

Local system homology of arrangement complements

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Abstract.

We use the critical points of a multivalued holomorphic function and Morse theory to find a basis for a local system homology group defined on the complement of an arrangement of hyperplanes. This generalizes results of Kohno [4] and Douai-Terao [2] from complexified real arrangements to all arrangements. We also show that the set of critical points satisfies the same recursion with respect to deletion and restriction as the β nc set of the arrangement.

§1. Introduction

A finite set of hyperplanes in \mathbb{C}^ℓ is called an affine arrangement, \mathcal{A} . Let

$$N(\mathcal{A}) = \cup_{H \in \mathcal{A}} H, \quad M(\mathcal{A}) = \mathbb{C}^\ell - N(\mathcal{A})$$

be the divisor and the complement of \mathcal{A} . We assume that \mathcal{A} is essential: \mathcal{A} contains ℓ linearly independent hyperplanes. For each $H \in \mathcal{A}$ choose an exponent $\lambda_H \in \mathbb{C}$ and let $\lambda = \{\lambda_H \mid H \in \mathcal{A}\}$. Let γ_H be the standard generator of $\pi_1(M)$ linking H . Define a rank one local system \mathcal{L}_λ on M by $\gamma_H \mapsto \exp(-2\pi i \lambda_H)$ and let \mathcal{L}_λ^\vee denote its dual. Let z_1, \dots, z_ℓ be coordinates in \mathbb{C}^ℓ . For each hyperplane $H \in \mathcal{A}$, choose a polynomial of degree one, α_H , with $H = \ker \alpha_H$. Call $Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$ a defining polynomial for \mathcal{A} . Define a multivalued holomorphic function on M by

$$\Phi_\lambda = \prod_{H \in \mathcal{A}} \alpha_H^{\lambda_H}.$$

The hypergeometric pairing

$$H_\ell(M; \mathcal{L}_\lambda^\vee) \times H^\ell(M; \mathcal{L}_\lambda) \longrightarrow \mathbb{C}$$

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is defined by $(\delta, \phi) \mapsto \int_{\delta} \Phi_{\lambda} \phi$. Under suitable nonresonance conditions on the exponents λ , the groups in the hypergeometric pairing have rank $\beta = (-1)^{\ell} \chi(M)$, where χ denotes Euler characteristic. Falk and Terao [3] constructed a basis for $H^{\ell}(M; \mathcal{L}_{\lambda})$, but a basis for $H_{\ell}(M; \mathcal{L}_{\lambda}^{\vee})$ is known only for complexified real arrangements [4], [2], defined in terms of the bounded chambers of the underlying real arrangement. Our main objective is to use Morse theory to construct a geometric homology basis for all arrangements.

Deletion and restriction is a basic tool for various induction arguments concerning arrangements [5]. Given a linear order on the hyperplanes of \mathcal{A} , Ziegler [8] associated with it a combinatorially defined set of cardinality β , called β nbcb, and proved that β nbcb satisfies a simple recursion with respect to deletion and restriction. We call a basis for either group in the hypergeometric pairing a β nbcb-basis if its elements are labeled by β nbcb and they satisfy the β nbcb recursion of Theorem 4.2. The cohomology basis constructed by Falk and Terao [3] is a β nbcb-basis. The homology basis constructed by Douai and Terao [2] for complexified real arrangements is also a β nbcb-basis. We show that our homology basis is also a β nbcb-basis.

§2. A Morse theoretic argument

A point $p \in M$ is a critical point of Φ_{λ} if and only if $d(\log \Phi_{\lambda})(p) = 0$. Properties of this critical set were established in [6], [7]:

Proposition 2.1. *For a dense open set of exponents λ , every critical point of Φ_{λ} in M is nondegenerate and has index ℓ . The number of these critical points is $\beta = (-1)^{\ell} \chi(M)$. Denote a critical set satisfying these conditions*

$$C(\mathcal{A}, \lambda) = \{p \in M \mid d(\log \Phi_{\lambda})(p) = 0\}. \quad Q.E.D.$$

We would like to apply a Morse theoretic argument analogous to [7, Sect. 5]. This argument relies on separating the divisors of zeroes and of poles of a certain generalized meromorphic function whose orders along its divisors generally are complex numbers, the exponents in our case. However, this procedure is only possible when the exponents are rational numbers or rational multiples of the same real number. Our strategy is to prove first the required homotopy result working with strictly positive rational exponents and subsequently extend its validity by perturbing the relevant Morse function slightly around the rational point. Since any open set in the space of exponents contains a rational point, the result

will thus have been proven for all exponents with strictly positive real part and small imaginary part.

