_j(p) \in P_{cl}^+$ and its level is $\langle c, \lambda \rangle + lev B$.

Combining the results in section 2.4, Theorem 3.7 and Proposition 3.12, we obtain

Theorem 3.13. *Let $\mathcal{P}(\mathbf{p}, B)$ and $\mathcal{P}(\mathbf{p}^\dagger, B^\dagger)$ be two sets of paths. If for certain $\lambda \in P_{cl}^+$, there exists a bijection*

$$\begin{array}{rccc} (3.3) \qquad & \mathcal{P}(\mathbf{p}, B)_0 & \longrightarrow & \mathcal{P}^{(\lambda)}(\mathbf{p}^\dagger, B^\dagger) \\ & p & \mapsto & p^\dagger \end{array}$$

such that $W(p) = \lambda + W(p^\dagger)$, then we have an isomorphism of P-weighted crystals

$$\mathcal{P}(\mathbf{p}, B) \simeq B(\lambda) \otimes \mathcal{P}(\mathbf{p}^\dagger, B^\dagger).$$

They are isomorphic to a direct sum of crystals of integrable highest weight $U_q(\mathfrak{g})$-modules, and their highest weight elements are parametrized by (3.3).

§4. Examples

We shall give two examples to which we can apply Theorem 3.13 efficiently.

4.1. Example 1

We present a useful proposition first. Similar to $O^{\le\lambda}$ we define $B^{\le\lambda}$ for a finite crystal B and $\lambda \in P_{cl}^+$ by

$$B^{\le\lambda} = \{b \in B \mid \tilde{e}_i^{\langle h_i, \lambda \rangle + 1} b = 0 \text{ for any } i\}.$$

Note that if $lev B = l$, then $B_{\min} = \bigsqcup_{\lambda \in (P_{cl}^+)_l} B^{\le\lambda}$.

Proposition 4.1. *Let B and $B^\dagger$ be finite crystals such that $lev B \ge lev B^\dagger$, and $\mathbf{p} = \cdots \otimes \mathbf{b}_j \otimes \cdots \otimes \mathbf{b}_1$ be a reference path for B. Suppose there exists a map $t : B_{\min} \to B^\dagger$ satisfying the following conditions:*

(1) *For any $\mu \in (P_{cl}^+)_l$ $(l = lev B)$, $t|_{B^{\le\mu}}$ is a bijection onto $(B^\dagger)^{\le\mu}$.*
(2) *$\mathrm{wt}\, t(b) = \mathrm{wt}\, b$ for any $b \in B_{\min}$.*
(3) *$H_{B^\dagger B^\dagger}(t(b_1) \otimes t(b_2)) = H_{BB}(b_1 \otimes b_2)$ up to global additive constant for any $(b_1, b_2) \in B_{\min}^2$ such that $\varphi(b_1) = \varepsilon(b_2)$.*

(4) $\mathbf{p}^\dagger = \cdots \otimes t(\mathbf{b}_j) \otimes \cdots \otimes t(\mathbf{b}_1)$ *is a reference path for* $B^\dagger$. *Then setting* $\lambda = \varphi(\mathbf{b}_1) - \varphi(t(\mathbf{b}_1))$, *we have*

$$\mathcal{P}(\mathbf{p}, B) \simeq B(\lambda) \otimes \mathcal{P}(\mathbf{p}^\dagger, B^\dagger).$$

Proof. Consider the following map.

$$\begin{array}{rcl} \mathcal{P}(\mathbf{p}, B)_0 & \longrightarrow & \mathcal{P}(\mathbf{p}^\dagger, B^\dagger) \\ p = \cdots \otimes b_j \otimes \cdots \otimes b_1 & \mapsto & p^\dagger = \cdots \otimes t(b_j) \otimes \cdots \otimes t(b_1) \end{array}$$

