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Deformations of Singularities, Complex Manifolds and CR–Structures

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Dedicated to Prof. M. Kuranishi on the occasion of his seventieth birthday

Abstract.

This note is an expanded version of the author's lecture at the conference on Several Complex Variables held in Osaka, Dec. 1994. We consider deformations of complex spaces and their relationship to deformations of CR–structures. An invariant is introduced which measures the change in the algebra of CR–functions under a deformation. These issues are then considered in the context of the deformations of the total space of the tangent bundle of a Riemann surface. The last section contains problems in deformation theory.

§1. Introduction

In the 1960s and 1970s a great deal of progress was made in the study of the deformations of complex analytic spaces. A complex structure on a manifold has two fundamentally different descriptions as: 1. A holomorphic coordinate atlas, 2. A formally integrable subbundle of the complexified tangent bundle. We refer to 1. as the "holomorphic description" and 2. as the "real description." The equivalence of these descriptions is the content of the Newlander–Nirenberg theorem. The holomorphic description is more general as it can also be used for (possibly non–reduced) analytic spaces. These two representations lead to different descriptions for the deformations of the given structure. In the holomorphic description one fixes a coordinate cover, the deformations then appear as families of holomorphic gluing maps. In the second case one represents the deformed subbundle of the complexified tangent bundle as a graph over the reference structure. This is conveniently

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parametrized by a vector valued $(0,1)$–form which satisfies a non–linear partial differential equation. In actual practice these descriptions are very difficult to directly compare. Instead one defines a universal object, the versal deformation, to which the various descriptions are compared.

The real description seems to be somewhat limited as it appears to require a smooth underlying space in order for the integrability conditions to be expressible in terms of a PDE. Kuranishi showed that this technique could be applied to study the deformations of an isolated singular point by considering the CR-structure induced on a (smooth) link of the singularity, see [Ku]. If the complex dimension of the singular space is at least 3 then there is a "real" description of the integrable deformations of this structure as the solution space of a non–linear PDE on the link. Once this space is obtained two problems remain in order to return to the original question of deforming a singularity: 1. One needs to show that the deformed CR–structures arise as the boundaries of complex spaces which are, in an appropriate sense, deformations of the initial singularity and 2. One needs to show that the solution space of the PDE has the structure of a complex space. Theorems of Boutet de Monvel and Harvey and Lawson assure that strictly pseudoconvex CR–structures on compact manifolds of dimension at least 5 do in fact arise as the boundaries of complex spaces, see [BdM, HaLa, Ro]. Additional geometric hypotheses on the initial singularity are needed in order to conclude that these spaces are deformations of the reference space. This problem has been treated in dimension greater than 3 by Buchwietz and Millson and Akahori and Miyajima, see [AkMi, BuMi, Mi].

If the dimension of initial variety is 2 then the situation is entirely different: there is no integrability condition and it is no longer the case that every CR–structure arises as the boundary of a complex space. This is equivalent to existence of an embedding of the manifold in $\mathbb{C}^n$ whose coordinate functions belong to the algebra of CR–functions. Such a CR–structure is called embeddable. The $\mathcal{C}^\infty$–generic structure is not embeddable, see [Ni, JT, Ep3]. Thus the central problem in obtaining a "real" description of the deformations of an isolated surface singularity is the problem of embeddability for CR–structures on compact 3–dimensional manifolds: give a criterion in terms of the deformation tensor of the CR–structure which characterizes the embeddable CR–structures. For the purposes of analysis one would like the criterion to be expressed in terms of a pseudodifferential equation. Of course a differential condition would be preferable but it is known from examples that the property of embeddability is non–local in nature. In general there may not be a simple condition which characterizes embeddability. Instead we seek a description of the general features of the set of embeddable structures intrinsically and as a subset of the space of all structures. For example are there situations when the embeddable structures form a submanifold or subvariety of the set of all structures? Does it make sense to say that the set of embeddable structures has infinite codimension? Indeed, with the current state of knowledge, it is not known in general if the set of embeddable structures is a closed subset in a reasonable topology.

In this lecture we give a description of the set of deformations of an embeddable CR–structure on a three dimensional manifold. We then define a stratification of the set of embeddable structures. This stratification has closed strata and is defined by formally analytic equations. We next consider the deformations of the total space of the tangent bundle of a Riemann surface and show how the real and holomorphic descriptions can be compared in this case. Finally we close with a collection of problems which bear on the problem of describing the space of embeddable CR–structures on a compact manifold.

Acknowledgments

Many of the ideas in this lecture were developed in collaborative efforts with Dan Burns, John Bland and Gennadi Henkin. I have also learned a lot in conversations with and from the papers of László Lempert. Several of the problems in §6 are motivated by his work, see [Le1-2]. I would also like to thank Ron Donagi for explaining the affine bundles which appear in §5 and Sascha Voronov for his help with algebraic geometry.

§2. Deformations of CR–structures

Let M denote a compact three manifold. A CR–structure on M is given by selecting a smoothly varying complex line $T^{0,1}_pM \subset T_pM \otimes \mathbb{C}$, $p \in M$. We require a non–degeneracy condition:

$$T^{0,1}_pM \cap \overline{T^{0,1}_pM} = \{0\} \text{ for every } p \in M. \tag{2.1}$$

As the fiber dimension in this case is one, the integrability condition is vacuous. The canonical example of a CR–structure arises on a real hypersurface in a complex manifold. If $M \hookrightarrow X$ is a real hypersurface then the CR–structure **induced** by the embedding is given by:

$$T^{0,1}_pM = T^{0,1}_pX \cap T_pM \otimes \mathbb{C}, \quad \text{for } p \in M.$$

The non–degeneracy condition implies that there is a smooth plane field $H \subset TM$ such that for every $p \in M$:

$$T^{1,0}_pM \oplus T^{0,1}_pM = H_p \otimes \mathbb{C}.$$

The plane field, H defines a contact structure on M if and only if the CR–structure is strictly pseudoconvex. We assume that this condition holds throughout the paper.

The CR–structure defines a pair of operators analogous to $\partial, \overline{\partial}$ by the formulæ :

$$\partial_b f = df \restriction_{T^{1,0}M}, \quad \bar{\partial}_b f = df \restriction_{T^{0,1}M} .$$

The operator $\bar{\partial}_b$ takes values in $\mathcal{C}^\infty(M; (T^{0,1}M)^*)$. This is a quotient of $(TM \otimes \mathbb{C})^*$; to represent $\bar{\partial}_b$ as a differential operator a non–canonical choice needs to be made. For example, selecting a one form defining H determines a representation of $\bar{\partial}_b$. We often use the notation $(M, \bar{\partial}_b)$ to denote the manifold M with the CR–structure which induces $\bar{\partial}_b$. The kernel of $\bar{\partial}_b$ is called the algebra of CR–functions. An embedding of the CR–manifold $(M, \bar{\partial}_b)$ is given by an embedding $F : M \to \mathbb{C}^N$ where the coordinate functions of F belong to $\ker \bar{\partial}_b$. The geometric description of a CR–embedding is that

$$F_* T^{0,1}M = T^{0,1}F(M),$$

where the CR–structure on the right hand side is that induced from the embedding.

The deformations of the CR–structure are given by sections of the endomorphism bundle:

$$\phi \in \mathcal{C}^\infty(M; \mathrm{End}(T^{0,1}M, T^{1,0}M)),$$

with

$${}^{\phi}T_p^{0,1}M = \{\bar{Z} + \phi_p(\bar{Z}) : \bar{Z} \in T_p^{0,1}M\}.$$

In order for (2.1) to hold we require:

$$\|\phi\|_{L^\infty} < 1. \tag{2.2}$$

We denote the set of endomorphisms satisfying (2.2) by $\mathrm{Def}(M, \bar{\partial}_b)$. Using the natural isomorphism:

$$\mathrm{End}(T^{0,1}M, T^{1,0}M) \simeq T^{1,0}M \otimes (T^{0,1}M)^*$$

the $\bar{\partial}_b$–operator defined by the deformation, ϕ can be represented as:

$$\bar{\partial}_b^{\phi} f = (\bar{\partial}_b + \phi \circ \partial_b) f. \tag{2.3}$$

Note that all such CR–structures have the same underlying plane field, *a priori* one might have expected that this should also be allowed to vary. However according to a theorem of A. Gray, contact fields are rigid and hence any small deformation of H is diffeomorphically equivalent to H by a diffeomorphism isotopic to the identity, see [Gy]. Thus no generality is lost in supposing that the underlying plane field is fixed.