Consider $\mathbb{C}^\ell \subset \mathbb{P}^\ell$ as the complement of the infinite hyperplane, P , defined by $z_0 = 0$. Assign it the exponent $\lambda_P = -\sum_{H \in \mathcal{A}} \lambda_H$. Let \hat{H} denote the projective closure of H and define $\mathcal{A}_\infty = \{\hat{H} \mid H \in \mathcal{A}\} \cup \{P\}$. Then $A = \hat{N} \cup P$ is the divisor of the projective arrangement \mathcal{A}_∞ and we have $\mathbb{P}^\ell - A = M = \mathbb{C}^\ell - N$. Let $\hat{\alpha}_H$ be the homogenized α_H and define

$$\hat{\Phi}_\lambda = z_0^{\lambda_P} \prod (\hat{\alpha}_H)^{\lambda_H}.$$

Assume that the exponents associated with the hyperplanes are rational and strictly positive, $\lambda_H \in \mathbb{Q}_{>0}$ for all $H \in \mathcal{A}$. The exponent λ_P at infinity is thus rational and strictly negative. There is an integer $m > 0$ so that $m\lambda_H$ is an integer for all $H \in \mathcal{A}$. The function $(\hat{\Phi}_\lambda)^m$ is a meromorphic function on \mathbb{P}^ℓ . Its set of critical points in $\mathbb{P}^\ell - A = M$ is $C(\mathcal{A}, \lambda)$ and the same holds for $|\hat{\Phi}_\lambda^m|^2$. We apply exactly the same blow-up procedure as in [7]. We form a birational map $\sigma : X \rightarrow \mathbb{P}^\ell$ by recursively blowing up points on A until $\sigma^{-1}(A) = D$ has normal crossings. Then $D = D_0 \cup D_\infty \cup D'$ where the divisor D_0 of zeroes of the meromorphic function $\sigma^* \hat{\Phi}_\lambda^m$ is disjoint from the divisor D_∞ of poles and D' consists of exceptional divisors connecting D_0 with D_∞ along whose generic points $\sigma^* \hat{\Phi}_\lambda^m$ neither vanishes nor diverges. Note that $\sigma^{-1}(N) \subset D_0$ and that in fact there are generally more irreducible components in D_0 than in $\sigma^{-1}(N)$.

Let T_0 and T_∞ respectively be sufficiently small open regular neighborhoods of D_0 and D_∞ in X and let $\bar{T}_0, \bar{T}_\infty$ be their closures. By the same argument as that leading to Theorem 5.3 of [7], the C^∞ extension F of $\sigma^* |\hat{\Phi}_\lambda^m|^2$ to $X - D_\infty$ defines a Morse function on the compact manifold $X - (T_0 \cup T_\infty)$ with boundary $\partial \bar{T}_0 \cup \partial \bar{T}_\infty$ none of whose critical points lies on $(X - (T_0 \cup T_\infty)) \cap D'$. There is a homotopy equivalence

$$X - (T_0 \cup T_\infty) \cong \partial \bar{T}_0 \cup e_1 \cup \dots \cup e_\beta,$$

with the following properties:

(1) Each e_i is an ℓ -cell constructed in terms of the level sets of F and it contains exactly one critical point of F so the number of cells is β .

(2) The right hand side is in fact a deformation retract of $X - (T_0 \cup T_\infty)$, hence of $X - (D_0 \cup D_\infty)$ as well.