From Theorem 3.13 it suffices to show that this map is a bijection onto $\mathcal{P}^{(\lambda)}(\mathbf{p}^\dagger, B^\dagger)$ such that $W(p) = \lambda + W(p^\dagger)$. Preservation of weight is immediate. To show the bijectivity one has to notice that $wt\, p^\dagger[j] - wt\, p[j]$ does not depend on j. Thus one has $wt\, p^\dagger[j] - wt\, p[j] = wt\, p^\dagger - wt\, p = -\lambda$, and hence

$$\lambda_j(p^\dagger) = \lambda + wt\, p^\dagger[j] = wt\, p[j] = \varphi(b_{j+1}) = \varepsilon(b_j).$$

Note that $p \in \mathcal{P}(\mathbf{p}, B)_0$ (cf. Proposition 3.9 & Corollary 3.10). In view of (1) this equality concludes the bijectivity. ∎

We now consider the $C_n^{(1)}$ case. For an odd positive integer l, consider a finite crystal $B^{1,l}$ given by

$$B^{1,l} = \left\{ (x_1, \ldots, x_n, \overline{x}_n, \ldots, \overline{x}_1) \;\middle|\; \begin{array}{l} x_i, \overline{x}_i \in \mathbf{Z}_{\geq 0} \, \forall i = 1, \cdots, n \\ \sum_{i=1}^n (x_i + \overline{x}_i) \in \{l, l-2, \ldots, 1\} \end{array} \right\}.$$

The crystal structure of $B^{1,l}$ is given by

$$\begin{aligned}
\tilde{e}_0 b &= \begin{cases} (x_1 - 2, x_2, \ldots, \overline{x}_2, \overline{x}_1) & \text{if } x_1 \geq \overline{x}_1 + 2, \\ (x_1 - 1, x_2, \ldots, \overline{x}_2, \overline{x}_1 + 1) & \text{if } x_1 = \overline{x}_1 + 1, \\ (x_1, x_2, \ldots, \overline{x}_2, \overline{x}_1 + 2) & \text{if } x_1 \leq \overline{x}_1, \end{cases} \\
\tilde{e}_i b &= \begin{cases} (x_1, \ldots, x_i + 1, x_{i+1} - 1, \ldots, \overline{x}_1) & \text{if } x_{i+1} > \overline{x}_{i+1}, \\ (x_1, \ldots, \overline{x}_{i+1} + 1, \overline{x}_i - 1, \ldots, \overline{x}_1) & \text{if } x_{i+1} \leq \overline{x}_{i+1}, \end{cases} \\
\tilde{e}_n b &= (x_1, \ldots, x_n + 1, \overline{x}_n - 1, \ldots, \overline{x}_1), \\
\tilde{f}_0 b &= \begin{cases} (x_1 + 2, x_2, \ldots, \overline{x}_2, \overline{x}_1) & \text{if } x_1 \geq \overline{x}_1, \\ (x_1 + 1, x_2, \ldots, \overline{x}_2, \overline{x}_1 - 1) & \text{if } x_1 = \overline{x}_1 - 1 \\ (x_1, x_2, \ldots, \overline{x}_2, \overline{x}_1 - 2) & \text{if } x_1 \leq \overline{x}_1 - 2, \end{cases} \\
\tilde{f}_i b &= \begin{cases} (x_1, \ldots, x_i - 1, x_{i+1} + 1, \ldots, \overline{x}_1) & \text{if } x_{i+1} \geq \overline{x}_{i+1}, \\ (x_1, \ldots, \overline{x}_{i+1} - 1, \overline{x}_i + 1, \ldots, \overline{x}_1) & \text{if } x_{i+1} < \overline{x}_{i+1}, \end{cases} \\
\tilde{f}_n b &= (x_1, \ldots, x_n - 1, \overline{x}_n + 1, \ldots, \overline{x}_1),
\end{aligned}$$

where $b = (x_1, \dots, x_n, \overline{x}_n, \dots, \overline{x}_1)$ and $i = 1, \dots, n-1$. If some component becomes negative upon application, it should be understood as 0. The values of ε_i, φ_i read

