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Quantum Matroids

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Abstract.

We define a *quantum matroid* to be any finite nonempty poset P satisfying the conditions R, SL, M, AU below.

R: P is ranked.
SL: P is a (meet) semilattice.
M: For all $x \in P$, the interval $[0, x]$ is a modular atomic lattice.
AU: For all $x, y \in P$ satisfying $\text{rank}(x) < \text{rank}(y)$, there exists an atom $a \in P$ such that $a \leq y$, $a \not\leq x$, and such that $x \vee a$ exists in P.

Condition AU is the *augmentation axiom.*

We develop a theory of quantum matroids. Although we deal at length with the general case, our emphasis is on quantum matroids P with the following extra structure: We say P is *nontrivial* if P has rank $D \geq 2$, and P is not a modular atomic lattice. In what follows suppose P is nontrivial. We say P is *q-line regular* whenever each rank 2 element in P covers exactly $q+1$ elements of P. We say P is *β-dual-line regular* whenever each element in P with rank $D-1$ is covered by exactly $\beta+1$ elements of P. We say P is *α-zig-zag regular* whenever for all pairs $x, y \in P$ such that $\text{rank}(x) = D-1$, $\text{rank}(y) = D$, and such that x covers $x \wedge y$, there exists exactly $\alpha+1$ pairs $x', y' \in P$ such that y' covers x, y' covers x', and such that y covers x'. We say P is *regular* whenever P is line regular, dual-line regular, and zig-zag regular. We prove the following theorem.

Theorem. *Let D denote an integer at least 4. Then a poset P is a nontrivial regular quantum matroid of rank D if and only if P is isomorphic to one of the following:*

(i) *A truncated Boolean algebra $B(D, N)$, $(D < N)$.*
(ii) *A Hamming matroid $H(D, N)$, $(2 \leq N)$.*

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(iii) *A truncated projective geometry* $L_q(D,N)$, $(D<N)$.
(iv) *An attenuated space* $A_q(D,N)$, $(D<N)$.
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§1. The definition of a $\mathcal{P}$-matroid

In this paper, we only consider finite structures.

We begin by recalling the classical notion of a (finite) matroid.

Definition 1.1. Let A denote a finite set. By an *A-matroid*, we mean a collection P of distinct subsets of A, that satisfies the following axioms NT, LI, AU:

NT: $P \neq \emptyset$.

LI: For all subsets $x, y \subseteq A$, $x \in P$ and $y \subseteq x$ implies $y \in P$.

AU: For all $x, y \in P$ such that $|x| < |y|$, there exists an element $a \in A$ such that $a \in y$, $a \notin x$, and such that $x \cup a \in P$.

AU is referred to as the *augmentation axiom.*

Here is the standard example of an A-matroid: let A denote a finite set of vectors taken from a fixed vector space, and let P denote the collection of all linearly independent subsets of A. Then P is an A-matroid.

Two more examples follow, which we refer to later in the paper.

Example 1.2. (i) The truncated Boolean algebra $B(D, N)$ $(0 \leq D \leq N)$: Let A denote a set of cardinality N, and set

$$P := \{ x \subseteq A \mid \ |x| \leq D \}.$$

Then P is an A-matroid.

(ii) $C(N_1, N_2, \ldots, N_D)$ $(0 < D,\ N_1,\ N_2,\ \ldots,\ N_D)$: Set

$$A := A_1 \cup A_2 \cup \cdots \cup A_D \qquad \text{(disjoint union)},$$

where

$$|A_i| = N_i \qquad\qquad (1 \le i \le D).$$

Set

$$P := \{\, x \subseteq A \mid \ \ |x \cap A_i| \le 1 \ \text{ for all } \ i \ (1 \le i \le D)\}.$$

Then P is an A-matroid.

Proof. Routine.

Note 1.3. In Example 1.2(ii), the case $N_i = N \ \ (1 \le i \le D)$ will turn out to have special significance. We refer to this matroid as the *Hamming matroid* $H(D, N)$.

We now generalize Definition 1.1, replacing subsets of a set A with subspaces of a vector space V.

Definition 1.4. Let V denote a finite vector space. By a *V-matroid*, we mean a collection P of distinct subspaces of V, that satisfies the following axioms NT, LI, AU:

NT: $P \neq \emptyset$.
LI: For all subspaces $x, y \subseteq V$, $x \in P$ and $y \subseteq x$ implies $y \in P$.
AU: For all $x, y \in P$ such that $\dim(x) < \dim(y)$, there exists a one dimensional subspace $a \subseteq V$ such that $a \subseteq y$, $a \not\subseteq x$, and such that $x + a \in P$.

We now consider some examples of V-matroids.

Example 1.5. The truncated projective geometry $L_q(D, N)$ $(0 \le D \le N)$: Let V denote a vector space of dimension N over the finite field $GF(q)$, and set

$$P := \{\, x \mid x \ \text{ is a subspace of } \ V, \quad \dim(x) \le D\}.$$

Then P is a V-matroid.

Proof. Routine.

Example 1.6. The attenuated space $A_q(D, N)$ $(0 \le D \le N)$: Let V denote an N-dimensional vector space over the finite field $GF(q)$. Fix a subspace $w \subseteq V$ such that $\dim(w) = N - D$, and set

$$P \ := \ \{\, x \mid x \ \text{ is a subspace of } \ V, \quad x \cap w = 0\}.$$

Then P is a V-matroid.

Proof. We check NT, LI, AU in Definition 1.4. NT holds since $0 \in P$, and LI holds trivially, so consider AU. Recall that for all subspaces $u,\, v \subseteq V$,

$$\dim(u) + \dim(v) = \dim(u \cap v) + \dim(u+v). \tag{1.1}$$

Pick any $x,\, y \in P$ such that $\dim(x) < \dim(y)$. We find a one dimensional subspace $a \subseteq V$ such that $a \subseteq y$, $a \not\subseteq x$, and such that $x + a \in P$. Observe

$$\begin{aligned} \dim(x+w) &= \dim(x) + \dim(w) \\ &< \dim(y) + \dim(w) \\ &= \dim(y+w), \end{aligned}$$

so $y + w \not\subseteq x + w$. In particular,

$$y \not\subseteq x + w. \tag{1.2}$$

By (1.2), there exists a one dimensional subspace $a \subseteq y$ such that $a \not\subseteq x + w$. Observe $a \not\subseteq x$, so it remains to show $x + a \in P$. Observe by (1.1) and the construction that

$$\begin{aligned} \dim\bigl((x+a) \cap w\bigr) &= \dim(x+a) + \dim(w) - \dim(x+a+w) \\ &= \dim(x) + 1 + \dim(w) - \dim(x+w) - 1 \\ &= 0, \end{aligned}$$

and we conclude $x + a \in P$, as desired.

Example 1.7. Polar spaces over $GF(q)$: Let V denote a finite dimensional vector space over the finite field $GF(q)$. Endow V with a form $\langle\, ,\, \rangle : V \times V \to GF(q)$ such that

$$\begin{aligned} \langle u+v,\, w\rangle &= \langle u,\, w\rangle + \langle v,\, w\rangle && (\forall u, v, w \in V), \\ \langle u,\, v+w\rangle &= \langle u,\, v\rangle + \langle u,\, w\rangle && (\forall u, v, w \in V), \\ \langle \alpha u,\, v\rangle &= \alpha\langle u,\, v\rangle && (\forall u, v \in V, \quad \forall \alpha \in GF(q)). \end{aligned}$$

Further assume

$$\langle u,\, v\rangle = \langle v,\, u\rangle \qquad (\forall u, v \in V) \quad \text{(the symmetric bilinear case)},$$

or

$$\langle u,\, u\rangle = 0 \qquad (\forall u \in V) \quad \text{(the alternating bilinear case)},$$

or

$$\langle u,\, v\rangle = \overline{\langle v,\, u\rangle} \qquad (\forall u, v \in V) \quad \text{(the Hermitean case)}.$$

(In the last case $^-$ denotes a field automorphism of $GF(q)$ of order 2.) A subspace $x \subseteq V$ is said to be *totally isotropic* (abbreviated *t.i.*) whenever $\langle u, v\rangle = 0$ for all $u, v \in x$. The set

$$P := \{x \mid x \text{ is a t.i. subspace of } V\} \tag{1.3}$$

is a V-matroid.

Proof. We verify NT, LI, AU in Definition 1.4. NT holds since $0 \in P$. LI is trivial, so consider AU. For all subspaces $x \subseteq V$, let $x^\perp$ denote the orthogonal complement

$$x^\perp := \{u \in V \mid \langle u, v\rangle = 0 \text{ for all } v \in x\}. \tag{1.4}$$

Observe x is t.i. if and only if $x \subseteq x^\perp$. By elementary linear algebra,

$$\dim(x^\perp) + \dim(x) = \dim(x \cap V^\perp) + \dim(V). \tag{1.5}$$

Pick any t.i. subspaces $x, y \subseteq V$ such that $\dim(x) < \dim(y)$. We find a one dimensional subspace $a \subseteq V$ such that $a \subseteq y$, $a \not\subseteq x$, and such that $x + a$ is t.i. To obtain a, we claim

$$x^\perp \cap y \not\subseteq x. \tag{1.6}$$

Suppose (1.6) fails. Then by the construction,

$$x^\perp \cap y = x \cap y. \tag{1.7}$$

Observe $x^\perp$, y are both contained in $(x \cap y)^\perp$, so

$$x^\perp + y \subseteq (x \cap y)^\perp. \tag{1.8}$$

Now by (1.5), (1.7), (1.8),

$$\begin{aligned}
\dim(x \cap V^\perp) + \dim(V) &= \dim(x^\perp) + \dim(x) \\
&< \dim(x^\perp) + \dim(y) \\
&= \dim(x^\perp + y) + \dim(x^\perp \cap y) \\
&\leq \dim\big((x \cap y)^\perp\big) + \dim(x \cap y) \\
&= \dim(x \cap y \cap V^\perp) + \dim(V),
\end{aligned}$$

an impossibility. Hence (1.6) holds. By (1.6), there exists a one dimensional subspace $a \subseteq x^\perp \cap y$ such that $a \not\subseteq x$. Observe $a \subseteq y$, and y is t.i., so a is t.i. Also x is t.i. and $a \subseteq x^\perp$, so $x + a$ is t.i. Now a

has the desired properties, so AU holds. Now P is a V-matroid, and we are done.

Example 1.8. More polar spaces over $GF(q)$: Let V denote a finite dimensional vector space over the finite field $GF(q)$. Endow V with a quadratic form, i.e., a function $f : V \to GF(q)$ satisfying

$$f(\alpha v) = \alpha^2 f(v) \qquad\qquad (\forall \alpha \in GF(q),\ \forall v \in V),$$

$$f(u+v) = f(u) + f(v) + \langle u, v\rangle \qquad\qquad (\forall u, v \in V),$$

where $\langle\,,\,\rangle = \langle\,,\,\rangle_f$ is a symmetric bilinear form from Example 1.7. A subspace $x \subseteq V$ is said to be *totally singular* (abbreviated *t.s.*) whenever $f(v) = 0$ for all $v \in x$. The set

$$P := \{x \mid x \text{ is a t.s. subspace of } V\}$$

is a V-matroid.

Proof. We verify NT, LI, AU in Definition 1.4. NT, LI hold as in Example 1.7, so consider AU. First observe that any t.s. subspace $z \subseteq V$ is t.i. (with respect to $\langle\,,\,\rangle$, in the sense of Example 1.7). Now pick any t.s. subspaces $x, y \subseteq V$ such that $\dim(x) < \dim(y)$. We find a one dimensional subspace $a \subseteq V$ such that $a \subseteq y$, $a \not\subseteq x$, and such that $x + a$ is t.s. By Example 1.7 and our preliminary comment, there exists a one dimensional subspace $a \subseteq V$ such that $a \subseteq y$, $a \not\subseteq x$, and such that $x + a$ is t.i. In fact $x + a$ is t.s. To see this, we pick any $v \in x + a$ and show $f(v) = 0$. Observe $v = v_1 + v_2$ for some $v_1 \in x$ and some $v_2 \in a$. Observe $f(v_1) = 0$, since x is t.s. Observe $f(v_2) = 0$, since $a \subseteq y$, and since y is t.s. Observe $\langle v_1, v_2\rangle = 0$, since $x + a$ is t.i. Now

$$\begin{aligned} f(v) &= f(v_1 + v_2) \\ &= f(v_1) + f(v_2) + \langle v_1, v_2\rangle \\ &= 0, \end{aligned}$$

as desired. Now a has the desired properties, so AU holds. Now P is a V-matroid, and we are done.

Note 1.9. In the nondegenerate case (Definitions 26.6, 26.8), the examples in 1.7, 1.8 are often referred to as *classical polar spaces*. This distinguishes them from the closely related *Tits polar spaces*, which we will encounter in Section 30. See [Ar], [Ca2], [Mu], [Ti] for information on the classical polar spaces.

We now seek a common generalization of Definitions 1.1, 1.4. We will use the language of partially ordered sets, so first we review some basic concepts from this area.

Let P denote a finite set. By a *partial order* on P, we mean a binary relation $\leq$ on P such that

$$
\begin{array}{llll}
\text{(i)} & x \leq x & & (\forall x \in P), \\
\text{(ii)} & x \leq y \text{ and } y \leq z & \rightarrow \quad x \leq z & (\forall\, x,\, y,\, z \in P), \\
\text{(iii)} & x \leq y \text{ and } y \leq x & \rightarrow \quad x = y & (\forall\, x,\, y \in P).
\end{array}
$$

By a *partially ordered set* (or *poset*, for short), we mean a pair $(P, \leq)$, where P is a finite set, and where $\leq$ is a partial order on P. Abusing notation, we will suppress reference to $\leq$, and just write P instead of $(P, \leq)$.

Let P, Q denote any posets. A map $\phi : P \rightarrow Q$ is said to be an *isomorphism of posets* whenever ϕ is a bijection, and for all $x, y \in P$,

$$x \leq y \quad (\text{in } P) \qquad \leftrightarrow \qquad \phi(x) \leq \phi(y) \qquad (\text{in } Q).$$

P and Q are said to be *isomorphic* whenever there exists an isomorphism of posets $\phi : P \rightarrow Q$. We do not distinguish between isomorphic posets.

Let P denote a poset, with partial order $\leq$, and let x and y denote any elements in P. As usual, we write $x < y$ whenever $x \leq y$ and $x \neq y$. We say y *covers* x whenever $x < y$, and there is no $z \in P$ such that $x < z < y$. An element $x \in P$ is said to be *maximal* (resp. *minimal*) whenever there is no $y \in P$ such that $x < y$ (resp. $y < x$). Let $\max(P)$ (resp. $\min(P)$) denote the set of all maximal (resp. minimal) elements in P. Whenever $\max(P)$ (resp. $\min(P)$) consists of a single element, we denote that element by 1 (resp. 0), and we say P *has a* 1 (resp. P *has a* 0).

Suppose P has a 0. By an *atom* (or *point*) in P, we mean an element in P that covers 0. We let A_P denote the set of atoms in P.

Suppose P has a 0. By a *rank function* on P, we mean a function $\text{rank} : P \rightarrow \mathbb{Z}$ such that $\text{rank}(0) = 0$, and such that for all $x, y \in P$,

$$(1.9) \qquad y \text{ covers } x \qquad \rightarrow \qquad \text{rank}(y) - \text{rank}(x) = 1.$$

Observe the rank function is unique if it exists. P is said to be *ranked* whenever P has a rank function. In this case, we set

$$(1.10) \qquad \text{rank}(P) := \max\{\, \text{rank}(x) \mid x \in P\},$$

$$(1.11) \qquad P_i := \{x \in P \mid \text{rank}(x) = i\} \qquad (i \in \mathbb{Z}),$$

and observe $P_0 = \{0\}$, $P_1 = A_P$. We refer to the elements of P_2 as the *lines* of P. For notational convenience set

$$\text{top}(P) := P_D, \tag{1.12}$$

where $D = \text{rank}(P)$. Observe

$$\text{top}(P) \subseteq \max(P), \tag{1.13}$$

but we might not have equality in (1.13).

Let P denote any poset, and let S denote any subset of P. Then there is a unique partial order on S such that for all $x, y \in S$,

$$x \leq y \quad (\text{in } S) \qquad \leftrightarrow \qquad x \leq y \quad (\text{in } P). \tag{1.14}$$

This partial order is said to be *induced* from P. By a *subposet* of P, we mean a subset of P, together with the partial order induced from P. Pick any $x, y \in P$ such that $x \leq y$. By the *interval* $[x, y]$, we mean the subposet

$$[x, y] := \{z \in P \mid x \leq z \leq y\}$$

of P.

Let P denote any poset, and pick any $x, y \in P$. By a *lower bound* for x, y, we mean an element $z \in P$ such that $z \leq x$ and $z \leq y$. Suppose the subposet of lower bounds for x, y has a unique maximal element. In this case we denote this maximal element by $x \wedge y$, and say $x \wedge y$ *exists*. The element $x \wedge y$ is known as the *meet* of x and y. P is said to be a (meet) *semilattice* whenever P is nonempty, and $x \wedge y$ exists for all $x, y \in P$. A semilattice has a 0. Suppose P is a semilattice, and pick any $x, y \in P$. By an *upper bound* for x, y, we mean an element $z \in P$ such that $z \geq x$ and $z \geq y$. Observe the subposet of upper bounds for x, y is closed under $\wedge$; in particular, it has a unique minimal element if and only if it is nonempty. In this case we denote this minimal element by $x \vee y$, and say $x \vee y$ *exists*. The element $x \vee y$ is known as the *join* of x and y. By a *lattice*, we mean a semilattice P such that $x \vee y$ exists for all $x, y \in P$. A lattice has a 1.

Suppose P is a semilattice. Then every interval in P is a lattice.

Suppose P is a semilattice. Then P is said to be *atomic* whenever each element of P is a join of atoms. A semilattice P is atomic if and only if each element of P that is not 0 and not an atom covers at least 2 elements of P.

Suppose P is a lattice. Then P is said to be *modular* whenever for all $x, y \in P$,

$$x, y \text{ cover } x \wedge y \qquad \leftrightarrow \qquad x \vee y \text{ covers } x, y. \tag{1.15}$$

P is modular if and only if P is ranked, and for all $x, y \in P$,

$$\text{rank}(x) + \text{rank}(y) = \text{rank}(x \wedge y) + \text{rank}(x \vee y) \tag{1.16}$$

[St, p104]. Suppose P is a modular atomic lattice. Then any interval in P is a modular atomic lattice.

We mention two examples of modular atomic lattices. (A full classification is given in Theorems 1.12, 1.13.)

Example 1.10. Let A denote a finite set. The *Boolean algebra* B_A is the poset of all subsets of A, ordered by inclusion. B_A is a modular atomic lattice. Moreover, for all x, $y \in B_A$,

$$x \wedge y = x \cap y, \tag{1.17}$$

$$x \vee y = x \cup y, \tag{1.18}$$

$$\text{rank}(x) = |x|. \tag{1.19}$$

We often write $B(D)$ to denote B_A, where $D = |A|$.

Proof. Routine.

Example 1.11. Let V denote a finite vector space. The *projective geometry* L_V is the poset of all subspaces of V, ordered by inclusion. L_V is a modular atomic lattice. Moreover, for all x, $y \in L_V$,

$$x \wedge y = x \cap y, \tag{1.20}$$

$$x \vee y = x + y, \tag{1.21}$$

$$\text{rank}(x) = \dim(x). \tag{1.22}$$

We often write $L_q(D)$ to denote L_V, where V is over the field $GF(q)$ and where $D = \dim(V)$.

Proof. Routine.

There is a classification of all modular atomic lattices essentially due to Veblen and Young, which we present below without proof.

Let q denote an integer at least 2. By a *projective plane* of order q, we mean a ranked lattice P of rank 3 such that each line in P covers exactly $q+1$ points in P, each point in P is covered by exactly $q+1$ lines in P, $a \vee b$ is a line for any distinct points $a, b \in P$, and $x \wedge y$ is a point for any distinct lines $x, y \in P$.

Let P denote a ranked poset with 0. A line $x \in P$ is said to be *thick* whenever x covers at least three points in P.

In the following theorem, we consider the modular atomic lattices that have all lines thick. In Theorem 1.13, we consider the general case.

Theorem 1.12 ([Ca2, Theorems 3.3.1, 3.4.1], [V-Y]). *For each nonnegative integer D, let Ω_D denote the class of all modular atomic lattices that have rank D and have all lines thick. Then for any poset P,*

$P \in \Omega_0$ if and only if $P = \{0\}$.
$P \in \Omega_1$ if and only if $P = \{0,1\}$.
$P \in \Omega_2$ if and only if P is a ranked lattice with rank 2 and P has at least 3 points.
$P \in \Omega_3$ if and only if P is a projective plane of order q for some integer $q \geq 2$.
For $D \geq 4$, $P \in \Omega_D$ if and only if P is isomorphic to $L_q(D)$ for some (prime power) integer $q \geq 2$.

Let P, Q denote any posets. By the *Cartesian product* $P \times Q$, we mean the poset on the set

$$P \times Q := \{xy \mid x \in P, \ \ y \in Q\}, \tag{1.23}$$

such that for all x, $x' \in P$ and all y, $y' \in Q$,

(1.24) $xy \leq x'y'$ (in $P \times Q$) $\leftrightarrow$ $x \leq x'$ (in P) and $y \leq y'$ (in Q).

Theorem 1.13 ([Ca2, Theorems 3.3.3, 3.4.1], [V-Y]). *Let Ω denote the class of modular atomic lattices that have all lines thick. Then for any poset P, the following are equivalent.*

(i) *P is a modular atomic lattice.*
(ii) *There exists an integer $r \geq 1$ and there exists $P_1, P_2, \ldots, P_r \in \Omega$ such that $P = P_1 \times P_2 \times \cdots \times P_r$.*

A modular atomic lattice is sometimes referred to as a *generalized projective geometry*.

Let P denote any lattice. Elements $x, y \in P$ are said to be *complements* whenever $x \wedge y = 0$ and $x \vee y = 1$. P is said to be *complemented* whenever each element in P has a complement. Let P denote any semilattice. Then P is said to be *relatively complemented* whenever each interval in P is complemented. A modular atomic lattice is relatively complemented. Let P denote any semilattice, and let $I = [x, y]$ denote any interval. Then elements $u, v \in I$ are said to be *relative complements* in I whenever

$$u \wedge v = x \qquad \text{and} \qquad u \vee v = y. \tag{1.25}$$

Let P denote any poset. By a *lower ideal* in P, we mean a subposet $S \subseteq P$ such that for all $x, y \in P$,

$$(1.26) \qquad x \in S \quad \text{and} \quad y \leq x \quad \rightarrow \quad y \in S.$$

An *upper ideal* of P is defined similarly.

Definition 1.14. Let $\mathcal{P}$ denote a modular atomic lattice. By a *$\mathcal{P}$-matroid*, we mean any subposet $P \subseteq \mathcal{P}$ satisfying conditions NT, LI, AU below.

NT: $P \neq \emptyset$.
LI: P is a lower ideal in $\mathcal{P}$.
AU: For all $x, y \in P$ such that $\text{rank}(x) < \text{rank}(y)$, there exists an atom $a \in \mathcal{P}$ such that $a \leq y$, $a \not\leq x$, and such that $x \vee a \in P$.

In view of Example 1.10, for any finite set A and any subset $P \subseteq B_A$, P is an A-matroid if and only if P (together with the partial order induced from B_A) is a B_A-matroid. Similarly, in view of Example 1.11, for any finite vector space V and any subset $P \subseteq L_V$, P is a V-matroid if and only if P (together with the partial order induced from L_V) is a L_V-matroid.

We end this section with a fundamental fact about $\mathcal{P}$-matroids.

Lemma 1.15. *Let $\mathcal{P}$ denote a modular atomic lattice, and let P denote a $\mathcal{P}$-matroid. Then*

$$\max(P) = \text{top}(P).$$

Proof. The inclusion $\supseteq$ is clear, so consider the inclusion $\subseteq$. Pick $x \in \max(P)$, and suppose $x \notin \text{top}(P)$. Pick $y \in \text{top}(P)$. Then $\text{rank}(x) < \text{rank}(y)$, so by AU, there exists an atom $a \in \mathcal{P}$ such that $a \leq y$, $a \not\leq x$, and such that $x \vee a \in P$. Now $x < x \vee a$, so $x \notin \max(P)$, a contradiction.

§2. $\mathcal{P}$-basis systems

Let P denote any poset. Elements $x, y \in P$ are said to be *comparable* whenever $x \leq y$ or $y \leq x$, and *incomparable* otherwise. By an *antichain* in P, we mean a subset $S \subseteq P$, where any two distinct elements of S are incomparable. There is a natural bijection from the set of all antichains in P to the set of all lower ideals in P. Indeed, for any subset $S \subseteq P$, let S^- denote the subposet

$$(2.1) \qquad S^- := \{x \in P \mid x \leq s \ \text{ for some } \ s \in S\}.$$

It is clear S^- is a lower ideal in P.

Lemma 2.1. *For any poset P, the map*

$$A \to A^- \tag{2.2}$$

induces a bijection from the set of all antichains of P to the set of all lower ideals of P. The inverse map is

$$L \to \max(L). \tag{2.3}$$

Proof. Routine.

Let $\mathcal{P}$ denote a modular atomic lattice. We have already considered one set of lower ideals in $\mathcal{P}$, namely the $\mathcal{P}$-matroids. The $\mathcal{P}$-matroids correspond to what set of antichains under (2.2), (2.3)? As we show in Theorem 2.5, this set consists of the $\mathcal{P}$-*basis systems*, defined as follows.

Definition 2.2. Let $\mathcal{P}$ denote a modular atomic lattice. By a $\mathcal{P}$-*basis system*, we mean any subset $B \subseteq \mathcal{P}$ that satisfies the conditions NT, AC, BA below.

NT: $B \neq \emptyset$.
AC: B is an antichain.
BA: For all $x, y \in \mathcal{P}$ such that $x \leq y$, if there exists b_1, $b_2 \in B$ such that $x \leq b_1$ and $b_2 \leq y$, then there exists $b_3 \in B$ such that $x \leq b_3 \leq y$.

The following lemma gives a second equivalent definition of a $\mathcal{P}$-basis system.

Lemma 2.3. *Let $\mathcal{P}$ denote a modular atomic lattice, and let B denote a nonempty antichain in $\mathcal{P}$. Then the following are equivalent.*

BA: *For all $x, y \in \mathcal{P}$ such that $x \leq y$, if there exists b_1, $b_2 \in B$ such that $x \leq b_1$ and $b_2 \leq y$, then there exists $b_3 \in B$ such that $x \leq b_3 \leq y$.*
BA': *For all b_1, $b_2 \in B$ and all $x \in \mathcal{P}$ such that b_1 covers x, there exists $b_3 \in B$ such that b_3 covers x and such that $b_3 \wedge b_2 > x \wedge b_2$.*

Proof. BA $\to$ BA'. We first claim that for all b_1, $b_2 \in B$ such that b_1 covers $b_1 \wedge b_2$, then b_2 covers $b_1 \wedge b_2$. To see this, observe $b_2 > b_1 \wedge b_2$ since B is an antichain, so there exists $z \in [b_1 \wedge b_2, b_2]$ such that z covers $b_1 \wedge b_2$. To prove the claim, it suffices to show

$b_2 = z$. Since b_1, z cover $b_1 \wedge b_2 = b_1 \wedge z$, we find by modularity that $b_1 \vee z$ covers b_1, z. In particular

$$b_1 \leq b_1 \vee z \geq z \leq b_2,$$

so by BA there exists $b_3 \in B$ such that

$$z \leq b_3 \leq b_1 \vee z.$$

But $b_3 \neq b_1 \vee z$ since B is an antichain, so $b_3 = z$. Now b_2 equals $b_3 = z$, since $b_3 \leq b_2$ and B is an antichain. This proves the claim.

Now pick any $b_1, b_2 \in B$, and pick any $x \in \mathcal{P}$ such that b_1 covers x. We must find $b_3 \in B$ such that b_3 covers x and $b_3 \wedge b_2 > x \wedge b_2$. We may assume $x \wedge b_2 = b_1 \wedge b_2$; otherwise we are done with $b_3 := b_1$. Set $y := x \vee b_2$. Then

$$b_1 \geq x \leq y \geq b_2,$$

so by BA, there exists $b_3 \in B$ such that $x \leq b_3 \leq y$. Observe $b_1 \not\leq y$; otherwise x and b_1 are both relative complements of b_2 in $[x \wedge b_2, y]$, contradicting (1.16). In particular $b_1 \not\leq b_3$. Now $x = b_1 \wedge b_3$, so b_3 covers x by our preliminary claim. Also $b_3 \wedge b_2 > x \wedge b_2$; otherwise x and b_3 are both relative complements of b_2 in $[x \wedge b_2, y]$, contradicting (1.16).

BA' $\rightarrow$ BA. Suppose we are given $x, y \in \mathcal{P}$ and $b_1, b_2 \in B$ such that

$$b_1 \geq x \leq y \geq b_2.$$

Of all the elements $b_3 \in B$ such that $b_3 \geq x$, pick one where $\operatorname{rank}(b_2 \wedge b_3)$ is maximal. BA will follow if we can show $b_3 \leq y$. Suppose $b_3 \not\leq y$. Then $b_3 > b_3 \wedge y$, so there exists $z \in [b_3 \wedge y, b_3]$ such that b_3 covers z. By BA', there exists $b_3' \in B$ such that b_3' covers z and $b_3' \wedge b_2 > z \wedge b_2$. Now

$$\begin{aligned} b_3' \wedge b_2 &> z \wedge b_2 \\ &\geq (b_3 \wedge y) \wedge b_2 \\ &= b_3 \wedge b_2, \end{aligned}$$

and

$$\begin{aligned} b_3' &\geq z \\ &\geq b_3 \wedge y \\ &\geq x, \end{aligned}$$

contradicting the construction of b_3. Hence $b_3 \leq y$, as desired. This proves Lemma 2.3.

Lemma 2.4. *Let $\mathcal{P}$ denote a modular atomic lattice, and let B denote a $\mathcal{P}$-basis system. Then for all $b \in B$, $D = \mathrm{rank}(b)$ is independent of b. We refer to D as the rank of B.*

Proof. Suppose there exists $b_1, b_2 \in B$ such that $\mathrm{rank}(b_1) \neq \mathrm{rank}(b_2)$. Of all such b_1, b_2, pick a pair such that $\mathrm{rank}(b_1 \wedge b_2)$ is maximal. Observe $b_1 \wedge b_2 < b_1$, since B is an antichain, so there exists $x \in [b_1 \wedge b_2, b_1]$ such that b_1 covers x. By BA', there exists $b_3 \in B$ such that b_3 covers x and such that $b_3 \wedge b_2 > x \wedge b_2$. Now

$$\begin{aligned} b_3 \wedge b_2 &> x \wedge b_2 \\ &= b_1 \wedge b_2, \end{aligned}$$

so $\mathrm{rank}(b_2) = \mathrm{rank}(b_3)$ by the construction. Also $\mathrm{rank}(b_1) = \mathrm{rank}(b_3)$, since both b_1, b_3 cover x, so $\mathrm{rank}(b_1) = \mathrm{rank}(b_2)$, contradicting our assumptions. This proves Lemma 2.4.

Theorem 2.5. *Let $\mathcal{P}$ denote a modular atomic lattice.*

(i) *Let B denote a $\mathcal{P}$-basis system. Then B^- is a $\mathcal{P}$-matroid.*
(ii) *Let P denote a $\mathcal{P}$-matroid. Then $\max(P)$ is a $\mathcal{P}$-basis system.*

In particular, the map $B \to B^-$ is a bijection from the set of all $\mathcal{P}$-basis systems to the set of all $\mathcal{P}$-matroids.

Proof. (i) Observe B^- certainly satisfies NT, LI in Definition 1.14. To verify condition AU in that definition, pick any $x, y \in B^-$ such that

$$\mathrm{rank}(x) < \mathrm{rank}(y). \tag{2.4}$$

By the construction, there exists $b_1, b_2 \in B$ such that $x \le b_1$, $y \le b_2$. Observe

$$b_1 \ \ge\ x\ \le b_2 \vee x\ \ge\ b_2,$$

so by BA, there exists $b_3 \in B$ such that

$$x\ \le\ b_3\ \le b_2 \vee x.$$

It is immediate from the left inequality above that $x \wedge y \le b_3 \wedge y$. We claim that in fact

$$x \wedge y < b_3 \wedge y. \tag{2.5}$$

To see (2.5), observe $b_3 \vee y \le b_2 \vee x$ by the construction, so

$$\mathrm{rank}(b_3 \vee y) \le \mathrm{rank}(b_2 \vee x). \tag{2.6}$$

Observe $x \wedge y \leq x \wedge b_2$ since $y \leq b_2$, so

$$\text{rank}(x \wedge y) \leq \text{rank}(b_2 \wedge x). \tag{2.7}$$

From (1.16) and Lemma 2.4, we also have

$$\text{rank}(b_2) = \text{rank}(b_3), \tag{2.8}$$

$$\text{rank}(b_3) + \text{rank}(y) = \text{rank}(b_3 \wedge y) + \text{rank}(b_3 \vee y), \tag{2.9}$$

$$\text{rank}(b_2 \wedge x) + \text{rank}(b_2 \vee x) = \text{rank}(x) + \text{rank}(b_2). \tag{2.10}$$

Summing (2.4), (2.6)–(2.10), we obtain

$$\text{rank}(x \wedge y) < \text{rank}(b_3 \wedge y),$$

and (2.5) follows. Now by (2.5), there exists an atom $a \in \mathcal{P}$ such that $a \leq b_3 \wedge y$ but $a \not\leq x \wedge y$. Now $a \leq y$ and $a \not\leq x$ by the construction. Also $a \vee x \in B^-$, since $a \vee x \leq b_3$ by the construction. We have now verified AU in Definition 1.14, so B^- is a $\mathcal{P}$-matroid.

(ii) Certainly $B := \max(P)$ satisfies conditions NT, AC in Definition 2.2. To show B is $\mathcal{P}$-basis system, it suffices to show BA'. First, we remark by Lemma 1.15 that $\text{rank}(b)$ is independent of $b \in B$. Now pick any $b_1, b_2 \in B$, and any $x \in \mathcal{P}$ such that b_1 covers x. We must find $b_3 \in B$ such that b_3 covers x, and such that $b_3 \wedge b_2 > x \wedge b_2$. Observe $\text{rank}(x) < \text{rank}(b_2)$ by our remark, so by AU in Definition 1.14, there exists an atom $a \in \mathcal{P}$ such that $a \leq b_2$, $a \not\leq x$, and such that $x \vee a \in P$. Observe a covers $0 = x \wedge a$, so $x \vee a$ covers x by modularity. In particular $\text{rank}(x \vee a) = \text{rank}(b_1)$, forcing $x \vee a \in B$ by our remark. Set $b_3 := x \vee a$. Since $b_3 \geq x$ we have

$$b_3 \wedge b_2 \geq x \wedge b_2.$$

In fact

$$b_3 \wedge b_2 > x \wedge b_2,$$

since $a \leq b_3 \wedge b_2$ but $a \not\leq x \wedge b_2$. Now BA' holds, so B is a $\mathcal{P}$-basis system by Lemma 2.3. This proves Theorem 2.5.