The group Cont_H consists of diffeomorphisms which preserve the contact field, i.e.

$$\psi \in \mathrm{Cont}_H \iff \psi_* H_p = H_{\psi(p)},\ \forall p \in M.$$

This group has the topology of a smooth tame lie group as explained in [ChLe]. It acts on $\mathrm{Def}(M, \bar{\partial}_b)$ as follows: if $\phi \in \mathrm{Def}(M, \bar{\partial}_b)$, $\psi \in \mathrm{Cont}_H$ then we define $\psi \cdot \phi$ by

$$^{\psi\cdot\phi}T^{0,1}_{\psi(p)}M = \psi_* {}^{\phi}T^{0,1}_p M. \tag{2.4}$$

As $\ker \bar{\partial}^{\phi}_b = \psi^*(\ker \bar{\partial}^{\psi\cdot\phi}_b)$ the structures in a Cont_H–orbit should be considered geometrically equivalent. We define a moduli space for CR–structures as the quotient

$$\mathcal{M}(M, [\bar{\partial}_b]) = \mathrm{Def}(M, \bar{\partial}_b) / \mathrm{Cont}_H\,.$$

In analogy with the case of Riemann surfaces we define a "Teichmüller" space by

$$\mathcal{T}(M, [\bar{\partial}_b]) = \mathrm{Def}(M, \bar{\partial}_b) / \mathrm{Cont}^0_H,$$

where Cont^0_H denotes the identity component of Cont_H.

In a recent paper, Cheng and Lee have shown that one can construct a slice for the contact action on $\mathrm{Def}(M, \bar{\partial}_b)$ for any three manifold, see [ChLe]. Taking advantage of the S^1–action, Bland constructed a different slice for structures on S^3 near to the structure induced on the unit sphere, see [Bl]. The former slice is a real manifold whereas the latter has the structure of a smooth bundle with complex fibers over a real manifold. As $\mathrm{Def}(M, \bar{\partial}_b)$ is an open set in a complex Frechet manifold whereas Cont_H is a real group, it seems unlikely that either $\mathcal{T}(M, [\bar{\partial}_b])$ or $\mathcal{M}(M, [\bar{\partial}_b])$ has a natural structure as a complex space. In these proceedings Lempert has shown that the Teichmüller space of structures near to the structure induced on a strictly pseudoconvex hypersurface in $\mathbb{C}^2$ has a natural Frechet manifold structure. It should be emphasized that in the work of Cheng and Lee and Bland only the action of a neighborhood of the identity in Cont_H is considered whereas Lempert takes the quotient by the full identity component.

§3. The relative index and the stratification of the moduli space

The issue of embeddability is intimately connected with the stability properties of the $\ker\bar{\partial}_b^\phi$ under deformations. In order to better understand this question we recall a theorem of Kohn, see [Ko]:

Theorem [Kohn]. *A compact pseudoconvex CR–manifold $(M, \bar{\partial}_b)$ is embeddable if and only if the range $\bar{\partial}_b$ is closed in $L^2(M)$.*

In order to understand the content of this theorem it is useful to introduce a second order, self adjoint operator with the same kernel as $\bar{\partial}_b$. Once a contact form is fixed we can represent $\bar{\partial}_b$ as closed operator on L^2 and define its formal adjoint, $\bar{\partial}_b^*$. The operator

$$\Box_b f = \bar{\partial}_b^* \bar{\partial}_b f$$

has a self adjoint extension to $L^2(M)$. The range of $\bar{\partial}_b$ is closed if and only the range of $\Box_b$ is closed. Since $\Box_b$ is self adjoint its range is closed if and only if 0 is an isolated point in the $\operatorname{spec}(\Box_b)$.

If $\bar{\partial}_b$ defines an embeddable CR–structure then

$$\operatorname{spec}(\Box_b) = \{0 < \lambda_1 \leq \lambda_2 \leq \dots \quad \}$$

Zero is an eigenvalue of infinite multiplicity as this eigenspace is simply the L^2–closure of $\ker\bar{\partial}_b$; the sequence $\{\lambda_i\}$ is discrete and tends to infinity. If $\bar{\partial}_b$ is non–embeddable then in addition to the sequence, $\{\lambda_i\}$ tending to infinity $\operatorname{spec}(\Box_b) \supset \{\mu_i\}$ where

$$\mu_i > 0 \text{ and } \mu_i = O(i^{-N}),\ \forall N > 0.$$

If $\bar{\partial}_b'$ is a small embeddable deformation of an embeddable structure, $\bar{\partial}_b$ it is possible that $\Box_b'$ has a finite number of eigenvalues $\{\mu_1, \dots, \mu_m\}$ which satisfy: $0 < \mu_i << \lambda_1$. The eigenfunctions of $\Box_b'$ corresponding to the $\{\mu_i\}$ are small perturbations of functions in $\ker\Box_b$. The presence or absence of "small eigenvalues" is therefore a measure of the size of $\ker\bar{\partial}_b'$ relative to $\ker\bar{\partial}_b$.

Unfortunately this reasoning can only be carried out in a small neighborhood of the reference structure and appears to depend very strongly on the non–canonical choices made in the definition of $\bar{\partial}_b$. In order to obtain something more robust we need to modify our point of view. The starting point is the following theorem:

Theorem 3.1. *Let $(M, \bar{\partial}_b)$ denote a compact, strictly pseudoconvex, embeddable, 3–dimensional CR–manifold. Let $\phi \in \mathrm{Def}(M, \bar{\partial}_b)$ and let $\mathcal{S}$ denote the orthogonal projection onto $\ker \bar{\partial}_b$ relative to some choice of smooth volume form on M. The CR–manifold $(M, \bar{\partial}_b^{\phi})$ is embeddable if and only if*

$$\mathcal{S} : \ker \bar{\partial}_b^{\phi} \longrightarrow \ker \bar{\partial}_b$$

is a Fredholm operator.

Remark. The results in this section are proved in [Ep1-2]. Observe that this characterization of embeddability holds globally in $\mathrm{Def}(M, \bar{\partial}_b)$. Note also that if $\mathcal{S}'$ denotes the projection onto $\ker \bar{\partial}_b'$ $\mathcal{S} - \mathcal{S}'$ is a compact operator if and only if $\bar{\partial}_b = \bar{\partial}_b'$.

The proof of Theorem 3.1 uses a fortuitous representation for $\mathcal{S} \restriction_{\ker \bar{\partial}_b'}$, some functional analysis and the theorem of Kohn stated above. This result suggests that we define a relative invariant for a pair of embeddable structures:

Definition. *If $\bar{\partial}_b$ and $\bar{\partial}_b'$ are two embeddable structures with the same underlying plane field and orientation then define the relative index*

$$\mathrm{Ind}(\bar{\partial}_b, \bar{\partial}_b') = \mathrm{ind}(\mathcal{S} : \ker \bar{\partial}_b' \to \ker \bar{\partial}_b).$$

Unfortunately many non–canonical choices were made in the definition of the relative index. In order for $\mathrm{Ind}(\bar{\partial}_b, \bar{\partial}_b')$ to be an interesting invariant we need to show that its value is independent of these choices. Obviously a volume form was chosen to define the orthogonal projector. A little subtler is the fact that we would like the invariant to be geometric in nature and hence constant along orbits of $\mathrm{Cont}_H \times \mathrm{Cont}_H$. Independence of the choice of volume form is easily established, constancy along orbits of the contact group requires considerably more effort. Nonetheless we can prove the following result:

Theorem 3.2. *The relative index $\mathrm{Ind}(\bar{\partial}_b, \bar{\partial}_b')$ is independent of the choice of volume form and is constant along the orbits of $\mathrm{Cont}_H^0 \times \mathrm{Cont}_H^0$*

There are two results which are of interest in their own right used in the proof of Theorem 3.2

Proposition 3.1. *If $\mathcal{F} \subset \mathrm{Def}(M, \bar{\partial}_b)$ and there is a constant $C >$ 0 such that for $\phi \in \mathcal{F}$*

$$\lambda_1(\Box_b^{\phi}) > C$$

then the map $\phi \to \mathcal{S}^\phi$ is continuous from the $\mathcal{C}^\infty$–topology on $\mathcal{F}$ to the norm topology on bounded operators on $L^2(M)$.