The next step consists of deleting D' , or actually an arbitrarily small neighborhood T' of D' . It is fairly obvious that deletion of T' from

$\partial\bar{T}_0 \cup e_1 \cup \dots \cup e_\beta$ yields a deformation retract of $X - (T_0 \cup T_\infty \cup T')$. Note that the critical points of F are not on D' . In view of the manner the cells e_i are constructed, one may choose the e_i so that $e_i \cap T' = \emptyset$ for all $i = 1, \dots, \beta$. This granted, we have the homotopy equivalence

$$X - (T_0 \cup T_\infty \cup T') \cong (\partial\bar{T}_0 - \partial\bar{T}_0 \cap T') \cup e_1 \cup \dots \cup e_\beta.$$

Note that $T = \sigma(T_0 \cup T_\infty \cup T')$ is an open tubular neighborhood of A . Since σ is an isomorphism outside D , the left hand side is isomorphic to the complement $\mathbb{P}^\ell - T$. It follows that its affine part, $T_a = T \cap \mathbb{C}^\ell$, is an open tubular neighborhood of N . Let \bar{T}_a be its closure in \mathbb{C}^ℓ . Then $\partial\bar{T}_0 - \partial\bar{T}_0 \cap T' \cong \partial\bar{T}_a$.

Theorem 2.2. *Let \mathcal{A} be an essential ℓ -arrangement. Suppose the exponents λ_H have strictly positive real parts and small imaginary parts. Then there is a homotopy equivalence*

$$\mathbb{C}^\ell - T_a = \mathbb{P}^\ell - T \cong \partial\bar{T}_a \cup \bigcup_{p \in C(\mathcal{A}, \lambda)} e_p,$$

where e_p is an ℓ -cell attached to $\partial\bar{T}_a$ and “centered” at the critical point $p \in C(\mathcal{A}, \lambda)$.

Proof. We proved the result for any rational point λ in the construction preceding the theorem. It remains to prove it for nearby points λ' . The location of the critical points varies analytically with λ' . If λ' is close to λ , then the cell attached to a critical point p' of $\Phi_{\lambda'}$ is at most a slight deformation of the cell centered at the nearby critical point p of Φ_λ . Q.E.D.

§3. Locally finite homology

Let \mathcal{L} be any rank one complex local system on $\mathbb{P}^\ell - A = M = \mathbb{C}^\ell - N$. We consider the locally finite homology $H_*^{lf}(M; \mathcal{L})$ of M with coefficients in \mathcal{L} , defined as the dual to the compactly supported cohomology of $\mathbb{P}^\ell - A$ with coefficients in the dual local system \mathcal{L}^\vee :

$$H_i^{lf}(M; \mathcal{L}) = H_c^i(M; \mathcal{L}^\vee)^*.$$

It may be thought of as the homology of the complex of chains of the form $\sum_\sigma a_\sigma \sigma$. These are possibly infinite linear combinations of oriented simplices σ of a given triangulation of M whose coefficients are local sections $a_\sigma \in \mathcal{L}(\sigma) = H^0(\sigma, \mathcal{L})$.

In order to compute the locally finite homology we first identify it with a relative homology group in Proposition 3.1. Its proof is related to ideas in [1, pp. 10–12]. The Morse theoretic result of Theorem 2.2 on the homotopy of M permits explicit determination which part of the homology comes from the vicinity of A and which part from the attached ℓ -cells.

Proposition 3.1. *Let T be a small open neighborhood of A in \mathbb{P}^ℓ , let \bar{T} denote its closure, and let $\partial\bar{T}$ denote the boundary of \bar{T} . There is an isomorphism*

$$\iota_*\alpha : H_i(\mathbb{P}^\ell - T, \partial(\mathbb{P}^\ell - T); \mathcal{L}) \rightarrow H_i^{lf}(\mathbb{P}^\ell - A; \mathcal{L}).$$