We can use Theorem 2.5 to get examples of $\mathcal{P}$-matroids.

Example 2.6. Let D denote a nonnegative integer. Let $\mathcal{P}$ denote a modular atomic lattice with rank $D+1$, and let B denote any non empty subset of $\mathcal{P}_D$. Then B is a $\mathcal{P}$-basis system of rank D. Moreover, B^- is a $\mathcal{P}$-matroid of rank D.

Proof. Routine application of Definition 2.2, Theorem 2.5.

We mention one other fact about basis systems in a modular atomic lattice.

Lemma 2.7. *Let $\mathcal{P}$ denote a modular atomic lattice, and let B denote a $\mathcal{P}$-basis system. Pick any $x, y \in \mathcal{P}$ such that $x \leq y$ and such that $B \cap [x, y] \neq \emptyset$. Then $B \cap [x, y]$ is a $[x, y]$-basis system.*

Proof. $B \cap [x, y]$ certainly satisfies conditions NT, AC in Definition 2.2. To verify BA, pick $u, v \in [x, y]$ and pick $b_1, b_2 \in B \cap [x, y]$ such that

$$b_1 \geq u \leq v \geq b_2.$$

We must find $b_3 \in B \cap [x, y]$ such that $u \leq b_3 \leq v$. Applying BA to B, we find there exists $b_3 \in B$ such that $u \leq b_3 \leq v$. But now $x \leq b_3 \leq y$ by the construction, so in fact $b_3 \in B \cap [x, y]$, as desired.

§3. The dual of a $\mathcal{P}$-matroid

Let P denote any poset. By the *poset-dual* of P, we mean the poset P^* on the same set as P, such that for all elements x, y,

$$x \leq y \ (\text{in } P^*) \quad \leftrightarrow \quad x \geq y \ (\text{in } P). \tag{3.1}$$

More generally, let S denote any subset of P. Then S^* will denote the subposet of P^* induced on S.

We mention that P is a modular atomic lattice if and only if P^* is a modular atomic lattice [St, Theorem 3.3.3].

Lemma 3.1. *Let $\mathcal{P}$ denote a modular atomic lattice. Then for all subsets $B \subseteq \mathcal{P}$, the following are equivalent.*

(i) *B is a $\mathcal{P}$-basis system.*
(ii) *B is a $\mathcal{P}^*$-basis system.*

Proof. This is immediate from the symmetry in the axioms NT, AC, BA from Definition 2.2.

Let P denote any poset. For any subset $S \subseteq P$, let S^+ denote the subposet

$$S^+ = \{ x \in P \mid x \geq s \ \text{for some} \ \ s \in S \}. \tag{3.2}$$

Observe S^+ is an upper ideal in P.

Definition 3.2. Let $\mathcal{P}$ denote a modular atomic lattice, and let P denote a $\mathcal{P}$-matroid. By the *matroid-dual* of P (with respect to $\mathcal{P}$), we mean the $\mathcal{P}^*$-matroid $(B^+)^*$, where $B = \max(P)$.

§4. The definition of a quantum matroid

In this section, we introduce the notion of a quantum matroid, and consider the examples with rank at most 2.

Definition 4.1. By a *quantum matroid*, we mean any nonempty poset P satisfying the conditions R, SL, M, AU below.

R: P is ranked.
SL: P is a (meet) semilattice.
M: For all $x \in P$, the interval $[0, x]$ is a modular atomic lattice.
AU: For all $x, y \in P$ satisfying $\text{rank}(x) < \text{rank}(y)$, there exists an atom $a \in P$ such that $a \leq y$, $a \not\leq x$, and such that $x \vee a$ exists in P.

Let $\mathcal{P}$ denote a modular atomic lattice, and let P denote a $\mathcal{P}$-matroid. Then the subposet P is a quantum matroid. In particular, any modular atomic lattice is a quantum matroid. We now consider the quantum matroids of rank at most 2.

A poset P is a quantum matroid of rank 0 if and only if P consists of a single element. A poset P is a quantum matroid of rank 1 if and only if P has a 0 and at least one other element, and all nonzero elements of P cover 0. The example below characterizes the quantum matroids of rank 2.

Example 4.2. A poset P is a quantum matroid of rank 2 if and only if P has a 0, and satisfies the following four conditions:

R: P is ranked and $\text{rank}(P) = 2$.
SL: For any distinct points $x, y \in P$, there exists at most one line $z \in P$ such that $x \leq z$, $y \leq z$.
M: Each line in P covers at least 2 points in P.
AU: For each point $x \in P$ and each line $y \in P$ such that $x \not\leq y$, there exists a point $x' \in P$ and a line $y' \in P$ such that $x \leq y' \geq x' \leq y$.

We have already proved some facts about $\mathcal{P}$-matroids. Are there corresponding results about the more general quantum matroids? Lemma 1.15 can certainly be extended to this level.

Lemma 4.3. *Let P denote a quantum matroid. Then*

$$\max(P) = \text{top}(P). \tag{4.1}$$

Proof. Similar to the proof of Lemma 1.15.

Problem 4.4. Extend the notion of the dual of a $\mathcal{P}$-matroid (Def. 3.2) to the level of quantum matroids.

§5. Prematroids and their subposets

Definition 5.1. By a *pre-quantum matroid* (or simply, a *prematroid*), we mean a nonempty poset P that satisfies conditions R, SL, M in Definition 4.1.

We will have occasion to consider subposets of prematroids that possess the following properties.

Definition 5.2. Let P denote any poset, and let S denote any subposet of P.

(i) S is said to be $\wedge$-*closed* in P whenever for all $x, y \in S$,

$$x \wedge_P y \quad \text{exists} \quad \rightarrow \quad x \wedge_P y \in S. \tag{5.1}$$

The notion of $\vee$-closure is defined similarly.

(ii) S is said to be *convex* in P whenever for all $x, y, z \in P$,

$$x, y \in S \quad \text{and} \quad x \leq z \leq y \quad \rightarrow \quad z \in S. \tag{5.2}$$

Lemma 5.3. *Let P denote a poset with 0, and let S denote any nonempty subposet of P. Then the following are equivalent.*

(i) *S is a lower ideal in P.*

(ii) *S is convex in P, and $0_P \in S$.*

If (i)–(ii) *hold, then S is $\wedge$-closed in P, and $0_S = 0_P$.*

Proof. (i) $\rightarrow$ (ii). Routine.

(ii) $\rightarrow$ (i). Pick any $x \in S$ and any $y \in P$ such that $y \leq x$. Then $0_P \leq y \leq x$, so $y \in S$ by convexity.

Now assume (i)–(ii). To see S is $\wedge$-closed in P, pick any $x, y \in S$ such that $x \wedge_P y$ exists. Certainly $x \wedge_P y \leq x$, so $x \wedge_P y \in S$ by the definition of a lower ideal. Hence S is $\wedge$-closed in P. It is clear that S, P have the same 0. This proves Lemma 5.3.

Lemma 5.4. *Let P denote a semilattice, and let S denote a nonempty $\wedge$-closed subposet of P. Then S is a semilattice. Moreover, for all $x, y \in S$,*

$$x \wedge_S y = x \wedge_P y. \tag{5.3}$$

Proof. Pick any $x,\, y \in S$. Then it suffices to show

$$\max(L) = \{\, x \wedge_P y \}, \tag{5.4}$$

where

$$L := \{\, z \in S \mid z \le x, \quad z \le y \}.$$

Certainly $x \wedge_P y \in L$, since $x \wedge_P y \in S$ by $\wedge$-closure. Also $x \wedge_P y \ge z$ for all $z \in L$, so $x \wedge_P y$ is the unique maximal element of L. this proves Lemma 5.4.

Lemma 5.5. *Let P denote a semilattice, and let S denote a convex subposet of P. Then for all $x,\, y \in S$, the following are equivalent.*

(i) $x \vee_S y$ *exists.*
(ii) $x \vee_P y$ *exists, and* $x \vee_P y \in S$.

If (i)–(ii) *hold, then*

$$x \vee_S y = x \vee_P y. \tag{5.5}$$

Proof. (i) $\to$ (ii). $x \vee_P y$ exists, since $x \vee_S y$ is an upper bound for $x,\, y$ in P. Also $x \vee_P y \in S$ by convexity, since $x \le x \vee_P y \le x \vee_S y$.
(ii) $\to$ (i). Clear.

Now suppose (i), (ii). We have observed $x \vee_P y \le x \vee_S y$. Also $x \vee_P y$ is an upper bound for $x,\, y$ in S, so $x \vee_P y \ge x \vee_S y$. Hence $x \vee_P y$ and $x \vee_S y$ are identical. This proves Lemma 5.5.

Lemma 5.6. *Let P denote a ranked semilattice, and let S denote a nonempty $\wedge$-closed, convex subposet of P. Then S is ranked. Moreover, for all $x \in S$,*

$$\mathrm{rank}_S(x) = \mathrm{rank}_P(x) \; - \; \mathrm{rank}_P(0_S). \tag{5.6}$$

Proof. S is a semilattice by Lemma 5.4; in particular S has a 0. We show the function $R : S \to \mathbb{Z}$ defined by

$$R(x) = \mathrm{rank}_P(x) \; - \; \mathrm{rank}_P(0_S) \qquad\qquad (x \in S) \tag{5.7}$$

is a rank function for S. Certainly $R(0_S) = 0$. Also, for any $x,\, y \in S$ such that y covers x (in S), then y covers x (in P) by the convexity of S, forcing

$$\mathrm{rank}_P(y) \; - \; \mathrm{rank}_P(x) = 1.$$

Now

$$R(y) \; - \; R(x) = 1$$

by (5.7), so R is a rank function for S by (1.9). This proves Lemma 5.6.

Corollary 5.7. *Let P denote a prematroid. Then any nonempty $\wedge$-closed, convex subposet of P is a prematroid. In particular, any nonempty lower ideal of P is a prematroid.*

Proof. Let S denote a nonempty $\wedge$-closed, convex subposet of P. Then S satisfies axiom SL by Lemma 5.4, and axiom R by Lemma 5.6. To see that S satisfies axiom M, pick any $x \in S$. Then the interval $[0_S, x]$ of S may be viewed as an interval in the modular atomic lattice $[0_P, x]$, and is therefore a modular atomic lattice. We have now shown S is a prematroid. The last line of the present corollary follows from Lemma 5.3.

§6. Embeddable Posets

In this section, we define the notion of an *embedding* of a poset, and consider posets that are embeddable into a modular atomic lattice.

Definition 6.1. Let P and $\mathcal{P}$ denote posets. By a *$\mathcal{P}$-embedding* of P, we mean an injection $\sigma : P \to \mathcal{P}$ that satisfies (i), (ii) below.

(i) $x \leq y \quad \leftrightarrow \quad \sigma(x) \leq \sigma(y) \qquad\qquad (\forall\, x,\, y \in P).$
(ii) $\sigma(P)$ is a lower ideal in $\mathcal{P}$.

Lemma 6.2. *Let $\mathcal{P}$ denote a modular atomic lattice.*

(i) *Let P denote a quantum matroid, and let $\sigma : P \to \mathcal{P}$ denote a $\mathcal{P}$-embedding of P. Then $\sigma(P)$ is a $\mathcal{P}$-matroid.*
(ii) *Let Q denote a $\mathcal{P}$-matroid, and let $\sigma : Q \to \mathcal{P}$ denote the identity map on Q. Then σ is an embedding of Q.*

Proof. Immediate from Definitions 1.14, 4.1.

Definition 6.3. A poset P is said to be *embeddable* whenever $P \neq \emptyset$, and there exists a pair $\mathcal{P}$, σ, where $\mathcal{P}$ is a modular atomic lattice, and σ is a $\mathcal{P}$-embedding of P.

Lemma 6.4. *Let P denote an embeddable poset. Then P is isomorphic to a lower ideal in a modular atomic lattice. In particular, P is a prematroid.*

Proof. Immediate from Definition 6.1, Corollary 5.7, and the observation that any modular atomic lattice is a prematroid.

We end this section with a conjecture.

Conjecture 6.5. *Let P denote a quantum matroid with rank at least* 4. *Then P is embeddable.*

We will see in Corollary 39.8 that the above conjecture is true for the *regular* quantum matroids. See also [C-J-P], [Sp1], [Ti, Theorem 8.21].

§7. The distance function ∂

For the next several sections, we will develop a theory of prematroids. We will use the following notation.

Let P denote any poset, and pick any $x, y \in P$. Let us say x, y are *adjacent* whenever x covers y or y covers x. Pick any nonnegative integer d. By a *path of length* d connecting x, y, we mean any sequence $x = x_0, x_1, \ldots, x_d = y$ $(x_0, x_1, \ldots, x_d \in P)$, such that x_i, x_{i+1} are adjacent for all i $(0 \leq i \leq d-1)$. P is said to be *connected* whenever for all $x, y \in P$, there exists a path in P connecting x and y. Suppose P has a 0. Then P is connected, since for all $x \in P$, there exists a path in P connecting x, 0.

Let P denote an arbitrary connected poset. For any $x, y \in P$, define the *distance*

$$\partial(x, y) := \min\{\, d \mid \text{there exists a path of length } d \text{ connecting } x, y\}.$$

Then for all x, y, $z \in P$,

$$\partial(x, y) + \partial(y, z) \geq \partial(x, z). \tag{7.1}$$

If P is ranked, we can say a bit more.

Lemma 7.1. *Let P denote a ranked poset with* 0, *and pick any $x, y \in P$. Then*

(i) *We have*

$$\partial(x, y) \geq \operatorname{rank}(y) - \operatorname{rank}(x). \tag{7.2}$$

(ii) *Equality holds in* (7.2) *if and only if $y \geq x$.*

(iii) *For all $z \in P$ such that z is adjacent to x,*

$$\partial(x, y) - \partial(z, y) \in \{1, -1\}. \tag{7.3}$$

Proof. Lines (i), (ii) are immediate from (1.9). Line (iii) follows from the observation that the graph structure on P is bipartite.

Let P denote an arbitrary connected poset. A path $x_0, x_1, \ldots, x_d$ in P is said to be *geodesic* whenever $\partial(x_0, x_d) = d$. More generally, any sequence $x_0, x_1, \ldots, x_d$ of elements from P is said to be *geodesic* whenever

$$\sum_{i=0}^{d-1} \partial(x_i, x_{i+1}) = \partial(x_0, x_d). \tag{7.4}$$

Let P denote a ranked poset with 0. Then for all $x_0, x_1, \ldots, x_d \in P$,

$$x_0 \leq x_1 \leq \cdots \leq x_d \qquad \rightarrow \qquad x_0, x_1, \ldots, x_d \text{ is geodesic.} \tag{7.5}$$

Let P denote a ranked poset with 0, and let $p := (x_0, x_1, \ldots, x_d)$ denote a path in P. By the *shape* of P, we mean the sequence

$$\mathrm{shape}(p) := (\mathrm{rank}(x_0), \ \mathrm{rank}(x_1), \ldots, \mathrm{rank}(x_d)). \tag{7.6}$$

By the *weight* of p, we mean the scalar

$$\mathrm{weight}(p) := \sum_{i=0}^{d} \mathrm{rank}(x_i). \tag{7.7}$$

Lemma 7.2. *Let P denote a prematroid. Pick any nonnegative integer d, and pick any path $p = (x_0, x_1, \ldots, x_d)$ $(x_0, x_1, \ldots, x_d \in P)$. Then the following are equivalent.*

(i) *There does not exist an integer i $(1 \leq i \leq d-1)$ such that*

$$x_{i-1} < x_i > x_{i+1}. \tag{7.8}$$

(ii) *There exists an integer e $(0 \leq e \leq d)$ such that*

$$x_0 > x_1 > x_2 > \cdots > x_e,$$
$$x_e < x_{e+1} < x_{e+2} < \cdots < x_d.$$

Suppose (i)–(ii) *hold. Then we say p is down-up. We call x_e the base of p.*

Proof. Routine.

Lemma 7.3. *Let P denote a prematroid, and pick any $x, y \in P$. Then there exists a geodesic down-up path connecting x, y.*

Proof. Set $d := \partial(x,y)$, and pick a geodesic path

$$p \;:=\; (x = x_0,\, x_1,\, \ldots,\, x_d = y) \qquad\qquad (x_0,\, x_1,\, \ldots,\, x_d \in P)$$

with minimal weight in the sense of (7.7). We claim p is down-up. Suppose not. Then by Lemma 7.2, there exists an integer i $(1 \le i \le d-1)$ such that

$$(7.9) \qquad\qquad x_{i-1} < x_i > x_{i+1}.$$

Observe $x_{i-1} \ne x_{i+1}$ since p is geodesic, and of course x_i covers both x_{i-1} and x_{i+1}, so $x_i = x_{i-1} \vee x_{i+1}$. It follows $x_{i-1} \vee x_{i+1}$ covers x_{i-1}, x_{i+1}, so by modularity, x_{i-1}, x_{i+1} both cover $x_{i-1} \wedge x_{i+1}$. Now the sequence

$$p' \;:= (x = x_0,\, x_1,\, \ldots,\, x_{i-1},\, x_{i-1} \wedge x_{i+1},\, x_{i+1},\, \ldots,\, x_d = y)$$

is a path. Observe p' is geodesic, since p, p' have the same length, and

$$\text{weight}(p') = \text{weight}(p) - 2.$$

This contradicts our construction, and we conclude p is down-up. This proves Lemma 7.3.

Lemma 7.4. *Let P denote a prematroid, and pick any $x, y, z \in P$. Then the following are equivalent:*

(i) *The sequence xzy is geodesic, and $z \le x$, $z \le y$.*
(ii) $z = x \wedge y$.

Proof. We set

$$N := \{\, u \in P \mid xuy \text{ is geodesic}, \quad u \le x, \;\; u \le y\},$$

and show $N = \{x \wedge y\}$. To do this, it suffices to show

$$(7.10) \qquad\qquad N \subseteq \{x \wedge y\},$$

and

$$(7.11) \qquad\qquad N \ne \emptyset.$$

To obtain (7.10), pick any $u \in N$. Observe u is a lower bound for x and y, so $u \le x \wedge y$. Now $u \le x \wedge y \le x$, so u, $x \wedge y$, x is geodesic by (7.5). Similarly $u \le x \wedge y \le y$, so u, $x \wedge y$, y is geodesic. Recall xuy is geodesic, so x, $x \wedge y$, u, $x \wedge y$, y is geodesic. In particular $x \wedge y$, u, $x \wedge y$ is geodesic, so $u = x \wedge y$. We now have (7.10). To obtain (7.11), recall by Lemma 7.3, there exists a geodesic down-up path p connecting x, y.

Let u denote the base of p, in the sense of Lemma 7.2. Then $u \in N$ by construction, and (7.11) follows. This proves Lemma 7.4.

Replacing $\wedge$ by $\vee$ in Lemma 7.4, we get the following result.

Lemma 7.5. *Let P denote a prematroid, and pick any $x, y, z \in P$. Then the following are equivalent:*

(i) *The sequence xzy is geodesic, and $z \geq x$, $z \geq y$.*
(ii) *The join $x \vee y$ exists, and $z = x \vee y$.*

Proof. Similar to Lemma 7.4.

Corollary 7.6. *Let P denote a prematroid, and pick any $x, y \in P$.*

(i)

$$\partial(x,y) = \mathrm{rank}(x) + \mathrm{rank}(y) - 2\,\mathrm{rank}(x \wedge y). \tag{7.12}$$

(ii) *Suppose $x \vee y$ exists. Then*

$$\partial(x,y) = 2\,\mathrm{rank}(x \vee y) - \mathrm{rank}(x) - \mathrm{rank}(y). \tag{7.13}$$

(iii) *Suppose $x \vee y$ exists. Then*

$$\mathrm{rank}(x) + \mathrm{rank}(y) = \mathrm{rank}(x \wedge y) + \mathrm{rank}(x \vee y). \tag{7.14}$$

Proof. To see (i), observe by Lemma 7.1(i),(ii) and Lemma 7.4 that

$$\begin{aligned}\partial(x,y) &= \partial(x, x \wedge y) + \partial(x \wedge y, y)\\ &= \mathrm{rank}(x) - \mathrm{rank}(x \wedge y) + \mathrm{rank}(y) - \mathrm{rank}(x \wedge y).\end{aligned}$$

The proof of (ii) is similar. To get (iii), equate (7.12), (7.13).

Corollary 7.7. *Let P denote a prematroid, and let S denote a nonempty $\wedge$-closed, convex subposet of P. Then for all x, $y \in S$,*

$$\partial_S(x,y) = \partial_P(x,y). \tag{7.15}$$

Proof. By Corollary 7.6(i),

$$\partial_P(x,y) = \mathrm{rank}_P(x) + \mathrm{rank}_P(y) - 2\mathrm{rank}_P(x \wedge_P y). \tag{7.16}$$

Observe S is a prematroid by Corollary 5.7. Applying Corollary 7.6(i) to S, we obtain

$$\partial_S(x,y) = \mathrm{rank}_S(x) + \mathrm{rank}_S(y) - 2\mathrm{rank}_S(x \wedge_S y). \tag{7.17}$$

Evaluating the right hand side of (7.17) using (5.3), (5.6), we find the right hand sides of (7.16), (7.17) are equal. Line (7.15) follows, and Corollary 7.7 is proved.

Lemma 7.8. *Let P denote a prematroid, and pick any $x, y, z \in P$. Then the following are equivalent:*

(i) *The sequence xzy is geodesic.*

(ii) *$z \geq x \wedge y$, and $x \wedge z$, $y \wedge z$ are relative complements in the interval $[x \wedge y, z]$.*

(iii) *There exists $u \in [x \wedge y, x]$ and there exists $v \in [x \wedge y, y]$ such that $z = u \vee v$.*

Moreover, if (i)–(iii) *hold, then*

$$u = x \wedge z, \tag{7.18}$$

$$v = y \wedge z. \tag{7.19}$$

Proof. (i) $\rightarrow$ (ii). Set $u := x \wedge z$, $v := y \wedge z$. Then it suffices to show $z = u \vee v$ and $x \wedge y = u \wedge v$. Observe xuz, zvy are each geodesic by Lemma 7.4, so $xuzvy$ is geodesic. In particular uzv is geodesic. Since $z \geq u$ and $z \geq v$ by the construction, we find $z = u \vee v$ by Lemma 7.5. Observe by our remarks above that $xuvy$ is geodesic. Observe by Lemma 7.4 that u, $u \wedge v$, v is geodesic, so x, u, $u \wedge v$, v, y is geodesic. In particular x, $u \wedge v$, y is geodesic. But $u \wedge v \leq u \leq x$ and $u \wedge v \leq v \leq y$, so $u \wedge v = x \wedge y$ by Lemma 7.4.

(ii) $\rightarrow$ (iii). Set $u := x \wedge z$, $v := y \wedge z$. Observe $u \geq x \wedge y$ since $z \geq x \wedge y$, and $u \leq x$ by construction, so $u \in [x \wedge y, x]$. Similarly $v \in [x \wedge y, y]$. Also $z = u \vee v$ by (ii) and (1.25).

(iii) $\rightarrow$ (i). Observe x, $x \wedge y$, y is geodesic by Lemma 7.4. Observe both x, u, $x \wedge y$ and $x \wedge y$, v, y are geodesic by the construction, so x, u, $x \wedge y$, v, y is geodesic. In particular, $xuvy$ is geodesic. Also u, $u \vee v$, v is geodesic by Lemma 7.5, so x, u, $u \vee v$, v, y is geodesic. In particular, x, $u \vee v$, y is geodesic, so (i) holds.

Now suppose (i)–(iii) hold. We have observed in the proof of (iii) $\rightarrow$ (i) that $xuzvy$ is geodesic. Now (7.18) holds by Lemma 7.4, since xuz is geodesic, and since $u \leq x$, $u \leq z$. Line (7.19) is similar. This proves Lemma 7.8.

Interchanging the roles of $\vee$, $\wedge$ in the above lemma, we obtain the following result.

Lemma 7.9. *Let P denote a prematroid, and pick any $x, y, z \in P$ such that $x \vee y$ exists. Then the following are equivalent:*

(i) *The sequence xzy is geodesic.*

(ii) $z \leq x \vee y$, *and* $x \vee z$, $y \vee z$ *are relative complements in the interval* $[z, x \vee y]$.

(iii) *There exists* $u \in [x, x \vee y]$ *and there exists* $v \in [y, x \vee y]$ *such that* $z = u \wedge v$.

Moreover, if (i)–(iii) *hold, then*

$$u = x \vee z, \tag{7.20}$$

$$v = y \vee z. \tag{7.21}$$

Proof. Similar to Lemma 7.8.

Lemma 7.10. *Let* P *denote a prematroid, and pick any* x, y, z, $z' \in P$ *such that both* xzy *and* $xz'y$ *are geodesic, and such that* $z \vee z'$ *exists. Then* x, $z \vee z'$, y *is geodesic.*

Proof. By Lemma 7.8, there exists u, $u' \in [x \wedge y, x]$ and there exists v, $v' \in [x \wedge y, y]$ such that $z = u \vee v$ and $z' = u' \vee v'$. Now

$$z \vee z' = (u \vee u') \vee (v \vee v'),$$
$$u \vee u' \in [x \wedge y, x],$$
$$v \vee v' \in [x \wedge y, y],$$

so x, $z \vee z'$, y is geodesic by Lemma 7.8.

Lemma 7.11. *Let* P *denote a prematroid, and pick any* x, y, z, $z' \in P$ *such that both* xzy *and* $xz'y$ *are geodesic, and such that* $x \vee y$ *exists. Then* x, $z \wedge z'$, y *is geodesic.*

Proof. Similar to Lemma 7.10.

Lemma 7.12. *Let* P *denote an embeddable poset, and pick any* $x, y, z, z' \in P$ *such that both* xzy *and* $xz'y$ *are geodesic. Then* x, $z \wedge z'$, y *is geodesic.*

Proof. By Lemma 6.4 , we may identify P with a lower ideal in some modular atomic lattice $\mathcal{P}$. By Lemmas 5.3, 5.4, $z \wedge z'$ is the same as computed in P or $\mathcal{P}$. By Lemma 5.3 and Corollary 7.7, the distance function for P equals the restriction to P of the distance function for $\mathcal{P}$. In particular, both xzy and $xz'y$ are geodesic in $\mathcal{P}$. Applying Lemma 7.11 to $\mathcal{P}$, we find $x, z \wedge z', y$ is geodesic in $\mathcal{P}$. By our above remark, $x, z \wedge z', y$ is geodesic in P. This proves Lemma 7.12.

Conjecture 7.13. *Let* P *denote a prematroid such that*

(i) *the rank of* P *is at least* 3,

(ii) *for all* $x, y, z, z' \in P$ *such that both* xzy *and* $xz'y$ *are geodesic, then* x, $z \wedge z'$, y *is geodesic.*

Then P *is embeddable.*

§8. Geodesically closed subposets in a prematroid

In this section, we introduce the notion of a geodesically closed subposet in a prematroid, and characterize these subposets in terms of the meet and join operation.

Definition 8.1. Let P denote a prematroid. A subposet $G \subseteq P$ is said to be *geodesically closed* in P, whenever G is nonempty, and for all x, y, $z \in P$,

$$x, y \in G \quad \text{and} \quad xzy \text{ geodesic in } P \quad \to \quad z \in G.$$

Lemma 8.2. *Let* P *denote a prematroid. Then for any subposet* $G \subseteq P$*, the following are equivalent.*

(i) G *is geodesically closed in* P.
(ii) G *is nonempty,* $\wedge$*-closed,* $\vee$*-closed, and convex in* P.

Proof. (i) $\to$ (ii). G is nonempty by Definition 8.1. Given $x, y \in G$, observe x, $x \wedge y$, y is geodesic in P by Lemma 7.4, so $x \wedge y \in G$. Suppose $x \vee y$ exists in P. Then x, $x \vee y$, y is geodesic in P by Lemma 7.5, so $x \vee y \in G$. Suppose $x \leq y$, and pick any $z \in [x, y]$. Then xzy is geodesic in P by (7.5), so $z \in G$. We now have (ii).

(ii) $\to$ (i). Suppose we are given $x, y \in G$ and $z \in P$ such that xzy is geodesic in P. Then by Lemma 7.8(i),(iii), $z = u \vee v$ for some $u \in [x \wedge y, x]$ and some $v \in [x \wedge y, y]$. Observe $x \wedge y \in G$ by $\wedge$-closure, so now $u, v \in G$ by convexity, and now $z \in G$ by $\vee$-closure. This proves Lemma 8.2.

Corollary 8.3. *Let* P *denote a prematroid, and let* G *denote a geodesically closed subposet of* P. *Then* G *is a prematroid.*

Proof. G is $\wedge$-closed and convex by Lemma 8.2, and is therefore a prematroid by Corollary 5.7.

Lemma 8.4. *Let* P *denote a prematroid, and pick any* $x \in P$.

(i) *The subposet* $x^+ := \{ y \in P \mid y \geq x \}$ *is geodesically closed in* P.
(ii) *For all* $y \in P$ *such that* $y \geq x$*, the interval* $[x, y]$ *is geodesically closed in* P.

Proof. The subposets x^+, $[x,y]$ satisfy the condition Lemma 8.2(ii), and are therefore geodesically closed in P by that lemma.

Lemma 8.5. *Let P denote a prematroid, and let G denote a geodesically closed subposet of P that is contained in an interval of P. Then G is an interval. In particular, G is a modular atomic lattice.*

Proof. Observe by Lemma 8.2 that $G = [x,y]$, where $x = \bigwedge_{w\in G} w$ and $y = \bigvee_{w\in G} w$.

§9. Submatroids and subspaces in a prematroid

In this section we introduce the notions of a *submatroid* and a *subspace* in a prematroid, and show these objects are in 1–1 correspondence.

Lemma 9.1. *Let P denote a prematroid. Then for any subposet $G \subseteq P$, the following are equivalent.*

(i) *G is a nonempty $\vee$-closed lower ideal in P.*
(ii) *G is geodesically closed in P, and $0_G = 0_P$.*

If (i)–(ii) *hold, we say G is a subprematroid of P,* (*or simply, a submatroid*)*.*

Proof. (i) $\to$ (ii). G is convex and $\wedge$-closed in P by Lemma 5.3, so G is geodesically closed in P by Lemma 8.2. G, P share the same 0 by Lemma 5.3.

(ii) $\to$ (i). G is nonempty by Definition 8.1. G is $\vee$-closed in P by Lemma 8.2. G is convex in P by the same lemma, so G is a lower ideal in P by Lemma 5.3. This proves Lemma 9.1.

Lemma 9.2. *Let P denote a prematroid. Then for all subposets $G \subseteq P$, and for all $x \in P$, the following are equivalent.*

(i) *G is geodesically closed in P, and $0_G = x$.*
(ii) *G is a submatroid of x^+.*

Proof. (i) $\to$ (ii). G is geodesically closed in P and contained in x^+, so G is geodesically closed in x^+. The result now follows from Lemma 9.1.

(ii) $\to$ (i). By Lemma 9.1, $0_G = x$, and G is geodesically closed in x^+. But x^+ is geodesically closed in P by Lemma 8.4(i), so G is geodesically closed in P. This proves Lemma 9.2.

Let P denote a poset with 0, and recall A_P denotes the set of all atoms of P. For all $x \in P$, define

$$\text{Shadow}(x) := \{\, a \in A_P \mid a \leq x \}. \tag{9.1}$$

Observe by Definition 5.1 that any prematroid is atomic.

Lemma 9.3. *Let P denote an atomic semilattice. Then*

(i) *for all $x \in P$,*

$$x = \bigvee_{a \in \mathrm{Shadow}(x)} a. \tag{9.2}$$

(ii) *for all $x, y \in P$,*

$$x \leq y \quad \leftrightarrow \quad \mathrm{Shadow}(x) \subseteq \mathrm{Shadow}(y). \tag{9.3}$$

(iii) *for all $x, y \in P$,*

$$\mathrm{Shadow}(x \wedge y) = \mathrm{Shadow}(x) \cap \mathrm{Shadow}(y). \tag{9.4}$$

Proof. (i) Immediate from the definition of an atomic semilattice. (ii), (iii) Immediate from (i).

Lemma 9.4. *Let P denote a prematroid, and let x, y denote incomparable elements in P such that $x \vee y$ exists. Then*

$$\mathrm{Shadow}(x \vee y) \backslash \mathrm{Shadow}(x \wedge y) = \bigcup \mathrm{Shadow}(a \vee b), \tag{9.5}$$

where the union is over all $a \in \mathrm{Shadow}(x) \backslash \mathrm{Shadow}(y)$ and all $b \in \mathrm{Shadow}(y) \backslash \mathrm{Shadow}(x)$.

Proof. $\supseteq$: Pick any $a \in \mathrm{Shadow}(x) \backslash \mathrm{Shadow}(y)$, any $b \in \mathrm{Shadow}(y) \backslash \mathrm{Shadow}(x)$, and any $c \in \mathrm{Shadow}(a \vee b)$. Observe $a \leq x \leq x \vee y$ and $b \leq y \leq x \vee y$, so $a \vee b \leq x \vee y$. Now $c \leq a \vee b \leq x \vee y$, so $c \in \mathrm{Shadow}(x \vee y)$. Observe $c \not\in \mathrm{Shadow}(x \wedge y)$; otherwise

$$b \leq a \vee b = a \vee c \leq x,$$

a contradiction.

$\subseteq$: Pick $c \in \mathrm{Shadow}(x \vee y) \backslash \mathrm{Shadow}(x \wedge y)$. We find $a \in \mathrm{Shadow}(x) \backslash \mathrm{Shadow}(y)$ and $b \in \mathrm{Shadow}(y) \backslash \mathrm{Shadow}(x)$ such that $c \in \mathrm{Shadow}(a \vee b)$. We may assume $c \not\leq x$; otherwise we are done with $a := c$, and with b an arbitrary element in $\mathrm{Shadow}(y) \backslash \mathrm{Shadow}(x)$. Similarly, we may assume $c \not\leq y$.

Observe $x \vee y$ is an upper bound for c, y, so $c \vee y$ exists. Set $y' := c \vee y$. Then y' covers y by modularity, and $y' \in [y, x \vee y]$ by the construction. Observe $x \wedge y' \geq x \wedge y$. In fact $x \wedge y' > x \wedge y$; otherwise y, y' are both relative complements of x in $[x \wedge y, x \vee y]$, contradicting

(7.14). Now $\text{Shadow}(x \wedge y')$ properly contains $\text{Shadow}(x \wedge y)$ by (9.3), so there exists an element $a \in \text{Shadow}(x \wedge y') \backslash \text{Shadow}(x \wedge y)$. Observe $a \in \text{Shadow}(x) \backslash \text{Shadow}(y)$. Observe y' is an upper bound for a, c, so $a \vee c$ exists. set $z := a \vee c$. Observe $a \neq c$ since $a \leq x$ and $c \not\leq x$, so z is a line by modularity. We mentioned y' is an upper bound for a, c, so $z = a \vee c \leq y'$. Now $y \vee z = y'$ covers y, so z covers $y \wedge z$ by modularity. In particular $y \wedge z$ is an atom. Set $b := y \wedge z$. Observe $b \in \text{Shadow}(y)$, so $a \neq b$. Observe z covers a, b, so $z = a \vee b$. Observe

$$\begin{aligned} c &\in \text{Shadow}(z) \\ &= \text{Shadow}(a \vee b). \end{aligned}$$

Observe $b \notin \text{Shadow}(x)$; otherwise $c \leq a \vee b \leq x$, a contradiction. this proves Lemma 9.4.