Remark. Here $\lambda_1(\Box_b^\phi) = \inf\{\operatorname{spec}\Box_b^\phi \setminus \{0\}\}$.

The second result shows that the relative index defines, in a sense, a 1–cocycle on the space of embeddable structures:

Proposition 3.2. *If $\bar{\partial}_b^1$, $\bar{\partial}_b^2$, $\bar{\partial}_b^3$ are embeddable structures in* $\operatorname{Def}(M, \bar{\partial}_b)$ *such that there exists a continuous family of embeddable structures $\bar{\partial}_b^t$ joining $\bar{\partial}_b^2$ to $\bar{\partial}_b^3$ then:*

$$\operatorname{Ind}(\bar{\partial}_b^1, \bar{\partial}_b^3) = \operatorname{Ind}(\bar{\partial}_b^1, \bar{\partial}_b^2) + \operatorname{Ind}(\bar{\partial}_b^2, \bar{\partial}_b^3).$$

Remark. We believe that the cocycle relation should hold in complete generality for any triple of embeddable structures in $\operatorname{Def}(M, \bar{\partial}_b)$.

Using the relative index we can define a stratification of the space of embeddable structures:

$$\mathfrak{S}_n = \{\phi \in \operatorname{Def}(M, \bar{\partial}_b) : \operatorname{Ind}(\bar{\partial}_b, \bar{\partial}_b^\phi) \geq -n\}.$$

From the invariance under the action of Cont_H^0 it follows that the stratification descends to the Teichmüller space, $\mathcal{T}(M, \bar{\partial}_b)$. Large subsets of the strata are closed in the $\mathcal{C}^\infty$–topology:

Theorem 3.3. *For any $\epsilon > 0$ the set*

$$\mathfrak{S}_n \cap \{\phi : \|\phi\|_{L^\infty} \leq \frac{1}{2} - \epsilon\}$$

is closed in the $\mathcal{C}^\infty$–topology.

Remark. It follows from Theorem 3.3 that a strategy for showing that the set of embeddable CR–structures is closed is to show that there is a k, N and $\epsilon > 0$ so that, for $n \geq N$

$$\mathfrak{S}_n \cap \{\|\phi\|_{C^k} < \epsilon\} = \mathfrak{S}_N \cap \{\|\phi\|_{C^k} < \epsilon\}$$

In case the reference structure is that induced on a strictly pseudoconvex hypersurface in $\mathbb{C}^2$, the results in [Le2] and [Ep1–2] imply that this conjecture holds, with $N = 0$. Using a recent result of Eliashberg this has been improved for the case of S^3. In [Ep1] it is shown that the set of embeddable structures on S^3 coincides with $\mathfrak{S}_0$.

If V is an affine variety and M_1 and M_2 are nearby strictly pseudoconvex hypersurfaces bounding compact domains in V then it is reasonable that $\mathrm{Ind}(\bar{\partial}_b^1, \bar{\partial}_b^2) = 0$. Here $\{\bar{\partial}_b^i\}$ are the CR–structures induced by the embeddings $M_i \hookrightarrow V$.

Theorem 3.4. *If V is a variety and M_t, $t \in [0,1]$ is a continuous family of smooth strictly pseudoconvex hypersurfaces in $V \backslash \mathrm{sing}(V)$ which bound compact domains then*

$$\mathrm{Ind}(\bar{\partial}_b^t, \bar{\partial}_b^s) = 0, \ \forall s, t \in [0,1].$$

Here $\bar{\partial}_b^t$ is the CR–structure induced from the embedding $M_t \hookrightarrow V$.

Remark. We call the family $\{M_t : t \in [0,1]\}$ a **strictly pseudoconvex isotopy** of M_0 to M_1.

Recall the construction of the Kuranishi space for an isolated singular point, (V,p) : Intersect V with a sphere of small radius centered on the singular point to obtain a smooth, strictly pseudoconvex hypersurface $M \hookrightarrow V$. This embedding induces a CR–structure, $\bar{\partial}_b$ on M. One then considers (in higher dimensions) the integrable deformations of $(M, \bar{\partial}_b)$ modulo an equivalence relation. The equivalence relation is, roughly speaking that

$$(M, \bar{\partial}_b^1) \sim (M, \bar{\partial}_b^2), \tag{3.1}$$

if they are in the same strictly pseudoconvex isotopy class.

In the actual construction of the Kuranishi space one requires that this condition hold only to first order. From Theorem 3.4 it follows that if we define a space $\mathfrak{D}([M, \bar{\partial}_b])$ to be the set of embeddable deformations of $(M, [\bar{\partial}_b])$ modulo (3.1) then the stratification of $\mathrm{Def}(M, \bar{\partial}_b)$ actually descends to this space.

In [La] an interesting situation is considered. Let X be a compact strictly pseudoconvex domain in a smooth modification of a Stein space. Assume moreover that the maximal compact analytic set $A \subset X$, is a union of smooth curves with only normal crossings. Of course the deformation space for the complex manifold X is infinite dimensional, however if one considers the deformation space modulo the relation (3.1) it becomes finite dimensional. In fact Laufer showed that the base space for the versal deformation is a manifold parametrized by $H^1(X; \Theta)$. Here Θ is the sheaf of germs of holomorphic vector fields. In the next section we consider the example of X a domain in the $T^{1,0}\Sigma$, where Σ is a Riemann surface of genus at least 2. We compare the "holomorphic" description of the deformation space as $H^1(X,\Theta)$ with a real description derived from considering deformations of the CR–structure on ∂X. In particular we consider the stratification of $H^1(X,\Theta)$ defined by the relative index.

§4. The CR–geometry of the tangent bundle

Let Σ denote a Riemann surface of genus at least 2. The complex structure is defined by specifying a coordinate cover $\{(V_\alpha, z_\alpha)\}$ and holomorphic transition functions $\{f_{\alpha\beta}\}$ where

$$z_\alpha = f_{\alpha\beta}(z_\beta) \text{ on } z_\beta(V_\alpha \cap V_\beta). \tag{4.1}$$

Let $\pi : T^{1,0}\Sigma \to \Sigma$ denote the holomorphic tangent space of Σ. Using the coordinate vector fields, $\{\partial_{z_\alpha}\}$ to locally trivialize $T^{1,0}\Sigma$ we obtain coordinates, $\{U_\alpha, (z_\alpha, w_\alpha)\}$ for the tangent space where $U_\alpha = \pi^{-1}(V_\alpha)$ and

$$w_\alpha = f'_{\alpha\beta}(z_\beta)w_\beta$$

on the overlaps. The total space is compactified by adding the curve "at infinity." Denote this space by $\widehat{T}\Sigma$. A Riemannian metric on Σ defines a function, homogeneous of degree 2, on the fibers of $T^{1,0}\Sigma$. In a local coordinate system this function is represented by $h(z_\alpha, w_\alpha) = e^{2u^\alpha}|w_\alpha|^2$. Define a hypersurface $M \hookrightarrow T^{1,0}\Sigma$ by

$$M = \{p \in T^{1,0}\Sigma : h(p) = 1\}.$$

Let D denote the unit disk bundle in $T^{1,0}\Sigma$ bounded by M. It is clear from the form of h that M is invariant under the natural action of S^1 on $T^{1,0}\Sigma$:

$$U_\phi(z_\alpha, w_\alpha) = (z_\alpha, e^{i\phi}w_\alpha), \phi \in [0, 2\pi).$$