$$\begin{aligned} \varepsilon_0(b) &= \tfrac{l-s(b)}{2} + (x_1 - \overline{x}_1)_+, & \varphi_0(b) &= \tfrac{l-s(b)}{2} + (\overline{x}_1 - x_1)_+, \\ \varepsilon_i(b) &= \overline{x}_i + (x_{i+1} - \overline{x}_{i+1})_+, & \varphi_i(b) &= x_i + (\overline{x}_{i+1} - x_{i+1})_+, \\ \varepsilon_n(b) &= \overline{x}_n, & \varphi_n(b) &= x_n. \end{aligned}$$

Here $s(b) = \sum_{i=1}^n (x_i + \overline{x}_i), (x)_+ = \max(x, 0)$ and $i = 1, \cdots, n-1$. $B^{1,l}$ is a level $\frac{l+1}{2}$ non-perfect crystal. Now for a fixed l set $B = B^{1,l}$. The minimal elements of B are grouped as $B_{\min} = \bigsqcup_{\mu \in (P_{cl}^+)_{\frac{l+1}{2}}} B^{\leq \mu}$, where for $\mu = \mu_0 \Lambda_0 + \cdots + \mu_n \Lambda_n$. The set $B^{\leq \mu}$ is given by

$$\begin{aligned} B^{\leq \mu} &= \{b_k^\mu \mid \mu_{k-1} > 0, 1 \leq k \leq n\} \cup \{b_{\overline{k}}^\mu \mid \mu_k > 0, 1 \leq k \leq n\}, \\ b_k^\mu &= (\mu_1, \dots, \mu_{k-1} - 1, \mu_k + 1, \dots, \mu_n, \mu_n, \dots, \mu_{k-1} - 1, \dots, \mu_1), \\ b_{\overline{k}}^\mu &= (\mu_1, \dots, \mu_k - 1, \dots, \mu_n, \mu_n, \dots, \mu_1). \end{aligned}$$

Next consider $B^\dagger = B^{1,1}$ by taking l to be 1. Setting

$$b_k^\dagger = (x_i = \delta_{ik}, \overline{x}_i = 0), \quad b_{\overline{k}}^\dagger = (x_i = 0, \overline{x}_i = \delta_{ik})$$

for $1 \leq k \leq n$, one has

$$(B^\dagger)^{\leq \mu} = \{b_k^\dagger \mid \mu_{k-1} > 0, 1 \leq k \leq n\} \cup \{b_{\overline{k}}^\dagger \mid \mu_k > 0, 1 \leq k \leq n\}$$

for μ as above. Define the map $t : B_{\min} \longrightarrow B^\dagger$ by

$$t|_{B^{\leq \mu}} : b_k^\mu \mapsto b_k^\dagger \qquad \text{for } k \in \{1, \dots, n, \overline{n}, \dots, \overline{1}\}.$$

We are to show that this t satisfies the conditions (1) – (4) in Proposition 4.1. For our purpose fix a dominant integral weight $\lambda \in (P_{cl}^+)_{\frac{l-1}{2}}$ and define $\boldsymbol{p} = \cdots \otimes \boldsymbol{b}_j \otimes \cdots \otimes \boldsymbol{b}_1$ by

$$\boldsymbol{b}_j = \begin{cases} b_{\overline{i}}^{\lambda + \Lambda_i} & \text{if } j \equiv i \,(\text{mod } 2n) \text{ for some } i\,(1 \leq i \leq n), \\ b_i^{\lambda + \Lambda_{i-1}} & \text{if } j \equiv 1 - i \,(\text{mod } 2n) \text{ for some } i\,(1 \leq i \leq n). \end{cases}$$