Definition 9.5. Let P denote a prematroid. By a *subspace* of P, we mean a subset $S \subseteq A_P$ such that for all lines $x \in P$,

$$|\text{Shadow}(x) \cap S| \geq 2 \quad \rightarrow \quad \text{Shadow}(x) \subseteq S. \tag{9.6}$$

Let P denote a prematroid. Our purpose for the rest of this section is to establish a 1–1 correspondence between the set $\mathcal{G}$ of all submatroids of P, and the set $\mathcal{S}$ of all subspaces of P. We proceed as follows. In Lemma 9.6, we find a map $G \rightarrow A_G$ from $\mathcal{G}$ to $\mathcal{S}$, and a map $S \rightarrow G_S$ from $\mathcal{S}$ to $\mathcal{G}$. In Theorem 9.7, we show these maps are inverses, establishing our 1–1 correspondence.

Lemma 9.6. *Let P denote a prematroid.*

(i) *Let G denote a submatroid of P. Then the set of atoms $A_G = A_P \cap G$ of G is a subspace of P.*
(ii) *Let S denote a subspace of P, and set*

$$G_S := \{\, x \in P \mid \text{Shadow}(x) \subseteq S\}. \tag{9.7}$$

Then G_S is a submatroid of P.

Proof. (i) Pick any line $x \in P$, and suppose there exists distinct points

$$y, z \in \text{Shadow}(x) \cap A_G.$$

We show

$$\text{Shadow}(x) \subseteq A_G. \tag{9.8}$$

To see (9.8), observe

$$x = y \vee z \in G$$

since G is $\vee$-closed, so now

$$\text{Shadow}(x) \subseteq [0, x] \subseteq G$$

since G is convex, and now

$$\text{Shadow}(x) \subseteq A_P \cap G = A_G,$$

as desired.

(ii) By Lemma 9.1, it suffices to show G_S is a nonempty, $\vee$-closed lower ideal in P. Observe $\text{Shadow}(0) = \emptyset \subseteq S$, so $0 \in G_S$ by (9.7). In particular $G_S \neq \emptyset$. G_S is a lower ideal in P by the construction. To see that G_S is $\vee$-closed in P, we pick any $x, y \in G_S$ such that $x \vee y$ exists in P, and show $x \vee y \in G_S$. This will occur if $\text{Shadow}(x \vee y) \subseteq S$, so we pick any $c \in \text{Shadow}(x \vee y)$ and show $c \in S$. We may assume x, y are incomparable, and that $c \notin \text{Shadow}(x \wedge y)$; otherwise the result is trivial. Now by Lemma 9.4, there exists $a \in \text{Shadow}(x) \backslash \text{Shadow}(y)$ and there exists $b \in \text{Shadow}(y) \backslash \text{Shadow}(x)$ such that $c \in \text{Shadow}(a \vee b)$. Observe $a \vee b$ is a line, and

$$|\text{Shadow}(a \vee b) \cap S| \geq |\{a, b\}| = 2,$$

so

$$\text{Shadow}(a \vee b) \subseteq S$$

by (9.6). Now

$$c \in \text{Shadow}(a \vee b) \subseteq S,$$

as desired. Now G_S is $\vee$-closed. We have now shown G_S is a nonempty, $\vee$-closed lower ideal in P, so G_S is a submatroid of P by Lemma 9.1.

Theorem 9.7. *Let P denote a prematroid. let $\mathcal{G}$ denote the set of all submatroids of P, and let $\mathcal{S}$ denote the set of all subspaces of P. Then the maps*

$$\begin{aligned} \mathcal{G} &\rightarrow \mathcal{S} \\ G &\rightarrow A_G \end{aligned}$$

and

$$\begin{aligned} \mathcal{S} &\rightarrow \mathcal{G} \\ S &\rightarrow G_S \end{aligned}$$

are inverses. In particular, They are both bijections.

Proof. First, let S denote a subspace of P, and write $G = G_S$. Then it is immediate from the construction that $A_G = S$. Secondly, let G denote any submatroid in P, and write $S = A_G$. We show $G = G_S$. To see $G \subseteq G_S$, pick any $x \in G$. Observe $\text{Shadow}(x) \subseteq S$ since G is a lower ideal in P, so $x \in G_S$. Hence $G \subseteq G_S$. To see $G \supseteq G_S$, pick any $x \in G_S$. Then $\text{Shadow}(x) \subseteq S$. Now

$$x = \bigvee_{a \in \text{Shadow}(x)} a \in G,$$

since G is $\vee$-closed by Lemma 9.1, and since $S \subseteq G$ by the construction. Hence $G \supseteq G_S$, so $G = G_S$. We have now established the given maps are inverses. This proves Theorem 9.7.

§10. Singular subspaces

Definition 10.1. Let P denote a prematroid. A subspace S of P is said to be *singular* whenever

(10.1) $$x \vee_P y \text{ exists for all } x, y \in S.$$

Lemma 10.2. *Let P denote a prematroid, and pick any $x \in P$. Then* $\text{Shadow}(x)$ *is a singular subspace of P.*

Proof. To show $\text{Shadow}(x)$ is a subspace of P, consider the interval $G = [0, x]$. Observe G is a submatroid of P by Lemma 8.4(ii), Lemma 9.1(ii), so A_G is a subspace of P by Lemma 9.6(i). Observe $\text{Shadow}(x) = A_G$ by construction, so $\text{Shadow}(x)$ is a subspace of P. It is clear that $\text{Shadow}(x)$ is singular. This proves Lemma 10.2.

Let P denote a prematroid, and let S denote a singular subspace of P. Must there exist an element $x \in P$ such that $\text{Shadow}(x) = S$? The answer is "no" in general, but "yes" in the following special case.

Theorem 10.3. *Let P denote a prematroid such that: for all $a \in A_P$, and all $u \in P$,*

(10.2) $$\textit{if } a \vee_P b \textit{ exists for all } b \in \text{Shadow}(u), \textit{ then } a \vee_P u \textit{ exists.}$$

Then for all singular subspaces S of P, there exists an element $x \in P$ such that

(10.3) $$\text{Shadow}(x) = S.$$

Proof. Let the singular subspace S be fixed, and let $G = G_S$ be the corresponding submatroid from (9.7). Pick any $x \in \max(G)$, and recall

$$\text{Shadow}(x) \subseteq S \tag{10.4}$$

by (9.7). We show equality holds in (10.4). Suppose not. Then there exists a point $a \in S \backslash \text{Shadow}(x)$. Since a, Shadow(x) are contained in a common singular subspace, $a \vee_P b$ exists for all $b \in \text{Shadow}(x)$. It follows by (10.2) that $a \vee_P x$ exists. Observe $a \vee_P x \in G$, (since G is $\vee$-closed by Lemma 9.1(i)), and $a \vee_P x > x$ (since $a \not\leq x$), and we have contradicted the maximality of x in G. We conclude equality holds in (10.4), and the theorem is proved.

§11. More on the distance function

Lemma 11.1. *Let P denote a prematroid, and pick any x, x', $y, y' \in P$ such that both $xx'y$ and $xy'y$ are geodesic. Then the following* (i)–(iv) *are equivalent.*

(i) $xx'y'y$ *is geodesic.*
(ii) $x \wedge x' \geq x \wedge y'$ *and* $y \wedge y' \geq y \wedge x'$.
(iii) $x' \wedge y' \geq (x \wedge y') \vee (x' \wedge y)$.
(iv) $x' \wedge y' = (x \wedge y') \vee (x' \wedge y)$.

Proof. (i) $\rightarrow$ (ii). Applying Lemma 7.8(ii) to the geodesic sequence $xx'y'$, we find $x' \geq x \wedge y'$. Of course $x \geq x \wedge y'$, so $x \wedge x' \geq x \wedge y'$. Interchanging the roles of x, y, we find $y \wedge y' \geq y \wedge x'$.

(ii) $\rightarrow$ (iii). Observe $x' \geq x \wedge x' \geq x \wedge y'$ by (ii), and certainly $y' \geq x \wedge y'$, so $x' \wedge y' \geq x \wedge y'$. Interchanging the roles of x, y, we find $x' \wedge y' \geq x' \wedge y$, and line (iii) follows.

(iii) $\rightarrow$ (iv). The sequence x, $x \wedge y'$, y' is geodesic by Lemma 7.4, and we assume $xy'y$ is geodesic, so x, $x \wedge y'$, y', y is geodesic. Since

$$x \wedge y' \leq (x \wedge y') \vee (x' \wedge y) \leq x' \wedge y' \leq y',$$

the four elements in the above line form a geodesic sequence. Now

$$x,\ x \wedge y',\ (x \wedge y') \vee (x' \wedge y),\ x' \wedge y',\ y',\ y \qquad \text{is geodesic,}$$

so in particular,

$$x,\ (x \wedge y') \vee (x' \wedge y),\ x' \wedge y',\ y \qquad \text{is geodesic.} \tag{11.1}$$

Interchanging the roles of x, y, we find

$$(11.2) \qquad x,\ x'\wedge y',\ (x\wedge y')\vee(x'\wedge y),\ y \qquad \text{is geodesic,}$$

and (11.1), (11.2) imply (iv).

(iv) $\rightarrow$ (i). By Lemma 7.4 and our assumptions, we observe $x,\ x\wedge y',\ y',\ y$ is geodesic. Observe $x\wedge y' \le x'\wedge y' \le y'$, so $x,\ x\wedge y', x'\wedge y',\ y',\ y$ is geodesic. In particular $x,\ x'\wedge y',\ y',\ y$ is geodesic. Interchanging the roles of x, y, we find $x,\ x',\ x'\wedge y',\ y$ is geodesic. Combining the above information, we find

$$(11.3) \qquad x,\ x',\ x'\wedge y',\ y',\ y \qquad \text{is geodesic.}$$

In particular, $xx'y'y$ is geodesic, so (i) holds. We have now proved Lemma 11.1.

Corollary 11.2. *Let P denote a prematroid, and pick $x, x', y, y' \in P$ such that $x \le x'$ and $y \le y'$. Then the following are equivalent.*

(i) *$xx'y'y$ is geodesic.*
(ii) *$xx'y$ and $xy'y$ are both geodesic.*

Proof. (i) $\rightarrow$ (ii). Clear.

(ii) $\rightarrow$ (i). We assume $x \le x'$, so

$$\begin{aligned} x\wedge x' &= x \\ (11.4) \qquad &\ge x\wedge y'. \end{aligned}$$

Interchanging the roles of x, y, we obtain

$$(11.5) \qquad y\wedge y' \ge y\wedge x'.$$

The condition in Lemma 11.1(ii) is now satisfied by (11.4), (11.5), so $xx'y'y$ is geodesic by that lemma.

§12. The function δ and the sets $x \star y$

Definition 12.1. Let P denote a prematroid. For all $x, y \in P$, define

$$(12.1) \qquad \delta(x,y) := \min\{\, \partial(x,z) \mid z \in P, \quad z\vee y \ \text{exists}\},$$

$$(12.2) \qquad x\star y := \{z \in P \mid \partial(x,z) = \delta(x,y), \quad z\vee y \ \text{exists}\}.$$

To get a feel for the above definition, we consider a very special case.

Lemma 12.2. *Let P denote a prematroid, and pick any $x, y \in P$. Then the following are equivalent.*

(i) $x \vee y$ *exists.*
(ii) $\delta(x,y) = 0$.
(iii) $x \star y = \{x\}$.
(iv) $\delta(y,x) = 0$.
(v) $y \star x = \{y\}$.

Proof. Immediate from Definition 12.1.

Let P denote a prematroid, and pick any $x, y \in P$. In general

$$\delta(x,y) \neq \delta(y,x). \tag{12.3}$$

It turns out (as we will show in Sections 18, 19) that δ is symmetric in its arguments precisely when P satisfies the augmentation axiom, and in this case $x \star y$ is a $[x \wedge y, x]$-basis system. For now, we will lay some groundwork with a general fact about $x \star y$, and an interpretation of δ.

Theorem 12.3. *Let P denote a prematroid, and pick any x, $y \in P$. Then*

$$x \star y \subseteq [x \wedge y,\ x]. \tag{12.4}$$

Proof. Pick any $z \in x \star y$. We first show $z \leq x$. Suppose $z \not\leq x$. We will obtain a contradiction by showing

$$(x \wedge z) \vee y \qquad \text{exists} \tag{12.5}$$

and

$$\partial(x, x \wedge z) \ < \ \partial(x,z). \tag{12.6}$$

To see (12.5), observe $x \wedge z \leq z \leq y \vee z$ and $y \leq y \vee z$, so $x \wedge z$, y have an upper bound. To see (12.6), recall $x, x \wedge z, z$ is geodesic by Lemma 7.4, and $x \wedge z \neq x$ by the construction. Line (12.6) follows. Now (12.5), (12.6) contradict Definition 12.1, so $z \leq x$. It remains to show $x \wedge y \leq z$. Suppose $x \wedge y \not\leq z$. We will obtain a contradiction by showing

$$\big((x \wedge y) \vee z\big) \vee y \qquad \text{exists} \tag{12.7}$$

and

$$\partial\big(x,\ (x \wedge y) \vee z\big) \ < \ \partial(x,z). \tag{12.8}$$

To see (12.7), observe $x \wedge y \leq y \leq y \vee z$ and $z \leq y \vee z$, so $(x \wedge y) \vee z \leq y \vee z$. Of course $y \leq y \vee z$, so $y \vee z$ is an upper bound for $(x \wedge y) \vee z$, y. We now have (12.7). To see (12.8), observe x is an upper bound for $x \wedge y$, z, so $(x \wedge y) \vee z \leq x$. Now $z \leq (x \wedge y) \vee z \leq x$, so $z, (x \wedge y) \vee z, x$ is geodesic. $z \neq (x \wedge y) \vee z$ by construction, and (12.8) follows. Now (12.7), (12.8) contradict Definition 12.1, so $x \wedge y \leq z$. We conclude $z \in [x \wedge y,\, x]$, and the theorem is proved.

We now establish an interpretation of δ. We begin with a technical lemma, and proceed to our main result Theorem 12.5.

Lemma 12.4. *Let P denote a prematroid, and pick any $x, y \in P$. Then for all $x' \in x^+$ and all $y' \in y^+$,*

(i) $\partial(x', x' \wedge y') \geq \delta(x, y)$,
(ii) $\partial(y', x' \wedge y') \geq \delta(y, x)$,
(iii) $\partial(x', y') \geq \delta(x, y) + \delta(y, x)$.

Proof. (i) Observe y' is an upper bound for $x \wedge y'$, y, so $(x \wedge y') \vee y$ exists. Now

$$\partial(x, x \wedge y') \geq \delta(x, y) \tag{12.9}$$

by Definition 12.1. Observe x' is an upper bound for x, $x' \wedge y'$, so $x \vee (x' \wedge y')$ exists. Clearly

$$x' \wedge y' \quad \leq \quad x \vee (x' \wedge y') \quad \leq \quad x',$$

so

$$\begin{aligned} \partial(x', x' \wedge y') &\geq \partial(x \vee (x' \wedge y'),\, x' \wedge y') & \\ &= \partial(x,\, x \wedge (x' \wedge y')) & \text{(modularity)} \\ &= \partial(x,\, x \wedge y') & \text{(since } x \leq x') \\ &\geq \delta(x, y) & \end{aligned}$$

by (12.9).

(ii) Similar.

(iii) Recall x', $x' \wedge y'$, y' is geodesic by Lemma 7.4, so

$$\begin{aligned} \partial(x', y') &= \partial(x', x' \wedge y') + \partial(x' \wedge y', y') \\ &\geq \delta(x, y) + \delta(y, x) \end{aligned}$$

by (i), (ii). This proves Lemma 12.4.

Theorem 12.5. *Let P denote a prematroid. Then for all $x, y \in P$,*

$$(12.10) \qquad \delta(x,y) + \delta(y,x) \;=\; \min\{\,\partial(x',y') \mid x' \in x^+,\;\; y' \in y^+\}.$$

Proof. In view of Lemma 12.4, it suffices to find some $x' \in x^+$ and some $y' \in y^+$ such that

$$(12.11) \qquad \partial(x',y') \;=\; \delta(x,y) \;+\; \delta(y,x).$$

To this end, pick $u \in x \star y$ and $v \in y \star x$, and set

$$x' := x \vee v,$$
$$y' := y \vee u.$$

Clearly $x' \in x^+$, $y' \in y^+$, so it remains to check (12.11). To obtain it, we decompose $\partial(x,y)$ in two ways. First, observe both $xx'y$ and $xy'y$ are geodesic by Lemma 7.8(i), (iii). Now $xx'y'y$ is geodesic by Corollary 11.2, so

$$(12.12) \qquad \partial(x,y) = \partial(x,x') \;+\; \partial(x',y') \;+\; \partial(y',y).$$

Second, observe $x, u, x \wedge y, v, y$ is geodesic by Lemma 7.4 and Theorem 12.3, so

$$(12.13) \qquad \partial(x,y) = \partial(x,u) \;+\; \partial(u, x \wedge y) \;+\; \partial(x \wedge y, v) \;+\; \partial(v,y).$$

We now evaluate the terms on the right in (12.13). By Definition 12.1,

$$(12.14) \qquad \partial(x,u) = \delta(x,y),$$
$$(12.15) \qquad \partial(v,y) = \delta(y,x).$$

Observe u, y are relative complements in $[x \wedge y, y']$ by Lemma 7.8(ii),(iii), so by modularity

$$(12.16) \qquad \partial(u, x \wedge y) = \partial(y', y).$$

Similarly,

$$(12.17) \qquad \partial(x \wedge y, v) = \partial(x, x').$$

Subtracting (12.12) from the sum of (12.13)–(12.17), we obtain (12.11), as desired. This proves the theorem.

§13. The functions ρ and γ

Definition 13.1. Let P denote a prematroid. For all $x, y \in P$, define

(i) $\rho(x,y) :\doteq \operatorname{rank}(x \wedge y)$,
(ii) $\gamma(x,y) := \partial(x \wedge y, z)$,

where z is any element of $x \star y$. (Observe $\gamma(x,y)$ is independent of the choice of z by Definition 12.1, Theorem 12.3.)

Let P denote a prematroid, and pick any $x, y \in P$. In this section, we consider the triple $\rho(x,y), \gamma(x,y), \delta(x,y)$. In our main result Theorem 13.5, we determine how this triple changes as we replace x by an element in P adjacent x.

Lemma 13.2. *Let P denote a prematroid. Then for all $x, y \in P$,*

$$\operatorname{rank}(x) = \rho(x,y) + \gamma(x,y) + \delta(x,y). \tag{13.1}$$

Proof. Pick any $z \in x \star y$. Observe $0,\ x \wedge y,\ z,\ x$ is geodesic by Theorem 12.3, so

$$\begin{aligned} \operatorname{rank}(x) &= \partial(0, x \wedge y) + \partial(x \wedge y, z) + \partial(z, x) \\ &= \rho(x,y) + \gamma(x,y) + \delta(x,y) \end{aligned}$$

by Lemma 7.1(ii) and Definitions 12.1, 13.1.

Lemma 13.3. *Let P denote a prematroid of rank D. Then for all $x, y \in P$,*

(i) *$\rho(x,y),\ \gamma(x,y),\ \delta(x,y)$ are nonnegative integers,*
(ii) *$\rho(x,y) + \gamma(x,y) + \gamma(y,x) + \delta(x,y) \leq D$.*

Proof. (i) Immediate.

(ii) Pick any $v \in y \star x$. Then $x \vee v$ exists by Definition 12.1. Observe x, v are relative complements in the interval $[x \wedge y, x \vee v]$ by Lemma 7.8(ii),(iii) (with $u := x$), so

$$\begin{aligned} \operatorname{rank}(x \vee v) - \operatorname{rank}(x) &= \operatorname{rank}(v) - \operatorname{rank}(x \wedge y) \\ &= \gamma(y,x) \end{aligned} \tag{13.2}$$

by Lemma 7.1(ii). Now by (13.2), Lemma 13.2, and the construction,

$$\begin{aligned} D &\geq \operatorname{rank}(x \vee v) \\ &= \operatorname{rank}(x) + \gamma(y,x) \\ &= \rho(x,y) + \gamma(x,y) + \delta(x,y) + \gamma(y,x), \end{aligned}$$

as desired. This proves Lemma 13.3.

Before proceeding to the main theorem of this section, we mention a result about ρ.

Lemma 13.4. *Let P denote a prematroid. Then for all $x, y, z \in P$, the following are equivalent.*

(i) $z \in [x \wedge y, x]$.
(ii) $z \leq x$, *and* xzy *is geodesic.*
(iii) $z \leq x$, *and* $z \wedge y = x \wedge y$.
(iv) $z \leq x$, *and* $\rho(z, y) = \rho(x, y)$.

Proof. (i) $\to$ (ii). Clearly $z \leq x$. We may view $z = u \vee v$, where $u := z$, $v := x \wedge y$, so xzy is geodesic by Lemma 7.8 (i),(iii).

(ii) $\to$ (iii). By Lemma 7.8(i),(ii), $x \wedge z$, $z \wedge y$ are relative complements in the interval $[x \wedge y, z]$. But $x \wedge z = z$, so $z \wedge y = x \wedge y$.

(iii) $\to$ (i). Observe $x \wedge y = z \wedge y \leq z$.

(iii) $\to$ (iv). Immediate from Definition 13.1(i).

(iv) $\to$ (iii). Observe $z \wedge y \leq z \leq x$ and $z \wedge y \leq y$, so $z \wedge y \leq x \wedge y$. But $z \wedge y, x \wedge y$ have the same rank, so they are equal. This proves Lemma 13.4.

Theorem 13.5. *Let P denote a prematroid, and pick any x, y, $z \in P$ such that x, z are adjacent. Then*

(i)

$$\operatorname{rank}(x) - \operatorname{rank}(z) = \Delta\rho + \Delta\gamma + \Delta\delta, \tag{13.3}$$

(ii)

$$\partial(x, y) - \partial(z, y) = -\Delta\rho + \Delta\gamma + \Delta\delta, \tag{13.4}$$

(iii)

$$|\Delta\rho| + |\Delta\gamma| + |\Delta\delta| = 1, \tag{13.5}$$

where

$$\Delta\rho = \rho(x, y) - \rho(z, y), \tag{13.6}$$

$$\Delta\gamma = \gamma(x, y) - \gamma(z, y), \tag{13.7}$$

$$\Delta\delta = \delta(x, y) - \delta(z, y). \tag{13.8}$$

Proof. (i) Immediate from Lemma 13.2.

(ii) By Corollary 7.6(i),

$$\partial(x,y) = \operatorname{rank}(x) + \operatorname{rank}(y) - 2\operatorname{rank}(x \wedge y), \tag{13.9}$$

$$\partial(z,y) = \operatorname{rank}(z) + \operatorname{rank}(y) - 2\operatorname{rank}(z \wedge y). \tag{13.10}$$

Subtracting (13.10) from (13.9), and evaluating $\operatorname{rank}(x)-\operatorname{rank}(z)$ using (13.3), we obtain (13.4).

(iii) Since $\delta(x,y)$, $\delta(z,y)$ measure the distance from x, z to the same set, and since x, z are adjacent, we have

$$\Delta\delta \in \{-1,\ 0,\ 1\} \tag{13.11}$$

by the triangle inequality. First assume $\Delta\delta \neq 0$. Interchanging x, z if necessary, we may assume that

$$\Delta\delta = 1. \tag{13.12}$$

We show

$$\Delta\rho = 0, \qquad \Delta\gamma = 0. \tag{13.13}$$

This will follow by Definition 13.1 if we can show

$$z \star y \subseteq x \star y, \tag{13.14}$$

$$z \wedge y = x \wedge y. \tag{13.15}$$

To see (13.14), pick any $w \in z \star y$. Then $w \vee y$ exists, so

$$\partial(x,w) \geq \delta(x,y). \tag{13.16}$$

Also $\partial(z,w) = \delta(z,y)$, so

$$\begin{aligned} \partial(x,w) &\leq \partial(x,z) \ + \ \partial(z,w) && (13.17)\\ &= 1 + \delta(z,y) \\ &= \delta(x,y) && (13.18) \end{aligned}$$

by (13.12). We must now have equality in (13.16)–(13.18). In particular $w \in x \star y$ by the construction. We now have (13.14). To see (13.15), pick any $w \in z \star y$. Then $w \in x \star y$ by (13.14), so $w \in [x \wedge y, x]$ by Theorem 12.3. Also xzw is geodesic by (13.17), so $z \in [x \wedge y, x]$ by Lemma 8.4. Now $z \wedge x = x \wedge y$ by Lemma 13.4(i),(iii), so (13.15) holds. Line (13.13) follows from (13.14), (13.15), and (13.5) follows from (13.12), (13.13). Next assume $\Delta\delta = 0$. By (7.3), the pair

$$(\operatorname{rank}(x) - \operatorname{rank}(z),\ \partial(x,y) - \partial(z,y))$$

is one of $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$. In each of these four cases, one readily solves (13.3), (13.4) for $\Delta\rho$, $\Delta\gamma$, and finds (13.5) holds in each case. We have now proved (iii), and the theorem.

Corollary 13.6. *Let P denote a prematroid, and pick $x, y, z \in P$ such that $z \geq x$. Then*

(i) $\rho(x,y) \leq \rho(z,y)$,
(ii) $\gamma(x,y) \leq \gamma(z,y)$,
(iii) $\delta(x,y) \leq \delta(z,y)$,
(iv) $\delta(y,x) \leq \delta(y,z)$,
(v) $\gamma(y,x) \geq \gamma(y,z)$.

Proof. (i)–(iii) It suffices to assume z covers x. By Theorem 13.5(i),(iii), exactly one of $\rho(x,y)-\rho(z,y)$, $\gamma(x,y)-\gamma(z,y)$, $\delta(x,y)-\delta(z,y)$ equals -1, and the other two equal 0. The inequalities follow.

(iv) Pick any $w \in y \star z$. Then by Definition 12.1,

$$\partial(y,w) = \delta(y,z) \tag{13.19}$$

and $z \vee w$ exists. Observe $x \vee w$ exists, since $x \leq z \leq z \vee w$ and $w \leq z \vee w$, so

$$\partial(y,w) \geq \delta(y,x) \tag{13.20}$$

by Definition 12.1. Our result follows from (13.19), (13.20).

(v) By Lemma 13.2, parts (i), (iv) above, and since ρ is symmetric in its arguments,

$$\begin{aligned} \gamma(y,x) &= \operatorname{rank}(y) - \rho(y,x) - \delta(y,x) \\ &\geq \operatorname{rank}(y) - \rho(y,z) - \delta(y,z) \\ &= \gamma(y,z), \end{aligned}$$

as desired. This proves Corollary 13.6.

Lemma 13.7. *Let P denote a prematroid, and let G denote a geodesically closed subposet of P. Then for all x, $y \in G$,*

(i) $x \star_G y = x \star_P y$,
(ii) $\delta_G(x,y) = \delta_P(x,y)$,
(iii) $\gamma_G(x,y) = \gamma_P(x,y)$,
(iv) $\rho_G(x,y) = \rho_P(x,y) - \operatorname{rank}(0_G)$.

Proof. (i), (ii) Immediate from Lemma 8.2 and Definition 12.1.
(iii) Immediate from Lemma 5.4, Definition 13.1(ii).
(iv) Immediate from Lemma 5.6, Definition 13.1(i).

§14. The posets $[x \wedge y, x \star y]$, $[x \star y, x]^*$

For notational convenience, we expand our notion of an interval in a prematroid.

Definition 14.1. Let P denote a prematroid. For any nonempty subsets $H \subseteq P$, $K \subseteq P$, define the subposet

$$[H,\, K] \;:=\; \{\, z \in P \mid \exists x \in H,\; \exists y \in K \quad \text{such that} \quad xzy \quad \text{is geodesic}\}.$$

Let P denote a prematroid, and pick any x, $y \in P$. In this section, we consider the posets $[x \wedge y, x \star y]$, $[x \star y, x]^*$. (Recall from (3.1), the $*$ means we *reverse* the usual partial order. As we mentioned in Section 12, it will turn out that if P satisfies the augmentation axiom, then $x \star y$ is a $[x \wedge y, x]$-basis system. In this case $[x \wedge y, x \star y]$ becomes a $[x \wedge y, x]$-matroid, $[x \star y, x]^*$ becomes a $[x \wedge y, x]^*$-matroid, and these matroids are duals in the sense of Definition 3.2. In this section, we establish a few facts about these posets.

Lemma 14.2. *Let P denote a prematroid, and pick any x, $y \in P$.*

(i) *The zero of $[x \wedge y, x \star y]$ is $x \wedge y$.*
(ii) $\mathrm{top}\big([x \wedge y, x \star y]\big) = x \star y$.
(iii) $\mathrm{rank}\big([x \wedge y, x \star y]\big) = \gamma(x, y)$.
(iv) *The zero of $[x \star y, x]^*$ is x.*
(v) $\mathrm{top}\big([x \star y, x]^*\big) = x \star y$.
(vi) $\mathrm{rank}\big([x \star y, x]^*\big) = \delta(x, y)$.

Proof. Immediate from Definitions 12.1, 13.1, 14.1, and Theorem 12.3.

Lemma 14.3. *Let P denote a prematroid, and fix any $x, y \in P$. Then for all $z \in P$, the following are equivalent.*

(i) $z \in [x \star y, x]^*$.
(ii) $z \leq x$, *and*

$$\delta(x, y) - \delta(z, y) = \mathrm{rank}(x) - \mathrm{rank}(z). \tag{14.1}$$

(iii) $z \leq x$, *and* $z \star y \subseteq x \star y$.
(iv) $z \leq x$, *and*

$$\rho(x, y) = \rho(z, y), \qquad\qquad \gamma(x, y) = \gamma(z, y). \tag{14.2}$$

Proof. (i) $\rightarrow$ (ii). Observe $z \leq x$ by Theorem 12.3. By assumption, there exists $u \in x \star y$ such that $u \leq z \leq x$. Now

$$\begin{aligned} \operatorname{rank}(x) - \operatorname{rank}(z) &= \partial(x,z) \\ &= \partial(x,u) - \partial(z,u) \\ &= \delta(x,y) - \delta(z,y) \end{aligned}$$

since $u \in z \star y$.

(ii) $\rightarrow$ (iii). Pick any $w \in z \star y$. We show $w \in x \star y$. Observe $w \vee y$ exists by assumption, so it suffices to show $\partial(x,w) = \delta(x,y)$. Observe $w \leq z \leq x$ by Theorem 12.3 and the construction, so

$$\begin{aligned} \partial(x,w) &= \partial(x,z) + \partial(z,w) \\ &= \operatorname{rank}(x) - \operatorname{rank}(z) + \delta(z,y) \\ &= \delta(x,y), \end{aligned}$$

as desired.

(iii) $\rightarrow$ (i). Pick any $w \in z \star y$. Then $w \leq z \leq x$ by Theorem 12.3, and $w \in x \star y$ by assumption, so $z \in [x \star y, x]^*$.

(ii) $\leftrightarrow$ (iv). The three scalars $\rho(x,y) - \rho(z,y)$, $\gamma(x,y) - \gamma(z,y)$, $\delta(x,y) - \delta(z,y)$ are non-negative by Corollary 13.6, and sum to $\operatorname{rank}(x)$ $-\operatorname{rank}(z)$ by Theorem 13.5(i). The result is now immediate. This proves Lemma 14.3.

§15. The posets x_y^+, x_y^-

Let P denote a prematroid. For all x, $y \in P$, define the subposets $x_y^+, x_y^- \subseteq P$ by

(15.1) $$x_y^+ := \{ z \in P \mid x \leq z, \quad xzy \text{ is geodesic}\},$$

(15.2) $$x_y^- := \{ z \in [x \wedge y, x] \mid z \vee y \text{ exists in } P\}.$$

Observe x is the zero of x_y^+, and $x \wedge y$ is the zero of x_y^-. In particular x_y^+, x_y^- are not empty.

Example 15.1. Let P denote a prematroid, and pick any $x, y \in P$ such that $x \vee y$ exists. Then

(i) $x_y^+ = [x, x \vee y]$,
(ii) $x_y^- = [x \wedge y, x]$.

Proof. (i) First consider the inclusion $\subseteq$. Pick any $z \in x_y^+$. Then xzy is geodesic, so $z \leq x \vee y$ by Lemma 7.9(i),(ii). Also $x \leq z$, so $z \in [x, x \vee y]$. Now consider the inclusion $\supseteq$. Pick any $z \in [x, x \vee y]$. Then x, z, $x \vee y$ is geodesic by (7.5). Also x, $x \vee y$, y is geodesic by Lemma 7.5, so x, z, $x \vee y$, y is geodesic. In particular, xzy is geodesic, so $z \in x_y^+$.

(ii) Immediate from (15.2).

Lemma 15.2. *Let P denote a prematroid, and pick any $x, y \in P$.*

(i) *x_y^+ is a submatroid of x^+.*
(ii) *x_y^+ is geodesically closed in P.*

Proof. (i) First, we claim x_y^+ is a lower ideal in x^+. Pick any $z \in x_y^+$ and any $w \in x^+$ such that $w \leq z$. We show $w \in x_y^+$. Of course $x \leq w$, so it remains to show xwy is geodesic. to this end, observe xwz is geodesic, since $x \leq w \leq z$, and xzy is geodesic by assumption, so $xwzy$ is geodesic. In particular xwy is geodesic, as desired. Hence x_y^+ is a lower ideal in x^+.

Next, we claim x_y^+ is $\vee$-closed in P. To see this, pick any $z, z' \in x_y^+$ such that $z \vee z'$ exists in P. Recall xzy, $xz'y$ are both geodesic, so $x, z \vee z', y$ is geodesic by Lemma 7.10. Recall $x \leq z$, $x \leq z'$, so $x \leq z \vee z'$. Now $z \vee z' \in x_y^+$, so x_y^+ is $\vee$-closed in P. Now x_y^+ is a nonempty $\vee$-closed lower ideal in x^+, so x_y^+ is a submatroid of x^+ by Lemma 9.1.

(ii) Immediate from (i) above and Lemma 9.2.

Lemma 15.3. *Let P denote a prematroid, and fix any x, $y \in P$. Then for all $z \in P$, the following are equivalent.*

(i) *$z \in x_y^+$.*
(ii) *$x \leq z$, and x, $y \wedge z$ are relative complements in the interval $[x \wedge y, z]$.*
(iii) *There exists an element $v \in [x \wedge y, y]$ such that $z = x \vee v$.*
(iv) *$x \leq z$, and*

$$\rho(z,y) - \rho(x,y) = \operatorname{rank}(z) - \operatorname{rank}(x). \tag{15.3}$$

(v) *$x \leq z$, and*

$$\gamma(x,y) = \gamma(z,y), \qquad \delta(x,y) = \delta(z,y). \tag{15.4}$$

Moreover, if (i)–(v) *hold, then*

$$v = y \wedge z. \tag{15.5}$$

Proof. (i) $\leftrightarrow$ (ii) $\leftrightarrow$ (iii). This is a special case of Lemma 7.8.