For $k \in \mathbb{Z}$ set

$$F_k = \{f \in \mathcal{C}^\infty(M) : U_\phi^* f = e^{ik\phi}\},$$

and

$$\mathcal{F}_k = \bigoplus_{j=k}^{\infty} F_j.$$

A contact form is defined on M by setting

$$\theta = -i\bar{\partial}\log h \restriction_M .$$

For the remainder of this section we suppose that h is defined by the constant curvature, -1 metric on Σ. A simple calculation shows that the one forms,

$$\theta^1 = \frac{\sqrt{2}dz_\alpha}{w_\alpha} \text{ on } M \cap U_\alpha,$$

piece together to give a globally defined one form, θ^1 on M which satisfies

$$d\theta = i\theta^1 \wedge \theta^{\bar{1}} \text{ and } U_\phi^* \theta^1 = e^{i\phi}\theta^1.$$

The section of $T^{0,1}M$ dual to $\theta^{\bar{1}}$ is of course globally defined and given in local coordinates by

$$\bar{Z} \restriction_{U_\alpha} = \frac{\bar{w}_\alpha}{\sqrt{2}}(\partial_{\bar{z}_\alpha} - 2u^\alpha_{\bar{z}_\alpha} \bar{w}_\alpha \partial_{\bar{w}_\alpha}).$$

The local coordinate representation of the $\bar{\partial}_b$–operator is:

$$\bar{\partial}_b f = \bar{Z}_\alpha f \theta^{\bar{1}}_\alpha.$$

A consequence of using the constant curvature metric to define h is that

$$\mathcal{L}_{\bar{Z}}\theta \wedge d\theta = 0,$$

thus the adjoint of $\bar{\partial}_b$, with these normalizations is

$$\bar{\partial}_b^*(g\theta^{\bar{1}}) = -Zg. \tag{4.2}$$

For details of these arguments see [Ep3]. The commutator $T = [\bar{Z}, Z]$ is a purely imaginary vector field that satisfies

$$T \restriction_{F_k} = -k. \tag{4.3}$$

Since $T^{0,1}M$ has a global non–vanishing section we identify $\mathrm{Def}(M, \bar{\partial}_b)$ with $\{\phi \in \mathcal{C}^\infty(M) : \|\phi\|_{L^\infty} < 1\}$ by setting

$${}^\phi T_p^{0,1}M = \{\lambda(\bar{Z}_p + \phi(p)Z_p) : \quad \lambda \in \mathbb{C}\}.$$

are S^1–invariant and so we say that deformations in $\mathcal{F}_{-2}$ ($\mathcal{F}_{-1}$) have non–negative (positive) Fourier coefficients. In [Ep3] it is shown that all deformations in $\mathcal{F}_{-2}$ are embeddable.

Using a formalism introduced by Bland and Duchamp in [BlDu] we can actually extend deformations lying in $\mathcal{F}_{-2}$ to integrable almost complex structures defined on the disk bundle in $T^{1,0}\Sigma$ bounded by M. In local coordinates $\phi \in \mathcal{F}_{-2}$ has a Fourier expansion

$$\phi(z_\alpha, w_\alpha) = \sum_{j=-2}^{\infty} a_j(z_\alpha, \bar{z}_\alpha) w_\alpha^j. \tag{4.4}$$

Define a vector field on $D \cap U_\alpha$ by:

$$\bar{\mathcal{Z}}_\alpha^\phi = (\partial_{\bar{z}_\alpha} - 2u^\alpha_{\bar{z}_\alpha}\bar{w}_\alpha\partial_{\bar{w}_\alpha}) + e^{2u^\alpha}\sum_{j=-2}^{\infty} a_j w_\alpha^{j+2}(\partial_{z_\alpha} - 2u^\alpha_{z_\alpha} w_\alpha \partial_{w_\alpha}). \tag{4.5}$$

Simple calculations show that

$$e^{2u^\alpha} w_\alpha(\bar{Z}_\alpha + \phi Z_\alpha) = \bar{\mathcal{Z}}_\alpha^\phi \restriction_M \text{ and}$$
$$\bar{\mathcal{Z}}_\beta^\phi = \bar{f}'_{\alpha\beta}\bar{\mathcal{Z}}_\alpha,$$

on the overlaps. As a consequence of the second relation it follows that $\{\bar{\mathcal{Z}}_\alpha d\bar{z}_\alpha\}$ is a globally defined vector valued $(0,1)$–form. In the next section we show that the $(1,0)$–part can be identified as the Dolbeault representative of a class in $H^1(D;\Theta)$. If we take $\bar{\mathcal{W}}_\alpha = \partial_{\bar{w}_\alpha}$ then

$$(\bar{\mathcal{Z}}_\alpha^\phi, \bar{\mathcal{W}}_\alpha)$$

is a globally defined, integrable, almost complex structure on the unit disk bundle which induces the CR–structure ${}^\phi T^{0,1}M$ on ∂D. This is a "real" representation of deformations of the complex structure on the disk bundle.

In [BlEp] a formalism is presented for studying the deformations of a surface singularity in terms of CR–structures on a link. Part of this program is to recognize when a deformation is, to first order, a wiggle of a hypersurface within a variety. A second order operator $\mathcal{P}$ is defined such that a deformation, ϕ is a wiggle, to first order if and only if $\phi = \mathcal{P}\varphi$. Results in [ChLe] show that $\mathcal{P}$ has a closed range and therefore it is reasonable to expect that the "versal deformation" of the isolated singular point will be found in $\ker \mathcal{P}^*$. Here the adjoint is defined relative to some choice of volume form. Note that the operator $\mathcal{P}^*$ is defined intrinsically on M. The analysis in [BlEp] applies to embedded CR–manifolds and the category of embedded deformations. So if we consider all embeddable families of deformations which lie in $\ker \mathcal{P}^*$ then we may be considering deformations of several families of singularities which share the reference CR–manifold as a link. A second caveat is that in the cited work it is assumed that the singularities are normal. In the next section some of the singularities to be considered are non–normal, so the results in [BlEp] cannot be applied directly but serve rather as motivation. In the case at hand one easily computes that $\mathcal{P}^* = Z^2$.

To complete our discussion of the geometry of M we relate the $\bar{\partial}_b$-operator to the $\overline{\partial}$–operator on Σ. Let κ denote the canonical bundle of Σ. A form of weight k is a section of $\kappa^{\otimes k}$, in local coordinates it is represented by $u = u^\alpha(z_\alpha, \bar{z}_\alpha)dz_\alpha^k$. A form of weight k defines a function $U \in F_k$, with local coordinate representation:

$$U(z_\alpha, w_\alpha) = u_a w_\alpha^k.$$

Moreover $\bar{\partial}_b U = 0$ if and only if $\overline{\partial} u = 0$. In this way we see that the $\ker \bar{\partial}_b \subset \mathcal{F}_0$ and the Fourier transform defines an isomorphism:

$$\ker \bar{\partial}_b = \bigoplus_{k=0}^{\infty} H^0(\Sigma; \kappa^k). \tag{4.6}$$

It is also useful to consider sections of $\overline{\kappa}^{-k}$; locally we have $v = v_\alpha d\bar{z}_\alpha^k$. In this case set $V \restriction_{U_\alpha \cap M} = v_\alpha e^{2ku^\alpha} w_\alpha^k$ to obtain a globally defined function on M. As before the equation $\partial_b V = 0$ is equivalent to $\partial v = 0$.

