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# Level-Rank Duality of Witten's 3-Manifold Invariants

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## §1. Introduction

The main object of this paper is to establish a duality satisfied by Witten's 3-manifold invariants for $sl(n, \mathbf{C})$ at level $k$ and those for $sl(k, \mathbf{C})$ at level $n$. This type of duality, which is called the level-rank duality, has been encountered in several contexts in solvable lattice models and conformal field theory — the Boltzmann weights of solvable lattice models [JMO], quantum groups at roots of unity [SA], link invariants related to Chern-Simons gauge theory [NRS], fusion algebras [KN], and the space of conformal blocks in Wess-Zumino-Witten conformal field theory [NT]. This subject has been also treated by many other authors from different viewpoints. More recently, Witten [W2] described the relationship between the fusion algebra and the quantum cohomology of the Grassmann manifold and explained the level-rank duality from this point of view. However, a precise formulation for the level-rank duality of Witten's 3-manifold invariants has not appeared in the literature, as far as the authors know.

Let $M$ be a closed oriented 3-manifold and we denote by $Z_k(M, SU(n))$ Witten's 3-manifold invariant for $sl(n, \mathbf{C})$ at level $k$ discovered in the seminal article [W1]. Subsequently these invariants were studied in detail in [RT] and [KM]. Our notation corresponds to $\tau_r(M)$ in [KM] with $r = k + 2$ in the case $n = 2$. To describe the duality between $Z_k(M, SU(n))$ and $Z_n(M, SU(k))$ we first factorize the invariants by the Dynkin diagram automorphism. This is the $sl(n, \mathbf{C})$ counterpart of the symmetry principle discovered in [KM] for $sl(2, \mathbf{C})$. In this paper we assume that the integers $n$ and $k$ are relatively prime. Let us suppose that $M$ is obtained by the Dehn surgery on a framed link $L$ in $S^3$. We recall that the invariant $Z_k(M, SU(n))$ is written as a weighted sum of link invariants obtained by associating with each component of the link

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a dominant integral weight for $sl(n, \mathbf{C})$ at level $k$ (see 4.1.1). By means of the $\mathbf{Z}_n$ action on the above set of weights, we show that the invariant $Z_k(M, SU(n))$ can be written in the form $\xi_{n,k}(M) Z_k(M, PSU(n))$, where $\xi_{n,k}(M)$ is an invariant of $M$ defined by the linking matrix.

The level-rank duality holds for the above $PSU(n)$ invariant. In Theorem 4.2.7, we prove the duality relation

$$Z_k(M, PSU(n)) = \overline{Z_n(M, PSU(k))}.$$

The argument of our proof also permits us to describe the level-rank duality of representations of the mapping class groups on the space of conformal blocks. We show that the two representations of the mapping class groups, one for $sl(n, \mathbf{C})$ at level $k$ and one for $sl(k, \mathbf{C})$ at level $n$ are contragredient to each other.

The paper is organized in the following way. In Section 2, we start from recalling basic properties of the fusion algebra of type $A$. We introduce the fusion algebra as a truncated representation ring of the Lie algebra $sl(n, \mathbf{C})$. Then, we define the cyclic group action on the set of dominant integral weights at level $k$, which is induced from the Dynkin diagram automorphism of the corresponding affine Lie algebra. By considering the orbit of this action we describe the level-rank duality for the fusion algebra. Although it is not directly used in the article, for the reader's convenience, we explain briefly the appearance of the fusion algebra in conformal field theory at the end of Section 2. In Section 3, we explain the level-rank duality of link invariants. Given a framed link in $S^3$ we associate a dominant integral weight at level $k$ for each component. We define an invariant of the above colored framed link, which is a generalization of the Jones polynomial at roots of unity. We describe the level-rank duality for the invariants of colored framed links. Section 4 is devoted to the proof of the level-rank duality of $PSU(n)$ invariants of 3-manifolds. In the case $n$ and $k$ are relatively prime we can factorize the $SU(n)$ invariant at level $k$ by the action of the Dynkin diagram automorphism to get the $PSU(n)$ invariant. We prove the duality for the $PSU(n)$ invariant at level $k$ and the $PSU(k)$ invariant at level $n$.