(ii) $\rightarrow$ (iv). Immediate from (7.14) and Definition 13.1(i).

(iv) $\rightarrow$ (i). By (7.12) and the observation $x \wedge z = x$, we find $\partial(x,z) + \partial(z,y) - \partial(x,y)$ equals twice

$$\mathrm{rank}(z) \; - \; \mathrm{rank}(x) \; + \; \rho(x,y) - \rho(z,y),$$

and is therefore 0 by (15.3). Now xzy is geodesic by (7.4), and (i) follows.

(iv) $\leftrightarrow$ (v). The scalars $\rho(z,y)-\rho(x,y)$, $\gamma(z,y)-\gamma(x,y)$, $\delta(z,y)-\delta(x,y)$ are nonnegative by Corollary 13.6, and sum to $\mathrm{rank}(z)-\mathrm{rank}(x)$ by Theorem 13.5(i). The result is now immediate.

Now suppose (i)–(v) hold. Then (15.5) holds by (7.19). This proves Lemma 15.3.

We now turn to x_y^-.

Lemma 15.4. *Let P denote a prematroid, and pick any $x, y \in P$. Then*

(i) *x_y^- is a lower ideal in the interval $[x \wedge y, x]$,*
(ii) $\mathrm{top}(x_y^-) = x \star y$,
(iii) $[x \wedge y, x \star y] \subseteq x_y^-$.

Proof. (i) Suppose we are given some $z \in x_y^-$ and some $z' \in [x \wedge y, x]$ such that $z' \leq z$. We show $z' \in x_y^-$. To do this, we must show $y \vee z'$ exists in P. But this is the case, since $y \vee z \geq y$ and $y \vee z \geq z \geq z'$.

(ii) x_y^- is a lower ideal in $[x \wedge y, x]$ by (i), so we may regard $\mathrm{top}(x_y^-)$ as the set of elements $z \in x_y^-$ with $\partial(x,z)$ minimal. Recall $x \star y$ consists of the elements $z \in P$ such that $z \vee y$ exists, and such that $\partial(x,z)$ is minimal subject to this existence. Observe $x \star y \subseteq x_y^-$ by Theorem 12.3 and the construction, and $w \vee y$ exists for all $w \in x_y^-$, so we may regard $x \star y$ as the set of elements $z \in x_y^-$ with $\partial(x,z)$ minimal. Our result follows.

(iii) Immediate from (i), (ii).

Our next goal is to show the posets x_y^+, y_x^- are isomorphic.

Theorem 15.5. *Let P denote a prematroid, and pick any $x, y \in P$. Then there exists poset isomorphisms $\sigma : x_y^+ \to y_x^-$, $\varepsilon : y_x^- \to x_y^+$ such that*

$$\sigma(z) = y \wedge z \qquad (\forall z \in x_y^+), \tag{15.6}$$

$$\varepsilon(v) = x \vee v \qquad (\forall v \in y_x^-). \tag{15.7}$$

Moreover, σ, ε are inverses.

Proof. Pick any $z \in x_y^+$. By Lemma 15.3(i),(iii), and (15.5),

$$y \wedge z \in y_x^- \tag{15.8}$$

and

$$x \vee (y \wedge z) = z. \tag{15.9}$$

Pick any $v \in y_x^-$. Then by Lemma 15.3(i),(iii) and (15.5),

$$x \vee v \in x_y^+ \tag{15.10}$$

and

$$y \wedge (x \vee v) = v. \tag{15.11}$$

By (15.8), there exists a map $\sigma : x_y^+ \to y_x^-$ satisfying (15.6). By (15.10), there exists a map $\varepsilon : y_x^- \to x_y^+$ satisfying (15.7). Observe σ, ε are inverses by (15.9), (15.11); in particular, these maps are bijections. It remains to check σ, ε respect the partial order. But this follows, since for all $z, z' \in x_y^+$,

$$z \leq z' \quad \to \quad y \wedge z \leq y \wedge z',$$

and for all $v, v' \in y_x^-$,

$$v \leq v' \quad \to \quad x \vee v \leq x \vee v'.$$

This proves Theorem 15.5.

Corollary 15.6. *Let P denote a prematroid. Then for all $x, y \in P$, the subposet x_y^+ is embeddable.*

Proof. It is clear y_x^- is embeddable. Indeed the identity map is an embedding of y_x^- into the modular atomic lattice $[x \wedge y, y]$. Now x_y^+ is embeddable, since x_y^+, y_x^- are isomorphic by Theorem 15.5.

Corollary 15.7. *Let P denote a prematroid. Then for all $x, y \in P$,*

(i) $\operatorname{rank}(x_y^-) = \gamma(x, y)$,
(ii) $\operatorname{rank}(x_y^+) = \gamma(y, x)$,
(iii) $|\operatorname{top}(x_y^+)| = |y \star x|$.

Proof. (i) Immediate from Lemma 14.2(iii) and Lemma 15.4(i),(ii).

(ii) Immediate from (i) above and the fact that x_y^+, y_x^- are isomorphic.

(iii) Immediate from Lemma 15.4(ii) and the fact that x_y^+, y_x^- are isomorphic.

This proves Corollary 15.7.

We finish this section with two technical results.

Lemma 15.8. *Let P denote a prematroid. Pick any $x, y \in P$ and pick any $z \in x_y^+$. Then*

(i) $z_y^+ = z^+ \cap x_y^+$,
(ii) $y_z^- = [y \wedge z, y] \cap y_x^-$,
(iii) $y_z^+ = y_x^+$.

Proof. (i) Recall $x \leq z$ and xzy is geodesic. To see the inclusion $\subseteq$, pick any $w \in z_y^+$. Then $z \leq w$, so $w \in z^+$. We show $w \in x_y^+$. Certainly $x \leq z \leq w$. Observe xzy and zwy are geodesic, so $xzwy$ is geodesic. In particular xwy is geodesic, so $w \in x_y^+$. To see the inclusion $\supseteq$, pick any $w \in z^+ \cap x_y^+$. Observe $x \leq z \leq w$ so xzw is geodesic. Also xwy is geodesic, so $xzwy$ is geodesic. In particular zwy is geodesic, so $w \in z_y^+$.

(ii) By Theorem 15.5, the map $\sigma : w \to y \wedge w$ induces an isomorphism of posets $x_y^+ \to y_x^-$ such that $\sigma(z_y^+) = y_z^-$. Now by (i) above,

$$\begin{aligned} y_z^- &= \sigma(z_y^+) \\ &= \sigma(z^+ \cap x_y^+) \\ &= (y \wedge z)^+ \cap y_x^- \\ &= [y \wedge z, y] \cap y_x^-, \end{aligned}$$

as desired.

(iii) This is immediate from Corollary 11.2. This proves Lemma 15.8.

Lemma 15.9. *Let P denote a prematroid, and pick any $x, y \in P$. Then for all $u \in y^+$,*

$$x_y^+ \text{ is a submatroid of } x_u^+. \tag{15.12}$$

Proof. By Lemma 15.2(i), it suffices to show $x_y^+ \subseteq x_u^+$, and without loss, we may assume u covers y. By (7.3), there are two possibilities;

$$\partial(x,u) = \partial(x,y) - 1 \tag{15.13}$$

or

$$\partial(x,u) = \partial(x,y) + 1. \tag{15.14}$$

First suppose (15.13). Then xuy is geodesic, so $u \in y_x^+$. Now $x_y^+ = x_u^+$ by Lemma 15.8(iii). Next assume (15.14). To see $x_y^+ \subseteq x_u^+$ in this case, we pick any $w \in x_y^+$ and show $w \in x_u^+$. Observe xwy is geodesic by assumption, and xyu is geodesic by (15.14), so $xwyu$ is geodesic. In particular xwu is geodesic, so $w \in x_u^+$, as desired. This proves Lemma 15.9.

§16. Projection into a submatroid

Lemma 16.1. *Let P denote a prematroid, and let G, H denote submatroids of P. Then $G \cap H$ is a submatroid of P.*

Proof. $G \cap H$ is geodesically closed in P, and contains 0_P, so we are done by Lemma 9.1.

Lemma 16.2. *Let P denote a prematroid. Let G denote a submatroid of P, and pick any $x \in P$. Then there exists a unique element $p = p(x, G)$ in G such that*

$$[0,x] \cap G = [0,p]. \tag{16.1}$$

We call p the projection of x into G, and write

$$p = \mathrm{proj}_G x. \tag{16.2}$$

Proof. $[0,x]$ is a submatroid of P by Lemma 8.4(ii), Lemma 9.1, so $[0,x] \cap G$ is a submatroid of P by Lemma 16.1. $[0,x] \cap G$ is contained in the interval $[0, x]$, and is therefore an interval by Lemma 8.5.

Lemma 16.3. *Let P denote a prematroid. Let G denote any submatroid of P, and pick any $x, y \in P$.*

(i) $\text{proj}_G x \leq x$.
(ii) *Equality holds in* (i) *if and only if* $x \in G$.
(iii) $\text{proj}_G(x \wedge y) = x \wedge \text{proj}_G y$.
(iv) $\text{proj}_G(x \wedge y) = \text{proj}_G x \wedge \text{proj}_G y$.

Proof. (i), (ii) Immediate from Lemma 16.2.
(iii) By (16.1), (16.2),

$$\begin{aligned}
[0, \text{proj}_G(x \wedge y)] &= [0, x \wedge y] \cap G \\
&= [0, x] \cap [0, y] \cap G \\
&= [0, x] \cap [0, \text{proj}_G y],
\end{aligned}$$

and our result follows.
(iv) Interchanging the roles of x, y in (iii),

$$\text{proj}_G(x \wedge y) = y \wedge \text{proj}_G x.$$

By this and (iii), we may view

$$\begin{aligned}
\text{proj}_G(x \wedge y) &= \text{proj}_G(x \wedge y) \wedge \text{proj}_G(x \wedge y) \\
&= x \wedge \text{proj}_G x \wedge y \wedge \text{proj}_G y \\
&= \text{proj}_G x \wedge \text{proj}_G y,
\end{aligned}$$

since $\text{proj}_G x \leq x$, $\text{proj}_G y \leq y$. This proves Lemma 16.3.

Theorem 16.4. *Let P denote a prematroid, and let G denote a submatroid of P. Then for all $x \in P$, and for all $y \in G$, the sequence*

$$x, \text{proj}_G x, y \qquad \textit{is geodesic.} \tag{16.3}$$

Proof. G is a lower ideal in P that contains y, and $x \wedge y \leq y$, so $x \wedge y \in G$. Certainly $x \wedge y \leq x$, so

$$x \wedge y \in [0, x] \cap G = [0, \text{proj}_G x],$$

forcing $x \wedge y \leq \text{proj}_G x$. Now $x \wedge y \leq \text{proj}_G x \leq x$ by the construction, so $x, \text{proj}_G x, y$ is geodesic by Lemma 13.4(i),(ii). This proves Theorem 16.4.

§17. The projection $x^+ \to x_y^+$

Let P denote a prematroid. Pick any $x, y \in P$, and write $G = x_y^+$. Observe x^+ is a prematroid by Corollary 8.3, Lemma 8.4, and G is a submatroid of x^+ by Lemma 15.2(i), so there exists a projection map $\mathrm{proj}_G : x^+ \to G$ by Lemma 16.2.

Lemma 17.1. *Let P denote a prematroid. Pick any $x, y \in P$, and write $G = x_y^+$. Then for all $z \in x^+$,*

$$\mathrm{proj}_G z = x \vee (y \wedge z). \tag{17.1}$$

Proof. We first show

$$\mathrm{proj}_G z \geq x \vee (y \wedge z). \tag{17.2}$$

Observe $x \leq z$, so $x \wedge y \leq y \wedge z$. Also $y \wedge z \leq y$, so $y \wedge z \in [x \wedge y, y]$. Observe z is an upper bound for $x, y \wedge z$, so $x \vee (y \wedge z)$ exists. Now $y \wedge z \in y_x^-$ by (15.2), so

$$x \vee (y \wedge z) \in G \tag{17.3}$$

by Theorem 15.5. Observe

$$x \vee (y \wedge z) \leq z \tag{17.4}$$

by our remarks above, and (17.2) follows from (17.3), (17.4), and Lemma 16.2. Next, we show

$$\mathrm{proj}_G z \leq x \vee (y \wedge z). \tag{17.5}$$

Write $p := \mathrm{proj}_G z$. Then $p \leq z$ by Lemma 16.3(i), so $y \wedge p \leq y \wedge z$. Also $p \in G = x_y^+$ by Lemma 16.2, so by Theorem 15.5,

$$p = x \vee (y \wedge p) \leq x \vee (y \wedge z).$$

Line (17.5) follows, and we are done by (17.2), (17.5).

Theorem 17.2. *Let P denote a prematroid. Pick any $x, y \in P$, and write $G = x_y^+$. Then for all $z \in x^+$, and for all $p \in P$, the following are equivalent.*

(i) $p = \mathrm{proj}_G z$.
(ii) $p \in [x, z]$, *and wpy is geodesic for all $w \in [x, z]$.*
(iii) $p \in [x, z]$, *and both xpy and zpy are geodesic.*
(iv) $p \in [x, z]$, *and*

$$\rho(p, y) = \rho(z, y), \qquad \gamma(p, y) = \gamma(x, y), \qquad \delta(p, y) = \delta(x, y). \tag{17.6}$$

Proof. (i) $\to$ (ii). Observe $p \in [x, z]$ by Lemma 16.3(i). Pick any $w \in [x, z]$. We show wpy is geodesic. By Lemma 7.8(i),(iii), it suffices to find

$$u \in [y \wedge w, w], \tag{17.7}$$

$$v \in [y \wedge w, y], \tag{17.8}$$

such that

$$p = u \vee v. \tag{17.9}$$

Set

$$u := \mathrm{proj}_G w.$$

Then (17.7) holds, since $u \le w$ by Lemma 16.3(i), and since

$$\begin{aligned} u &= x \vee (y \wedge w) \\ &\ge y \wedge w \end{aligned} \tag{17.10}$$

by Lemma 17.1. Set

$$v := y \wedge z. \tag{17.11}$$

Then $v \le y$ by construction. Recall $w \le z$, so

$$y \wedge w \le v \tag{17.12}$$

by (17.11). Now (17.8) holds. Now

$$\begin{aligned} p &= x \vee (y \wedge z) && \text{(Lemma 17.1)} \\ &= x \vee v && (17.11) \\ &= x \vee (y \wedge w) \vee v && (17.12) \\ &= u \vee v, && (17.10) \end{aligned}$$

so (17.9) holds. Now wpy is geodesic by (17.7)–(17.9), and we are done.

(ii) $\to$ (iii). Clear.

(iii) $\to$ (i). We assume $x \le p$ and xpy is geodesic, so $p \in x_y^+$ by (15.1). Now

$$p = x \vee (y \wedge p) \tag{17.13}$$

by Theorem 15.5. We assume $p \le z$ and zpy is geodesic, so

$$y \wedge p = y \wedge z \tag{17.14}$$

by Lemma 13.4(ii),(iii). Now

$$p = x \vee (y \wedge z) \tag{17.15}$$

by (17.13), (17.14), so $p = \mathrm{proj}_G z$ by Lemma 17.1.

(iii) $\leftrightarrow$ (iv). We assume $p \leq z$, so by Lemma 13.4(ii),(iv), zpy is geodesic if and only if

$$\rho(p, y) = \rho(z, y).$$

We assume $x \leq p$, so by Lemma 15.3(i),(v), xpy is geodesic if and only if

$$\gamma(p, y) = \gamma(x, y), \qquad \delta(p, y) = \delta(x, y).$$

This proves Theorem 17.2.

Theorem 17.3. *With the notation of Theorem* 17.2, *suppose* (i)–(iv) *hold. Then for all* $w \in [x, z]$, *the following are equivalent.*

(i) wxy *and* wzy *are both geodesic.*
(ii) wxp *and* wzp *are both geodesic.*
(iii) w, p *are relative complements in the interval* $[x, z]$.
(iv) $\rho(w, y) = \rho(x, y), \qquad \gamma(w, y) = \gamma(z, y), \qquad \delta(w, y) = \delta(z, y).$

Proof. (i) $\rightarrow$ (ii). Observe xpy is geodesic by Theorem 17.2(iii), and wxy is geodesic, so $wxpy$ is geodesic. In particular, wxp is geodesic. Similarly, zpy is geodesic by Theorem 17.2(iii), and wzy is geodesic, so $wzpy$ is geodesic. In particular, wzp is geodesic.

(ii) $\rightarrow$ (i). Recall wpy is geodesic by Theorem 17.2(ii), and wxp is geodesic, so $wxpy$ is geodesic. In particular wxy is geodesic. Similarly wzp is geodesic, so $wzpy$ is geodesic. In particular wzy is geodesic.

(ii) $\leftrightarrow$ (iii). We assume $x \leq w$, $x \leq p$, so by Lemma 7.4, $x = w \wedge p$ if and only if wxp is geodesic. We assume $w \leq z$, $p \leq z$, so by Lemma 7.5, $z = w \vee p$ if and only if wzp is geodesic.

(i) $\leftrightarrow$ (iv). We assume $x \leq w$, so by Lemma 13.4(ii),(iv), wxy is geodesic if and only if

$$\rho(w, y) = \rho(x, y).$$

We assume $w \leq z$, so by Lemma 15.3(i),(v), wzy is geodesic if and only if both

$$\gamma(w, y) = \gamma(z, y), \qquad \delta(w, y) = \delta(z, y).$$

This proves Theorem 17.3.

§18. The augmentation axiom

Let P denote a prematroid, and recall by Definition 4.1 that P is a quantum matroid if and only if P satisfies the augmentation axiom AU from that definition. In this section, we show this occurs if and only if the function δ from Definition 12.1 is symmetric in its arguments.

We begin with some notation.

Lemma 18.1. *Let P denote a prematroid. Pick any nonnegative integer d, and pick any path $p = (x_0, x_1, \ldots, x_d)$ $(x_0, x_1, \ldots, x_d \in P)$. Then the following* (i), (ii) *are equivalent.*

(i) *There does not exist an integer i $(0 \le i \le d-3)$ such that*

$$x_i > x_{i+1} < x_{i+2} < x_{i+3}, \tag{18.1}$$

or such that

$$x_i > x_{i+1} > x_{i+2} < x_{i+3}. \tag{18.2}$$

(ii) *There exists integers e, f $(0 \le e \le f \le d,\ \ f-e$ is even), such that*

$$x_0 < x_1 < \cdots < x_{e-1} < x_e,$$

$$x_e > x_{e+1} < x_{e+2} > x_{e+3} < \cdots > x_{f-3} < x_{f-2} > x_{f-1} < x_f,$$

$$x_f > x_{f+1} > \cdots > x_{d-1} > x_d.$$

Suppose (i), (ii) *hold. Then we say p is up-flat-down. If $e = 0$ and $f = d$, we say p is flat.*

Proof. Routine.

Theorem 18.2. *Let P denote a prematroid of rank D. Then the following* (i)–(vi) *are equivalent.*

(i) *P satisfies the augmentation axiom AU in Definition* 4.1.

(ii) *For all integers i $(2 \le i \le D)$, and for all $x, y \in P$ such that* $\operatorname{rank}(x) = i-1$, $\operatorname{rank}(y) = i$, *and $\partial(x, y) = 3$, there exists a path in P with endpoints x, y and shape $(i-1, i, i-1, i)$.*

(iii) *For all integers i $(2 \le i \le D)$, and all geodesic paths p in P of shape $(i-1, i-2, i-1, i)$, there exists a path p' in P that has the same endpoints as p, and has shape $(i-1, i, i-1, i)$.*

(iv) *For all $x, y \in P$, there exists a geodesic up-flat-down path connecting x, y.*

(v) *For all $x, y \in P$, for all $x' \in \operatorname{top}(x_y^+)$, and for all $y' \in \operatorname{top}(y_x^+)$, there exists a geodesic flat path connecting x', y'.*

(vi) *$\delta(x, y) = \delta(y, x)$ for all $x, y \in P$.*

Proof. (i) $\to$ (ii). Observe $\operatorname{rank}(x) < \operatorname{rank}(y)$, so by assumption, there exists an atom $a \in P$ such that $a \leq y$, $a \not\leq x$, and such that $x \vee a$ exists in P. Set $u := x \vee a$. Observe a covers $0 = x \wedge a$, so u covers x by modularity. In particular $\operatorname{rank}(u) = i$. We show $\partial(u, y) = 2$. Suppose not. Then $\partial(u, y) = 4$ by (7.3), so uxy is geodesic. In this case $u \wedge y = x \wedge y$ by Lemma 13.4(ii), (iii), contradicting the fact that $a \leq u \wedge y$, $a \not\leq x \wedge y$. We have now shown $\partial(u, y) = 2$, so u, y cover $u \wedge y$ by Lemma 7.4. Now $x, u, u \wedge y, y$ is a path with shape $(i-1, i, i-1, i)$, as desired.

(ii) $\to$ (iii). Immediate.

(iii) $\to$ (iv). Set $d := \partial(x, y)$, and pick a geodesic path

$$p := (x = x_0,\, x_1,\, \ldots,\, x_d = y) \qquad\qquad (x_0,\, x_1,\, \ldots,\, x_d \in P)$$

with maximal weight in the sense of (7.7). We claim p is up-flat-down. Suppose not. Then by Lemma 18.1, there exists an integer i $(0 \leq i \leq d-3)$ such that either

$$x_i > x_{i+1} < x_{i+2} < x_{i+3} \tag{18.3}$$

or

$$x_i > x_{i+1} > x_{i+2} < x_{i+3}. \tag{18.4}$$

Interchanging the roles of x, y if necessary, we may assume (18.3). The path x_i, x_{i+1}, x_{i+2}, x_{i+3} is geodesic, with shape $(j-1, j-2, j-1, j)$ for an appropriate integer j $(2 \leq j \leq D)$, so by (iii), there exists a path x_i, x'_{i+1}, x'_{i+2}, x_{i+3} of shape $(j-1, j, j-1, j)$. Observe the sequence

$$p' := (x_0,\, x_1,\, \ldots,\, x_i,\, x'_{i+1},\, x'_{i+2},\, x_{i+3},\, \ldots,\, x_{d-1},\, x_d = y)$$

is a path. p' is geodesic, since p, p' have the same length, and

$$\operatorname{weight}(p') = \operatorname{weight}(p) + 2.$$

This contradicts our construction, and we conclude p is up-flat-down.

(iv) $\to$ (v). Let x, y, x', y' be given. By (iv), there exists a geodesic up-flat-down path p connecting x', y'. We claim p is flat. Set $d := \partial(x', y')$, and write $p = (x' = x_0,\, x_1,\, \ldots,\, x_d = y')$. Suppose p is not flat. Then $d \geq 1$, and either $x' < x_1$ or $x_{d-1} > y'$. Interchanging the roles of x, y if necessary, we may assume $x' < x_1$. Recall $xx'y'y$ is geodesic by Corollary 11.2. Now $xx'x_1y'y$ is geodesic by the construction, and in particular xx_1y is geodesic. Also $x \leq x' < x_1$, so $x_1 \in x_y^+$ by (15.1). But this is inconsistent with $x' < x_1$ and the assumption $x' \in \operatorname{top}(x_y^+)$. We conclude p is flat.

(v) $\to$ (vi). Let x, y be given, and pick any $x' \in \mathrm{top}(x_y^+)$, $y' \in \mathrm{top}(y_x^+)$. By (v) there exists a flat path connecting x', y', so

$$\mathrm{rank}(x') = \mathrm{rank}(y'). \tag{18.5}$$

By Lemma 13.2, Lemma 15.2(i), and Corollary 15.7(ii),

$$\begin{aligned} \mathrm{rank}(x') &= \mathrm{rank}(x) + \mathrm{rank}(x_y^+) \\ &= \rho(x,y) + \gamma(x,y) + \delta(x,y) + \gamma(y,x), \end{aligned} \tag{18.6}$$

and similarly

$$\mathrm{rank}(y') = \rho(y,x) + \gamma(y,x) + \delta(y,x) + \gamma(x,y). \tag{18.7}$$

The result now follows from (18.5)–(18.7), since ρ is symmetric in its arguments.

(vi) $\to$ (i). Pick any $x, y \in P$ such that $\mathrm{rank}(x) < \mathrm{rank}(y)$. We find an atom $a \in P$ such that $a \le y$, $a \not\le x$, and such that $x \vee a$ exists in P. To this end, observe by Lemma 13.2, Corollary 15.7(i), and (vi) above that

$$\begin{aligned} \mathrm{rank}(y_x^-) &= \gamma(y,x) \\ &= \mathrm{rank}(y) - \rho(y,x) - \delta(y,x) \\ &> \mathrm{rank}(x) - \rho(x,y) - \delta(x,y) \\ &= \gamma(x,y) \\ &\ge 0, \end{aligned}$$

so there exists $v \in y_x^-$ such that $v > x \wedge y$. By Lemma 9.3(ii), there exists an atom $a \in P$ such that $a \le v$ but $a \not\le x \wedge y$. Observe $a \le v \le y$. Observe $a \not\le x$; otherwise $a \le x \wedge y$, a contradiction. Observe $x \vee a$ exists in P, since $x \vee v$ is an upper bound for a, x. The element a now has the desired properties, so we are done. This proves Theorem 18.2.

§19. $x \star y$ is a $[x \wedge y, x]$-basis system

Let P denote a quantum matroid, and pick any $x, y \in P$. Our goal in this section is to establish the related facts that $x \star y$ is a $[x \wedge y, x]$-basis system, $[x \wedge y, x \star y]$ is a $[x \wedge y, x]$-matroid, and that $[x \star y, x]^*$ is a $[x \wedge y, x]^*$-matroid.

Lemma 19.1. *Let P denote a quantum matroid, and let G denote a geodesically closed subposet of P. Then G is a quantum matroid.*

Proof. G is a prematroid by Corollary 8.3, so it remains to show G satisfies the augmentation axiom. By Theorem 18.2, it suffices to show the function δ_G is symmetric in its arguments. But this is the case, since δ_G is a restriction of δ_P by Lemma 13.7(ii), and since δ_P is symmetric in its arguments by Theorem 18.2(vi). This proves Lemma 19.1.

Corollary 19.2. *Let P denote a quantum matroid, and pick any $x, y \in P$. Then x_y^+, x^+ are both quantum matroids.*

Proof. These subposets are geodesically closed in P by Lemma 8.4, Lemma 15.2(ii), so they are quantum matroids by Lemma 19.1. This proves Corollary 19.2.

Theorem 19.3. *Let P denote a quantum matroid, and pick any $x, y \in P$.*

(i) $x_y^- = [x \wedge y, x \star y]$.
(ii) $x \star y$ *is a* $[x \wedge y, x]$*-basis system.*
(iii) $[x \wedge y, x \star y]$ *is a* $[x \wedge y, x]$*-matroid.*
(iv) $[x \star y, x]^*$ *is a* $[x \wedge y, x]^*$*-matroid.*

Proof. (i) Observe the posets x_y^-, y_x^+ are isomorphic by Theorem 15.5, and y_x^+ is a quantum matroid by Lemma 19.2, so x_y^- is a quantum matroid. Now by Lemma 4.3 and Lemma 15.4(ii),

$$\max(x_y^-) = \mathrm{top}(x_y^-) = x \star y,$$

and our result follows since x_y^- is a lower ideal in $[x \wedge y, x]$.

(ii) We have seen x_y^- is both a quantum matroid and a lower ideal in $[x \wedge y, x]$, so x_y^- is a $[x \wedge y, x]$-matroid. Now $x \star y = \max(x_y^-)$ is a $[x \wedge y, x]$-basis system by Theorem 2.5(ii).

(iii) Immediate from (ii) and Theorem 2.5(i).

(iv) Immediate from (ii), Theorem 2.5(i), and Lemma 3.1.

§20. The notion of relative closeness

Lemma 20.1. *Let P denote a quantum matroid. Then for all $x, y \in P$,*

$$\rho(x, y) + \delta(x, y) \leq \mathrm{rank}(y). \tag{20.1}$$

Proof. Observe by Definition 13.1(i), Lemma 13.2, and Theorem 18.2(vi) that

$$\begin{aligned} \rho(x,y)+\delta(x,y) &= \rho(y,x)+\delta(y,x) \\ &= \operatorname{rank}(y)-\gamma(y,x) \\ &\le \operatorname{rank}(y). \end{aligned} \tag{20.2}$$

We now consider the case of equality.

Lemma 20.2. *Let P denote a quantum matroid, and pick any $x, y \in P$. Then the following* (i)–(v) *are equivalent.*

(i) *Equality holds in (20.1).*
(ii) $\gamma(y,x)=0$.
(iii) *x is the unique element in x_y^+.*
(iv) *$x \wedge y$ is the unique element in y_x^-.*
(v) $y \star x = x \wedge y$.
(vi) *zxy is geodesic for all $z \in x^+$.*

If (i)–(vi) *hold, we say x is relatively close to y.*

Proof. (i) $\leftrightarrow$ (ii). Immediate from (20.2).

(ii) $\leftrightarrow$ (iii) $\leftrightarrow$ (iv). The posets x_y^+, y_x^- both have rank $\gamma(y,x)$ by Corollary 15.7.

(iv) $\leftrightarrow$ (v). Recall $y_x^- = [x\wedge y, y\star x]$ by Theorem 19.3(i).

(iii) $\rightarrow$ (vi). Pick any $z \in x^+$, and write $p := x \vee (y \wedge z)$. Then zpy is geodesic by Lemma 17.1, Theorem 17.2(iii). But $p \in x_y^+$ by Lemma 17.1, so $p = x$ by (iii). We conclude zxy is geodesic, as desired.

(vi) $\rightarrow$ (iii). Pick any $z \in x_y^+$. Then certainly $z \in x^+$, so zxy is geodesic by (vi). Also xzy is geodesic by (15.1), so $z = x$. This proves Lemma 20.2.

Lemma 20.3. *Let P denote a quantum matroid, and pick any $x, y, z \in P$.*

(i) *If x is relatively close to y, and $x \le z$, then z is relatively close to y.*
(ii) *Suppose $z \in \operatorname{top}(x_y^+)$. Then z is relatively close to y.*

Proof. (i) Observe $0 \le \gamma(y,z) \le \gamma(y,x)$ by Lemma 13.3(i), Corollary 13.6(v), and $\gamma(y,x)=0$ by Lemma 20.2(ii), so $\gamma(y,z)=0$. Now z is relatively close to y by Lemma 20.2.

(ii) z is the unique element in z_y^+ by Lemma 15.8(i), so z is relatively close to y by Lemma 20.2(iii).

Theorem 20.4. *Let P denote a quantum matroid, and pick any $x, y \in P$ such that x is relatively close to y. Then for all $z \in x^+$, the following* (i)–(iii) *hold.*

(i) $\rho(z, y) = \rho(x, y)$,
(ii) $\delta(z, y) = \delta(x, y)$,
(iii) $\gamma(z, y) - \gamma(x, y) = \text{rank}(z) - \text{rank}(x)$.

Proof. (i) $x \leq z$ by assumption, and zxy is geodesic by Lemma 20.2(vi), so $\rho(z, y) = \rho(x, y)$ by Lemma 13.4(ii),(iv).

(ii) z is relatively close to y by Lemma 20.3(i), so by Lemma 20.2(i) and (i) above,

$$\begin{aligned}\delta(z, y) &= \text{rank}(y) - \rho(z, y) \\ &= \text{rank}(y) - \rho(x, y) \\ &= \delta(x, y).\end{aligned}$$

(iii) By Lemma 13.2,

$$\text{rank}(x) = \rho(x, y) + \gamma(x, y) + \delta(x, y), \tag{20.3}$$

$$\text{rank}(z) = \rho(z, y) + \gamma(z, y) + \delta(z, y). \tag{20.4}$$

Our result is immediate upon subtracting (20.3) from (20.4), and evaluating the result using (i), (ii) above.

§21. The staircase theorem

In this section, we describe a quantum matroid in a way that may help the reader visualize its structure. Theorem 21.3 is our main result. First, we need a few definitions.

Recall a *directed graph* (or *di-graph*) is a pair $\mathcal{D} := (V\mathcal{D}, E\mathcal{D})$, where $V\mathcal{D}$ is a nonempty finite set (of *vertices*) and $E\mathcal{D} \subseteq V\mathcal{D} \times V\mathcal{D}$ (the *edges*). For all $u, v \in V\mathcal{D}$, we write $u \to v$ whenever $uv \in E\mathcal{D}$. Observe possibly both $u \to v$ and $v \to u$, possibly only one of these occurs, or possibly neither. We may also have $u \to u$.

Definition 21.1. Let $\mathcal{D}$ denote any di-graph, and let P denote any poset. By a *$\mathcal{D}$-partition* of P, we mean a map $\sigma : P \to V\mathcal{D}$, such that (i), (ii) hold below:

(i) For all $x, y \in P$, if x, y are adjacent then $\sigma(x) \to \sigma(y)$ or $\sigma(y) \to \sigma(x)$ (or both).

(ii) For all $u, v \in V\mathcal{D}$ such that $u \to v$, and for all $x \in P$ such that $\sigma(x) = u$, then there exists $y \in P$ such that x, y are adjacent and such that $\sigma(y) = v$.

(Caution: We do not require σ be onto $V\mathcal{D}$).

The quantum matroids have $\mathcal{D}$-partitions for certain di-graphs $\mathcal{D}$, described below.

Definition 21.2. For any nonnegative integers a, b, define the di-graph $\mathcal{D} = \mathcal{D}(a, b)$ as follows: The vertex set $V\mathcal{D}$ is the set of three tuples

(21.2) $V\mathcal{D} := \{(\rho, \gamma, \delta) | \rho, \gamma, \delta \in \mathbb{Z}, 0 \leq \rho, 0 \leq \delta, \rho + \delta \leq a, 0 \leq \gamma \leq b\}.$

For all pairs of vertices (ρ, γ, δ), $(\rho', \gamma', \delta') \in V\mathcal{D}$, there is an edge $(\rho, \gamma, \delta) \to (\rho', \gamma', \delta')$ in $\mathcal{D}$ whenever one of the following rows holds:

(21.2)	$\rho' = \rho + 1$	$\gamma' = \gamma$	$\delta' = \delta$
(21.3)	$\rho' = \rho - 1$	$\gamma' = \gamma$	$\delta' = \delta$
(21.4)	$\rho' = \rho$	$\gamma' = \gamma + 1$	$\delta' = \delta$
(21.5)	$\rho' = \rho$	$\gamma' = \gamma - 1$	$\delta' = \delta = 0$
(21.6)	$\rho' = \rho$	$\gamma' = \gamma$	$\delta' = \delta - 1$

Observe the "shape" of $\mathcal{D}(a, b)$ resembles that of a staircase of height a and width b. For example, $\mathcal{D}(2, 3)$ looks as follows (we abbreviate $u \leftrightarrow v$ whenever $u \to v$ and $v \to u$):

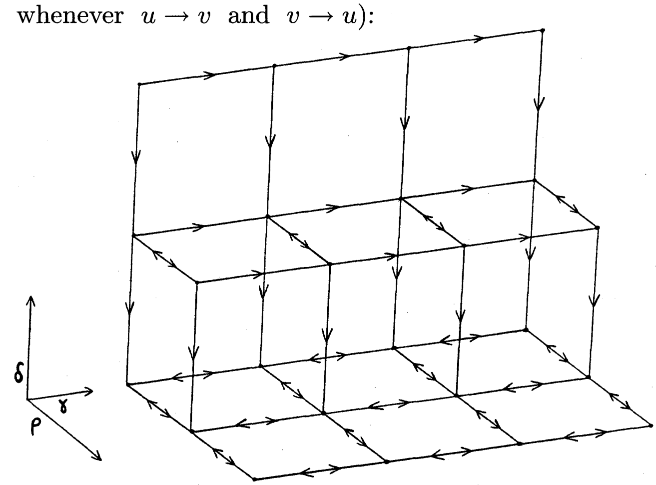

Fig. 1.