Finally we compute the $\overline{\partial}$–operator on $(1,0)$–vector fields defined in D. Let

$$p_{1,0} : TX \otimes \mathbb{C} \longrightarrow T^{1,0}X$$

denote the canonical projection defined by the complex structure. If V_α is a $(1,0)$–vector field defined in a subset of U_α then

$$\overline{\partial} V_\alpha = p_{1,0}[V_\alpha, \partial_{\bar{z}_\alpha}]d\bar{z}_\alpha + p_{1,0}[V_\alpha, \partial_{\bar{w}_\alpha}]d\bar{w}_\alpha. \tag{4.7}$$

§5. Deformations of the tangent bundle

In [La] the following theorem is proved:

Theorem [Laufer]. *Let X be a strictly pseudoconvex manifold with a one dimensional exceptional set A. Then there is a strictly pseudoconvex neighborhood of A in X and a deformation, $\omega : \mathcal{X} \to Q$ of $X = \omega^{-1}(0)$, with Q a manifold such that the Kodaira–Spencer map $\rho_0 : {}_Q T_0 \to H^1(X, \Theta)$ is an isomorphism.*

We apply this theorem to study the deformations of the disk bundle $D \subset T^{1,0}\Sigma$, modulo the equivalence relation (3.1). Using the Fourier transform we reduce the computation of $H^1(D; \Theta)$ to computations on Σ. This introduces a grading on $H^1(D; \Theta)$. Using the Bland–Duchamp extension and the Dolbeault isomorphism, we identify elements of $H^1(D;\Theta)$ with elements of $\mathrm{Def}(M,\bar{\partial}_b)\cap\ker\mathcal{P}^*$. Using the CR–representation we then give a description of the geometry of the different types of deformations.

Choose coordinates, $\{(z_\alpha,V_\alpha)\}$ for Σ so that $\mathfrak{V}=\{V_\alpha\}$ is a Leray cover then $\{\pi^{-1}(V_\alpha)\}$ is a Leray cover of $T^{1,0}\Sigma$. Their intersections with D define a Leray cover of the disk bundle, which we denote by $\mathfrak{U}=\{U_\alpha\}$ and therefore

$$H^1(D;\Theta)\simeq H^1(\mathfrak{U};\Theta). \tag{5.1}$$

On the sets U_α the coordinate vector fields $\{\partial_{z_\alpha},\partial_{w_\alpha}\}$ trivialize the holomorphic tangent bundle. Locally, holomorphic sections take the form: $a_\alpha(z_\alpha,w_\alpha)\partial_{z_\alpha}+b_\alpha(z_\alpha,w_\alpha)\partial_{w_\alpha}$ where a_α and b_α have Taylor expansions

$$a_\alpha=\sum_{j=0}^{\infty}a_{\alpha j}(z_\alpha)w_\alpha^j,\quad b_\alpha=\sum_{j=0}^{\infty}b_{\alpha j}(z_\alpha)w_\alpha^j. \tag{5.2}$$

An elementary computation shows that on the overlaps:

$$b_{\alpha 0}=f'_{\alpha\beta}\,b_{\beta 0},\quad \begin{pmatrix} a_{\alpha j}\\ b_{\alpha(j+1)}\end{pmatrix}=[f'_{\alpha\beta}]^{-j}\begin{pmatrix} f'_{\alpha\beta} & 0\\ \frac{f''_{\alpha\beta}}{f'_{\alpha\beta}} & 1\end{pmatrix}\begin{pmatrix} a_{\beta j}\\ b_{\beta(j+1)}\end{pmatrix},\ j\geq 0. \tag{5.3}$$

Let $\mathcal{V}$ denote the vector bundle defined on Σ by the 2×2–matrix in (5.3). From (5.2) and (5.3) it follows that

$$H^1(D;\Theta)\simeq H^1(\Sigma;\kappa^{-1})\bigoplus_{j=0}^{\infty}H^1(\Sigma;\mathcal{V}\otimes\kappa^j). \tag{5.4}$$

The groups appearing on the right hand side can easily be computed. By Serre duality

$$H^1(\Sigma;\mathcal{V}\otimes\kappa^j)\simeq(H^0(\Sigma;\mathcal{V}'\otimes\kappa^{1-j}))'. \tag{5.5}$$

Here $\mathcal{V}'$ is the vector bundle dual to $\mathcal{V}$; it fits into a short exact sequence:

$$0\longrightarrow\kappa\longrightarrow\mathcal{V}'\longrightarrow\mathcal{O}\longrightarrow 0. \tag{5.6}$$

Since κ^{1-j} is locally free we can tensor in (5.6) to obtain the exact sequences of vector bundles

$$0\longrightarrow\kappa^{2-j}\longrightarrow\mathcal{V}'\otimes\kappa^{1-j}\longrightarrow\kappa^{1-j}\longrightarrow 0. \tag{5.7}$$

Using (5.5), (5.7) and the long exact sequence in cohomology one easily shows that

$$H^1(\Sigma;\mathcal{V}\otimes\kappa^j)=0\text{ for }j\geq 3. \tag{5.8}$$

The computations for $j \in \{0,1,2\}$ follow from the exact sequences

$$\begin{aligned}
&j=0 \quad 0 \longrightarrow H^0(\Sigma;\kappa^2) \longrightarrow H^0(\Sigma;\mathcal{V}'\otimes\kappa) \longrightarrow H^0(\Sigma;\kappa) \longrightarrow 0,\\
&j=1 \quad 0 \longrightarrow H^0(\Sigma;\kappa) \longrightarrow H^0(\Sigma;\mathcal{V}') \longrightarrow H^0(\Sigma;\mathcal{O}) \longrightarrow H^1(\Sigma;\kappa)\ldots,\\
&j=2 \quad 0 \longrightarrow H^0(\Sigma;\mathcal{O}) \longrightarrow H^0(\Sigma;\mathcal{V}'\otimes\kappa^{-1}) \longrightarrow 0.
\end{aligned} \tag{5.9}$$

Only the case $j = 1$ requires further comment: a simple calculation shows that the generator of $H^0(\Sigma;\mathcal{O})$ is mapped by l to the 1–cocycle $\{\frac{f''_{\alpha\beta}}{f'_{\alpha\beta}}\}$ which is non–trivial in the one dimensional group $H^1(\Sigma;\kappa)$. Hence the sequence in this case can be replaced with

$$j=1 \quad 0 \longrightarrow H^0(\Sigma;\kappa) \longrightarrow H^0(\Sigma;\mathcal{V}') \longrightarrow 0. \tag{5.10}$$

Dualizing the sequences in (5.9) and (5.10) we obtain:

Proposition 5.1. *The S^1–action defines a grading of the cohomology group $H^1(D;\Theta)$ over $\{-1,0,1,\ldots\}$. Denote the summands by H^1_j. We have $H^1_j = 0$ for $j \geq 3$ and*

$$\begin{aligned}
&0 \longrightarrow H^1(\Sigma;\kappa^{-1}) \longrightarrow H^1_{-1} \longrightarrow 0,\\
&0 \longrightarrow H^1(\Sigma;\mathcal{O}) \longrightarrow H^1_0 \longrightarrow H^1(\Sigma;\kappa^{-1}) \longrightarrow 0,\\
&0 \longrightarrow H^1_1 \longrightarrow H^1(\Sigma;\mathcal{O}) \longrightarrow 0\\
&0 \longrightarrow H^1_2 \longrightarrow H^1(\Sigma;\kappa) \longrightarrow 0.
\end{aligned} \tag{5.11}$$

Remark. A cohomology group occurring to the left of H^1_j in (5.11) corresponds to vector fields in the subbundle of Θ spanned by $\{\partial_{w_\alpha}\}$ whereas a group to the right corresponds to a section of the quotient of Θ by this subbundle. In what follows it is useful to observe that the quotient bundle has a non–holomorphic representation as the subbundle of $T^{1,0}(T^{1,0}\Sigma)$ spanned by $\{Z'_\alpha = \partial_{z_\alpha} - 2u^\alpha_{z_\alpha} w_\alpha \partial_{w_\alpha}\}$. This facilitates finding the Dolbeault representatives for the cohomology group $H^1(D;\Theta)$. We now show how these correspond to first order deformations in $\ker\mathcal{P}^*$.

Laufer's theorem states that all the first order deformations in $H^1(D;\Theta)$ are unobstructed and therefore correspond to genuine deformations of the complex structure on D. The Bland–Duchamp extension shows that CR–structures lying in $\mathcal{F}_{-2}$ are extensible as integrable complex structures on D. Thus for deformations in $H^1(D;\Theta)$, corresponding to CR–structures in $\ker\mathcal{P}^*\cap\mathcal{F}_{-2}$, the connection between the first order data and the actual complex structures is quite clear. As we shall see, there are first order deformations which correspond to CR–deformations in F_{-3}. In these cases the representation of the deformed CR–structure requires higher order data.