Note that $\varepsilon(b_{\overline{i}}^{\lambda+\Lambda_i}) = \varphi(b_i^{\lambda+\Lambda_{i-1}}) = \lambda + \Lambda_i, \varepsilon(b_i^{\lambda+\Lambda_{i-1}}) = \varphi(b_{\overline{i}}^{\lambda+\Lambda_i}) = \lambda + \Lambda_{i-1}$. $\boldsymbol{p}$ becomes a reference path. Let us check (1) – (4) in Proposition 4.1. (1),(2) and (4) are straightforward. To check (3) one can use the formula for H_{BB} in [KKM] section 5.7. (In [KKM] our non-perfect case is not considered. However, the formula itself is valid. Since the formula in [KKM] contains some misprints, we rewrite it below.)

$$H_{B^{1,l}B^{1,l}}(b \otimes b') = \max_{1 \leq j \leq n} (\theta_j(b \otimes b'), \theta'_j(b \otimes b'), \eta_j(b \otimes b'), \eta'_j(b \otimes b')),$$

$$\theta_j(b\otimes b') = \sum_{k=1}^{j-1}(\overline{x}_k-\overline{x}'_k)+\frac{1}{2}(s(b')-s(b)),$$

$$\theta'_j(b\otimes b') = \sum_{k=1}^{j-1}(x'_k-x_k)+\frac{1}{2}(s(b)-s(b')),$$

$$\eta_j(b\otimes b') = \sum_{k=1}^{j-1}(\overline{x}_k-\overline{x}'_k)+(\overline{x}_j-x_j)+\frac{1}{2}(s(b')-s(b)),$$

$$\eta'_j(b\otimes b') = \sum_{k=1}^{j-1}(x'_k-x_k)+(x'_j-\overline{x}'_j)+\frac{1}{2}(s(b)-s(b')),$$

where $b=(x_1,\cdots,x_n,\overline{x}_n,\cdots,\overline{x}_1), b'=(x'_1,\cdots,x'_n,\overline{x}'_n,\cdots,\overline{x}'_1)$.

Therefore, the isomorphism in Proposition 4.1 holds with notations above.

4.2. Example 2

We consider the $A^{(1)}_{n-1}$ case. Let $B^{1,l}$ be the crystal base of the symmetric tensor representation of $U'_q(A^{(1)}_{n-1})$ of degree l. As a set it reads

$$B^{1,l}=\{(a_0,a_1,\cdots,a_{n-1})\mid a_i\in\mathbf{Z}_{\geq 0},\sum_{i=0}^{n-1}a_i=l\}.$$

For convenience we extend the definition of a_i to $i\in\mathbf{Z}$ by setting $a_{i+n}=a_i$ and use a simpler notation (a_i) for $(a_0,a_1,\cdots,a_{n-1})$. For instance, (a_{i-1}) means $(a_{n-1},a_0,\cdots,a_{n-2})$. The actions of $\tilde{e}_r,\tilde{f}_r$ $(r=0,\cdots,n-1)$ are given by

$$\tilde{e}_r(a_i)=(a_i-\delta^{(n)}_{i,r}+\delta^{(n)}_{i,r-1}),\quad \tilde{f}_r(a_i)=(a_i+\delta^{(n)}_{i,r}-\delta^{(n)}_{i,r-1}).$$

Here $\delta^{(n)}_{ij}=1$ $(i\equiv j \bmod n)$, $=0$ (otherwise). If some component becomes negative upon application, it should be understood as 0. The values of ε,φ read as follows.

$$\varepsilon((a_i))=\sum_{i=0}^{n-1}a_i\Lambda_i,\quad \varphi((a_i))=\sum_{i=0}^{n-1}a_{i-1}\Lambda_i.$$

Thus $lev\,B^{1,l}=l$ and all elements are minimal. We introduce a $\mathbf{Z}$-linear automorphism σ on P_{cl} by $\sigma\Lambda_i=\Lambda_{i-1}$ $(\Lambda_{-1}=\Lambda_{n-1})$.