Proof. Consider the pair $(\mathbb{P}^\ell - T, \partial(\mathbb{P}^\ell - T))$ consisting of the (oriented) topological manifold $\mathbb{P}^\ell - T$ together with its boundary. Notice that $\partial(\mathbb{P}^\ell - T)$ is $\partial\bar{T}$ up to orientation and that removing the boundary from $\mathbb{P}^\ell - T$ gives the open topological manifold $\mathbb{P}^\ell - T \cup \partial(\mathbb{P}^\ell - T) = \mathbb{P}^\ell - \bar{T}$. The Lefschetz duality isomorphism $H_i(\mathbb{P}^\ell - T, \partial(\mathbb{P}^\ell - T); G)^* \cong H^{2\ell-i}(\mathbb{P}^\ell - \bar{T}; G)$ is well known for any constant abelian group of coefficients G . It generalizes to local coefficients in the form

$$H_i(\mathbb{P}^\ell - T, \partial(\mathbb{P}^\ell - T); \mathcal{L}) \cong H^{2\ell-i}(\mathbb{P}^\ell - \bar{T}; \mathcal{L})^*.$$

Combined with Poincaré duality,

$$\begin{aligned} H^{2\ell-i}(\mathbb{P}^\ell - \bar{T}; \mathcal{L}^\vee) &\xrightarrow{\sim} H_c^i(\mathbb{P}^\ell - \bar{T}; \mathcal{L}^\vee) \\ &\parallel \\ &H_i^{lf}(\mathbb{P}^\ell - \bar{T}; \mathcal{L})^* \end{aligned}$$

we obtain an isomorphism

$$\alpha : H_i(\mathbb{P}^\ell - T, \partial(\mathbb{P}^\ell - T); \mathcal{L}) \xrightarrow{\sim} H_i^{lf}(\mathbb{P}^\ell - \bar{T}; \mathcal{L}).$$

The inclusion map $\iota : \mathbb{P}^\ell - \bar{T} \hookrightarrow \mathbb{P}^\ell - A$ is a homotopy equivalence. It induces the isomorphism

$$\iota_* : H_i^{lf}(\mathbb{P}^\ell - \bar{T}; \mathcal{L}) \xrightarrow{\sim} H_i^{lf}(\mathbb{P}^\ell - A; \mathcal{L}).$$

Thus we have

$$\begin{aligned} \iota^* : H_c^i(\mathbb{P}^\ell - A; \mathcal{L}^\vee) &\xrightarrow{\sim} H_c^i(\mathbb{P}^\ell - \bar{T}; \mathcal{L}^\vee) \\ &\parallel \\ &H_i^{lf}(\mathbb{P}^\ell - A; \mathcal{L})^* \qquad H_i^{lf}(\mathbb{P}^\ell - \bar{T}; \mathcal{L})^*, \end{aligned}$$

and the conclusion follows.

Q.E.D.

The isomorphism in Proposition 3.1 can be described concretely as follows. A class in $H_i(\mathbb{P}^\ell - T, \partial(\mathbb{P}^\ell - T); \mathcal{L})$ is represented by a relative cycle of the form $c = c' + c''$, where $c' \in C_i(\mathbb{P}^\ell - T; \mathcal{L})$ with boundary $\partial c' \in C_{i-1}(\partial\bar{T}; \mathcal{L})$ and $c'' \in C_i(\partial\bar{T}; \mathcal{L})$. Then $\alpha(c) = c' - \partial c'$ is the interior of c' . Finally, $\iota_*\alpha(c) = \iota_*(c' - \partial c')$ is obtained by retracting $\mathbb{P}^\ell - \bar{T}$ to $\mathbb{P}^\ell - A$ by letting $\partial\bar{T}$ “collapse” onto A .

Definition 3.2. Let $\delta_p = \iota_*\alpha(e_p)$ be the locally finite cycle obtained from the interior of the cell e_p of Theorem 2.2 by letting $\partial\bar{T}$ “collapse” onto A and define

$$\Delta(\mathcal{A}, \lambda) = \{\delta_p \mid p \in C(\mathcal{A}, \lambda)\}.$$

Lemma 3.3. Let \mathcal{L} be a rank one complex local system on M . Then $\dim H_\ell^{lf}(M; \mathcal{L}) \geq \beta$ and $\Delta(\mathcal{A}, \lambda)$ provides β linearly independent cycles.