If $\phi \in \mathrm{Def}(M, \bar{\partial}_b)$ then, as observed in §4, we can extend ϕ to D as a mermorphic function in w_α. If $\phi \in \mathcal{F}_{-2}$ the extended function is actually holomorphic and the vector valued $(0,1)$–forms

$$\omega_\alpha^\phi = e^{2u^\alpha} \sum_{j=-2}^{\infty} a_j w_\alpha^{j+2} (\partial_{z_\alpha} - 2u_{z_\alpha}^\alpha w_\alpha \partial_{w_\alpha}) d\bar{z}_\alpha \tag{5.12}$$

patch together to give a globally defined $T^{1,0}$–valued, $(0,1)$–form. We denote this form by ω^ϕ.

With the normalizations given in §4 the operator $\mathcal{P}^* = Z^2$ and therefore

$$\ker \mathcal{P}^* \subset \bigoplus_{j=-\infty}^{0} F_j. \tag{5.13}$$

The $\ker \mathcal{P}^*$ has the following simple description:

Proposition 5.2. *The* $\ker \mathcal{P}^* \cap \mathcal{C}^\infty(M)$ *is equal to* $\ker Z \oplus \bar{Z} \ker Z$ *and the* $\ker Z = \overline{\ker \bar{Z}}$.

Proof. Using the L^2–inner product we can decompose

$$\ker Z^2 = (\ker Z^2 \ominus \ker Z) \oplus \ker Z. \tag{5.14}$$

From (4.2) we conclude that range $\bar{Z}$ is the orthogonal complement to $\ker Z$; to establish the first claim in the proposition it is only necessary to verify that

$$Z^2 \bar{Z} u = 0 \text{ for } u \in \ker Z. \tag{5.15}$$

We decompose $u \in \ker Z$ into its Fourier components:

$$u = \sum_{j=-\infty}^{0} u_j.$$

Since $u_j \in \ker Z$ it follows from (4.3) that

$$Z \bar{Z} u_j = [Z, \bar{Z}] u_j = j u_j.$$

This implies that (5.15) holds for each of the Fourier components of u and consequently for u itself. The second statement in the proposition is obvious.

Now we can give the correspondence between the "non–negative" classes in $H^1(D; \Theta)$ and first order deformations of the CR–structure on M.

Theorem 5.1. *The equivalence classes of "non–negative" first order deformations of the complex structure on D are represented by first order deformations of the CR–structure on M via the following correspondences:*

$$\begin{aligned} H^1_0 &\leftrightarrow \ker Z^2 \cap F_{-2}, \\ H^1_1 &\leftrightarrow \ker Z^2 \cap F_{-1} = \ker Z \cap F_{-1}, \\ H^1_2 &\leftrightarrow \ker Z^2 \cap F_0 = \ker Z \cap F_0. \end{aligned} \tag{5.16}$$

The $T^{1,0}$–valued, $(0,1)$–forms defined in (5.12) *are Dolbeault representatives of the corresponding classes in $H^1(D;\Theta)$.*

Remark. The remaining case of $H^1_{-1} \leftrightarrow \bar{Z}(\ker Z \cap F_{-2})$ is discussed below.

Proof. Using Proposition 5.2 and (4.6) it is a simple matter to show that indicated pairs of vector spaces in (5.16) have the same dimensions and are therefore abstractly isomorphic. Because they depend holomorphically on w_α, the vector valued $(0,1)$–forms defined in (5.12) are $\overline{\partial}$–closed and hence define Dolbeault cohomology classes in $H^{0,1}(D;T^{1,0}D) \simeq H^1(D;\Theta)$. The isomorphism goes as follows: begin with a $\phi \in \mathrm{Def}(M,\bar{\partial}_b)$ and find local solutions to

$$\overline{\partial} V_\alpha = \omega^\phi_\alpha.$$

The 1–cocycle $\{V_\alpha - V_\beta\}$ represents the corresponding class in $H^1(D;\Theta)$. It is a simple matter to check that the vector fields $\{V_\alpha\}$ can be selected to respect the grading and hence define the maps in (5.16). The only point that remains is to show that the map from $\phi \longrightarrow [\omega^\phi]$ is injective.

We give the details of this argument for some representative cases, beginning with the easiest case, $H^1_2 \simeq H^1(\Sigma;\kappa) \simeq \mathbb{C}$. The $\ker Z \cap F_0$ is easily seen to consist of exactly the constant functions. The class defined by $\lambda \in \mathbb{C}$ is trivial if and only if we can find $V \in \mathcal{C}^\infty(D;T^{1,0}D)$ such that

$$\overline{\partial} V = \omega^\lambda. \tag{5.17}$$

We let $V \restriction_{U_\alpha} = a_\alpha Z'_\alpha + b_\alpha \partial_{w_\alpha}$.

Using formula (4.7) we see that (5.17) is equivalent to:

$$\begin{aligned} \partial_{\bar{w}_\alpha} a_\alpha &= \partial_{\bar{w}_\alpha} b_\alpha = 0, \\ \partial_{\bar{z}_\alpha} a_\alpha &= -\lambda e^{2u^\alpha} w^2_\alpha, \\ \partial_{\bar{z}_\alpha} b_\alpha &= 2 w_\alpha a_\alpha u^\alpha_{z_\alpha \bar{z}_\alpha}. \end{aligned} \tag{5.18}$$

The w_α dependence follows immediately from (5.18):

$$a_\alpha = w_\alpha^2 A_\alpha(z_\alpha), \quad b_\alpha = w_\alpha^3 B_\alpha(z_\alpha).$$

This leaves the equations on Σ :

$$\begin{aligned} \partial_{\bar z_\alpha} A_\alpha &= -\lambda e^{2u^\alpha}, \\ \partial_{\bar z_\alpha} B_\alpha &= 2A_\alpha u^\alpha_{z_\alpha \bar z_\alpha}. \end{aligned} \tag{5.19}$$

In order for V to be globally defined it is necessary that $\{A_\alpha dz_\alpha\}$ piece together to define a global smooth section of κ. We rewrite the equation for A as

$$\bar\partial(A_\alpha dz_\alpha) = -\lambda e^{2u^\alpha} dz_\alpha \otimes d\bar z_\alpha. \tag{5.20}$$

$$d(A_\alpha dz_\alpha) = -\lambda e^{2u^\alpha} dz_\alpha \wedge d\bar z_\alpha. \tag{5.21}$$

It follows from Stokes theorem that this equation is solvable if and only if

$$\int_\Sigma \lambda e^{2u^\alpha} dz_\alpha \wedge d\bar z_\alpha = 0,$$

i.e. if and only if $\lambda = 0$. This completes the case $j = 2$. The case $j = 1$ is quite similar and is left to the reader.