Now consider the finite crystal $B=B^{1,l}\otimes B^{1,m}$ $(l\geq m)$ and set $B^\dagger=B^{1,m}$. From Lemma 2.7 (1) the level of B is l. Fix two dominant integral weights $\lambda = \sum_{i=0}^{n-1} \lambda_i \Lambda_i \in (P_{cl}^+)_{l-m}, \mu = \sum_{i=0}^{n-1} \mu_i \Lambda_i \in (P_{cl}^+)_m$. From (λ, μ) we define a path

$$\mathbf{p}^{(\lambda,\mu)}(j) = (\lambda_{i+j} + \mu_{i+2j}) \otimes (\mu_{i+2j-1}) \in B.$$

From Lemma 2.7 (2) we see $\mathbf{p}^{(\lambda,\mu)}(j) \in B_{\min}$ and by (2.5),(2.6) we obtain $\varepsilon(\mathbf{p}^{(\lambda,\mu)}(j)) = \sigma^j \lambda + \sigma^{2j}\mu = \varphi(\mathbf{p}^{(\lambda,\mu)}(j+1))$. Therefore $\mathbf{p}^{(\lambda,\mu)}$ is a reference path.

We would like to show

$$(4.1) \qquad \mathcal{P}(\mathbf{p}^{(\lambda,\mu)}, B) \simeq B(\lambda) \otimes \mathcal{P}(\mathbf{p}^{(\mu)}, B^\dagger) \quad \text{as } P\text{-weighted crystals}$$

with $\mathbf{p}^{(\mu)}(j) = (\mu_{i+j})$. To do this, consider the following map

$$(4.2) \qquad \begin{array}{ccc} \mathcal{P}(\mathbf{p}^{(\lambda,\mu)}, B)_0 & \longrightarrow & \mathcal{P}(\mathbf{p}^{(\mu)}, B^\dagger) \\ p & \mapsto & p^\dagger \end{array}$$

given by $p^\dagger(j) = (b_{i-j+1}^{(j)})$ for $p(j) = (a_i^{(j)}) \otimes (b_i^{(j)})$. Note that $\mathbf{p}^{(\lambda,\mu)}$ is sent to $\mathbf{p}^{(\mu)}$ under this map. By Theorem 3.13 it suffices to check the following items:

(i) The map (4.2) is a bijection onto $\mathcal{P}^{(\lambda)}(\mathbf{p}^{(\mu)}, B^\dagger)$.
(ii) $wt\,p - wt\,p^\dagger = \lambda$.
(iii) $E(p) = E(p^\dagger)$.

Since $p \in \mathcal{P}(\mathbf{p}^{(\lambda,\mu)}, B)_0$, one obtains (cf. Lemma 2.7 (2), Proposition 3.9)

$$(4.3) \qquad \varphi_i((a_i^{(j)})) = a_{i-1}^{(j)} \geq b_i^{(j)} = \varepsilon_i((b_i^{(j)}))$$

$$(4.4) \qquad \varphi_i(p(j)) = a_{i-1}^{(j)} + b_{i-1}^{(j)} - b_i^{(j)} = a_i^{(j-1)} = \varepsilon_i(p(j-1))$$

for any i, j. Taking sufficiently large J and using (4.4), one has

$$\begin{aligned} wt\,p^\dagger[j] &= \sum_i b_{i-J+1}^{(J)} \Lambda_i + \sum_{k=j+1}^{J} \sum_i (b_{i-k}^{(k)} - b_{i-k+1}^{(k)}) \Lambda_i \\ &= \sum_i (b_{i-J+1}^{(J)} - a_{i-J}^{(J)} + a_{i-j}^{(j)}) \Lambda_i \\ &= \sum_i a_{i-j}^{(j)} \Lambda_i - \lambda. \end{aligned}$$

Thus the condition $\varepsilon(p^\dagger(j)) \leq \lambda_j(p^\dagger)$ is equivalent to saying $b_{i-j+1}^{(j)} \leq a_{i-j}^{(j)}$ for any i, which is guaranteed by (4.3). This proves (i). For (ii) one only has to notice that $wt\,p[j] = \varphi(p(j+1)) = \sum_i a_i^{(j)} \Lambda_i$.