Proof. The boundaries of the the ℓ -cells e_p of Theorem 2.2. lie on $\partial\bar{T}$. Since e_p is simply connected, the sections of any local system \mathcal{L} on $\mathbb{P}^\ell - A$ over the cell e_p are $H^0(e_i, \mathcal{L}) = \mathbb{C}$, the constant functions on e_p . It follows that any linear combination $\sum a_p e_p$ with constant coefficients $a_p \in \mathbb{C} = H^0(e_p, \mathcal{L})$ is a relative cycle in $H_\ell(\mathbb{P}^\ell - T, \partial(\mathbb{P}^\ell - T); \mathcal{L})$. Since the e_p are mutually disjoint, the space of such cycles has dimension β . Next we argue that the cells e_p are homologically independent. Recall from Theorem 2.2 that the space $\mathbb{P}^\ell - T$ is homotopically retractible to $\partial\bar{T}_a \cup_{p \in C(\mathcal{A}, \lambda)} e_p$. Thus every $(\ell + 1)$ -chain in $C_*(\mathbb{P}^\ell - T)$ may be retracted to lie within $\partial\bar{T}$. Hence the boundaries $\partial C_{\ell+1}(\mathbb{P}^\ell - T, \partial(\mathbb{P}^\ell - T); \mathcal{L})$ form a subspace of $C_\ell(\partial(\mathbb{P}^\ell - T); \mathcal{L})$.

Here we should note that there does not have to be any relationship between the local system \mathcal{L} and the function Φ_λ in terms of which the cells e_p are defined. Q.E.D.

Theorem 3.4. If \mathcal{L} is generic, then $\Delta(\mathcal{A}, \lambda)$ is a basis for $H_\ell^{lf}(M; \mathcal{L}) = H_\ell(M; \mathcal{L})$. Thus $\Delta(\mathcal{A}, \lambda)$ is a basis for $H_\ell(M; \mathcal{L}_\lambda^\vee)$ for generic λ .

Proof. It is known [4] that for a sufficiently generic local system \mathcal{L} , $H_i(M; \mathcal{L}) = 0 = H_{2\ell-i}^{lf}(M; \mathcal{L})$ for $i \neq \ell$. In this case the dimension of the only nonvanishing homology group is $\dim H_\ell^{lf}(M; \mathcal{L}) = \beta$. Q.E.D.

§4. The β ncb homology basis

We review first some combinatorial constructions for arrangements. A set $S \subseteq \mathcal{A}$ is dependent if $\bigcap_{H \in S} H \neq \emptyset$ and $\text{codim}(\bigcap_{H \in S} H) < |S|$.

A subset of \mathcal{A} which has nonempty intersection and is not dependent is called independent. Maximal independent sets are called frames. Since \mathcal{A} is essential, every frame has cardinality ℓ and intersection a point. Introduce a linear order in the arrangement by writing $\mathcal{A} = \{H_1, \dots, H_n\}$ and setting $H_i < H_j$ if $i < j$. An inclusion-minimal dependent set is called a circuit. A broken circuit is a subset S of \mathcal{A} for which there exists $H < \min(S)$ such that $H \cup S$ is a circuit. The non-broken circuit complex of \mathcal{A} is the collection of subsets of \mathcal{A} which have nonempty intersection and contain no broken circuits. Maximal sets of this complex are frames of \mathcal{A} called nbc-frames. We call an ordered frame $(H_{i_1}, \dots, H_{i_\ell})$ standard if $i_1 < \dots < i_\ell$.

Definition 4.1. *An nbc-frame B is called a β nbc-frame if for every $H \in B$ there exists $H' < H$ in \mathcal{A} such that $(B - \{H\}) \cup \{H'\}$ is a frame. The set of standard ordered β nbc-frames of \mathcal{A} is denoted $\beta\text{nbc}(\mathcal{A})$.*