Theorem 21.3. *Let P denote a quantum matroid with rank D. Pick any integer a $(0 \leq a \leq D)$, fix any $y \in P$ such that* $\operatorname{rank}(y) = a$*, and set*

$$\sigma(x) := \big(\rho(x,y),\ \gamma(x,y),\ \delta(x,y)\big) \qquad (\forall x \in P). \tag{21.7}$$

Then σ is a $\mathcal{D}(a,\ D-a)$-partition of P.

Proof. Abbreviate $\mathcal{D} := \mathcal{D}(a, D-a)$. Pick any $x \in P$, and abbreviate

$$\begin{aligned} \rho &:= \rho(x,y), \\ \gamma &:= \gamma(x,y), \\ \delta &:= \delta(x,y). \end{aligned}$$

Let us first check $\sigma(x) \in V\mathcal{D}$. To do this, we verify ρ, γ, δ satisfy the inequalities in (21.1) (with $b = D-a$). Observe $0 \leq \rho$, $0 \leq \gamma$, $0 \leq \delta$ by Lemma 13.3(i), and $\rho + \delta \leq a$ by Lemma 20.1. To see $\gamma \leq D-a$, observe

$$\begin{aligned} \gamma(x,y) &\leq D - \rho(x,y) - \gamma(y,x) - \delta(x,y) \\ &= D - \operatorname{rank}(y) \end{aligned}$$

by Lemmas 13.2, 13.3 and Theorem 18.2(vi). We have now shown $\sigma(x) \in V\mathcal{D}$.

Next, let us verify that σ satisfies (i) of Definition 21.1. To this end, pick any $z \in P$ such that x, z are adjacent, and set

$$\begin{aligned} \rho' &:= \rho(z,y), \\ \gamma' &:= \gamma(z,y), \\ \delta' &:= \delta(z,y). \end{aligned}$$

We must show $(\rho, \gamma, \delta) \to (\rho', \gamma', \delta')$ or $(\rho', \gamma', \delta') \to (\rho, \gamma, \delta)$. Interchanging x, z if necessary, we may assume x covers z. But then by Theorem 13.5(i),(iii), the three tuple $(\rho - \rho', \gamma - \gamma', \delta - \delta')$ equals either

$$(1,0,0) \quad \text{(in which case } (\rho,\gamma,\delta) \to (\rho',\gamma',\delta') \text{ by (21.3))},$$

or

$$(0,1,0) \quad \text{(in which case } (\rho',\gamma',\delta') \to (\rho,\gamma,\delta) \text{ by (21.4))},$$

or

$$(0,0,1) \quad \text{(in which case } (\rho,\gamma,\delta) \to (\rho',\gamma',\delta') \text{ by (21.6))}.$$

We have now shown σ satisfies (i) of Definition 21.1.

It remains to show σ satisfies part (ii) of Definition 21.1. To this end, let x, ρ, γ, δ be as above, and pick any $(\rho',\gamma',\delta') \in V\mathcal{D}$ such that $(\rho,\gamma,\delta) \to (\rho',\gamma',\delta')$. We must find $z \in P$ such that x, z are adjacent, and such that $\sigma(z) = (\rho',\gamma',\delta')$. We consider the 5 cases (21.2)–(21.6) in turn.

First assume $(\rho',\gamma',\delta') = (\rho+1,\gamma,\delta)$. Observe by Lemma 13.2 and Corollary 15.7(ii) that

$$\begin{aligned}\operatorname{rank}(x_y^+) &= \gamma(y,x)\\ &= a-\rho-\delta\\ &= a-\rho'-\delta'+1\\ &\geq 1,\end{aligned}$$

so there exists $z \in x_y^+$ such that z covers x. Observe $\sigma(z) = (\rho+1,\gamma,\delta)$ by Lemma 15.3(iv),(v).

Next assume $(\rho',\gamma',\delta') = (\rho-1,\gamma,\delta)$. Then

$$\rho = \rho' + 1 \geq 1,$$

so $x \wedge y \neq 0$. Let z' denote a relative complement of $x \wedge y$ in the interval $[0,x]$. Then $z' < x$ by modularity, so there exists $z \in [z',x]$ such that x covers z. Observe $x \in z'^+_y$ by Lemma 15.3(i),(ii), so $z'xy$ is geodesic. Observe $z'zx$ is geodesic by the construction, so $z'zxy$ is geodesic. In particular zxy is geodesic, so $x \in z_y^+$ by (15.1). Now $\sigma(z) = (\rho-1,\gamma,\delta)$ by Lemma 15.3(iv),(v).

Next assume $(\rho',\gamma',\delta') = (\rho,\gamma+1,\delta)$. Pick any $u \in \operatorname{top}(x_y^+)$, and observe

$$\begin{aligned}\operatorname{rank}(u) &= \operatorname{rank}(x) + \operatorname{rank}(x_y^+)\\ &= \rho(x,y)+\gamma(x,y)+\delta(x,y)+\gamma(y,x)\\ &= a+\gamma\\ &= a+\gamma'-1\\ &< D,\end{aligned}$$

so by Lemma 4.3, there exists $v \in P$ such that v covers u. Let z denote a relative complement of u is $[x,v]$. Then z covers x by modularity. We now compute $\sigma(z)$. Observe by Lemma 15.3(iv),(v) and our choice of u that

$$\sigma(u) = (\rho_1,\gamma,\delta), \tag{21.8}$$

where $\rho_1 = \rho + \operatorname{rank}(u) - \operatorname{rank}(x)$. Observe u is relatively close to y by Lemma 20.3(ii), so

$$\sigma(v) = (\rho_1, \gamma + 1, \delta) \tag{21.9}$$

by (21.8) and Theorem 20.4. Observe $u = x \vee (y \wedge v)$ by (21.8), (21.9), Lemma 17.1, and Theorem 17.2(i),(iv), so

$$\sigma(z) = (\rho, \gamma + 1, \delta)$$

by Theorem 17.3(iii),(iv).

Next assume $\delta = 0$ and $(\rho', \gamma', \delta') = (\rho, \gamma - 1, 0)$. Observe by Lemma 13.2 that

$$\begin{aligned} \operatorname{rank}(x) - \rho &= \gamma \\ &= \gamma' + 1 \\ &\geq 1, \end{aligned}$$

so $x > x \wedge y$. Hence there exists $z \in [x \wedge y, x]$ such that x covers z. Now $\sigma(z) = (\rho, \gamma - 1, 0)$ by Definition 13.1, since $z \wedge y = x \wedge y$ by Lemma 13.4(i),(iii) and $z \star y = z$ by Lemma 12.2.

Finally assume $(\rho', \gamma', \delta') = (\rho, \gamma, \delta - 1)$. Observe by Lemma 14.2(vi) that

$$\begin{aligned} \operatorname{rank}\bigl([x \star y, x]^*\bigr) &= \delta \\ &= \delta' + 1 \\ &\geq 1, \end{aligned}$$

so there exists $z \in [x \star y, x]^*$ that is covered by x (in the poset P). Observe $\sigma(z) = (\rho, \gamma, \delta - 1)$ by Lemma 14.3(ii),(iv). This proves Theorem 21.3.

§22. The graph on top(P)

In this section we consider a graph defined on the top of a quantum matroid.

Theorem 22.1. *Let P denote a quantum matroid with rank D.*

(i) *For all $x, y \in top(P)$, there exists a geodesic flat path in P connecting x, y.*

For (ii)–(iv) *below, we view* $\mathrm{top}(P)$ *as the vertex set of an undirected graph, where vertices* $x, y \in \mathrm{top}(P)$ *are declared adjacent whenever* x, y *cover* $x \wedge y$.

(ii) *The graph* $\mathrm{top}(P)$ *is connected.*

(iii) *For all* $x, y \in \mathrm{top}(P)$,

$$\partial_{\mathrm{top}}(x,y) = \partial(x,y)/2 \tag{22.1}$$
$$= D - \rho(x,y) \tag{22.2}$$
$$= \delta(x,y), \tag{22.3}$$

where ∂_{top} *denotes the path length distance function for the graph* $\mathrm{top}(P)$.

(iv)

$$\mathrm{diam}_{\mathrm{top}}(P) = \max\{\delta(x,y) \mid x, y \in P\} \tag{22.4}$$
$$\leq D, \tag{22.5}$$

where $\mathrm{diam}_{\mathrm{top}}(P)$ *denotes the diameter of the graph* $\mathrm{top}(P)$.

Proof. (i) By Theorem 18.2(iv), there exists a geodesic up-flat-down path p connecting x, y. But x, $y \in \mathrm{top}(P)$, so p is flat.

(ii) Immediate from (i).

(iii) Line (22.1) is immediate from (i) and the definition of a flat path in Lemma 18.1. To see (22.2), observe by Corollary 7.6(i) that

$$\begin{aligned}\partial(x,y) &= \mathrm{rank}(x) + \mathrm{rank}(y) - 2\mathrm{rank}(x \wedge y) \\ &= 2\big(D - \rho(x,y)\big).\end{aligned}$$

To see (22.3), observe x is the unique element in x_y^+, so x is relatively close to y by Lemma 20.2(iii). Now

$$\rho(x,y) + \delta(x,y) = \mathrm{rank}(y) = D$$

by Lemma 20.2(i).

(iv) To see (22.4), observe by (22.3) and Corollary 13.6(iii),(iv) that

$$\begin{aligned}\mathrm{diam}_{\mathrm{top}}(P) &= \max\{\, \partial_{\mathrm{top}}(x,y) \mid x, y \in \mathrm{top}(P)\} \\ &= \max\{\, \delta(x,y) \mid x, y \in \mathrm{top}(P)\} \\ &= \max\{\, \delta(x,y) \mid x, y \in P\}.\end{aligned}$$

We now have (22.4). Line (22.5) is immediate from (22.2). This proves Theorem 22.1.

For the remainder of this section, we investigate the quantum matroids P such that $\mathrm{diam}_{\mathrm{top}}(P) \leq 1$.

Lemma 22.2. *For any poset P, the following are equivalent.*

(i) *P is a quantum matroid, and*

$$\mathrm{diam}_{\mathrm{top}}(P) = 0. \tag{22.6}$$

(ii) *P is a quantum matroid, and*

$$|\mathrm{top}(P)| = 1. \tag{22.7}$$

(iii) *P is a modular atomic lattice.*

Proof. (i) $\leftrightarrow$ (ii). Clear

(ii) $\rightarrow$ (iii). Observe $\max(P) = \mathrm{top}(P)$ consists of a single element, so P has a 1. Now $P = [0, 1]$ is a modular atomic lattice by condition M in Definition 4.1.

(iii) $\rightarrow$ (ii). Clear.

Lemma 22.3. *For any poset P, the following are equivalent.*

(i) *P is a quantum matroid, and*

$$\mathrm{diam}_{\mathrm{top}}(P) \leq 1. \tag{22.8}$$

(ii) *P is a quantum matroid, and*

$$x,\ y \text{ cover } x \wedge y \text{ for all distinct } x,\ y \in \mathrm{top}(P). \tag{22.9}$$

(iii) *P is a prematroid, and*

$$x,\ y \text{ cover } x \wedge y \text{ for all distinct } x,\ y \in \max(P). \tag{22.10}$$

(iv) *P is a prematroid, and*

$$\delta(x, y) \leq 1 \qquad \text{for all } \ x,\ y \in P. \tag{22.11}$$

If (i)–(iv) *hold, we call P a design matroid.*

Proof. (i) $\rightarrow$ (ii). Let x, y denote distinct elements in $\mathrm{top}(P)$. Then $\partial_{\mathrm{top}}(x, y) = 1$ by (22.8), so $\partial(x, y) = 2$ by (22.1). Now x, y cover $x \wedge y$ by Lemma 7.4.

(ii) $\rightarrow$ (iii). It is clear P is a prematroid. Also $\max(P) = \mathrm{top}(P)$ by Lemma 4.3, so (22.10) follows from (22.9).

(iii) $\rightarrow$ (iv). Let x, $y \in P$ be given. We show $\delta(x, y) \leq 1$. There exists x', $y' \in \max(P)$ such that $x \leq x'$, $y \leq y'$. Observe $\delta(x, y) \leq \delta(x', y')$ by Corollary 13.6(iii),(iv), so it suffices to show $\delta(x', y') \leq 1$. Assume $\delta(x', y') \neq 0$; otherwise we are done. Then x', y' are distinct, so x', y' covers $x' \wedge y'$ by (22.10). Now $\delta(x', y') = 1$ by Definition 12.1, and we are done.

(iv) $\rightarrow$ (i). To show P is a quantum matroid, we show P satisfies the augmentation axiom. Pick any x, $y \in P$. By Theorem 18.2(vi), it suffices to show

$$\delta(x, y) = \delta(y, x). \tag{22.12}$$

First suppose $x \vee y$ exists. Then $\delta(x, y) = 0$, $\delta(y, x) = 0$ by Lemma 12.2, so (22.12) holds. Now suppose $x \vee y$ does not exist. Then $\delta(x, y) \neq 0$, $\delta(y, x) \neq 0$ by Lemma 12.2, so $\delta(x, y)$, $\delta(y, x)$ are both 1 by (22.11). Again (22.12) holds, so (22.12) holds in general. Now P satisfies the augmentation axiom by Theorem 18.2, so P is a quantum matroid. To see (22.8), observe by (22.4), (22.11) that

$$\begin{aligned}\mathrm{diam}_{\mathrm{top}}(P) &= \max\{\delta(x, y) \mid x,\, y \in P\} \\ &\leq 1.\end{aligned}$$

This proves Lemma 22.3.

Lemma 22.4. *Let P denote a quantum matroid. Then x_y^+, x_y^- are design matroids for all x, $y \in P$ such that $\delta(x, y) = 1$.*

Proof. We show x_y^- is a design matroid by showing it satisfies condition (ii) in Lemma 22.3. Observe x_y^- is a quantum matroid by Theorem 19.3(i),(iii), so it remains to show u, v cover $u \wedge v$ for all distinct u, $v \in \mathrm{top}(x_y^-)$. By Theorem 12.3, Lemma 15.4(ii), and our assumption $\delta(x, y) = 1$, we find x covers u, v. Now $u \vee v = x$ covers u, v, so u, v covers $u \wedge v$ by modularity. We have now shown x_y^- satisfies condition (ii) in Lemma 22.3, so x_y^- is a design-matroid.

To see that x_y^+ is a design-matroid, recall $\delta(y, x) = 1$ by Theorem 18.2(vi), so y_x^- is a design-matroid by our above remarks. Recall x_y^+ is isomorphic to y_x^- by Theorem 15.5, so x_y^+ is a design-matroid. We have now proved Lemma 22.4.

§23. Quantum matroids and diagram geometries

In this section we obtain a characterization of a quantum matroid that might be useful to people doing research on diagram geometries. We do not explicitly introduce the language of diagram geometries in order to avoid cumbersome terminology, but a reader familiar with these geometries should have no trouble translating our result into that language.

Theorem 23.1. *Let D denote a nonnegative integer. Then a poset P is a quantum matroid of rank D if and only if* (i)–(iv) *hold below.*

(i) *P is a prematroid of rank D.*
(ii) *For all $x \in P$, there exists $x' \in \mathrm{top}(P)$ such that $x \leq x'$.*
(iii) *For all $x \in P$ with $\mathrm{rank}(x) \leq D-2$, the subposet of P induced on $x^+ \backslash \{x\}$ is connected.*
(iv) *For all $x \in P$ with $\mathrm{rank}(x) = D-2$, the poset x^+ is a quantum matroid of rank* 2.

(The quantum matroids of rank 2 are described in Example 4.2.)

Proof. We first assume P is a quantum matroid, and verify (i)–(iv). Line (i) is immediate from Definition 5.1, and (ii) is just Lemma 4.3. To see (iii), pick any $x \in P$ such that $\mathrm{rank}(x) \leq D-2$. We show the subposet of P induced on $x^+ \backslash \{x\}$ is connected. By (ii) above, any element in $x^+ \backslash \{x\}$ is connected by a path in $x^+ \backslash \{x\}$ to some element in $\mathrm{top}(x^+)$. Hence it suffices to pick any u, $v \in \mathrm{top}(x^+)$, and show u, v are connected by a path in $x^+ \backslash \{x\}$. By Theorem 22.1(i), there exists a geodesic flat path p in P that connects u, v. Recall x^+ is geodesically closed in P by Lemma 8.4(i), so p is contained in x^+. The elements of p all have rank $D-1$ or D by Lemma 18.1, and $\mathrm{rank}(x) \leq D-2$, so x is not included in p. It follows p is contained in $x^+ \backslash \{x\}$, as desired. We now have (iii). To see (iv), recall x^+ is a quantum matroid by Corollary 19.2, and $\mathrm{rank}(x^+) = 2$ by part (ii) above. We have now proved the theorem in one direction, so we now consider the converse.

Let P denote a poset satisfying (i)–(iv) in the present theorem. We show P is a quantum matroid of rank D by induction on D. The case $D \leq 1$ is trivial, and the case $D = 2$ is immediate from assumption (iv), so from now on assume $D \geq 3$.

P is a prematroid of rank D by assumption (i), so we need only show P satisfies the augmentation axiom. To do this, we show P satisfies condition (iii) in Theorem 18.2. For the rest of this proof, we use the following terminology: For any paths p, p' in P, we say *p' replaces p* whenever p, p' share the same endpoints. For each integer i $(2 \leq i \leq D)$, let E_i denote the proposition that any geodesic path in P of shape $(i-1, i-2, i-1, i)$ can be replaced by a path in P that has shape $(i-1, i, i-1, i)$. The condition (iii) in Theorem 18.2 will follow if we can show E_2, E_3, $\ldots$, E_D. We do this in two steps.

Claim 1. E_3, E_4, $\ldots$, E_D *hold.*

Proof of Claim 1. Pick an integer i $(3 \leq i \leq D)$, and pick a path $xyzw$ in P of shape $(i-1, i-2, i-1, i)$. We show $xyzw$ can be replaced by a path in P that has shape $(i-1, i, i-1, i)$. The poset y^+ satisfies the conditions (i)–(iv) of the present theorem (with D replaced by $D-i+2$), and $D-i+2 \leq D-1$, so y^+ is a quantum matroid by induction. Observe the path $xyzw$ is contained in y^+, and has shape 1012 (in y^+). Applying Theorem 18.2(iii) to y^+, we find the path $xyzw$ can be replaced by a path $xy'z'w$ in y^+ that has shape 1212 (in y^+). Observe the path $xy'z'w$ has shape $(i-1, i, i-1, i)$ (in P), so we are done. This proves Claim 1.

Claim 2. *E_2 holds.*

Proof of Claim 2. Pick any $x, y \in P$ such that $\operatorname{rank}(x) = 1$, $\operatorname{rank}(y) = 2$, and $x \not\leq y$. We show x, y are connected by a path in P that has shape 1212. To do this, we show (i), (ii) below:

(i) There exists a path in P with endpoints x, y and shape $1212 \cdots 12$.
(ii) Any path in P with shape 121212 can be replaced by a path in P with shape 1212.

To see (i), recall by (iii) of the present theorem that there exists a path in $P \backslash \{0\}$ connecting x, y. Of all such paths, pick a path

$$p = (x = x_0, x_1, \ldots, x_d = y) \qquad (x_0, x_1, \ldots, x_d \in P)$$

with minimal weight in the sense of (7.7). Set

$$r := \max\{\operatorname{rank}(x_i) \mid 0 \leq i \leq d\},$$

and observe $r \geq \operatorname{rank}(x_d) = 2$. p will have the desired shape $1212 \cdots 12$ if we can show $r = 2$. Suppose $r \geq 3$, and pick any integer i $(0 \leq i \leq d)$ such that $\operatorname{rank}(x_i) = r$. Then $1 \leq i \leq d-1$, and $x_{i-1} < x_i > x_{i+1}$. Observe $x_0, x_1, \ldots, x_d$ are distinct by the construction, so $x_{i-1} \vee x_{i+1} = x_i$ covers x_{i-1}, x_{i+1}. Now x_{i-1}, x_{i+1} cover $x_{i-1} \wedge x_{i+1}$ by modularity. Now p can be replaced by a path

$$p' = (x = x_0, x_1, \ldots, x_{i-1}, x_{i-1} \wedge x_{i+1}, x_{i+1}, \ldots, x_d = y)$$

that is contained in $P \backslash \{0\}$, and has

$$\operatorname{weight}(p') = \operatorname{weight}(p) - 2.$$

This contradicts the construction, so $r = 2$. The path p now has the desired shape $1212 \cdots 12$, so (i) holds.

To see part (ii) in the present claim, consider the following sequences:

(s1) 121212
(s2) 12323212
(s3) 12323232
(s4) 12321232
(s5) 121232
(s6) 123232
(s7) 123212
(s8) 1232
(s9) 1212

To show part (ii) in the present claim, we show that any path p in P whose shape is one of (s1)–(s8), can be replaced by a path in P whose shape is included below shape(p) in the above list. We write $p = (x_0, x_1, x_2, \ldots)$ and consider each of the shapes (s1)–(s8) in turn.

Case s1. By assumption (ii) of the present theorem, and since $D \geq 3$, there exists $y \in P$ such that y covers x_3. first suppose $x_1 \leq y$. Then $(x_0, x_1, y, x_3, x_4, x_5)$ is a path of shape 123212 (s7). Next suppose $x_1 \not\leq y$. Then (x_1, x_2, x_3, y) is a geodesic path of shape 2123, so by E_3, there exists a path (x_1, z, w, y) in P of shape 2323. Now $(x_0, x_1, z, w, y, x_3, x_4, x_5)$ is a path of shape 12323212 (s2).

Case s2. First suppose $x_4 \geq x_7$. Then $(x_0, x_1, x_2, x_3, x_4, x_7)$ is a path of shape 123232 (s6). Next suppose $x_4 \not\geq x_7$. Then (x_7, x_6, x_5, x_4) is a geodesic path of shape 2123, so by E_3, there exists a path (x_7, y, z, x_4) in P of shape 2323. Now $(x_0, x_1, x_2, x_3, x_4, z, y, x_7)$ is a path of shape 12323232 (s3).

Case s3. First suppose $x_3 = x_5$. Then $(x_0, x_1, x_2, x_3, x_6, x_7)$ is a path of shape 123232 (s6). Next suppose $x_3 \neq x_5$. Then $x_3 \vee x_5 = x_4$ covers x_3, x_5, so x_3, x_5 cover $x_3 \wedge x_5$ by modularity. Now $(x_0, x_1, x_2, x_3, x_3 \wedge x_5, x_5, x_6, x_7)$ is a path of shape 12321232 (s4).

Case s4. First suppose $x_0 = x_4$. Then (x_0, x_5, x_6, x_7) is a path of shape 1232 (s8). Next suppose $x_0 \neq x_4$. Observe x_2 is an upper bound for x_0, x_4, so $x_0 \vee x_4$ exists. $x_0 \vee x_4$ covers x_0, x_4 by modularity, so $(x_0, x_0 \vee x_4, x_4, x_5, x_6, x_7)$ is a path of shape 121232 (s5).

Case s5. First suppose $x_1 \leq x_4$. Then (x_0, x_1, x_4, x_5) is a path of shape 1232 (s8). Next suppose $x_1 \not\leq x_4$. Then (x_1, x_2, x_3, x_4) is a geodesic path of shape 2123, so by E_3, there exists a path (x_1, y, z, x_4) in P of shape 2323. Now $(x_0, x_1, y, z, x_4, x_5)$ is a path of shape 123232 (s6).

Case s6. First suppose $x_3 = x_5$. Then (x_0, x_1, x_2, x_5) is a path of shape 1232 (s8). Next suppose $x_3 \neq x_5$. Then $x_3 \vee x_5 =$

x_4 covers x_3, x_5, so x_3, x_5 cover $x_3 \wedge x_5$ by modularity. Now $(x_0, x_1, x_2, x_3, x_3 \wedge x_5, x_5)$ is a path of shape 123212 (s7).

Case s7. First suppose $x_0 = x_4$. Then (x_0, x_5, x_0, x_5) is a path of shape 1212 (s9). Next suppose $x_0 \neq x_4$. Observe x_2 is an upper bound for x_0, x_4, so $x_0 \vee x_4$ exists. $x_0 \vee x_4$ covers x_0, x_4 by modularity, so $(x_0, x_0 \vee x_4, x_4, x_5)$ is a path of shape 1212 (s9).

Case s8. First suppose $x_1 = x_3$. Then (x_0, x_3, x_0, x_3) is a path of shape 1212 (s9). Next suppose $x_1 \neq x_3$. Then $x_1 \vee x_3 = x_2$ covers x_1, x_3, so x_1, x_3 cover $x_1 \wedge x_3$ by modularity. Now $(x_0, x_1, x_1 \wedge x_3, x_3)$ is a path of shape 1212 (s9).

We have now shown part (ii) in the present claim, so E_2 holds. Now the propositions $E_2, E_3, \ldots, E_D$ hold by Claims 1, 2, so P satisfies condition (iii) in Theorem 18.2. Now P satisfies the augmentation axiom by that theorem, and we conclude P is a quantum matroid of rank D.

§24. A Characterization of quantum matroids

In this section we obtain a characterization of a quantum matroid that is related to the material on Tits polar spaces in Sections 29, 30.

Definition 24.1. Let us say a prematroid P is *transversal* whenever $\max(P) = \mathrm{top}(P)$.

Theorem 24.2. *Let P denote a prematroid of rank D. Then the following are equivalent.*

(i) *P satisfies the augmentation axiom AU in Definition* 4.1.

(ii) *For all atoms $x \in P$, and for all $y \in \mathrm{top}(P)$ such that $x \not\leq y$, x_y^+ is transversal and has rank $D-1$.*

(iii) *For all $u \in P$, and for all $v \in top(P)$ such that u covers $u \wedge v$, there exists $v' \in \mathrm{top}(P)$ such that $u \leq v'$, and such that v, v' cover $v \wedge v'$.*

Proof. (i) $\rightarrow$ (ii). Let x, y be given. Observe x_y^+ is a quantum matroid by Corollary 19.2, so it is transversal by Lemma 4.3. Observe $\mathrm{top}(x_y^+) \subseteq \mathrm{top}(P)$ by Theorem 18.2(v), so $\mathrm{rank}(x_y^+) = D-1$.

(ii) $\rightarrow$ (iii). Pick any $u \in P$ and any $v \in \mathrm{top}(P)$ such that u covers $u \wedge v$. Let x denote a relative complement of $u \wedge v$ in $[0, u]$. Then x is an atom by modularity. Observe $x \not\leq v$; otherwise

$$\begin{aligned} u &= x \vee (u \wedge v) \\ &\leq v, \end{aligned}$$

a contradiction. Now $x \wedge v = 0$. We may now view $x, u \wedge v$ as relative complements in $[x \wedge v, u]$, so $u \in x_v^+$ by Lemma 15.3(i),(ii). Pick any $v' \in \max(x_v^+)$ such that $u \leq v'$. We check v' has the required properties. We mentioned x is an atom such that $x \not\leq v$, so by assumption x_v^+ is transversal and has rank $D-1$. It follows

$$\begin{aligned} v' &\in \max(x_v^+) \\ &= \operatorname{top}(x_v^+) \\ &\subseteq \operatorname{top}(P). \end{aligned}$$

Observe $x, v \wedge v'$ are relative complements in $[0, v']$ by Lemma 15.3(i),(ii), and x covers 0, so v' covers $v \wedge v'$ by modularity. In particular $\operatorname{rank}(v \wedge v') = D - 1$, and it follows v covers $v \wedge v'$.

(iii) $\rightarrow$ (i). To show P satisfies the augmentation axiom, we first show $\max(P) = \operatorname{top}(P)$. Suppose $\max(P) \neq \operatorname{top}(P)$. Then $[0, \operatorname{top}(P)]$ is a proper subset of P. Pick any element

$$u \in P \backslash [0, \operatorname{top}(P)] \tag{24.1}$$

with $\operatorname{rank}(u)$ minimal. Certainly $u \neq 0$, so there exists $x \in P$ such that u covers x. Of course $\operatorname{rank}(x) < \operatorname{rank}(u)$, so $x \in [0, \operatorname{top}(P)]$ by construction. Pick any $v \in \operatorname{top}(P)$ such that $x \leq v$. Observe $u \not\leq v$ by (24.1), so $u \wedge v = x$. Now u covers $u \wedge v$, so by (iii), there exists $v' \in \operatorname{top}(P)$ such that $u \leq v'$ (and such that v, v' cover $v \wedge v'$). Now $u \in [0, \operatorname{top}(P)]$, contradicting (24.1). We conclude $\max(P) = \operatorname{top}(P)$.

We are now ready to show P satisfies the augmentation axiom. To do this, we show P satisfies condition (iii) in Theorem 18.2. Pick any integer i $(2 \leq i \leq D)$, and pick any geodesic path $xyzw$ in P of shape $(i-1, i-2, i-1, i)$. We find y', $z' \in P$ such that $xy'z'w$ is a path of shape $(i-1, i, i-1, i)$. We may assume $x \vee w$ does not exist; otherwise, we are done with $y' := x \vee z$, $z' := z$. Since $\max(P) = \operatorname{top}(P)$, there exists $v \in \operatorname{top}(P)$ such that $w \leq v$. Observe $x \not\leq v$ (otherwise v is an upper bound for x, w), and it follows $x \wedge v = y$. Now x covers $x \wedge v = y$, so by assumption, there exists $v' \in \operatorname{top}(P)$ such that $x \leq v'$, and such that v, v' cover $v \wedge v'$. Observe $w \not\leq v \wedge v'$; otherwise $w \leq v \wedge v' \leq v'$, making v' an upper bound for x, w. Now $v = (v \wedge v') \vee w$, so $v \wedge v'$, w are relative complements in $[w \wedge v', v]$. Recall v covers $v \wedge v'$, so

$$w \text{ covers } w \wedge v' \tag{24.2}$$

by modularity. Observe v' is an upper bound for $x, w \wedge v'$, so $x \vee$

$(w \wedge v')$ exists. Observe x, $w \wedge v'$ cover $x \wedge (w \wedge v') = y$, so

$$x \vee (w \wedge v') \text{ covers } x,\ w \wedge v' \tag{24.3}$$

by modularity. Set

$$y' := x \vee (w \wedge v'),$$
$$z' := w \wedge v'.$$

Then $xy'z'w$ is a path of shape $(i-1, i, i-1, i)$ by (24.2), (24.3), and the construction. We have now shown P satisfies condition (iii) in Theorem 18.2, so P satisfies the augmentation axiom by that theorem. We have now proved Theorem 24.2.

§25. Any Cartesian product of quantum matroids is a quantum matroid

In this section, we show the property of being a quantum matroid is closed under the Cartesian product operation mentioned above line (1.23). First, a word about notation. Let P, Q denote any posets, and let S, T denote subposets of P, Q, respectively. Then the poset $S \times T$ is isomorphic to the subposet of $P \times Q$ induced on

$$\{xy \mid x \in S,\ \ y \in T\};$$

consequently, we do not distinguish between these posets.

We mention a few elementary facts about the Cartesian product. Let P, Q denote any nonempty posets. Pick any x, $y \in P$, and any x', $y' \in Q$. Then $xx' \wedge_{P\times Q} yy'$ exists if and only if both $x \wedge_P y$, $x' \wedge_Q y'$ exist. In this case,

$$xx' \wedge_{P\times Q} yy' \;=\; x \wedge_P y, x' \wedge_Q y'. \tag{25.1}$$

Similarly, $xx' \vee_{P\times Q} yy'$ exists if and only if both $x \vee_P y$, $x' \vee_Q y'$ exist, and in this case,

$$xx' \vee_{P\times Q} yy' \;=\; x \vee_P y, x' \vee_Q y'. \tag{25.2}$$

Let P, Q denote semilattices. Then $P \times Q$ is a semilattice.

Let P, Q denote posets with 0. Then $P \times Q$ has a 0. Moreover,

$$0_{P\times Q} \;=\; 0_P 0_Q. \tag{25.3}$$

Let P, Q denote ranked posets with 0. Then $P \times Q$ is ranked. Moreover, for all $x \in P$ and for all $x' \in Q$,

$$\operatorname{rank}_{P\times Q}(xx') \;=\; \operatorname{rank}_P(x) \;+\; \operatorname{rank}_Q(x'). \tag{25.4}$$

Let P, Q denote modular atomic lattices. Then $P \times Q$ is a modular atomic lattice.

Lemma 25.1. *Let P, Q denote prematroids. Then $P \times Q$ is a prematroid.*

Proof. It is immediate from our remarks above that $P\times Q$ satisfies R, SL. To see that $P \times Q$ satisfies M, pick any $x \in P$ and any $x' \in Q$. Then

$$[0_{P\times Q}, xx'] \;=\; [0_P, x] \times [0_Q, x']$$

is a Cartesian product of modular atomic lattices, and is therefore a modular atomic lattice.

Lemma 25.2. *Let P, Q denote prematroids. Then for all x, $y \in P$ and for all x', $y' \in Q$,*

(i) $\partial_{P\times Q}(xx', yy') \;=\; \partial_P(x, y) \;+\; \partial_Q(x', y')$,
(ii) $\delta_{P\times Q}(xx', yy') \;=\; \delta_P(x, y) \;+\; \delta_Q(x', y')$.

Proof. (i) Expand each side using (7.12), and evaluate the results using (25.1), (25.4).

(ii) First, we show the inequality $\leq$ holds. By Definition 12.1, there exists $z \in P$ such that $z \vee_P y$ exists, and such that

$$\partial_P(x, z) = \delta_P(x, y).$$

Similarly, there exists $z' \in Q$ such that $z' \vee_Q y'$ exists, and such that

$$\partial_Q(x', z') = \delta_Q(x', y').$$

Now $zz' \vee_{P\times Q} yy'$ exists by our preliminary remarks, so in view of Definition 12.1,

$$\begin{aligned}\delta_{P\times Q}(xx', yy') &\leq \partial_{P\times Q}(xx', zz')\\ &= \partial_P(x, z) + \partial_Q(x', z')\\ &= \delta_P(x, y) + \delta_Q(x', y'),\end{aligned}$$

as desired. Next, we show the inequality $\geq$ holds. By Definition 12.1, there exists an element $zz' \in P \times Q$ such that $zz' \vee_{P\times Q} yy'$ exists, and such that

$$\partial_{P\times Q}(xx', zz') = \delta_{P\times Q}(xx', yy').$$

Observe $z \vee_P y$ exists by our preliminary remarks, so

$$\partial_P(x, z) \geq \delta_P(x, y)$$

by Definition 12.1. Similarly $z' \vee_Q y'$ exists, so

$$\partial_Q(x', z') \geq \delta_Q(x', y').$$

Now

$$\begin{aligned}\delta_{P\times Q}(xx', yy') &= \partial_{P\times Q}(xx', zz') \\ &= \partial_P(x, z) + \partial_Q(x', z') \\ &\geq \delta_P(x, y) + \delta_Q(x', y'),\end{aligned}$$

as desired. We conclude equality holds in (ii), and we are done.