We give the argument for one further case: $\phi \in \bar Z \psi$ where $\psi \in \ker Z \cap F_{-1}$. In local coordinates $\phi = w_\alpha^{-2} c_\alpha$, $\psi = w_\alpha^{-1} d_\alpha$, the equations satisfied by ϕ and ψ are

$$\begin{aligned} \partial_{\bar z_\alpha} d_\alpha &= e^{2u^\alpha} c_\alpha, \\ \partial_{z_\alpha}(e^{2u^\alpha} d_\alpha) &= 0. \end{aligned} \tag{5.22}$$

We need to show that there exists no vector field V such that

$$\bar\partial V = \omega^\phi. \tag{5.23}$$

If $V = a_\alpha Z'_\alpha + b_\alpha \partial_{w_\alpha}$, in local coordinates then (5.23) is equivalent to

$$\begin{aligned} \partial_{\bar w_\alpha} a_\alpha &= \partial_{\bar w_\alpha} b_\alpha = 0, \\ \partial_{\bar z_\alpha} a_\alpha &= -e^{2u^\alpha} c_\alpha, \\ \partial_{\bar z_\alpha} b_\alpha &= \frac{1}{2} w_\alpha a_\alpha e^{2u^\alpha}. \end{aligned} \tag{5.24}$$

The collection $\{a_\alpha(z_\alpha)\partial_{z_\alpha}\}$ patch together to define a global vector field on Σ. From (5.22) it is clear that $a_\alpha = -d_\alpha$ is a global solution to the second equation in (5.24). The solution is unique as $\ker\overline{\partial} = 0$ on $\mathcal{C}^{\infty}(\Sigma; T^{1,0}\Sigma)$. Setting $b_{\alpha}(z_{\alpha}, w_{\alpha}) = B_{\alpha}(z_{\alpha})w_{\alpha}$, the last equation in (5.24) becomes:

$$\partial_{\bar{z}_{\alpha}} B_{\alpha} = -\frac{1}{2} d_{\alpha} e^{2u^{\alpha}}. \tag{5.25}$$

A simple calculation show that the $\{B_{\alpha}\}$ must patch together to define a function, B on Σ which satisfies

$$\bar{\partial} B = d_{\alpha} e^{2u^{\alpha}} d\bar{z}_{\alpha}.$$

This equation is not solvable as (5.22) implies that $\{d_{\alpha} e^{2u^{\alpha}} d\bar{z}_{\alpha}\}$ patch together to define a global section in $\ker\overline{\partial}^{*}$.

The remaining cases are left to the interested reader.

Before considering H^{1}_{-1} we give a brief geometric description of each of the types of deformations with non–negative Fourier coefficients. In [Ep3] it was shown that $\bar{Z} + \phi Z$ defines an embeddable deformation of $(M, \bar{\partial}_{b})$ provided $\phi \in \mathcal{F}_{-2}$. However the possibility remains that $\mathrm{Ind}(\bar{\partial}_{b}, \bar{\partial}^{\phi}_{b}) \neq 0$. From the Bland–Duchamp extension it is evident that the zero section of $T^{1,0}\Sigma$ persists under these deformations as a negatively embedded curve diffeomorphic to Σ. Let D^{ϕ} denote the unit disk bundle with the complex structure defined by the Bland–Duchamp extension.

We begin with $\phi \in \ker\mathcal{P}^{*} \cap F_{-2}$. These structures are S^{1}–invariant and so can be embedded in the total space of a line bundle, L_{ϕ} of degree $2g-2$ over a Riemann surface, Σ_{ϕ} of genus g. If $\phi \in \ker Z \cap F_{-2} \simeq H^{0}(\Sigma; \kappa^{2})$ then the CR–structure defined by ϕ can be realized as a hypersurface in the holomorphic tangent bundle of a Riemann surface. An easy calculation shows that for such ϕ,

$$\mathrm{Ind}(\bar{\partial}_{b}, \bar{\partial}^{\phi}_{b}) = 0.$$

For small enough deformations this implies that $\dim H^{0}(\Sigma_{\phi}; L^{-1}_{\phi}) = \dim H^{0}(\Sigma; \kappa) = g$. The only line bundles of the given degree which have a g–dimensional space of holomorphic sections are the canonical bundles. Thus L^{-1}_{ϕ} is the canonical bundle of Σ_{ϕ}. Of course the holomorphic quadratic differentials are isomorphic to the tangent space of the Teichmüller space of the Riemann surface Σ. Thus $\ker Z \cap F_{-2}$ provides a CR–representation for the local moduli of the deformations of Σ.

The other deformations in H^{1}_{0} lie in $\bar{Z}(\ker Z \cap F_{-1}) \simeq H^{0}(\Sigma; \kappa)$. This group is classically identified with the holomorphic moduli for line bundles of a fixed degree over a Riemann surface. It is easy to show that for such $\phi \neq 0$ the relative index satisfies

$$\mathrm{Ind}(\bar{\partial}_b, \bar{\partial}_b^{\phi}) = -1.$$

Thus the line bundles obtained are not the tangent bundle as the $\dim(\Sigma_\phi ; L_\phi^{-1}) = g - 1$. It follows from Theorem 5.1 that all small deformations of the tangent bundle are found in this family. The complex structure on Σ_ϕ may also vary within this family though it varies $O(\phi^2)$ for small ϕ.

For the structures in $\ker Z \cap F_{-1}$ the complex structure on Σ as well as its normal bundle $[\Sigma] \restriction_\Sigma$ are unchanged. However D^ϕ is not a domain in the total space of a line bundle. This is proved as follows: one finds that the "first order" invariant defined in §2 of [MoRo] is non–zero. This is the obstruction to splitting the normal bundle sequence along Σ :

$$0 \longrightarrow \Theta_\Sigma \longrightarrow \Theta_{D^\phi} \longrightarrow N_\Sigma^\phi \longrightarrow 0.$$

Thus $\Sigma \hookrightarrow D^\phi$ is a holomorphic curve which is not the zero section of a line bundle. However if D_ϕ were a domain in a holomorphic line bundle then Σ would be homologous to the zero section. But this is impossible because Σ is negatively embedded and is therefore the unique holomorphic curve in its homology class. The structures in $\ker Z \cap F_0$ again cannot be embedded in the total space of a line bundle. In this case one computes that the "second order" obstruction defined in §3 of [MoRo] is non–vanishing and therefore the embedding of $\Sigma \hookrightarrow D^\phi$ is not equivalent to the embedding of Σ as the zero section in N_Σ^ϕ. The relative index in these cases satisfies:

$$\mathrm{Ind}(\bar{\partial}_b, \bar{\partial}_b^{\phi}) = 0.$$

We close with a discussion of $\phi \in \mathcal{F}_1$. Of course $\mathcal{F}_1 \subset \mathrm{range}\, \mathcal{P}$ so these are, to first order, wiggles of M within $T^{1,0}\Sigma$. Indeed it follows from the theorem of Grauert on lifting formal equivalences, [Gr] that there is a neighborhood of Σ in D^ϕ which is biholomorphic to a neighborhood of the zero section in $T^{1,0}\Sigma$. We conjecture that, for a reasonable notion of smallness, the small deformations in $\mathcal{F}_1$ can be realized by wiggling the hypersurface M within $T^{1,0}\Sigma$.

Finally we consider deformations in H^1_{-1}. Apparently the first order part should correspond to a CR–deformation in $\ker \mathcal{P}^* \cap F_{-3}$. The complex structures on D corresponding to deformations in H^1_{-1} are affine bundles which have no compact subvarieties. If the coefficients in the Bland–Duchamp extension were smooth on D then the zero section would persist as a holomorphic curve. This extension for structures in F_{-3} has a first order pole along the zero section of $T^{1,0}\Sigma$. In the present case, the singularity appears to be more in the nature of a "coordinate singularity" as opposed to an unfillable complex manifold.

Let $\phi \in \bar{Z}(\ker Z \cap F_{-2})$ and let ω^ϕ denote the vector valued $(0,1)$-form defined on $D \setminus \Sigma$ by (5.12). Clearly we can think of ω^ϕ as a first order deformation of the complex structure on the deleted space, $D \setminus \Sigma$. We show below that there is a Čech representative $\{\xi_{\alpha\beta}\} \in H^1(D \setminus \Sigma; \Theta)$ which extends to define a class in $H^1(D; \Theta)$. This provides an identification of the first order deformations of D in H^1_{-1} with these deformations of the CR–structure on M. The relationship in this case is not as transparent as in the previous cases, in part because the first order deformation of the CR–structure is not itself embeddable but requires higher order correction terms.