The notation and terminology reflect the fact that $|\beta\text{nbc}(\mathcal{A})| = \beta$. The set $\beta\text{nbc}(\mathcal{A})$ satisfies a recursion with respect to deletion and restriction [8, Thm. 1.5]. Given $H \in \mathcal{A}$, let $\mathcal{A}' = \mathcal{A} - \{H\}$ be the arrangement with H deleted and let $\mathcal{A}'' = \{H \cap K \mid K \in \mathcal{A}', H \cap K \neq \emptyset\}$ be the arrangement restricted to H . We will always choose the last hyperplane $H = H_n$ for the triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$. The linear order on \mathcal{A}' is inherited from \mathcal{A} . The linear order on \mathcal{A}'' is determined by labeling each hyperplane K of \mathcal{A}'' by the smallest hyperplane $\nu(K)$ of \mathcal{A} containing it. Clearly $\nu(K) < H_n$ for all $K \in \mathcal{A}''$. If $S'' = (K_{i_1}, \dots, K_{i_p})$ is a set of hyperplanes in \mathcal{A}'' , define $\nu(S'') = (\nu(K_{i_1}), \dots, \nu(K_{i_p}))$.

Theorem 4.2 (Ziegler). *Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple and assume that \mathcal{A}' is essential. Write $\overline{\beta\text{nbc}(\mathcal{A}'')} = \{(\nu(B''), H_n) \mid B'' \in \beta\text{nbc}(\mathcal{A}'')\}$. There is a disjoint union*

$$\beta\text{nbc}(\mathcal{A}) = \beta\text{nbc}(\mathcal{A}') \cup \overline{\beta\text{nbc}(\mathcal{A}'')}. \quad Q.E.D.$$

We refer to this formula as the β nbc recursion. When $\ell = 1$, $\mathcal{A} = \{H_1, \dots, H_n\}$ consists of $n \geq 1$ points in \mathbb{C} . Here $\beta\text{nbc}(\mathcal{A}) = \{H_2, \dots, H_n\}$. In order to interpret the β nbc recursion in this case, we agree that $\beta\text{nbc}(\mathcal{A}'')$ has one element, the empty set, so $\overline{\beta\text{nbc}(\mathcal{A}'')} = \{H_n\}$. Introduce a linear order in $\beta\text{nbc}(\mathcal{A})$ using the lexicographic order on the hyperplanes read from right to left. Note that in this order elements of $\beta\text{nbc}(\mathcal{A}')$ are always smaller than elements of $\overline{\beta\text{nbc}(\mathcal{A}'')}$.

Next we show that the homology basis constructed in Theorem 3.4 is a β nbc basis. Since $\Delta(\mathcal{A}, \lambda)$ is labeled by $C(\mathcal{A}, \lambda)$, it suffices to show

that the critical points may be labeled by βNBC and satisfy the βNBC recursion. The weights λ' of \mathcal{A}' are inherited from \mathcal{A} : if $H \in \mathcal{A}'$, then $\lambda'_H = \lambda_H$. The weights λ'' of \mathcal{A}'' are defined by $\lambda''_K = \sum \lambda_H$ for $K \subset H$ and $H \in \mathcal{A}'$. We may assume that λ' and λ'' are generic.

Lemma 4.3. *If \mathcal{A}' is essential, then the map $\lambda_n \mapsto 0$ induces a bijection onto the disjoint union:*

$$\tau : C(\mathcal{A}, \lambda) \longrightarrow C(\mathcal{A}', \lambda') \cup C(\mathcal{A}'', \lambda'').$$

Proof. Choose coordinates so that $H_n = \ker z_1$ and write $\lambda_n = t$. Then $Q = z_1 Q'$ where Q' defines the essential arrangement \mathcal{A}' . Similarly $\Phi = z_1^t \Phi'$ where Φ' is the corresponding multivalued holomorphic function for \mathcal{A}' whose critical points are $C(\mathcal{A}', \lambda')$. Let Φ'' denote the restriction of Φ' to H_n . The partial derivatives of Φ are

$$\begin{aligned} \partial_{z_1} \log \Phi &= g_1 / (z_1 Q') \\ \partial_{z_i} \log \Phi &= g_i / Q' \quad \text{for } 2 \leq i \leq \ell \text{ where} \\ g_1 &= tQ' + z_1 Q' \partial_{z_1} \log \Phi' \\ g_i &= Q' \partial_{z_i} \log \Phi' \quad \text{for } 2 \leq i \leq \ell \end{aligned}$$