In order to prove (iii), we set

$$E_L^{diff} = \sum_{j=1}^{L} j\{H_{BB}(((a_i^{(j+1)}) \otimes (b_i^{(j+1)})) \otimes ((a_i^{(j)}) \otimes (b_i^{(j)}))) \\ -H_{B^\dagger B^\dagger}((b_{i-(j+1)+1}^{(j+1)}) \otimes (b_{i-j+1}^{(j)}))\}.$$

We can assume $(a_i^{(j)}) \otimes (b_i^{(j)}) \in B_{\min}$ for $1 \leq j \leq L+1$. Under such assumption the isomorphism $B^{1,l} \otimes B^{1,m} \simeq B^{1,m} \otimes B^{1,l}$ sends $(a_i) \otimes (b_i)$ to $(b_{i+1}) \otimes (a_i - b_{i+1} + b_i)$ [NY]. Thus, from Proposition 3.1 we have

$$H_{BB}(((a_i) \otimes (b_i)) \otimes ((a_i') \otimes (b_i'))) = b_0 + a_0' + b_0' + H_{B^\dagger B^\dagger}((b_i) \otimes (b_{i+1}')).$$

Let us recall the following formula for $H_{B^{1,m}B^{1,m}}$ (cf. [KKM] section 5.1).

$$H_{B^{1,m}B^{1,m}}((b_i) \otimes (b_i')) = \max_{0 \leq j \leq n-1} (\sum_{k=0}^{j-1} (b_k' - b_k) + b_j')$$

From this one gets

$$H_{B^\dagger B^\dagger}((b_i^{(j+1)}) \otimes (b_{i+1}^{(j)})) - H_{B^\dagger B^\dagger}((b_{i-j}^{(j+1)}) \otimes (b_{i-j+1}^{(j)})) \\ = \sum_{k=1}^{j} (b_{k-j-1}^{(j+1)} - b_{k-j}^{(j)}).$$

Using above facts and (4.4) one obtains

$$E_L^{diff} = \sum_{j=1}^{L} \sum_{k=0}^{j-1} a_{-k}^{(L)} + L \sum_{k=0}^{L} b_{-k}^{(L+1)}.$$

This completes (iii). We have finished proving (4.1). It is also known [KMN2] that $\mathcal{P}(\boldsymbol{p}^{(\mu)}, B^{1,m}) \simeq B(\mu)$. Therefore we have

$$\mathcal{P}(\boldsymbol{p}^{(\lambda,\mu)}, B^{1,l} \otimes B^{1,m}) \simeq B(\lambda) \otimes B(\mu) \quad \text{as } P\text{-weighted crystals.}$$

The multi-component version is straightforward. Consider the finite crystal $B^{1,l_1} \otimes \cdots \otimes B^{1,l_s}$ $(l_1 \geq \cdots \geq l_s \geq l_{s+1} = 0)$. For $\lambda^{(i)} \in (P_{cl}^+)_{l_i - l_{i+1}}$ $(1 \leq i \leq s)$ we define a reference path $\boldsymbol{p}^{(\lambda_1, \cdots, \lambda_s)}$ by

$$\text{the } k\text{-th tensor component of } \boldsymbol{p}^{(\lambda_1,\cdots,\lambda_s)}(j) \\ = (\lambda_{i+kj-k+1}^{(k)} + \lambda_{i+(k+1)j-k+1}^{(k+1)} + \cdots + \lambda_{i+sj-k+1}^{(s)}).$$

Then we have

$$\mathcal{P}(\boldsymbol{p}^{(\lambda_1,\cdots,\lambda_s)}, B^{1,l_1} \otimes \cdots \otimes B^{1,l_s}) \simeq B(\lambda_1) \otimes \cdots \otimes B(\lambda_s).$$

The proof will be given elsewhere.