Let $\psi \in \ker Z \cap F_{-2}$ and set $\phi = \bar{Z}\psi$. In local coordinates $\phi = w_\alpha^{-3} c_\alpha$, $\psi = w_\alpha^{-2} d_\alpha$. To find the class in $H^1(D; \Theta)$ corresponding to ϕ we need to solve a system of equations,

$$\begin{aligned} \partial_{\bar{w}_\alpha} a_\alpha &= \partial_{\bar{w}_\alpha} b_\alpha = 0, \\ \partial_{\bar{z}_\alpha} a_\alpha &= -w_\alpha^{-1} e^{2u^\alpha} c_\alpha, \\ \partial_{\bar{z}_\alpha} b_\alpha &= \frac{1}{2} w_\alpha a_\alpha e^{2u^\alpha}. \end{aligned} \tag{5.26}$$

Let $a_\alpha = w_\alpha^{-1} A_\alpha$. As before the fact that $\phi = \bar{Z}\psi$ implies that $A_\alpha = -d_\alpha$ piece together to give a global solution to the second equation in (5.26). This leaves only the equation:

$$\partial_{\bar{z}_\alpha} b_\alpha = -\frac{1}{2} d_\alpha e^{2u^\alpha}. \tag{5.27}$$

This can again be interpreted as a $\overline{\partial}$–equation on Σ which cannot be solved because the right hand side belongs to $\ker \overline{\partial}^*$. The vector fields $\{w_\alpha^{-1} A_\alpha Z'_\alpha\}$ piece together to define a global vector field on $D \setminus \Sigma$. The Čech representative of ω^ϕ is therefore

$$\{b_\alpha \partial_{w_\alpha} - b_\beta \partial_{w_\beta}\}.$$

These evidently extend to define a holomorphic 1–cocycle on D which represents a class in H^1_{-1}. Among the first order deformations of $(M, \bar{\partial}_b)$ in $\ker \mathcal{P}^* \cap \oplus_{\{k<-2\}} F_k$ only those in $\bar{Z}(\ker Z \cap F_{-2})$ correspond to classes which extend to D.

The elements in H^1_{-1} have a simple geometric description as the affine bundles over Σ with $T^{1,0}\Sigma$ as their underlying vector bundle. If

$\{\xi_{\alpha\beta}\}$ represents a class in $H^1(\Sigma;\kappa^{-1})$ then we can define coordinates $\{U_\alpha,(z_\alpha,v_\alpha)\}$ with transition functions:

$$z_\alpha = f_{\alpha\beta}(z_\beta), \quad v_\alpha = f'_{\alpha\beta}(z_\beta)v_\beta + \xi_{\alpha\beta}. \tag{5.28}$$

The cocycle condition for $H^1(\Sigma;\kappa^{-1})$ implies that these relations are consistent. We denote the total space of the affine bundle by A_ξ. It is a Stein manifold which can be compactified by adding a curve "at infinity," denote the compactified space by $\widehat{A}_\xi$. Let N be a small neighborhood of $0 \in H^1(\Sigma;\kappa^{-1})$ and set:

$$\mathcal{A} = \bigcup_{\xi\in N} (\widehat{A}_\xi, \xi). \tag{5.29}$$

It is easy to see that $\mathcal{A}$ has a natural structure as a complex manifold and for sufficiently small N, $\pi : \mathcal{A} \to N$ is a deformation space of $\pi^{-1}(0) = \widehat{T}\Sigma$. Perhaps shrinking N further, the real analytic hypersurface $M \hookrightarrow \pi^{-1}(0)$ can be extended to a real analytic hypersurface $\mathcal{M} \hookrightarrow \mathcal{A}$, intersecting the fibers of π transversely.

Let $M_\xi = \pi^{-1}(\xi) \cap \mathcal{M}$ with the induced CR–structure denoted by $\bar\partial_b^\xi$. For $\xi \neq 0$ the hypersurface M_ξ bounds a domain, $D_\xi \subset A_\xi$. As A_ξ is a Stein manifold it follows from Hamilton's stability theorem that for $\xi \neq 0$ there is an open neighborhood $U_\xi \subset N$ such that for $\xi' \in U_\xi$ the complex manifold $D_{\xi'}$ can be realized as a small perturbation of D_ξ within A_ξ, see [Ha]. We can therefore apply Theorem 3.4 to conclude that

$$\operatorname{Ind}(\bar\partial_b^\xi, \bar\partial_b^{\xi'}) = 0 \text{ for } \xi' \in U_\xi. \tag{5.30}$$

Since $N\setminus\{0\}$ is connected we can apply Proposition 3.2 to conclude that (5.30) holds for any pair $\xi,\xi' \in N\setminus\{0\}$. In a forthcoming paper with Donagi a detailed analysis of the structure of the algebra of holomorphic functions on affine bundles will be given. A computation in that paper shows that if the genus of Σ is g then

$$\operatorname{Ind}(\bar\partial_b^0, \bar\partial_b^\xi) = -(g+1), \text{ for } \xi \in N\setminus\{0\}. \tag{5.31}$$

This is a larger value than attained for any non–negative deformation.

In [BlEp] the analogous deformations are studied for line bundles over $\mathbb{P}^1$. In this case the linear part of the CR–representative is again of the form $\bar Z g$ for a homogeneous function $g \in \ker Z$. The corresponding embeddable deformation is simply $\bar Z g - g^2$. We conjecture that, in the present case, the embeddable deformation with first order part $\bar Z\psi$ is $\bar Z\psi + \frac{1}{2}\psi^2$.

§6. Problems in deformation theory

In this section we propose six problems in deformation theory which are related to the problem of characterizing the space of embeddable CR-structures.

Problem 1. Let $(M, \bar{\partial}_b)$ be a compact 3–dimensional, strictly pseudoconvex embeddable CR–manifold. Is there a generalization of Bland's construction of a slice for the action of Cont_H which has the structure of a complex fiber bundle over a real manifold, see [Bl]?

Problem 2. Suppose V is an affine variety and M_1, $M_2 \hookrightarrow V$ are two strictly pseudoconvex hypersurfaces in $V \setminus \mathrm{sing}(V)$ which bound compact domains and are smoothly isotopic in $V \setminus \mathrm{sing}(V)$. Are M_1 and M_2 in the same strictly pseudoconvex isotopy class, see Theorem 3.4 ?

Problem 3. If $(M, \bar{\partial}_b)$ is an embeddable, 3–dimensional, CR–manifold are there integers k and N and a positive ϵ so that

$$\mathfrak{S}_n \cap \{\phi : \|\phi\|_{C^k} < \epsilon\} = \mathfrak{S}_N \cap \{\phi : \|\phi\|_{C^k} < \epsilon\} \text{ for } n \geq N?$$

Problem 4. Is there an effective way to translate between the real and holomorphic descriptions of the deformations of complex (or CR) structures?

Problem 5. Suppose X is a smooth complex surface with a compact, maximal, exceptional analytic subvariety, A. Let $\mathcal{I}$ denote the ideal of A and $\mathcal{J} \subset \mathcal{I}$, a sub–ideal for which there exists a neighborhood $U_{\mathcal{J}}$ of A and proper embedding $F_{\mathcal{J}} : U_{\mathcal{J}} \setminus A \to \mathbb{C}^n \setminus \{0\}$ where the coordinate functions of $F_{\mathcal{J}}$ belong to $\mathcal{J}$. We can of course extend $F_{\mathcal{J}}$ to A by 0 and this defines the germ of a singularity $(F_{\mathcal{J}}(U_{\mathcal{J}}), 0) \subset (\mathbb{C}^n, 0)$. If $\mathcal{J}_2 \subset \mathcal{J}_1$ are two such sub–ideals of $\mathcal{I}$ then a deformation of $(F_{\mathcal{J}_1}(U_1), 0)$ induces a deformation of $(F_{\mathcal{J}_2}(U_2), 0)$. Is there a "universal" sub–ideal $\mathfrak{U} \subset \mathcal{I}$ such that for every ideal, $\mathcal{J}$ as above the versal deformation space of $(F_{\mathcal{J}}(U_{\mathcal{J}}), 0)$ can be realized as a subspace of the versal deformation of $(F_{\mathfrak{U}}(U_{\mathfrak{U}}), 0)$?

Problem 6. In [MoRo] the formal neighborhoods of a complex curve, Σ with fixed positive, normal bundle, N are considered. It is shown that there is an infinite dimensional space of inequivalent neighborhoods. Is the subspace of formal neighborhoods which can be realized by an embedding in a compact projective variety finite dimensional?

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