Note that only g_1 depends on t . The zero set of the equations $\{g_i = 0 \mid 1 \leq i \leq \ell\}$ in M is $C(\mathcal{A}, \lambda)$ when $t \neq 0$. Setting $t = 0$ gives the solutions $C(\mathcal{A}', \lambda')$ of critical points of Φ' in M' , the complement of \mathcal{A}' . Next consider the points on H_n defined by $\{z_1 = 0, g_i = 0 \mid 1 \leq i \leq \ell\}$. For $t \neq 0$, these equations have no solution on $M'' = H_n \cap M'$, the complement of \mathcal{A}'' . Adding $t = 0$ to these equations gives exactly the critical points of Φ'' on M'' . Q.E.D.

Theorem 4.4. *There is a labeling of the critical points*

$$\rho : C(\mathcal{A}, \lambda) \longrightarrow \beta\text{NBC}(\mathcal{A})$$

which is a bijection and respects the βNBC recursion.

Proof. We argue by double induction on ℓ and n . We may assume that in each arrangement the first ℓ hyperplanes are linearly independent, so the induction starts with this essential arrangement and when we add the next hyperplane all labels come from the restriction where the induction hypothesis holds since it has lower dimension. If $\ell = 1$ and $n = 2$, then $\beta\text{NBC}(\mathcal{A}) = \{(H_2)\}$ and $C(\mathcal{A}, \lambda) = \{p\}$ is a singleton. Define $\rho(p) = (H_2)$. For the induction step we use the recursions in Theorem 4.2 and Lemma 4.4. We may assume that $\rho' : C(\mathcal{A}', \lambda') \rightarrow \beta\text{NBC}(\mathcal{A}')$

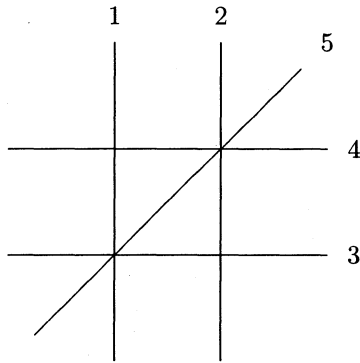


FIG 1. The Selberg arrangement

and $\rho'' : C(\mathcal{A}'', \lambda'') \rightarrow \beta\text{NBC}(\mathcal{A}'')$ satisfy the theorem. For $p \in C(\mathcal{A}, \lambda)$ define

$$\rho(p) = \begin{cases} \rho'(\tau(p)) & \text{if } \tau(p) \in C(\mathcal{A}', \lambda') \\ (\nu\rho''(\tau(p)), H_n) & \text{if } \tau(p) \in C(\mathcal{A}'', \lambda''). \end{cases}$$

This labeling respects the recursion by construction. Q.E.D.

Corollary 4.5. *The critical point labeling of the open cells*

$$\Delta(\mathcal{A}, \lambda) = \{\delta_{\rho(p)} \mid \rho(p) \in \beta\text{NBC}(\mathcal{A})\}$$

provides a βNBC basis for $H_\ell(M; \mathcal{L}_\lambda^\vee)$. Q.E.D.

It is interesting to note that if \mathcal{A} is a complexified real arrangement and the critical points lie in the bounded chambers, then the critical point labeling may differ from the bounded chamber labeling of [2].

Example 4.6. *Let $Q = (z_1 + 1)(z_1 - 1)(z_2 + 1)(z_2 - 1)(z_1 - z_2)$ be the Selberg arrangement with the given linear order, see Figure 1. Then $\beta\text{NBC}(\mathcal{A}) = \{(2, 4), (2, 5)\}$. The bounded chamber labeling of [2] is independent of λ . It assigns (2, 4) to the upper left chamber and (2, 5) to the lower right. The critical point labeling depends on λ . Let $\lambda_5 = t$ be a small positive real number. With $\lambda = (1.5, 0.5, 0.9, 0.3, t)$ the critical point labels are the same, but with $\lambda = (1.5, 0.5, 0.3, 0.9, t)$ the labels are reversed.*

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