Acknowledgements

The authors thank Professor Masaki Kashiwara and Professor Tetsuji Miwa for useful discussions and kind interest.

References

[AK] T. Akasaka and M. Kashiwara, Finite-dimensional representations of quantum affine algebras, Publ. Res. Inst. Math. Sci. **33** (1997) 839-867.

[AM] S. R. Aladim and M. J. Martins, Critical behaviour of integrable mixed spin chains, J. Phys. A**26** (1993) L529-534.

[DJKMO] E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, One dimensional configuration sums in vertex models and affine Lie algebra characters, Lett. Math. Phys. **17** (1989) 69-77.

[DJO] E. Date, M. Jimbo and M. Okado, Crystal base and q-vertex operators, Commun. Math. Phys. **155** (1993) 47-69.

[DMN] H. J. de Vega, L. Mezincescu and R. I. Nepomechie, Thermodynamics of integrable chains with alternating spins, Phys. Rev. B**49** (1994) 13223-13226.

[HKKOTY] G. Hatayama, A. N. Kirillov, A. Kuniba, M. Okado, T. Takagi and Y. Yamada, Character formulae of $\widehat{sl}_n$-modules and inhomogeneous paths, Nucl. Phys. **B536** [PM] (1998) 575-616.

[HKOTY] G. Hatayama, A. Kuniba, M. Okado, T. Takagi and Y. Yamada, Remarks on fermionic formula, Contemporary Mathematics **248** (1999) 243-291.

[HKMW] J. Hong, S-J. Kang, T. Miwa and R. Weston, Vertex models with alternating spins, math.QA/9811175.

[Kac] V. G. Kac, *Infinite dimensional Lie algebras*, 3rd edition, Cambridge Univ. Press. Cambridge (1990).

[KK] S-J. Kang and M. Kashiwara, Quantized affine algebras and crystals with core, Commun. Math. Phys. **195** (1998) 725-740.

[KKM] S-J. Kang, M. Kashiwara and K. C. Misra, Crystal bases of Verma modules for quantum affine Lie algebras, Compositio Math. **92** (1994) 299-325.

[KMN1] S-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima and A. Nakayashiki, Affine crystals and vertex models, Int. J. Mod. Phys. A **7** (suppl. 1A) (1992) 449-484.

[KMN2] S-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima and A. Nakayashiki, Perfect crystals of quantum affine Lie algebras, Duke Math. J. **68** (1992) 499-607.

[K1] M. Kashiwara, On crystal bases of the q-analogue of universal enveloping algebras, Duke Math. J. **63** (1991) 465-516.

[K2] M. Kashiwara, Crystal bases of modified quantized enveloping algebra, Duke Math. J. **73** (1994) 383-413.

[Ku] A. Kuniba, Thermodynamics of $U_q(X_r^{(1)})$ Bethe ansatz system with q a root of unity, Nucl. Phys. **B389** (1993) 209-244.

[NY] A. Nakayashiki and Y. Yamada, Kostka polynomials and energy functions in solvable lattice models, Selecta Mathematica, New Ser. **3** (1997) 547-599.

Goro Hatayama
Institute of Physics
Graduate School of Science
University of Tokyo
Komaba, Tokyo 153-8902
Japan

Yoshiyuki Koga
Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka, Osaka 560-0043
Japan

Atsuo Kuniba
Institute of Physics
Graduate School of Arts and Sciences
University of Tokyo
Komaba, Tokyo 153-8902
Japan

Masato Okado
Department of Informatics and Mathematical Science
Graduate School of Engineering Science
Osaka University
Toyonaka, Osaka 560-8531
Japan

Taichiro Takagi
Department of Mathematics and Physics
National Defense Academy
Yokosuka 239-8686
Japan