Theorem 25.3. *Let P, Q denote quantum matroids. Then the Cartesian product $P \times Q$ is a quantum matroid.*

Proof. Observe $P \times Q$ is a prematroid by Lemma 25.1, so it remains to show $P \times Q$ satisfies the augmentation axiom. To do this, it suffices by Theorem 18.2(vi) to show the function $\delta_{P\times Q}$ is symmetric in its arguments. But this is an immediate consequence of Lemma 25.2(ii), since δ_P, δ_Q are each symmetric in their arguments by Theorem 18.2(vi).

§26. The radical of a quantum matroid

Definition 26.1. Let P denote a quantum matroid. By the *radical* of P, we mean the element

$$\mathrm{Rad}(P) := \bigwedge_{x \in \mathrm{top}(P)} x. \tag{26.1}$$

We say P is *degenerate* whenever $\mathrm{Rad}(P) > 0$, and *nondegenerate* whenever $\mathrm{Rad}(P) = 0$.

Let P denote a quantum matroid, and write $R = \mathrm{Rad}(P)$. Recall by Corollary 19.2 that the subposet R^+ is a quantum matroid.

Lemma 26.2. *Let P denote a quantum matroid, and write $R =$ $\mathrm{Rad}(P)$. Then*

(i) *R^+ is nondegenerate,*
(ii) $\mathrm{top}(R^+) = \mathrm{top}(P)$.

Proof. Routine.

Lemma 26.3. *Let P denote a quantum matroid, and pick any $x \in P$. Then the following are equivalent.*

(i) $x \leq \mathrm{Rad}(P)$.

(ii) $x \leq y$ *for all* $y \in \mathrm{top}(P)$.
(iii) $x \vee y$ *exists for all* $y \in P$.

Proof. (i) $\rightarrow$ (ii). Observe $x \leq \mathrm{Rad}(P) \leq y$ for all $y \in \mathrm{top}(P)$.
(ii) $\rightarrow$ (iii). Pick any $y \in P$. By Lemma 4.3, there exists $u \in \mathrm{top}(P)$ such that $y \leq u$. Observe $x \leq u$ by assumption, so u is an upper bound for x, y.
(iii) $\rightarrow$ (i). Pick any $y \in \mathrm{top}(P)$. Observe $x \vee y$ exists by assumption, so $x \leq y$. Now $x \leq \mathrm{Rad}(P)$ by Definition 26.1. We have now proved Lemma 26.3.

Lemma 26.4. *Let* P *denote a quantum matroid, and suppose the condition* (10.2) *holds. Then*

(26.2) $$\mathrm{Shadow}(\mathrm{Rad}(P)) = \{x \in A_P | x \vee a \textit{ exists for all } a \in A_P\}.$$

Proof. $\subseteq$: Pick any $x \in \mathrm{Shadow}(\mathrm{Rad}(P))$. Then by Lemma 26.3(i), (iii), $x \vee a$ exists for all $a \in A_P$
$\supseteq$: Pick any $x \in A_P$, and assume $x \vee a$ exists for all $a \in A_P$. We show x satisfies condition (iii) of Lemma 26.3. Pick any $y \in P$. Certainly $x \vee a$ exists for all $a \in \mathrm{Shadow}(y)$, so $x \vee y$ exists by (10.2). Now x satisfies condition (iii) of Lemma 26.3, so $x \in \mathrm{Shadow}(\mathrm{Rad}(P))$ by Lemma 26.3(i),(iii).

Lemma 26.5. *Let* P *denote a quantum matroid with rank* D. *Suppose that for each* $x \in P$ *such that* $\mathrm{rank}(x) = D - 1$, x *is covered by at least two elements in* $\mathrm{top}(P)$. *Then* $\mathrm{Rad}(P) = 0$.

Proof. Suppose $R := \mathrm{Rad}(P) > 0$, and pick any $z \in \mathrm{top}(P)$. Since $[0, z]$ is relatively complemented, there exists $x \in P$ such that z covers x and $R \not\leq x$. By assumption, there exists $z' \in \mathrm{top}(P)$ such that z' covers x and $z' \neq z$. Observe z' is an upper bound for x, R, forcing $z' \geq x \vee R = z$, an impossibility. Hence $\mathrm{Rad}(P) = 0$.

We finish this section with some results concerning the polar spaces from Examples 1.7, 1.8.

Definition 26.6. Let V, $\langle\, ,\rangle$ be as in Example 1.7, but assume q is odd in the symmetric bilinear case.

(i) By the *radical* of $\langle\, ,\rangle$, we mean

$$\mathrm{Rad}(\langle\, ,\rangle) := \{\, u \in V \mid \langle u, v\rangle = 0 \quad \text{for all} \quad v \in V\}.$$

(ii) $\langle\, ,\rangle$ is said to be *degenerate* if $\mathrm{Rad}(\langle\, ,\rangle) = 0$, and *nondegenerate* otherwise.

Lemma 26.7. *Let* V, $\langle\, , \rangle$, P *be as in Example* 1.7, *but assume* q *is odd in the symmetric bilinear case.*

(i) $\mathrm{Rad}(\langle\, , \rangle) = \mathrm{Rad}(P)$.
(ii) $\langle\, , \rangle$ *is degenerate if and only if* P *is degenerate.*

Proof. Routine.

Definition 26.8. Let V, f, $\langle\, , \rangle_f$ be as in Example 1.8.

(i) By the *radical* of f, we mean

$$\mathrm{Rad}(f) := \{v \in \mathrm{Rad}(\langle\, , \rangle_f) \mid f(v) = 0\}.$$

(ii) f is said to be *degenerate* if $\mathrm{Rad}(f) = 0$, and *nondegenerate* otherwise.

Lemma 26.9. *Let* V, f, P *be as in Example* 1.8.

(i) $\mathrm{Rad}(f) = \mathrm{Rad}(P)$.
(ii) f *is degenerate if and only if* P *is degenerate.*

Proof. Routine.

§27. Line regularity and dual-line regularity

Definition 27.1. Let P denote a quantum matroid of rank D, and let q denote an integer.

(i) Suppose $D \geq 2$. Then P is said to be *q-line regular* whenever for all lines $x \in P$,

$$|\mathrm{Shadow}(x)| = q + 1. \tag{27.1}$$

(ii) Suppose $D \leq 1$. Then P is said to be *q-line regular* whenever q is positive.

Lemma 27.2. *Let* P *denote a* q*-line regular quantum matroid. Then*

$$q \geq 1. \tag{27.2}$$

Proof. If P has rank at least 2, then (27.2) follows from condition M in Definition 4.1. If P has rank 0 or 1, then (27.2) is immediate from Definition 27.1(ii).

Definition 27.3. Let q, j denote integers. We define $\left[\begin{smallmatrix} j \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} j \\ 1 \end{smallmatrix}\right]_q$ by

$$\begin{bmatrix} j \\ 1 \end{bmatrix} := \frac{q^j - 1}{q - 1} \qquad \text{if} \quad q \neq 1, \tag{27.3}$$

and

$$\begin{bmatrix} j \\ 1 \end{bmatrix} := j \qquad \text{if} \quad q = 1. \tag{27.4}$$

Lemma 27.4. *Let P denote a modular atomic lattice of rank D, and let q denote an integer. Then the following are equivalent.*

(i) *P is q-line-regular.*
(ii) *All intervals in P are q-line-regular.*
(iii) *P^* is q-line-regular.*

Suppose (i)-(iii) *hold. Then*

$$|A_P| = \begin{bmatrix} D \\ 1 \end{bmatrix}, \tag{27.5}$$

where [ ] *is from* (27.3), (27.4).

Proof. Routine.

Lemma 27.5. *Let P denote a q-line regular quantum matroid. Then for all intervals $I = [x, y]$ in P,*

$$\begin{bmatrix} i \\ 1 \end{bmatrix} = \big|\{z \in I \mid z \ \textit{covers} \ x\}\big|, \tag{27.6}$$

$$= \big|\{z \in I \mid y \ \textit{covers} \ z\}\big|, \tag{27.7}$$

where $i := \mathrm{rank}(y) - \mathrm{rank}(x)$, and where [ ] *is from* (27.3), (27.4).

Proof. The interval $[0, y]$ is q-line-regular by construction, so I, I^* are both q-line regular by Lemma 27.4. I, I^* each have $\left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right]$ atoms by (27.5), and (27.6), (27.7) follow.

Lemma 27.6. *Let P denote a quantum matroid with rank $D \geq 3$, and suppose all the lines of P are thick. Then P is q-line regular for some integer $q \geq 2$. If $D \geq 4$ then q is a prime power.*

Proof. Fix any $x \in \mathrm{top}(P)$. Then $[0, x]$ is a modular atomic lattice, all of whose lines are thick. Applying Theorem 1.12, we find there exists an integer $q \geq 2$ such that $[0,x]$ is isomorphic to a projective plane of order q (if $D = 3$) or $L_q(D)$ (if $D \geq 4$). In any case $[0,x]$ is q-line regular. We show P is q-line regular. To this end, suppose there exists $y \in \mathrm{top}(P)$ such that $\partial(x,y) = 2$, i.e. x, y are adjacent in the graph $\mathrm{top}(P)$. Then by our preliminary remarks, $[0,y]$ is q'-line regular for some integer $q' \geq 2$. But the intervals $[0,x]$, $[0,y]$ share at least one line in common since $\mathrm{rank}(x \wedge y) = D - 1 \geq 2$, so $q = q'$. Since the graph $\mathrm{top}(P)$ is connected by Theorem 22.1(ii), we conclude $[0,z]$ is q-line regular for all $z \in \mathrm{top}(P)$. Now pick any line $u \in P$. By Lemma 4.3, there exists $z \in \mathrm{top}(P)$ such that $u \leq z$. $[0,z]$ is q-line-regular by our above remarks, so u covers exactly $q+1$ points. We have now shown P is q-line regular. Now suppose $D \geq 4$. Then $[0,x]$ is isomorphic to $L_q(D)$, so q is a prime power. We have now proved Lemma 27.6.

Definition 27.7. Let P denote a quantum matroid with rank D.

(i) For all $x \in P$, define

$$\mathrm{Shadow}_D(x) := \{y \mid y \in \mathrm{top}(P),\ \ y \geq x\}. \tag{27.8}$$

(ii) By a *dual-line* in P, we mean any element $x \in P$ such that

$$\mathrm{rank}(x) = D - 1. \tag{27.9}$$

Definition 27.8. Let P denote a quantum matroid with rank D, and let β denote an integer.

(i) Suppose $D \geq 1$. Then P is said to be *β-dual-line regular* whenever

$$|\mathrm{Shadow}_D(x)| = \beta + 1 \tag{27.10}$$

for all dual-lines $x \in P$.

(ii) Suppose $D = 0$. Then P is said to be *β-dual-line regular* whenever β is nonnegative.

Lemma 27.9. *Let P denote a β-dual-line regular quantum matroid. Then*

$$\beta \geq 0. \tag{27.11}$$

Proof. Immediate from Definition 27.8(i),(ii).

In the next lemma, we consider the case of equality in (27.11).

Lemma 27.10. *Let P denote a quantum matroid. Then the following are equivalent.*

(i) *P is a 0-dual-line regular.*
(ii) *P is a modular atomic lattice.*

Proof. (i) $\to$ (ii). Suppose P is not a modular atomic lattice. Then by Lemma 22.2, there exists at least two elements in $\mathrm{top}(P)$. The graph on $\mathrm{top}(P)$ is connected by Theorem 22.1(ii), so there exists $x, y \in \mathrm{top}(P)$ that are adjacent in the graph on $\mathrm{top}(P)$. Observe x, y cover $x \wedge y$, so $x \wedge y$ is a dual-line. But x and y are both in $\mathrm{Shadow}_D(x \wedge y)$, contradicting our assumption that P is 0-dual-line regular. We conclude P is a modular atomic lattice.

(ii) $\to$ (i). Clear.

We close this section with a theorem concerning design matroids that are both line regular and dual-line regular.

Theorem 27.11. *Let P denote a q-line regular, β-dual-line regular design matroid of rank D. Then*

$$|\mathrm{top}(P)| = 1 + \beta \begin{bmatrix} D \\ 1 \end{bmatrix}, \tag{27.12}$$

where [ ] *is from* (27.3), (27.4).

Proof. Fix any element $x \in \mathrm{top}(P)$. Set

$$\Lambda := \{\, y \in P \mid x \text{ covers } y\},$$

and observe by Lemma 27.5 that

$$|\Lambda| = \begin{bmatrix} D \\ 1 \end{bmatrix}. \tag{27.13}$$

We now count adjacencies between $\mathrm{top}(P)\backslash\{x\}$ and Λ. Observe by Lemma 22.3(ii), and since P is a semilattice, each element $z \in \mathrm{top}(P)\backslash\{x\}$ covers exactly one element in Λ; namely $x \wedge z$. By (27.10), each element in Λ is covered by exactly β elements in $\mathrm{top}(P)\backslash\{x\}$. It follows

$$\begin{aligned} \beta|\Lambda| &= |\mathrm{top}(P)\backslash\{x\}| \\ &= |\mathrm{top}(P)| - 1, \end{aligned} \tag{27.14}$$

and the result follows from (27.13), (27.14).

§28. Zig-zag regularity

Definition 28.1. Let P denote a quantum matroid of rank $D \geq 2$. For each integer i $(2 \leq i \leq D)$, let Δ_i denote the set of ordered pairs

$$\begin{aligned}\Delta_i := \{xy | x, y \in P, \operatorname{rank}(x) = i-1, \operatorname{rank}(y) = i, \partial(x,y) = 3,\\ x \vee y \text{ does not exist}\}\\ = \{xy | x, y \in P, \rho(x,y) = i-2, \gamma(x,y) = 0, \delta(x,y) = 1, \gamma(y,x) = 1\}.\end{aligned}$$

In Lemma 28.4, we define a function *zig-zag* : $\Delta_i \to \mathbb{Z}$, but first, let us consider when $\Delta_i \neq \emptyset$.

Lemma 28.2. *Let P denote a quantum matroid with rank $D \geq 2$. For each integer i $(1 \leq i \leq D)$, let $\tilde{\Delta}_i$ denote the set of ordered pairs*

$$\tilde{\Delta}_i := \{xy \mid x, y \in P, \quad \operatorname{rank}(x) = i, \quad \operatorname{rank}(y) = i, \quad \partial(x,y) = 2, \\ x \vee y \textit{ does not exist}\}.$$

Then the following statements (i)–(iii) *hold.*

(i) $\Delta_i \neq \emptyset \quad \leftrightarrow \quad \tilde{\Delta}_i \neq \emptyset \qquad (2 \leq i \leq D)$.

(ii) $\tilde{\Delta}_{i-1} \neq \emptyset \quad \to \quad \Delta_i \neq \emptyset \qquad (2 \leq i \leq D)$.

(iii) *Suppose there exists atoms x, $y \in P$ such that $x \vee y$ does not exist. Then*

$$\Delta_i \neq \emptyset \qquad (2 \leq i \leq D). \tag{28.1}$$

Proof. (i) $\to$: Pick any $xy \in \Delta_i$. By Theorem 18.2(ii) and Definition 28.1, there exists elements $z, w \in P$ such that $xzwy$ is a path with shape $(i-1, i, i-1, i)$. Recall $\partial(x,y) = 3$ by Definition 28.1, so $\partial(z,y) = 2$. Recall $x \vee y$ does not exist by Definition 28.1. It follows $z \vee y$ does not exist; otherwise $z \vee y$ is an upper bound for x, y. Now $zy \in \tilde{\Delta}_i$.

$\leftarrow$: Pick any $uv \in \tilde{\Delta}_i$. Observe u covers at least two elements of P since $\operatorname{rank}(u) = i \geq 2$; in particular there exists an element $x \in P$ such that u covers x and $x \neq u \wedge v$. Observe $\partial(x,v) \in \{1,3\}$ by (7.3). In fact $\partial(x,v) = 3$; otherwise v covers x, making x a lower bound for u, v, and forcing $x = u \wedge v$. We claim $x \vee v$ does not exist. Suppose $x \vee v$ exists. Observe xuv is geodesic by our above remarks; it follows $u \leq x \vee v$ by Lemma 7.9(i),(ii). In this case $x \vee v$ is an upper bound for u, v, so $u \vee v$ exists, contradicting our assumptions. We conclude $x \vee v$ does not exist. Now $xv \in \Delta_i$ by Definition 28.1.

(ii) Pick $uv \in \tilde{\Delta}_{i-1}$. By Lemma 4.3, there exists $y \in P$ such that y covers v. Observe $u \vee y$ does not exist; otherwise $u \vee y$ is an upper bound for u, v. In particular $u \not\leq y$. Now $\partial(u, y) = 3$ by (7.3), and it follows $uy \in \Delta_i$ by Definition 28.1.

(iii) Observe $xy \in \tilde{\Delta}_1$, so $\tilde{\Delta}_1 \neq \emptyset$. The result now follows from (i), (ii).

Lemma 28.3. *Let P denote a quantum matroid with rank $D \geq 2$. Then the following are equivalent.*

(i) *P is a modular atomic lattice.*
(ii) *$x \leq y$ for all $x \in A_P$ and all $y \in \mathrm{top}(P)$.*
(iii) $\tilde{\Delta}_D = \emptyset$.
(iv) $\Delta_D = \emptyset$.

Proof. (i) $\rightarrow$ (ii). Clear since $\mathrm{top}(P) = \{1\}$.

(ii) $\rightarrow$ (iii). Suppose there exists $uy \in \tilde{\Delta}_D$, and let x denote a relative complement of $u \wedge y$ is $[0, u]$. Observe u covers $u \wedge y$ by the definition of $\tilde{\Delta}_D$, so x is an atom by modularity. Observe $x \not\leq y$; otherwise x is a lower bound for u, y, forcing $x \leq u \wedge y$, and contradicting the construction.

(iii) $\rightarrow$ (i). Suppose P is not a modular atomic lattice. Then P is not 0-dual-line regular by Lemma 27.10, so there exists a dual-line $w \in P$, and distinct elements $u, v \in \mathrm{top}(P)$ such that $w \leq u$, $w \leq v$. Observe $\partial(u, v) = 2$ by the construction, so $uv \in \tilde{\Delta}_D$.

(iii) $\leftrightarrow$ (iv). Immediate from Lemma 28.2(i).

Lemma 28.4. *Let P denote a quantum matroid with rank $D \geq 2$. For all integers i $(2 \leq i \leq D)$, and for all elements $xy \in \Delta_i$, the sets*

(i) $\mathrm{top}(x_y^+)$
(ii) $y \star x$
(iii) $\{p \mid p$ *is a path in P with endpoints x, y and shape* $(i-1, i, i-1, i)\}$

all have the same cardinality.

We denote this cardinality by *zig-zag*(x, y).

Proof. The sets (i), (ii) have the same cardinality by Corollary 15.7(iii). The sets (ii), (iii) also have the same cardinality, since the map $u \rightarrow x, x \vee u, u, y$ is a bijection from the set $y \star x$ to the set in (iii).

Definition 28.5. Let P denote a quantum matroid with rank D, and let α denote an integer.

(i) Assume $D \geq 2$, and that P is not a modular atomic lattice. Then P is said to be *α-zig-zag regular* whenever

$$\text{zig-zag}(x,y) = \alpha + 1 \tag{28.2}$$

for all $xy \in \Delta_D$.

(ii) Assume $D \leq 1$, or that P is a modular atomic lattice. Then P is said to be *α-zig-zag regular* whenever α is nonnegative.

Lemma 28.6. *Let P denote an α-zig-zag regular quantum matroid. Then*

$$\alpha \geq 0. \tag{28.3}$$

Proof. Immediate from Definition 28.5.

In the next section, we consider the case of equality in (28.3). For now, we mention a few other inequalities concerning α.

Lemma 28.7. *Let P denote an α-zig-zag regular quantum matroid with rank $D \geq 2$, and assume P is not a modular atomic lattice.*

(i) *Suppose P is q-line regular. Then*

$$\alpha \leq q. \tag{28.4}$$

(ii) *Suppose P is β-dual-line regular. Then*

$$\alpha \leq \beta. \tag{28.5}$$

Proof. By Lemma 28.3(i),(iv), there exists an element $xy \in \Delta_D$. To see (i), observe the interval $[x \wedge y, y]$ has rank 2, so by Lemma 28.4(ii), Corollary 27.5,

$$\begin{aligned}
\alpha + 1 &= \text{zig-zag}(x,y) \\
&= |y \star x| \\
&\leq \big|\{v \mid v \in P,\ \ x \wedge y < v < y\}\big| \\
&= q + 1.
\end{aligned}$$

To see (ii), observe by Lemma 28.4(i) that

$$\begin{aligned}
\alpha + 1 &= \text{zig-zag}(x,y) \\
&= |\text{top}(x_y^+)| \\
&\leq |\text{Shadow}_D(x)| \\
&= \beta + 1.
\end{aligned}$$

§29. The 0-zig-zag regular quantum matroids

The purpose of this section and the next is to establish that a nondegenerate 0-zig-zag regular quantum matroid is the same thing as a Tits polar space.

Theorem 29.1. *Let P denote a quantum matroid. Then the following are equivalent.*

(i) *P is 0-zig-zag regular.*
(ii) *x_y^+ is a modular atomic lattice for all $x, y \in P$.*
(iii) *$|x \star y| = 1$ for all $x, y \in P$.*

Proof. Let D denote the rank of P.

(i) $\to$ (ii). Suppose there exists a pair $x, y \in P$ such that x_y^+ is not a modular atomic lattice. We may assume

$$\text{rank}(y) \; - \; \partial(x, x \wedge y) \quad \text{is maximal} \tag{29.1}$$

among all such pairs.

We first claim $y \in \text{top}(P)$. Suppose not. Then by Lemma 4.3, there exists an element $u \in P$ such that $y < u$. Observe

$$x \wedge y \leq x \wedge u \leq x,$$

so

$$\partial(x, x \wedge u) \leq \partial(x, x \wedge y).$$

Now

$$\text{rank}(u) - \partial(x, x \wedge u) > \text{rank}(y) - \partial(x, x \wedge y),$$

so x_u^+ is a modular atomic lattice by (29.1). Observe x_y^+ is a submatroid of x_u^+ by Lemma 15.9, so x_y^+ is a modular atomic lattice by Lemma 8.5, Lemma 9.1. This contradicts our assumptions, so $y \in \text{top}(P)$.

Next, we claim $\partial(x, x \wedge y) \geq 2$. Certainly $\partial(x, x \wedge y) \neq 0$; otherwise $x \leq y$, implying $x_y^+ = [x, y]$ is a modular atomic lattice. Suppose $\partial(x, x \wedge y) = 1$. We obtain a contradiction to Lemma 27.10 by showing x_y^+ is 0-dual-line regular. To do this, we pick any dual-line w in x_y^+, and show

$$|\text{top}(x_y^+) \cap w^+| = 1.$$

Observe $\text{top}(x_y^+) \subseteq \text{top}(P)$ by Theorem 18.2(v), so

$$\text{rank}_P(w) = D - 1. \tag{29.2}$$

Observe $x, w \wedge y$ are relative complements in $[x \wedge y, w]$ by Lemma 15.3(ii), and x covers $x \wedge y$, so

$$w \text{ covers } w \wedge y \tag{29.3}$$

by modularity. Now $wy \in \Delta_D$ by (29.2), (29.3), so by Lemma 15.8(i),

$$\begin{aligned} |\text{top}(x_y^+) \cap w^+| &= |\text{top}(w_y^+)| \\ &= \text{zig-zag}(wy) \\ &= 1, \end{aligned}$$

as desired. We have now shown x_y^+ is 0-dual-line regular, so x_y^+ is a modular atomic lattice by Lemma 27.10. This contradicts the construction, so $\partial(x, x \wedge y) \neq 1$. We conclude $\partial(x, x \wedge y) \geq 2$.

Since $\partial(x, x \wedge y) \geq 2$, there exists an element $s \in P$ such that $x \wedge y < s < x$. Observe $s \wedge y = x \wedge y$ by Lemma 13.4(i),(iii), so

$$\begin{aligned} \text{rank}(y) \; - \; \partial(s, s \wedge y) \; &= \; \text{rank}(y) \; - \; \partial(s, x \wedge y) \\ &> \; \text{rank}(y) \; - \; \partial(x, x \wedge y), \end{aligned}$$

implying s_y^+ is a modular atomic lattice by (29.1). Let z denote a maximal element of s_y^+. Observe $z \in \text{top}(P)$ by Theorem 18.2(v), and $s = x \wedge z$ by Lemma 7.4, so

$$\begin{aligned} \text{rank}(z) \; - \; \partial(x, x \wedge z) \; &= \; \text{rank}(y) \; - \; \partial(x, s) \\ &> \; \text{rank}(y) \; - \; \partial(x, x \wedge y), \end{aligned}$$

implying x_z^+ is a modular atomic lattice by (29.1).

We claim $x_y^+ \subseteq x_z^+$ (in fact equality holds, but we will not need this). To prove the claim, we pick $u \in x_y^+$ and show $u \in x_z^+$. Set

$$p := s \vee (u \wedge y).$$

Observe $p \in s_y^+$ by Lemma 17.1, so spz is geodesic. Observe szy is geodesic since $z \in s_y^+$, so $spzy$ is geodesic. In particular pzy is geodesic. Observe upy is geodesic by Lemma 17.1, Theorem 17.2(i),(iii), so $upzy$ is geodesic. In particular uzy is geodesic. Observe xuy is geodesic since $u \in x_y^+$, so $xuzy$ is geodesic. In particular xuz is geodesic, so $u \in x_z^+$, as desired. We have now shown $x_y^+ \subseteq x_z^+$. Now x_y^+ is geodesically closed in x_z^+ by Lemma 15.2(ii), and we saw x_z^+ is a modular atomic lattice, so x_y^+ is a modular atomic lattice by Lemma 8.5. This contradicts our assumption, and we are done.

(ii) $\to$ (i). For all $xy \in \Delta_D$,

$$\begin{aligned}\text{zig-zag}(xy) &= |\text{top}(x_y^+)| \\ &= 1.\end{aligned}$$

(ii) $\leftrightarrow$ (iii). Recall by Lemma 22.2(ii),(iii) that for all $x, y \in P$, x_y^+ is a modular atomic lattice if and only if $|\text{top}(x_y^+)| = 1$. But $|\text{top}(x_y^+)| = |y \star x|$ by Lemma 15.7(iii), so the result follows. This proves Theorem 29.1.

We now modify Theorem 24.2 using the above theorem, to obtain a characterization of the 0-zig-zag regular quantum matroids.

Theorem 29.2. *Let P denote a prematroid of rank D. Then the following are equivalent.*

(i) *P satisfies the augmentation axiom AU in Definition* 4.1, *and P is* 0*-zig-zag regular.*

(ii) *For all atoms $x \in P$ and for all $y \in \text{top}(P)$ such that $x \not\leq y$, x_y^+ is a modular atomic lattice with rank $D-1$.*

(iii) *For all $u \in P$ and all $v \in \text{top}(P)$ such that u covers $u \wedge v$, there exists a unique $v' \in \text{top}(P)$ such that $u \leq v'$ and such that v, v' cover $v \wedge v'$.*

Proof. (i) $\to$ (ii). Let x, y be given. P is a quantum matroid by Definition 4.1, and 0-zig-zag regular by assumption, so x_y^+ is a modular atomic lattice by Theorem 29.1(ii). x_y^+ has rank $D-1$ by Theorem 24.2.

(ii) $\to$ (iii). Very similar to the proof of Theorem 24.2 (ii) $\to$ (iii).

(iii) $\to$ (i). Observe P satisfies condition (iii) of Theorem 24.2, so P satisfies the augmentation axiom by that theorem. We show P is 0-zig-zag regular. Pick $xy \in \Delta_D$. Then x covers $x \wedge y$ by Definition 28.1, so by assumption, there exists a unique $y' \in \text{top}(P)$ such that $x \leq y'$, and such that y, y' cover $y \wedge y'$. Put another way, there exists a unique path with endpoints x, y and shape $(D-1, D, D-1, D)$, so $\text{zig-zag}(x, y) = 1$. We have now shown P is 0-zig-zag regular. This proves Theorem 29.2.

Next, we consider when a 0-zig-zag regular quantum matroid is nondegenerate.

Theorem 29.3. *Let P denote a* 0*-zig-zag regular quantum matroid with rank D. Then the following are equivalent.*

(i) $\text{Rad}(P) = 0$.

(ii) *For all atoms* $a \in P$*, there exists an atom* $b \in P$ *such that* $a \vee b$ *does not exist.*
(iii) *For all* $x \in P$*, there exists* $y \in \mathrm{top}(P)$ *such that* $x \wedge y = 0$.
(iv) *For all dual-lines* $x \in P$*,* x *is covered by at least two elements in* $\mathrm{top}(P)$.
(v) *There exists* $x, y \in \mathrm{top}(P)$ *such that* $x \wedge y = 0$.

Proof. (i) $\rightarrow$ (ii). Let the atom a be given. Certainly $a \not\leq \mathrm{Rad}(P)$, so by Lemma 26.3(iii), there exists an element $y \in P$ such that $a \vee y$ does not exist. Observe $\delta(a, y) = 1$ by construction, so $\delta(y, a) = 1$ by Theorem 18.2(vi). Now y covers $y \star a$ by Definition 12.1 and Theorem 12.3. Let b denote a relative complement of $y \star a$ in $[0, y]$. Then b is an atom by modularity. To show $a \vee b$ does not exist, we show $b \notin y_a^-$. But this holds, since $y_a^- = [0, y \star a]$ by Theorem 19.3(i), and $b \notin [0, y \star a]$ by construction.

(ii) $\rightarrow$ (iii). Let x be given, and pick an element $z \in \mathrm{top}(P)$ with

$$\rho(x, z) \text{ minimal.} \tag{29.4}$$

We assume $\rho(x, z) > 0$ and get a contradiction. By construction, there exists an atom $a \in P$ such that $a \leq x \wedge z$. By (ii), there exists an atom $b \in P$ such that $a \vee b$ does not exist. Observe $b \not\leq z$; otherwise z is an upper bound for a, b. Now b covers $b \wedge z = 0$, so by Theorem 29.2(iii), there exists an element

$$y \in \mathrm{top}(P) \tag{29.5}$$

such that $b \leq y$, and such that y, z cover $y \wedge z$. Set $h := y \wedge z$. By Theorem 13.5, $\rho(x, z) - \rho(x, h)$ equals 0 or 1. Suppose for the moment $\rho(x, z) = \rho(x, h)$. Then $x \wedge z \leq h$ by Lemma 13.4(i),(iv), implying y is an upper bound for a, b, a contradiction. Hence

$$\rho(x, z) = \rho(x, h) + 1, \tag{29.6}$$

and this forces

$$z \in h_x^+ \tag{29.7}$$

by Theorem 15.3(i),(iv). Observe $y \vee z$ does not exist. Now by (29.7), and since h_x^+ is a modular atomic lattice by Theorem 29.1,

$$y \notin h_x^+. \tag{29.8}$$

By Theorem 13.5 and since y covers h, $\rho(x, y) - \rho(x, h)$ equals 0 or 1. Suppose for the moment $\rho(x, y) = \rho(x, h) + 1$. Then $y \in h_x^+$ by

Lemma 15.3(iv), contradicting (29.8). Hence

$$\rho(x,y) = \rho(x,h). \tag{29.9}$$

Now (29.5), (29.6), (29.9) contradict (29.4).

(iii) $\to$ (iv). Pick any dual-line $x \in P$. We show x is covered by at least two elements in $\operatorname{top}(P)$. By Lemma 4.3, there exists an element $y \in \operatorname{top}(P)$ that covers x. By assumption, there exists an element $z \in P$ such that $y \wedge z = 0$. By Theorem 29.1, x_z^+ is a modular atomic lattice. Let y' denote the unique maximal element in x_z^+. Then $y' \in \operatorname{top}(P)$ by Theorem 18.2(v). It remains to check $y \neq y'$. Observe $x \geq y \wedge z = 0$, so yxz is geodesic by Lemma 13.4(i),(ii). But $xy'z$ is geodesic by the construction and (15.1), so $y \neq y'$, as desired.

(iv) $\to$ (i). This is just Lemma 26.5.

(iii) $\to$ (v). Clear.

(v) $\to$ (i). Pick any $x, y \in P$ such that $x \wedge y = 0$. Then by Definition 26.1,

$$\begin{aligned} \operatorname{Rad}(P) &= \bigwedge_{u \in \operatorname{top}(P)} u \\ &\leq x \wedge y \\ &= 0, \end{aligned}$$

so $\operatorname{Rad}(P) = 0$. We have now proved Theorem 29.3.

§30. Tits polar spaces

In this section, we show that a nondegenerate 0-zig-zag regular quantum matroid is the same thing as a Tits polar space.

Our first result concerns atomic semilattices. It will allow us to shift our point of view a bit, bringing it into line with how Tits polar spaces are traditionally viewed.

Lemma 30.1. *Let P denote an atomic semilattice, and define a poset*

$$\tilde{P} := \{\operatorname{Shadow}(x) \mid x \in P\}, \tag{30.1}$$

with partial order by inclusion. Then the map

$$\begin{aligned} P &\to \tilde{P} \\ x &\to \operatorname{Shadow}(x) \end{aligned}$$

is an isomorphism of posets.

Proof. The map is clearly onto $\tilde{P}$. The map is 1–1, and respects the partial order, by Lemma 9.3(ii). This proves Lemma 30.1.

Passing from P to $\tilde{P}$, Lemma 30.1 allows us to view any atomic semilattice as a collection of distinct subsets of A_P, with partial order defined by inclusion. We adopt this point of view for the remainder of the section.

Definition 30.2 [Ti, p102]. A *Tits polar space* is a collection P of distinct subsets of a set A (of *points*), partially ordered by inclusion, such that the following axioms hold.

PS1: P is closed under taking intersections, and has all the single points of A as its collection of minimal non-empty members.

PS2: All unrefinable chains in P have the same length D.

PS3: If x is a maximal member of P, then x, together with all the elements of P which it contains, is a modular atomic lattice of rank D.

PS4: Given a point $x \in P$, and a maximal member y of P that does not contain x, there exists a unique maximal member y' of P such that y, y' cover $y \wedge y'$. $y \wedge y'$ contains all the elements of y that lie together with x in some element of P.

PS5: There exists two maximal elements in P that have empty intersection.

The scalar D is the *rank* of P.

We are now ready for the main theorem of this section.

Theorem 30.3. *For any nonnegative integer D, the following are equivalent.*

(i) *P is a nondegenerate, 0-zig-zag regular quantum matroid of rank D.*

(ii) *P is a Tits polar space of rank D.*

Proof. (i) → (ii). Recall by Lemma 4.3 that

$$\max(P) = \mathrm{top}(P). \tag{30.2}$$

To show P is a Tits polar space of rank D, we check P satisfies PS1–PS5.

PS1: P is closed under taking intersections by Lemma 9.3(iii). P has $A = A_P$ as its collection of minimal nonempty members by the definition of A_P.

PS2: Immediate since P is ranked, $\operatorname{rank}(P) = D$, and since (30.2) holds.

PS3: Pick any $x \in \max(P)$. Then $\operatorname{rank}(x) = D$ by (30.2), so $[0, x]$ is a modular atomic lattice of rank D by condition M of Definition 4.1.

PS4: Pick any point $x \in A$, and any element $y \in \max(P)$ such that $x \not\leq y$. Then x covers $x \wedge y = 0$, so by Theorem 29.2(iii) and (30.2), there exists a unique element $y' \in \max(P)$ such that $x \leq y'$, and such that y, y' cover $y \wedge y'$. To see that $y \wedge y'$ contains all the elements of y that lie together with x in some element of P, observe by Theorem 19.3(i), Theorem 29.1(iii) that

$$\begin{aligned}\{a \in A \mid a \leq y, \quad a \vee x \text{ exists in } P\} &= y_x^- \cap A \\ &= [0, y \star x] \cap A \\ &= [0, y \wedge y'] \cap A.\end{aligned}$$

PS5: Immediate from Theorem 29.3(i),(v), and (30.2).

We have now shown P satisfies PS1–PS5, so P is a Tits polar space of rank D.

(ii) $\rightarrow$ (i). We first show P is a prematroid, by showing P satisfies the conditions SL, R, M in Definition 4.1.

SL: P is a semilattice by PS1.

R: P is ranked by PS2.

M: Pick any $x \in P$, and pick any $y \in \max(P)$ such that $x \leq y$. Observe $[0, y]$ is a modular atomic lattice by PS3, so the interval $[0, x]$ is a modular atomic lattice.

We have now shown P is a prematroid. In fact P is transversal and has rank D by PS2.

We now show P satisfies the augmentation axiom, and is 0-zig-zag regular. To do this, we show P satisfies condition (ii) in Theorem 29.2. Pick any atom $x \in P$ and any element $y \in \operatorname{top}(P)$ such that $x \not\leq y$. We show x_y^+ is a modular atomic lattice of rank $D - 1$. By PS4, and since P is transversal, there exists a unique $y' \in \operatorname{top}(P)$ such that $x \leq y'$, and such that y, y' cover $y \wedge y'$. Moreover, $y \wedge y'$ contains all the elements of y that lie together with x in some element of P. It follows

$$y \star x = y \wedge y' \tag{30.3}$$

and

$$\operatorname{Shadow}(y \wedge y') = y_x^- \cap A. \tag{30.4}$$

We show $y_x^- = [0, y \wedge y']$. To see the inclusion $\supseteq$, observe by Lemma 15.4(iii) and (30.3) that

$$\begin{aligned} y_x^- &\supseteq [0, y \star x] \\ &= [0, y \wedge y']. \end{aligned}$$

To see the inclusion $\subseteq$, pick any $z \in y_x^-$. Then

$$\begin{aligned} \mathrm{Shadow}(z) &\subseteq y_x^- \cap A \\ &= \mathrm{Shadow}(y \wedge y') \end{aligned}$$

by (30.4), so $z \in [0, y \wedge y']$ by Lemma 9.3(ii). We have now shown $y_x^- = [0, y \wedge y']$. By this, and since y covers $y \wedge y'$, it follows y_x^- is a modular atomic lattice of rank $D-1$. Recall x_y^+, y_x^- are isomorphic by Theorem 15.5; in particular x_y^+ is a modular atomic lattice of rank $D-1$, as desired. We have now shown P satisfies condition (ii) in Theorem 29.2; it follows P satisfies the augmentation axiom, and is 0-zig-zag regular. Now P is a 0-zig-zag regular quantum matroid by Definition 4.1. P is nondegenerate by Theorem 29.3(i),(v) and PS5. This proves Theorem 30.3.

The Tits polar spaces (and hence the nondegenerate 0-zig-zag regular quantum matroids) are essentially classified by J. Tits. In the following two theorems we present the classification in the line regular case.

Theorem 30.4. *Let D denote an integer at least 2. Then the following are equivalent.*

(i) *P is a 1-line regular Tits polar space of rank D.*
(ii) *There exists integers $N_1, N_2, \ldots, N_D$, all at least 2, such that P is isomorphic to $C(N_1, N_2, \ldots, N_D)$.*

Suppose (i), (ii) *hold. Then P is β-dual-line regular if and only if*

$$N_i = \beta + 1 \qquad\qquad (1 \le i \le D).$$

In this case P is isomorphic to the Hamming matroid $H(D, \beta+1)$ listed in Example 40.1(2).

Proof. Routine.

Theorem 30.5([Ti]). *Let D denote an integer at least* 4, *and let q denote an integer at least* 2. *Then the following are equivalent.*

(i) *P is a q-line regular Tits polar space of rank D.*
(ii) *q is a prime power, and P is isomorphic to a classical polar space of rank D over the field $GF(q)$.*

(*See Example* 40.1(5).)

Suppose (i), (ii) *hold. Then* P *is* $q^{1+\varepsilon}$*-dual-line regular, where* ε *is given in Example* 40.1(5).

§31. The α-zig-zag regular quantum matroids with $\alpha > 0$

In the previous two sections, we considered the 0-zig-zag regular quantum matroids. In this section, we say a bit about the α-zig-zag regular quantum matroids with $\alpha > 0$.

Theorem 31.1. *Let* P *denote a quantum matroid with rank* $D \geq 2$. *Suppose* P *is* α*-zig-zag regular for some integer* $\alpha > 0$, *but assume* P *is not a modular atomic lattice. Then*

(i) P *is dual-line regular,*
(ii) $\mathrm{Rad}(P) = 0$.

Proof. (i) Pick any $x, y \in P$ such that $\mathrm{rank}(x) = D - 1$, $\mathrm{rank}(y) = D - 1$. To show P is dual-line regular, it suffices to show

$$|\mathrm{Shadow}_D(x)| = |\mathrm{Shadow}_D(y)|. \tag{31.1}$$

First, consider the special case where $x \vee y$ exists. Here we may assume $x \neq y$; otherwise (31.1) clearly holds. Observe by Lemma 28.4(i) and Definition 28.5 that for each $u \in \mathrm{Shadow}_D(x) \backslash \{x \vee y\}$, there exists exactly α elements $v \in \mathrm{Shadow}_D(y) \backslash \{x \vee y\}$ such that $\partial(u, v) = 2$. This remains true if we interchange the roles of x and y, so

$$|\mathrm{Shadow}_D(x) \backslash \{x \vee y\}| \alpha = |\mathrm{Shadow}_D(y) \backslash \{x \vee y\}| \alpha.$$

Line (31.1) is immediate since $\alpha > 0$. We now have (31.1) in our special case. To show (31.1) holds in general, we construct a path p with endpoints x, y and shape

$$(D-1, D, D-1, D, \ldots, D-1, D, D-1). \tag{31.2}$$

To obtain p, recall by Lemma 4.3 that there exists an element $y' \in \mathrm{top}(P)$ such that y' covers y. By Theorem 18.2(iv), there exists a geodesic up-flat-down path p' in P with endpoints x, y'. By the construction, p' must have shape $(D-1, D, D-1, D, \ldots, D-1, D)$. Appending y to the end of p', we obtain a path p with endpoints x, y and shape (31.2), as desired. Write $p = (x = x_0, x_1, \ldots, x_{d-1}, x_d = y)$ $(x_0, x_1, \ldots, x_d \in P)$, and observe by (31.2) that d is even. Moreover

$$x_i \vee x_{i+2} \text{ exists} \qquad (0 \leq i \leq d-2, \ i \text{ even}),$$

so by the above special case,

$$|\mathrm{Shadow}_D(x_i)| = |\mathrm{Shadow}_D(x_{i+2})| \qquad (0 \le i \le d, \ \ i \text{ even}).$$

Line (31.1) follows, so P is dual-line regular, as desired.

(ii) By (i), P is β-dual-line regular for some integer β. Observe $\beta \ge \alpha > 0$ by Lemma 28.7(ii), so $\mathrm{Rad}(P) = 0$ by Lemma 26.5.

§32. The definition of a regular quantum matroid

Definition 32.1. A quantum matroid P is said to be *regular*, with *parameters* (D, q, α, β), whenever the following four conditions hold.

(i) P has rank D.
(ii) P is q-line regular.
(iii) P is α-zig-zag regular.
(iv) P is β-dual-line regular.

Let us consider a few very special cases. Any quantum matroid with rank $D \le 1$ is regular. However, the parameters q, α are not uniquely defined in this case. Similarly, any q-line regular modular atomic lattice is regular, but the parameter α is not uniquely defined in this case. In contrast to the above two cases, consider a regular quantum matroid P with rank $D \ge 2$, that is not a modular atomic lattice. Then the parameters q, α, β are uniquely defined.

Some results concerning regular quantum matroids do not hold unless the parameters are uniquely defined, so we make the following definition.

Definition 32.2. A quantum matroid P is said to be *trivial* whenever P has rank $D \le 1$, or P is a modular atomic lattice.

In Example 4.2, we characterized the quantum matroids of rank 2. Below we present a similar result concerning the regular quantum matroids of rank 2.

Example 32.3. Let q, α, β denote integers, and let P denote a poset. Then P is a regular quantum matroid with parameters $(2, q, \alpha, \beta)$ if and only if P has a 0, and the following conditions (i)–(vi) hold.

(i) P is ranked and $\mathrm{rank}(P) = 2$.
(ii) For any distinct points $x, y \in P$, there exists at most one line $z \in P$ such that $x \le z$, $y \le z$.
(iii) Each line in P covers exactly $q + 1$ points in P.

(iv) Each point in P is covered by exactly $\beta+1$ lines in P.
(v) For each point $x \in P$ and each line $y \in P$ such that $x \not\leq y$, there exists exactly $\alpha+1$ pairs $x'y'$, such that x' is a point in P, y' is a line in P, and $x \leq y' \geq x' \leq y$.
(vi) $q \geq 1$, $\alpha \geq 0$, $\beta \geq 0$.

Note. A regular quantum matroid with parameters $(2, q, \alpha, \beta)$ is essentially the same thing as an (R, K, T)-*partial geometry*, where $R := \beta + 1$, $K := q + 1$, and $T := \alpha + 1$ [Bo].

The following theorem gives a characterization of the regular quantum matroids of arbitrary rank $D \geq 2$. Compare this with Theorem 23.1.

Theorem 32.4. *Let D, q, α, β denote integers with $D \geq 2$, and let P denote a poset. Then P is a regular quantum matroid with parameters (D, q, α, β) if and only if* (i)–(iv) *hold below.*

(i) *P is a prematroid of rank D.*
(ii) *For all $x \in P$, there exists $x' \in \mathrm{top}(P)$ such that $x \leq x'$.*
(iii) *For all $x \in P$ such that $\mathrm{rank}(x) \leq D-2$, the poset induced on $x^+ \backslash \{x\}$ is connected.*
(iv) *For all $x \in P$ such that $\mathrm{rank}(x) = D-2$, the poset x^+ is a regular quantum matroid with parameters $(2, q, \alpha, \beta)$.*

Proof. First suppose P is a regular quantum matroid with parameters (D, q, α, β). Then the above conditions (i)–(iii) hold by Theorem 23.1. To see that the above condition (iv) holds, pick any $x \in P$ such that $\mathrm{rank}(x) = D-2$. Then x^+ is a quantum matroid of rank 2 by Theorem 23.1(iv). Observe x^+ is q-line regular by Lemma 27.4(i),(ii), and since P is q-line regular. Observe x^+ is α-zig-zag regular by Lemma 28.4, Definition 28.5, and since P is α-zig-zag regular. Observe x^+ is β-dual-line regular by Definition 27.7, and since P is β-dual-line regular. Now x^+ is a regular quantum matroid with parameters $(2, q, \alpha, \beta)$, as desired.

Conversely, suppose the above conditions (i)–(iv) hold. Observe P satisfies the conditions (i)–(iv) in Theorem 23.1, so P is a quantum matroid by that theorem. We check P is regular. To show P is q-line regular, it suffices to show $[0, y]$ is q-line regular for all $y \in \mathrm{top}(P)$. Observe $[0, y]^*$ is q-line regular by condition (iv) above, so $[0, y]$ is q-line regular by Lemma 27.4(i),(iii). We have now shown P is q-line regular. It is immediate from the construction that P is α-zig-zag regular and β-dual-line regular, so P is a regular quantum matroid with parameters (D, q, α, β), as desired.

§33. Formulae for $|A_P|$, $|\mathbf{top}(P)|$

In this section, we assume P is a regular quantum matroid with parameters (D, q, α, β), and compute $|A_P|$, $|\mathrm{top}(P)|$ in terms of D, q, α, β. First, we introduce some notation.

Definition 33.1. For all integers j, q, α, we define $t_j = t_j(q, \alpha)$ by

$$t_j := 1 + \alpha \begin{bmatrix} j \\ 1 \end{bmatrix}, \tag{33.1}$$

where $\left[\begin{smallmatrix} j \\ 1 \end{smallmatrix}\right]$ is from (27.3), (27.4).

Lemma 33.2. *With the notation of Definition* 33.1:

(i) $t_{j+1} - t_j = \alpha q^j \qquad (j \in \mathbb{Z})$.
(ii) *Assume* $\alpha \geq 0$, $q \geq 1$. *Then*

$$1 = t_0 \leq t_1 \leq t_2 \leq \ldots. \tag{33.2}$$

Proof. (i) Immediate from (27.3), (27.4), (33.1).
(ii) Immediate from (33.1) and (i) above.

Lemma 33.3. *Let P denote a regular quantum matroid with parameters (D, q, α, β). Then for all $x \in P$, and for all $y \in$* top(P) *such that x covers $x \wedge y$,*

$$|\mathrm{top}(x_y^+)| = t_{D-i}, \tag{33.3}$$

where

$$i := \mathrm{rank}(x). \tag{33.4}$$

Proof. By Definition 33.1 and Theorem 27.11, it suffices to show x_y^+ is a q-line regular, α-dual-line regular design-matroid, with rank $D - i$. Observe by Lemma 27.4(i),(ii) that x_y^+ is q-line regular. To see that x_y^+ is α-dual-line regular, we pick any dual-line z of x_y^+, and show

$$|z^+ \cap \mathrm{top}(x_y^+)| = \alpha + 1. \tag{33.5}$$

Observe

$$\mathrm{top}(x_y^+) \subseteq \mathrm{top}(P) \tag{33.6}$$

by Theorem 18.2(v), so z is a dual-line of P. Now $\partial(y,z) = 3$ by the construction, so $zy \in \Delta_D$ by Definition 28.1. Now by Lemma 28.4(i) and Lemma 15.8(i),

$$\begin{aligned}\alpha + 1 &= \text{zig-zag}(zy)\\ &= |\mathrm{top}(z_y^+)|\\ &= |z^+ \cap \mathrm{top}(x_y^+)|,\end{aligned}$$

as desired. We have now shown x_y^+ is α-dual-line regular. Observe $\delta(x,y) = 1$ by (12.1) and the construction, so x_y^+ is a design-matroid by Lemma 22.4. It is clear from (33.6) and the construction that x_y^+ has rank $D - i$. We have now proved Lemma 33.3.

Theorem 33.4. *Let P denote a regular quantum matroid with parameters (D, q, α, β). Then for all integers i $(0 \le i < D)$, and for all $x \in P$ with* $\mathrm{rank}(x) = i$*,*

$$\Big|\{\, z \in P \mid z \text{ covers } x\}\Big| = \eta_i, \tag{33.7}$$

where

$$\eta_i := (1 + \beta\frac{q^{D-i-1}}{t_{D-i-1}})\begin{bmatrix} D-i \\ 1 \end{bmatrix}, \tag{33.8}$$

and where t_j is from (33.1)*. In particular,*

$$|A_P| = (1 + \beta\frac{q^{D-1}}{t_{D-1}})\begin{bmatrix} D \\ 1 \end{bmatrix} \tag{33.9}$$

if $D \ge 1$.

Proof. Pick any $x \in P$ such that $\mathrm{rank}(x) = i$. By Lemma 4.3, there exists $y \in \mathrm{top}(P)$ such that $x \le y$. To show (33.7), (33.8), it suffices to show

$$|X| = \begin{bmatrix} D-i \\ 1 \end{bmatrix}, \tag{33.10}$$

$$|Y| = \beta\frac{q^{D-i-1}}{t_{D-i-1}}\begin{bmatrix} D-i \\ 1 \end{bmatrix}, \tag{33.11}$$

where

$$X := \{z \in P \mid z \text{ covers } x, \quad z \le y\}, \tag{33.12}$$

$$Y := \{z \in P \mid z \text{ covers } x, \quad z \not\le y\}. \tag{33.13}$$

Line (33.10) is immediate from Lemma 27.5, so consider (33.11). Set

$$X' := \{z \in [x, y] \mid y \text{ covers } z\},$$

and observe

$$|X'| = \begin{bmatrix} D-i \\ 1 \end{bmatrix} \tag{33.14}$$

by Lemma 27.5. Set

$$Y' := \{z \in \operatorname{top}(P) \mid x \leq z, \ \ \partial(y, z) = 2\}.$$

Observe each element of X' is covered by exactly β elements of Y'. Also, observe each element of Y' covers a unique element of X'. It follows

$$|Y'| = |X'|\beta. \tag{33.15}$$

Next, we claim each element of Y is less than or equal to exactly t_{D-i-1} elements of Y'. To see this, pick any $z \in Y$. Observe z covers $z \wedge y = x$; in particular $\operatorname{rank}(z) = i+1$. Now by Lemma 33.3,

$$\begin{aligned} t_{D-i-1} &= |\operatorname{top}(z_y^+)| \\ &= |Y' \cap z^+|, \end{aligned}$$

as desired.

Next, we claim each element in Y' is greater than or equal to exactly q^{D-i-1} elements in Y. To see this, pick any $z \in Y'$. Then z is greater than or equal to exactly $\left[\begin{smallmatrix} D-i \\ 1 \end{smallmatrix}\right]$ elements in $X \cup Y$, but

$$\begin{aligned} \big|\{v \in X \mid v \leq z\}\big| &= \big|\{v \in X \mid v \leq z \wedge y\}\big| \\ &= \begin{bmatrix} D-i+1 \\ 1 \end{bmatrix} \end{aligned}$$

by Lemma 27.5. It follows z is greater than or equal to exactly

$$\begin{bmatrix} D-i \\ 1 \end{bmatrix} - \begin{bmatrix} D-i+1 \\ 1 \end{bmatrix} = q^{D-i-1}$$

elements in Y, as desired.

Combining our claims, we find

$$|Y'|q^{D-i-1} = |Y|t_{D-i-1}, \tag{33.16}$$

and (33.11) follows from (33.14), (33.15), (33.16). We have now established (33.10), (33.11), and (33.7), (33.8) follow.

To obtain (33.9), set $i=0$ in (33.7), (33.8). This proves Theorem 33.4.

Corollary 33.5. *Let P denote a regular quantum matroid with parameters (D,q,α,β). Then*

$$|\mathrm{top}(P)| = \prod_{i=0}^{D-1}(1+\beta q^i t_i^{-1}), \tag{33.17}$$

where t_i is from (33.1).

Proof. We compute the number of paths

$$x_0,\, x_1,\, \ldots,\, x_D \qquad (x_i \in P,\;\; \mathrm{rank}(x_i)=i,\;\; (0\le i\le D))$$

in two ways. Constructing these paths from left to right, we find by Theorem 33.4 that the number is

$$\eta_0\eta_1\cdots\eta_{D-1}. \tag{33.18}$$

Constructing the above paths from right to left, we find by Lemma 27.5 that the number is

$$|\mathrm{top}(P)|\prod_{i=1}^{D}\begin{bmatrix} i\\ 1\end{bmatrix}. \tag{33.19}$$

Now equate (33.18), (33.19), and evaluate the result using (33.8). This proves Corollary 33.5.

§34. The zig-zag function, revisited

Let P denote a regular quantum matroid, with parameters (D,q,α,β). In this section, we show the zig-zag function is constant over each Δ_i $(2\le i\le D)$, and we compute these constants in terms of D,q,α,β. First, we prove an extension of Lemma 33.3.

Theorem 34.1. *Let P denote a regular quantum matroid with parameters (D,q,α,β), and pick any $x,y\in P$. Assume*

$$x \text{ covers } x\wedge y, \tag{34.1}$$

and

$$x\vee y \text{ does not exist.} \tag{34.2}$$

Then

$$|\mathrm{top}(x_y^+)| = \frac{t_{D-i}}{t_{D-j}}, \tag{34.3}$$

where

$$i := \mathrm{rank}(x), \tag{34.4}$$

$$j := \mathrm{rank}(y). \tag{34.5}$$

Proof. By Lemma 4.3, there exists an element $u \in \mathrm{top}(P)$ such that $y \leq u$. To get our result, it suffices to show

$$|\mathrm{top}(x_u^+)| = t_{D-i} \tag{34.6}$$

and

$$|\mathrm{top}(x_y^+)| t_{D-j} = |\mathrm{top}(x_u^+)|. \tag{34.7}$$

Observe $x \not\leq u$; otherwise u is an upper bound for x, y, contradicting (34.2). Now x covers $x \wedge y = x \wedge u$ by (34.1), so (34.6) follows from Lemma 33.3.

Now consider (34.7). Recall by Lemma 15.9 that x_y^+ is a submatroid of x_u^+. We claim each element in $\mathrm{top}(x_y^+)$ is less than or equal to exactly t_{D-j} elements in $\mathrm{top}(x_u^+)$. To see this, pick any $z \in \mathrm{top}(x_y^+)$. Observe z covers $z \wedge u$ by Lemma 15.3(i),(ii), and $\mathrm{rank}(z) = j$ by Theorem 18.2(v), so by Lemma 33.3 and Lemma 15.8(i),

$$\begin{aligned} t_{D-j} &= |\mathrm{top}(z_u^+)| \\ &= |z^+ \cap \mathrm{top}(x_u^+)|, \end{aligned}$$

as desired.

Next, we claim each element in $\mathrm{top}(x_u^+)$ is greater than or equal to a unique element in $\mathrm{top}(x_y^+)$. To see this, pick any $z \in \mathrm{top}(x_u^+)$. We show

$$p := x \vee (y \wedge z)$$

is the unique element in $\mathrm{top}(x_y^+)$ that is less than or equal to z. Observe $p \in x_y^+$ by Lemma 17.1. In fact $p \in \mathrm{top}(x_y^+)$, since y covers $y \wedge z$ by the construction. Observe $p \leq z$ by Lemma 16.3(i), Lemma 17.1. Now suppose there exists an element $p' \in \mathrm{top}(x_y^+) \backslash \{p\}$ such that $p' \leq z$. Then z is an upper bound for p, p'; consequently $p \vee p'$ exists. But this is impossible, since x_y^+ is $\vee$-closed by Lemma 8.2, Lemma

15.2, and since p, p' are both in $\operatorname{top}(x_y^+)$. Combining the above two claims, we obtain (34.7). We have now established (34.6), (34.7), and (34.3) follows. This proves Theorem 34.1.

Corollary 34.2. *Let P denote a regular quantum matroid with parameters (D, q, α, β). Then for all integers j $(2 \leq j \leq D)$ and for all $x, y \in P$ such that $xy \in \Delta_j$,*

$$\text{zig-zag}(x,y) = \frac{t_{D-j+1}}{t_{D-j}} \tag{34.8}$$

$$= q + \frac{\alpha + 1 - q}{t_{D-j}}. \tag{34.9}$$

(*Caution*: Δ_j *may be empty*).

Proof. Observe x, y satisfy (34.1), (34.2) by the definition of Δ_j, so by Lemma 28.4(i), Theorem 34.1,

$$\begin{aligned}\text{zig-zag}(x,y) &= |\operatorname{top}(x_y^+)| \\ &= \frac{t_{D-j+1}}{t_{D-j}}.\end{aligned}$$

Line (34.9) follows from (33.1).

§35. The ζ-uniform $\mathcal{P}$-basis systems

Let $\mathcal{P}$ denote a modular atomic lattice, and let B denote a $\mathcal{P}$-basis system from Definition 2.2. Recall by Theorem 2.5(i) that B^- is a $\mathcal{P}$-matroid, and by Lemma 3.1 that B^{+*} is a $\mathcal{P}^*$-matroid. In this section we introduce a condition on B that forces both of these quantum matroids to be regular. This condition will play a role in our subsequent work on regular quantum matroids.

Definition 35.1. Let $\mathcal{P}$ denote a modular atomic lattice of rank N, let B denote a $\mathcal{P}$-basis system of rank D, and let ζ denote an integer.

(i) Suppose $1 \leq D \leq N-1$. Then B is said to be ζ-*uniform* whenever

$$\big|B \cap [x,y]\big| = \zeta + 1 \tag{35.1}$$

for all $x \in B^-$ and for all $y \in B^+$ such that

$$x \leq y, \tag{35.2}$$

$$\mathrm{rank}(x) = D - 1, \tag{35.3}$$

$$\mathrm{rank}(y) = D + 1. \tag{35.4}$$

(ii) Suppose $D = 0$ or $D = N$. Then B is said to be *ζ-uniform* whenever ζ is nonnegative.

Lemma 35.2. *Let $\mathcal{P}$ denote a modular atomic lattice, and let B denote a ζ-uniform $\mathcal{P}$-basis system. Then*

$$\zeta \geq 0. \tag{35.5}$$

Proof. With the notation of Definition 35.1, suppose $1 \leq D \leq N - 1$. Then (35.5) follows from condition BA in Definition 2.2. Next suppose $D = 0$ or $D = N$. Then (35.5) is immediate from Definition 35.1(ii).

Theorem 35.3. *Let $\mathcal{P}$ denote a q-line regular modular atomic lattice of rank N, and let B denote a ζ-uniform $\mathcal{P}$-basis system of rank D. Then*

(i) *The $\mathcal{P}$-matroid B^- is regular, with parameters*

$$\left(D,\ q,\ \zeta,\ \zeta \begin{bmatrix} N-D \\ 1 \end{bmatrix}\right).$$

(ii) *The $\mathcal{P}^*$-matroid B^{+*} is regular, with parameters*

$$\left(N-D,\ q,\ \zeta,\ \zeta \begin{bmatrix} D \\ 1 \end{bmatrix}\right).$$

Proof. (i) By the construction, B^- has rank D, and is q-line regular. To see that B^- is ζ-zig-zag regular, suppose there exists $x, y \in B^-$ such that $\mathrm{rank}(x) = D - 1$, $\mathrm{rank}(y) = D$, and $\partial(x, y) = 3$. We show

$$\text{zig-zag}(x, y) = \zeta + 1. \tag{35.6}$$

Observe $x \vee_{\mathcal{P}} y \geq y \in B$, so

$$x \vee_{\mathcal{P}} y \in B^+. \tag{35.7}$$

Observe x covers $x \wedge y$ by the construction, so $x \vee_{\mathcal{P}} y$ covers y by the modularity of $\mathcal{P}$. In particular

$$\operatorname{rank}_{\mathcal{P}}(x \vee_{\mathcal{P}} y) = D + 1. \tag{35.8}$$

Now by (35.7), (35.8), and Definition 35.1,

$$\begin{aligned} \zeta + 1 &= \big|B \cap [x, x \vee_{\mathcal{P}} y]\big| \\ &= \text{zig-zag}(x, y), \end{aligned}$$

as desired. We have now shown B^- is ζ-zig-zag regular. It remains so show B^- is $\zeta\left[{N-D \atop 1}\right]$-dual-line regular. To this end, suppose there exists $x \in B^-$ such that $\operatorname{rank}(x) = D - 1$. We show

$$|B \cap [x, 1_{\mathcal{P}}]| = 1 + \zeta \begin{bmatrix} N - D \\ 1 \end{bmatrix}. \tag{35.9}$$

To see (35.9), observe by Lemma 27.4(i),(ii), and Definition 35.1 that $(B^+ \cap [x, 1_{\mathcal{P}}])^*$ is a q-line regular, ζ-dual-line regular design-matroid of rank $N - D$. Applying Theorem 27.11 to this matroid, we get (35.9).

(ii) Observe $\mathcal{P}^*$ is q-line regular by Lemma 27.4. By Definition 35.1 and the construction, B is a ζ-uniform $\mathcal{P}^*$-basis system of rank $N - D$. The result now follows from part (i) above.

§36. $x \star y$ is a uniform $[x \wedge y, x]$-basis system

Let P denote a quantum matroid, and pick any x, $y \in P$. Recall by Theorem 19.3 that $x \star y$ is a $[x \wedge y, x]$-basis system. In this section, we assume P is regular, and show $x \star y$ is ζ-uniform, for some integer ζ that depends on $\rho(x,y)$, $\gamma(x,y)$, $\delta(x,y)$, $\gamma(y,x)$, and the parameters of P. We combine this information with results from Sections 33, 35 to compute the number of atoms in $[x \wedge y, x \star y]$, $[x \star y, x]^*$, and x_y^+.

Theorem 36.1. *Let P denote a regular quantum matroid with parameters (D, q, α, β). Pick $x, y \in P$, and set*

$$\rho := \rho(x, y), \tag{36.1}$$

$$\gamma := \gamma(x, y), \tag{36.2}$$

$$\delta := \delta(x, y), \tag{36.3}$$

$$\gamma^t := \gamma(y, x). \tag{36.4}$$

Then the $[x \wedge y, x]$-basis system $x \star y$ is ζ-uniform, where

$$\zeta := \alpha \frac{q^{D-\rho-\gamma-\delta-\gamma^t}}{t_{D-\rho-\gamma-\delta-\gamma^t}}, \tag{36.5}$$

and where t_j *is from* (33.1).

Proof. Recall by Lemma 13.2 and Theorem 18.2(vi) that

$$\operatorname{rank}(x) = \rho + \gamma + \delta, \tag{36.6}$$

$$\operatorname{rank}(y) = \rho + \gamma^t + \delta. \tag{36.7}$$

Pick $b \in x \star y$. We may assume $x \wedge y < b < x$; otherwise there is nothing to prove by Lemma 13.3(ii), Definition 35.1(ii). Now there exists $u, v \in [x \wedge y, x]$ such that b covers u and v covers b. We must show

$$\big|x \star y \cap [u, v]\big| = \zeta + 1. \tag{36.8}$$

Observe $u \in x_y^-$ by Theorem 19.3(i), so $u \vee y$ exists. Set $r := u \vee y$. Observe by Lemma 15.3(ii),(iii) that u, y are relative complements in the interval $[x \wedge y, r]$; if follows by Lemma 14.2(iii), (36.7), and the construction that

$$\begin{aligned} \operatorname{rank}(r) &= \operatorname{rank}(y) + \operatorname{rank}(u) - \operatorname{rank}(x \wedge y) \\ &= \rho + \gamma + \delta + \gamma^t - 1. \end{aligned} \tag{36.9}$$

Observe by the construction that $\delta(v, y) = 1$. Observe $r \in y_v^+$ by Lemma 15.3(i),(iii), so

$$\begin{aligned} \delta(v, r) &= \delta(v, y) \\ &= 1 \end{aligned}$$

by Lemma 15.3(v). Now there exists $w \in [u, r]$ such that r covers w and such that $v \vee w$ exists. Set $s := v \vee w$. Observe by Lemma 15.3(i),(ii) that v, w are relative complements in the interval $[u, s]$, so by (36.9),

$$\begin{aligned} \operatorname{rank}(s) &= \operatorname{rank}(w) + \operatorname{rank}(v) - \operatorname{rank}(u) \\ &= \operatorname{rank}(w) + 2 \\ &= \operatorname{rank}(r) + 1 \\ &= \rho + \gamma + \delta + \gamma^t. \end{aligned} \tag{36.10}$$

We compute $|\operatorname{top}(r_s^+)|$ in two ways. On one hand, observe r covers $r \wedge s = w$ and $r \vee s$ does not exist, so by (36.9), (36.10), and Theorem 34.1,

$$\begin{aligned} |\operatorname{top}(r_s^+)| &= \frac{t_{D-\rho-\gamma-\delta-\gamma^t+1}}{t_{D-\rho-\gamma-\delta-\gamma^t}} \\ &= \zeta + 1 \end{aligned} \tag{36.11}$$

by (33.1), (36.5). On the other hand, one finds by Theorem 15.5, Lemma 15.8(ii),(iii), and the observations $s \in v_r^+$, $r \in y_v^+$, that

$$\begin{aligned} |\mathrm{top}(r_s^+)| &= |\mathrm{top}(r_v^+)| \\ &= |\mathrm{top}(v_r^-)| \\ &= |\mathrm{top}(v_y^-) \cap [u,v]| \\ &= \big|x \star y \cap [u,v]\big|. \end{aligned} \tag{36.12}$$

Line (36.8) follows from (36.11), (36.12), so we are done.

Corollary 36.2. *With the assumptions and notation of Theorem 36.1:*

(i) *The $[x \wedge y, x]$-matroid $[x \wedge y, x \star y]$ is regular, with parameters*

$$\left(\gamma,\ q,\ \zeta,\ \zeta \begin{bmatrix} \delta \\ 1 \end{bmatrix}\right). \tag{36.13}$$

(ii) *The $[x \wedge y, x]^*$-matroid $[x \star y, x]^*$ is regular, with parameters*

$$\left(\delta,\ q,\ \zeta,\ \zeta \begin{bmatrix} \gamma \\ 1 \end{bmatrix}\right). \tag{36.14}$$

(iii) *The quantum matroid x_y^+ is regular, with parameters*

$$\left(\gamma^t,\ q,\ \zeta,\ \zeta \begin{bmatrix} \delta \\ 1 \end{bmatrix}\right). \tag{36.15}$$

Proof. (i), (ii) Immediate from Lemma 14.2(iii),(vi), Theorem 35.3, and Theorem 36.1.

(iii) Interchanging the roles of x, y in (i) above, we find $[x\wedge y, y\star x]$ is regular, with parameters (36.15). The result now follows, since x_y^+ is isomorphic to $y_x^- = [x \wedge y, y \star x]$ by Theorem 15.5 and Theorem 19.3(i). We have now proved Corollary 36.2.

Theorem 36.3. *With the assumptions and notation of Theorem 36.1:*

(i) *The number of atoms in the $[x \wedge y, x]$-matroid $[x \wedge y, x \star y]$ is*

$$\frac{t_{D-\rho-\gamma^t-1}}{t_{D-\rho-\delta-\gamma^t-1}} \begin{bmatrix} \gamma \\ 1 \end{bmatrix} \tag{36.16}$$

if $\gamma \geq 1$, and 0 if $\gamma = 0$.

(ii) *The number of atoms in the* $[x \wedge y, x]^*$*-matroid* $[x \star y, x]^*$ *is*

$$\frac{t_{D-\rho-\gamma^t-1}}{t_{D-\rho-\gamma-\gamma^t-1}} \begin{bmatrix} \delta \\ 1 \end{bmatrix} \tag{36.17}$$

if $\delta \geq 1$, *and* 0 *if* $\delta = 0$.

(iii) *The number of atoms in the quantum matroid* x_y^+ *is*

$$\frac{t_{D-\rho-\gamma-1}}{t_{D-\rho-\gamma-\delta-1}} \begin{bmatrix} \gamma^t \\ 1 \end{bmatrix} \tag{36.18}$$

if $\gamma^t \geq 1$, *and* 0 *if* $\gamma^t = 0$.

Proof. Apply the formula (33.9) in each of the three cases of Corollary 36.2, and simplify the result using (33.1), (36.5).

§37. The staircase theorem, revisited

Theorem 37.1. *Let* P *denote a regular quantum matroid with parameters* (D, q, α, β). *Pick any integer* a $(0 \leq a \leq D)$, *and fix an element* $y \in P$ *such that* $\operatorname{rank}(y) = a$. *Let* $\mathcal{D} = \mathcal{D}(a, D-a)$ *and* $\sigma : P \to V\mathcal{D}$ *be as in Theorem* 21.3. *Then for all* (ρ, γ, δ), $(\rho', \gamma', \delta') \in V\mathcal{D}$, *and for all* $x \in \sigma^{-1}(\rho, \gamma, \delta)$, *the number*

$$\left|\{z \in \sigma^{-1}(\rho', \gamma', \delta') \mid z \text{ is adjacent to } x\}\right| \tag{37.1}$$

is given as follows.

Case $(\rho', \gamma', \delta') = (\rho - 1, \gamma, \delta)$:

$$q^{\gamma+\delta} \begin{bmatrix} \rho \\ 1 \end{bmatrix} \tag{37.2}$$

Case $(\rho', \gamma', \delta') = (\rho + 1, \gamma, \delta)$:

$$\frac{t_{D-\rho-\gamma-1}}{t_{D-\rho-\gamma-\delta-1}} \begin{bmatrix} a-\rho-\delta \\ 1 \end{bmatrix} \tag{37.3}$$

Case $(\rho', \gamma', \delta') = (\rho, \gamma - 1, \delta)$:

$$q^{\delta} \frac{t_{D-a-\gamma-1}}{t_{D-a-\gamma+\delta-1}} \begin{bmatrix} \gamma \\ 1 \end{bmatrix} \tag{37.4}$$

if $\delta \geq 1$, *and*

$$\begin{bmatrix} \gamma \\ 1 \end{bmatrix} \tag{37.5}$$

if $\delta = 0$.

Case $(\rho', \gamma', \delta') = (\rho, \gamma + 1, \delta)$:

$$q^{a-\rho-\delta} \frac{t_{D-\rho-\gamma-1}}{t_{D-\rho-\gamma-\delta-1}} \frac{t_{D-a-\gamma-1}}{t_{D-a-\gamma+\delta-1}} \eta_{a+\gamma} \tag{37.6}$$

Case $(\rho', \gamma', \delta') = (\rho, \gamma, \delta - 1)$:

$$\frac{t_{D-a+\delta-1}}{t_{D-a-\gamma+\delta-1}} \begin{bmatrix} \delta \\ 1 \end{bmatrix} \tag{37.7}$$

Case $(\rho', \gamma', \delta') = (\rho, \gamma, \delta + 1)$:

$$\frac{t_{\delta-1}}{q^{\delta-1}} \frac{q^{D-\rho-\gamma-\delta-1}}{t_{D-\rho-\gamma-\delta-1}} \frac{q^{D-a-\gamma+\delta-1}}{t_{D-a-\gamma+\delta-1}} (\beta - \alpha \begin{bmatrix} \delta \\ 1 \end{bmatrix}) \begin{bmatrix} a - \rho - \delta \\ 1 \end{bmatrix} \tag{37.8}$$

if $a + \gamma < D$, *and*

$$\frac{q^{D-\rho-\gamma-\delta-1}}{t_{D-\rho-\gamma-\delta-1}} (\beta - \alpha \begin{bmatrix} \delta \\ 1 \end{bmatrix}) \begin{bmatrix} D - \rho - \gamma - \delta \\ 1 \end{bmatrix} \tag{37.9}$$

if $a + \gamma = D$.

The number (37.1) *equals* 0 *in all other cases.*

Proof. We set

$$\Lambda := \{z \in \sigma^{-1}(\rho', \gamma', \delta') \mid z \text{ is adjacent to } x\},$$

and compute $|\Lambda|$ in each of the above cases.

Case $(\rho', \gamma', \delta') = (\rho - 1, \gamma, \delta)$. By Lemma 13.4 and Lemma 27.5,

$$\begin{aligned} |\Lambda| &= \big|\{z \in P \mid x \text{ covers } z, \ \ z \not\geq x \wedge y\}\big| \\ &= \big|\{z \in P \mid x \text{ covers } z\}\big| \ - \ \big|\{z \in P \mid x \text{ covers } z, \ \ z \geq x \wedge y\}\big| \\ &= \begin{bmatrix} \rho + \gamma + \delta \\ 1 \end{bmatrix} - \begin{bmatrix} \gamma + \delta \\ 1 \end{bmatrix} \\ &= q^{\gamma+\delta} \begin{bmatrix} \rho \\ 1 \end{bmatrix}, \end{aligned}$$

as desired.

Case $(\rho', \gamma', \delta') = (\rho+1, \gamma, \delta)$. By Lemma 15.3, Λ consists of those elements in x_y^+ that cover x, i.e. the atoms of the poset x_y^+. Now $|\Lambda|$ is given in (36.18). Eliminating γ^t in (36.18) using $\gamma^t = a - \rho - \delta$, we obtain (37.3), as desired.

Case $(\rho', \gamma', \delta') = (\rho, \gamma, \delta - 1)$. By Lemma 14.3, Λ consists of the elements in $[x \star y, x]$ that are covered by x (in P), i.e., the atoms of the poset $[x \star y, x]^*$. Now $|\Lambda|$ is given in (36.17). Eliminating γ^t in (36.17) as in the previous case, we obtain (37.7), as desired.

Case $(\rho', \gamma', \delta') = (\rho, \gamma - 1, \delta)$. In this case Λ consists of the elements in $[x \wedge y, x]$ that are covered by x but are not in $[x \star y, x]$. If $\delta \geq 1$, then by the previous case,

$$\begin{aligned} |\Lambda| &= \begin{bmatrix} \gamma + \delta \\ 1 \end{bmatrix} - \frac{t_{D-a+\delta-1}}{t_{D-a-\gamma+\delta-1}} \begin{bmatrix} \delta \\ 1 \end{bmatrix} \\ &= q^{\delta} \frac{t_{D-a-\gamma-1}}{t_{D-a-\gamma+\delta-1}} \begin{bmatrix} \gamma \\ 1 \end{bmatrix}, \end{aligned}$$

as desired. If $\delta = 0$ then $x \star y = x$, so

$$|\Lambda| = \begin{bmatrix} \gamma \\ 1 \end{bmatrix}$$

by Lemma 14.2(iii) and Lemma 27.5.

Case $(\rho', \gamma', \delta') = (\rho, \gamma + 1, \delta)$. We show $|\Lambda|$ equals the expression in (37.6) by induction on

$$\begin{aligned} \gamma^t &:= \gamma(y, x) \\ &= a - \rho - \delta. \end{aligned}$$

First consider the case $\gamma^t = 0$. Here x is relatively close to y in the sense of Lemma 20.2, so by Theorem 20.4,

$$\Lambda = \{z \in P \mid z \text{ covers } x\}.$$

In particular by Theorem 33.4,

$$\begin{aligned} |\Lambda| &= \eta_i \qquad\qquad (i = \operatorname{rank}(x)), \\ &= \eta_{\rho+\gamma+\delta}, \end{aligned}$$

which is what (37.6) reduces to in this case. Now assume $\gamma^t > 0$. Set

$$\begin{aligned} \Omega &:= \{u \in x_y^+ \mid u \text{ covers } x\} \\ &= \{u \in P \mid u \geq x, \ \ \sigma(u) = (\rho + 1, \gamma, \delta)\}, \end{aligned}$$

and observe by (37.3) that

$$|\Omega| = \frac{t_{D-\rho-\gamma-1}}{t_{D-\rho-\gamma-\delta-1}} \begin{bmatrix} a - \rho - \delta \\ 1 \end{bmatrix}. \tag{37.10}$$

Set

$$\Psi := \{v \in P \mid v \geq x, \ \ \sigma(v) = (\rho+1, \gamma+1, \delta)\}.$$

Observe by induction and (37.6) (with $(a', \rho', \gamma', \delta') = (a, \rho+1, \gamma, \delta)$) that each element in Ω is covered by exactly

$$q^{a-\rho-\delta-1} \frac{t_{D-\rho-\gamma-2}}{t_{D-\rho-\gamma-\delta-2}} \frac{t_{D-a-\gamma-1}}{t_{D-a-\gamma+\delta-1}} \eta_{a+\gamma}$$

elements in Ψ. Also observe by Theorem 17.2(i),(iv) that each element $v \in \Psi$ covers a unique element in Ω, i.e., $x \vee (v \wedge y)$. From our above observations

$$|\Psi| = |\Omega| q^{a-\rho-\delta-1} \frac{t_{D-\rho-\gamma-2}}{t_{D-\rho-\gamma-\delta-2}} \frac{t_{D-a-\gamma-1}}{t_{D-a-\gamma+\delta-1}} \eta_{a+\gamma}. \tag{37.11}$$

Observe by (37.3) (with $(a', \rho', \gamma', \delta') = (a, \rho, \gamma+1, \delta)$) that each element in Λ is covered by exactly

$$\frac{t_{D-\rho-\gamma-2}}{t_{D-\rho-\gamma-\delta-2}} \begin{bmatrix} a-\rho-\delta \\ 1 \end{bmatrix}$$

elements in Ψ. Also observe by Theorem 17.3(iii),(iv) that each element in Ψ covers exactly q elements in Λ. From the above two observations

$$|\Lambda| \frac{t_{D-\rho-\gamma-2}}{t_{D-\rho-\gamma-\delta-2}} \begin{bmatrix} a-\rho-\delta \\ 1 \end{bmatrix} = |\Psi| q. \tag{37.12}$$

Combining (37.10)-(37.12), we find $|\Lambda|$ equals the expression (37.6), as desired.

Case $(\rho', \gamma', \delta') = (\rho, \gamma, \delta+1)$. First assume $a+\gamma < D$. In this case Λ consists of all the elements in P that cover x, but are not counted in (37.3) or (37.6). By (37.3), (37.6), and Theorem 33.4, we find

$$\begin{aligned} |\Lambda| \ = \ \eta_{\rho+\gamma+\delta} \ &- \ \frac{t_{D-\rho-\gamma-1}}{t_{D-\rho-\gamma-\delta-1}} \begin{bmatrix} a-\rho-\delta \\ 1 \end{bmatrix} \\ &- \ q^{a-\rho-\delta} \frac{t_{D-\rho-\gamma-1}}{t_{D-\rho-\gamma-\delta-1}} \frac{t_{D-a-\gamma-1}}{t_{D-a-\gamma+\delta-1}} \eta_{a+\gamma}. \end{aligned}$$

Evaluating this using (27.3), (27.4), (33.1), (33.8), we obtain (37.8).

Now assume $a+\gamma = D$. In this case, Λ consists of all the elements in P that cover x, but are not counted in (37.3). Proceeding as above, we find

$$|\Lambda| \ = \ \eta_{\rho+\gamma+\delta} \ - \ \frac{t_{D-\rho-\gamma-1}}{t_{D-\rho-\gamma-\delta-1}} \begin{bmatrix} D-\gamma-\rho-\delta \\ 1 \end{bmatrix}.$$

Evaluating this using (27.3), (27.4), (33.1), (33.8), we obtain (37.9), as desired.

In any other case, the expression (37.1) equals 0 by the staircase theorem. This proves Theorem 37.1.

Corollary 37.2. *Let P denote a regular quantum matroid with parameters (D, q, α, β). Pick any $x \in P$, pick any $y \in \operatorname{top}(P)$, and set*

$$\begin{aligned} \rho &:= \rho(x,y), \\ \delta &:= \delta(x,y) \\ &= \operatorname{rank}(x) - \rho. \end{aligned}$$

Then

$$\begin{aligned} &\text{(i)} \quad \big|\{z \in P \mid x \textit{ covers } z, \;\; \partial(z,y) = \partial(x,y)+1\}\big| = q^{\delta}\begin{bmatrix}\rho\\1\end{bmatrix}, \\ &\text{(ii)} \quad \big|\{z \in P \mid x \textit{ covers } z, \;\; \partial(z,y) = \partial(x,y)-1\}\big| = \begin{bmatrix}\delta\\1\end{bmatrix}. \end{aligned}$$

Suppose $x \notin \operatorname{top}(P)$. Then

(iii)

$$\big|\{z \in P \mid z \textit{ covers } x, \;\; \partial(z,y) = \partial(x,y)-1\}\big| = \frac{t_{D-\rho-1}}{t_{D-\rho-\delta-1}}\begin{bmatrix}D-\rho-\delta\\1\end{bmatrix},$$

(iv)

$$\big|\{z \in P \mid z \textit{ covers } x, \;\; \partial(z,y) = \partial(x,y)+1\}\big| = \frac{q^{D-\rho-\delta-1}}{t_{D-\rho-\delta-1}}\left(\beta - \alpha\begin{bmatrix}\delta\\1\end{bmatrix}\right)\begin{bmatrix}D-\rho-\delta\\1\end{bmatrix}.$$

Proof. This is the case $a = D$ in Theorem 37.1. With the notation of that theorem, observe $\mathcal{D} = \mathcal{D}(D,0)$, so by Definition 21.2, $\gamma = 0$ for all $(\rho, \gamma, \delta) \in V\mathcal{D}$. Setting $\Delta\gamma = 0$ in Theorem 13.5, we find the sets (i)–(iv) above equal the sets of vertices adjacent to x and contained in $\sigma^{-1}(\rho-1, 0, \delta)$, $\sigma^{-1}(\rho, 0, \delta-1)$, $\sigma^{-1}(\rho+1, 0, \delta)$, $\sigma^{-1}(\rho, 0, \delta+1)$, resp. The cardinality of these sets is given in (37.2), (37.7), (37.3), (37.9), respectively, (where $\gamma = 0$, $a = D$). This proves Corollary 37.2.

§38. The graph on top(P) is distance-regular

Definition 38.1. A finite, connected, undirected graph $\Gamma = (V\Gamma, E\Gamma)$ of diameter d is said to be *distance-regular*, with *intersection numbers* c_i, b_i $(0 \leq i \leq d)$, whenever for all integers i $(0 \leq i \leq d)$, and all $x, y \in V\Gamma$ at distance $\partial_\Gamma(x,y) = i$, the scalars

$$c_i := \left|\{z \in V\Gamma \mid xz \in E\Gamma,\ \ \partial_\Gamma(y,z) = i-1\}\right|,$$
$$b_i := \left|\{z \in V\Gamma \mid xz \in E\Gamma,\ \ \partial_\Gamma(y,z) = i+1\}\right|,$$

are independent of x, y.

Theorem 38.2. *Let P denote a nontrivial regular quantum matroid with parameters (D, q, α, β). Then* (i)–(iii) *hold below.*

(i) *The graph on* top(P) *is distance-regular, with intersection numbers*

(38.1)
$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} (1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}) \qquad (0 \leq i \leq d),$$

(38.2)
$$b_i = (\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix})(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}) \qquad (0 \leq i \leq d),$$

where $d := \mathrm{diam}_{\mathrm{top}}(P)$.

(ii) *Suppose* $d < D$. *Then*

(38.3)
$$\beta = \alpha \begin{bmatrix} d \\ 1 \end{bmatrix}.$$

(iii) *Assume* $d = D$. *Then the graph on* top(P) *has classical parameters* (D, q, α, β) *in the sense of Brouwer, Cohen, Neumaier* [B-C-N].

Proof. (i) Routine application of Corollary 37.2 using line (22.1).
(ii) Immediate from (i) and the observation $b_d = 0$.
(iii) Immediate from [B-C-N, p194].

§39. The classification of the regular quantum matroids with rank at least 4

In this section, we classify the nontrivial regular quantum matroids with rank at least 4. We do this as follows. Let P denote a nontrivial regular quantum matroid with parameters (D, q, α, β), and assume $D \geq 4$. First, we show $\alpha \in \{0, q-1, q\}$. In each case, we invoke a result from the literature to identify P, giving us our classification. Our main result is Theorem 39.6.

Lemma 39.1. *Let P denote a nontrivial regular quantum matroid with parameters (D, q, α, β), and assume $\alpha \neq q$. Then* (i)–(iii) *hold below.*

(i) $(q-1-\alpha)t_{D-i}^{-1} \geq 0 \qquad (2 \leq i \leq D)$.

(ii) $\Delta_i \neq \emptyset \qquad (2 \leq i \leq D)$.

(iii) $(q-1-\alpha)t_{D-i}^{-1} \in \mathbb{Z} \qquad (2 \leq i \leq D)$.

Proof. (i) Recall $\alpha \leq q$ by (28.4), and we assume $\alpha \neq q$, so $\alpha \leq q-1$. Also t_{D-i} is positive by Lemma 33.2(ii).

(ii) By Lemma 28.2(iii), it suffices to find $x, y \in A_P$

assume P is not a modular atomic lattice, there exists $x \in A_P$ and there exists $u \in \operatorname{top}(P)$ such that $x \not\leq u$. We show there exists an atom $y \in \operatorname{Shadow}(u)$ such that $x \vee y$ does not exist. To this end, recall by (15.2) and Theorem 19.3(i) that for all $y \in \operatorname{Shadow}(u)$, $x \vee y$ exists if and only if $y \in [0, u \star x]$. Hence, it suffices to show $\operatorname{Shadow}(u) \backslash [0, u \star x]$ is not empty. Observe by (27.5),

$$|\operatorname{Shadow}(u)| = \begin{bmatrix} D \\ 1 \end{bmatrix}. \tag{39.1}$$

To compute the number of atoms in $[0, u \star x]$, observe by Definitions 12.1, 13.1 that $\rho(u,x) = 0$, $\gamma(u,x) = D-1$, $\delta(u,x) = 1$, $\gamma(x,u) = 0$. By Theorem 36.3(i) (with $\rho = 0$, $\gamma = D-1$, $\delta = 1$, $\gamma^t = 0$), the number of atoms in $[0, u \star x]$ equals

$$\frac{t_{D-1}}{t_{D-2}} \begin{bmatrix} D-1 \\ 1 \end{bmatrix}. \tag{39.2}$$

Observe

$$\begin{aligned} \begin{bmatrix} D \\ 1 \end{bmatrix} - \frac{t_{D-1}}{t_{D-2}} \begin{bmatrix} D-1 \\ 1 \end{bmatrix} &= 1 + \frac{q-1-\alpha}{t_{D-2}} \begin{bmatrix} D-1 \\ 1 \end{bmatrix} \\ &\geq 1 \end{aligned}$$

by (33.1) and (i) above, so $\operatorname{Shadow}(u) \backslash [0, u \star x]$ is not empty, as desired.

(iii) Let the integer i be given. There exists $xy \in \Delta_i$ by part (ii) above, so by Corollary 34.2,

$$\frac{q-1-\alpha}{t_{D-i}} = q - \text{zig-zag}(x, y)$$

is an integer.

Theorem 39.2. *Let P denote a nontrivial q-line regular, α-zig-zag regular quantum matroid with rank $D \geq 3$.*

(i) *Suppose $D \geq 4$. Then $\alpha \in \{0, q-1, q\}$.*
(ii) *Suppose $D = 3$. Then $\alpha = q$ or $1+\alpha$ divides q.*

Proof. (i) We assume $\alpha \neq 0$, $\alpha \neq q-1$, $\alpha \neq q$, and get a contradiction. Observe $\alpha \geq 1$ by Lemma 28.6 and our assumptions, so P is dual-line regular by Theorem 31.1. It follows P is regular by Definition 32.1. Now on one hand, by Lemma 39.1(i),(iii) (with $i = D-2$), we find $(q-1-\alpha)t_2^{-1}$ is a positive integer. On the other hand, by (27.2), (33.1),

$$\begin{aligned}\frac{q-1-\alpha}{t_2} &= \frac{q-1-\alpha}{1+\alpha(q+1)} \\ &\leq \frac{q-2}{q+2} \\ &< 1,\end{aligned}$$

a contradiction.

(ii) Assume $\alpha \neq 0$, $\alpha \neq q$; otherwise the result is trivial. Then as in (i) above, P is regular. By Lemma 39.1(iii) (with $i = 2$, $D = 3$), we find $(q-1-\alpha)t_1^{-1}$ is an integer. By (33.1),

$$\begin{aligned}\frac{q-1-\alpha}{t_1} &= \frac{q-1-\alpha}{1+\alpha} \\ &= \frac{q}{1+\alpha} - 1,\end{aligned}$$

so $1+\alpha$ divides q, as desired. This proves Theorem 39.2.

Suppose P is a nontrivial regular quantum matroid with parameters (D, q, α, β), and assume $D \geq 4$. In each of the three cases in Theorem 39.2(i), we can identify P. If $\alpha = 0$, then P is known by Theorems 30.3, 30.4, 30.5. In each of the other cases $\alpha = q-1$, $\alpha = q$, there is a result in the literature of diagram geometries that identifies P. We quote these results below, translated into the language of quantum matroids via Theorem 23.1. First, we eliminate the easy case $q = 1$.

Theorem 39.3. *Let D denote an integer at least 2, and let P denote a poset. Then the following are equivalent.*

(i) *P is a nontrivial 1-line regular, 1-zig-zag regular quantum matroid with rank D.*

(ii) *There exists an integer* $N > D$ *such that* P *is isomorphic to the truncated Boolean algebra* $B(D, N)$.

Suppose (i), (ii) *hold. Then* P *is* $(N-D)$*-dual-line regular.* (See Example 40.1(1).)

Proof. Routine.

Theorem 39.4 ([Bu1, Theorem 8]). *Let* D *denote an integer at least* 3, *let* q *denote an integer at least* 2, *and let* P *denote a poset. Then the following are equivalent.*

(i) P *is a nontrivial* q*-line regular,* q*-zig-zag regular quantum matroid with rank* D.

(ii) q *is a prime power, and there exists an integer* $N > D$ *such that* P *is isomorphic to the truncated projective geometry* $L_q(D, N)$.

Suppose (i), (ii) *hold. Then* P *is* β*-dual-line regular, where*

$$\beta = q\frac{q^{N-D} - 1}{q - 1}.$$

See Example 40.1(3).

Theorem 39.5 ([Sp1, Theorem 3]). *Let* D *denote an integer at least* 3, *let* q *denote an integer at least* 2, *and let* P *denote a poset. Then the following are equivalent.*

(i) P *is a nontrivial* q*-line regular,* $(q-1)$*-zig-zag regular quantum matroid with rank* D.

(ii) q *is a prime power, and there exists an integer* $N > D$ *such that* P *is isomorphic to the attenuated space* $A_q(D, N)$.

Suppose (i), (ii) *hold. Then* P *is* $(q^{N-D} - 1)$*-dual-line regular.*

(See Example 40.1(4)).

We now arrive at the central theorem of this paper.

Theorem 39.6. *Let* D *denote an integer at least* 4, *and let* P *denote a poset. Then the following are equivalent.*

(i) P *is a nontrivial regular quantum matroid with rank* D.

(ii) P *is isomorphic to one of the following:*

- (iia) *A truncated Boolean algebra* $B(D, N)$, $(D < N)$.
- (iib) *A Hamming matroid* $H(D, N)$, $(2 \leq N)$.
- (iic) *A truncated projective geometry* $L_q(D, N)$, $(D < N)$.
- (iid) *An attenuated space* $A_q(D, N)$, $(D < N)$.
- (iie) *A classical polar space of rank* D.

Proof. (i) $\rightarrow$ (ii). Let (D, q, α, β) denote the parameters of P. Then

$$q \geq 1 \tag{39.3}$$

by Lemma 27.2,

$$\alpha \in \{0, q-1, q\} \tag{39.4}$$

by Theorem 39.2, and

$$\beta \geq 1 \tag{39.5}$$

by Lemma 27.9, Lemma 27.10, and Definition 32.2.

First assume $\alpha = 0$. In this case $\mathrm{Rad}(P) = 0$ by Theorem 29.3(i),(iv) and (39.5), so P is nondegenerate by Definition 26.1. Now P is a Tits polar space of rank D by Theorem 30.3. If $q = 1$ then by Theorem 30.4 and (39.5), P is isomorphic to $H(D, N)$, where $N = \beta + 1 \geq 2$. If $q \geq 2$ then by Theorem 30.5, P is isomorphic to a classical polar space of rank D.

Next assume $\alpha = q - 1$. In this case we may assume $q \geq 2$; otherwise $\alpha = 0$ by (39.3), and our previous remarks apply. Now by Theorem 39.5, q is a prime power, and P is isomorphic to $A_q(D, N)$ for some integer $N > D$.

Finally assume $\alpha = q$. If $q = 1$, then by Theorem 39.3, P is isomorphic to $B(D, N)$ for some integer $N > D$. If $q \geq 2$, then by Theorem 39.4, q is a prime power, and P is isomorphic to $L_q(D, N)$ for some integer $N > D$.

(ii) $\rightarrow$ (i). Assume P is isomorphic to $B(D, N)$, for some integer $N > D$. Then P is a nontrivial regular quantum matroid of rank D by Theorem 39.3.

Assume P is isomorphic to $H(D, N)$, for some integer $N \geq 2$. Then P is a Tits polar space by Theorem 30.4, so P is a nondegenerate quantum matroid of rank D by Theorem 30.3. In particular P is nontrivial. P is 1-line regular and $(N-1)$-dual-line regular by Theorem 30.4, and 0-zig-zag regular by Theorem 30.3.

Assume P is isomorphic to $L_q(D, N)$, for some integer $N > D$. Then P is a nontrivial, regular quantum matroid of rank D by Theorem 39.4.

Assume P is isomorphic to $A_q(D, N)$, for some integer $N > D$. Then P is a nontrivial, regular quantum matroid of rank D by Theorem 39.5.

Assume P is a classical polar space of rank D. Then P is a Tits polar space of rank D by Theorem 30.5, so P is a nondegenerate quantum matroid of rank D by Theorem 30.3. In particular P is nontrivial. P is line-regular by Theorem 30.5, 0-zig-zag regular by Theorem 30.3, and dual-line regular by Theorem 30.5. This proves Theorem 39.6.

Corollary 39.7. *Let D denote an integer at least 4. Then the following are equivalent statements concerning a finite, undirected graph Γ.*

(i) Γ *is isomorphic to the graph on* $\mathrm{top}(P)$, *where* P *is a nontrivial regular quantum matroid with rank* D.

(ii) Γ *is isomorphic to one of the following:*

- (iia) *The Johnson graph* $J(D,N)$, $(D < N)$.
- (iib) *The Hamming graph* $H(D,N)$, $(2 \leq N)$.
- (iic) *The q-Johnson graph* $J_q(D,N)$, $(D < N)$.
- (iid) *The bilinear forms graph* $H_q(D,N)$, $(D < N)$.
- (iie) *A dual polar space graph of diameter* D.

(See [B-I, p300] for the definitions of these graphs).

Proof. Immediate from Theorem 39.6.

Corollary 39.8. *Let P denote a regular quantum matroid with rank $D \geq 4$. Then P is embeddable (in the sense of Definition 6.3).*

Proof. Concerning the examples in Theorem 39.6, observe P is an A-matroid in cases (iia), (iib), and a V-matroid in cases (iic)–(iie).

§40. The examples of regular quantum matroids

Example 40.1. Let D denote an integer at least 2. In each of the following cases 1–5, P is a nontrivial regular quantum matroid of rank D. In each case, the parameters q, α, β are given. (See Definition 32.1). By Theorem 39.6, there are no other nontrivial regular quantum matroids with rank $D \geq 4$.

1. The truncated Boolean algebra $B(D,N)\,(D<N)$ **[Bu1], [Te].**

Let A denote a set of cardinality N.

$$P = \{x \subseteq A \mid \ |x| \leq D\},$$

$$x \leq y \text{ whenever } x \text{ is a subset of } y \ \ (x,y \in P),$$

$$\mathrm{rank}(x) = |x| \ \ (x \in P),$$

$$q = 1, \qquad \alpha = 1, \qquad \beta = N - D.$$

2. The Hamming matroid $H(D,N)$ $(2 \leq N)$ **[De], [Te].**
Set

$$A = A_1 \cup A_2 \cup \cdots \cup A_D \qquad \text{(disjoint union)},$$

where $|A_i| = N (1 \leq i \leq D)$.

$$P = \{x \subseteq A \mid \quad |x \cap A_i| \leq 1 \quad \text{for all } i \quad (1 \leq i \leq D)\},$$

$$x \leq y \quad \text{whenever} \quad x \quad \text{is a subset of} \quad y \ (x,\, y \in P),$$

$$\operatorname{rank}(x) = |x| \quad (x \in P),$$

$$q = 1, \qquad \alpha = 0, \qquad \beta = N - 1.$$

3. The truncated projective geometry $L_q(D,N)$ $(D < N)$ **[Bu1], [Sta], [Te].**

Let V denote a vector space of dimension N over the field $GF(q)$.

$$P \ = \{x \mid x \ \text{ is a subspace of } \ V, \quad \dim(x) \leq D\},$$

$$x \leq y \quad \text{whenever} \quad x \quad \text{is a subspace of} \quad y \ (x,\, y \in P),$$

$$\operatorname{rank}(x) = \dim(x) \quad (x \in P),$$

$$\alpha = q, \qquad \beta = q\frac{q^{N-D} - 1}{q - 1}.$$

4. The attenuated space $A_q(D,N)$ $(D < N)$ **[De], [Hu], [Sp1], [Sta], [Te].**

Let V denote a vector space of dimension N over the field $GF(q)$, and fix a subspace $w \subseteq V$ of dimension $N - D$.

$$P \ = \{x \mid x \ \text{ is a subspace of } \ V, \quad x \cap w = 0\},$$

$$x \leq y \quad \text{whenever} \quad x \quad \text{is a subspace of} \quad y \ (x, y \in P),$$

$$\operatorname{rank}(x) \ = \ \dim(x)\ , (x \in P),$$

$$\alpha = q - 1, \qquad \beta = q^{N-D} - 1.$$

5. The classical polar spaces of rank D **over** $GF(q)$ **[C-J-P], [Ca2], [Mu].**

Let V denote a vector space over the field $GF(q)$, and assume V possesses one of the following nondegenerate forms:

name	**dim V**	**form**	ε
$B_D(q)$	$2D+1$	quadratic	0
$C_D(q)$	$2D$	alternating	0
$D_D(q)$	$2D$	quadratic (Witt index D)	-1
${}^2D_{D+1}(q)$	$2D+2$	quadratic (Witt index D)	1
${}^2A_{2D}(r)$	$2D+1$	Hermitean ($q=r^2$)	$\frac{1}{2}$
${}^2A_{2D-1}(r)$	$2D$	Hermitean ($q=r^2$)	$-\frac{1}{2}$.

We call a subspace of V *totally isotropic* whenever the form vanishes completely on that subspace. In each of the above cases, the dimension of any maximal isotropic subspace is D.

$$P = \{x \mid x \text{ is an isotropic subspace of } V\},$$

$$x \le y \text{ whenever } x \text{ is a subspace of } y \ (x,\, y \in P),$$

$$\operatorname{rank}(x) = \dim(x) \ (x \in P),$$

$$\alpha = 0, \qquad \beta = q^{1+\varepsilon}.$$

§41. Directions for further research

In this section we give some conjectures and open problems concerning quantum matroids and related topics. See also Problem 4.4, Conjecture 6.5, and Conjecture 7.13 in the text.

Conjecture 41.1. *Let P denote a quantum matroid with rank $D \ge 2$. Let us say P is* thick line connected *whenever for all distinct atoms $x, y \in A_P$, there exists an integer $d \ge 1$ and a sequence $x = x_0, x_1, \ldots, x_d = y$ $(x_0, x_1, \ldots, x_d \in A_P)$ such that $x_i \vee x_{i+1}$ exists and is a thick line for all i $(0 \le i < d)$. We conjecture that if P is thick line connected and if D is sufficiently large, then every line in P is thick.*

Problem 41.2. Let us say a finite, undirected graph Γ is a *quantum matroid graph* whenever there exists a quantum matroid P such that Γ is isomorphic to the graph on $\operatorname{top}(P)$. Find a simple combinatorial property that characterizes the quantum matroid graphs among all the finite undirected graphs. See Corollary 39.7.

Problem 41.3. Let P denote a classical polar space (Example 40.1(5)). What quantum matroid is dual to P in the sense of Definition 3.2?

Problem 41.4. For a classical matroid (Definition 1.1), one has the dependency axioms, the hyperplane axioms, the circuit axioms, the bond axioms, etc. See for example [Wh, Chapter 2]. To what extent are there analogous axioms for the $\mathcal{P}$-matroids, where $\mathcal{P}$ is any modular atomic lattice ?

Conjecture 41.5. *Let P denote a quantum matroid. Pick any $x, y \in P$, and let G denote the minimal geodesically closed subposet of P containing x, y. We conjecture*

$$\mathit{rank}(G) = \gamma(x, y) + \gamma(y, x) + \delta(x, y).$$

(See Definitions 12.1, 13.1).

Problem 41.6. Which quantum matroids are Cohen-Macaulay? (See[B-G-S]).

Problem 41.7. Let P denote a quantum matroid of rank D. Let us call P *weakly zig-zag regular* whenever for all integers i $(1 \leq i \leq D-1)$, and for all $x, y \in P$ such that $\operatorname{rank}(x) = i$, $\operatorname{rank}(y) = i$, the number of paths in P with endpoints x, y and shape $(i, i-1, i, i+1, i)$ equals the number of paths in P with endpoints x, y and shape $(i, i+1, i, i-1, i)$. If P is regular then P is weakly zig-zag regular. Classify the weakly zig-zag regular quantum matroids.

Problem 41.8. Let D denote an integer at least 3, and let q denote an integer at least 2. Find a short, direct proof, not involving Theorem 39.5, that any nontrivial q-line regular, $(q-1)$-zig-zag regular quantum matroid is embeddable. (See Definition 6.3.)

Problem 41.9. Let N denote an integer at least 3, let q denote a prime power, and let V denote an N dimensional vector space over the field $GF(q)$. Pick an integer D $(2 \leq D < N)$, and let P denote a nontrivial $(q-1)$-zig-zag regular V-matroid of rank D that spans V. Find a short, direct proof, not involving Theorem 39.5, that there exists a subspace $w \subseteq V$ such that $\dim(w) = N - D$ and such that

$$P = \{\, x \mid x \text{ is a subspace of } V, \quad x \cap w = 0\}.$$

(See Example 40.1(4).)

Problem 41.10. Let $\mathcal{P}$ denote a modular atomic lattice. Classify the uniform $\mathcal{P}$-basis systems. Give a short, direct proof, that does not refer to Theorems 35.3, 39.6. (See Definition 35.1.)

Problem 41.11. Let P denote a quantum matroid, and pick any x, $y \in P$. What can be said about $\mathrm{Rad}(x_y^+)$? Under what conditions is u_v^+ nondegenerate for all u, $v \in P$ such that $u \vee v$ does not exist? (See line (15.1) and Definition 26.1.)

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