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## §2. Duality of fusion algebras

### 2.1. Definition of the fusion algebra of type $A$

In this section we summarize basic facts about the fusion algebra of type $A$. First, let us fix some notation. Let $\mathbf{h}$ denote the Cartan subalgebra of $sl(n, \mathbf{C})$ fixed as the set of diagonal matrices in $sl(n, \mathbf{C})$. The linear form $\varepsilon_i : \mathbf{h} \to \mathbf{C}$, $1 \leq i \leq n$, is defined by associating to $X \in \mathbf{h}$ its $(i,i)$ component $X_{ii}$. The set of roots of $sl(n, \mathbf{C})$ is given by

$$\Delta = \{\varepsilon_i - \varepsilon_j \ ; \ 1 \leq i \neq j \leq n\}.$$

We fix the set of fundamental roots as

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad 1 \leq i \neq j \leq n-1.$$

The Cartan-Killing form induces an inner product on $\mathbf{h}^*$ defined by

$$\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}.$$

The fundamental system of weights is

$$\Lambda_i = (\varepsilon_1 + \cdots + \varepsilon_i) - \frac{i}{n} \sum_{j=1}^{n} \varepsilon_j, \quad 1 \leq i \leq n,$$

which is characterized by

$$\langle \Lambda_i, \alpha_j \rangle = \delta_{ij}.$$

We denote by $\rho$ the half sum of the positive roots $\frac{1}{2}\sum_{i<j}(\varepsilon_i - \varepsilon_j)$, which is equal to $\sum_{i=1}^{n-1} \Lambda_i$.

By definition the set of dominant integral weights of $sl(n, \mathbf{C})$ is given by

$$P_+(n) = \{\sum_{i=1}^{n-1} a_i \Lambda_i \ ; \ a_i \in \mathbf{Z}, a_i \geq 0\}.$$

For $\lambda = \sum_{i=1}^{n-1} a_i \Lambda_i \in P_+(n)$ we put $|\lambda| = \sum_{i=1}^{n-1} i a_i$, which is the number of nodes in the corresponding Young diagram. Here $|\lambda|$ is considered modulo $n$. We set

$$P = \mathbf{Z}\Lambda_1 \oplus \cdots \oplus \mathbf{Z}\Lambda_{n-1},$$

which is called the weight lattice. The Weyl group $W$ acts on $P$ by means of the reflections $s_{\alpha_i}$, $1 \leq i \leq n-1$, defined by

$$s_{\alpha_i}(v) = v - \langle v, \alpha_i \rangle \alpha_i.$$

A dominant integral weight $\lambda$ is characterized by the property

$$\langle \lambda, \alpha_i \rangle \in \mathbf{Z}, \quad \langle \lambda, \alpha_i \rangle \geq 0, \quad 1 \leq i \leq n-1$$

and the set $P_+(n)$ is in one-to-one correspondence with the set of finite dimensional irreducible representations of $sl(n, \mathbf{C})$.

We denote by $V_\lambda$ the irreducible representation of $sl(n, \mathbf{C})$ with highest weight $\lambda \in P_+(n)$. As a representation space of $sl(n, \mathbf{C})$ the tensor product $V_\lambda \otimes V_\mu$ admits a decomposition

$$V_\lambda \otimes V_\mu = \sum_\nu M_{\lambda\mu}^\nu \otimes V_\nu,$$

where the multiplicity $n_{\lambda\mu}^\nu = \dim M_{\lambda\mu}^\nu$ is called the Littlewood-Richardson coefficient. Let us recall that the representation ring $R_n$ is a free $\mathbf{Z}$ module with basis $\lambda \in P_+(n)$ equipped with the product structure defined by

$$\lambda \cdot \mu = \sum_\nu n_{\lambda\mu}^\nu \ \nu \quad \text{in} \quad R_n.$$

It is well known that $R_n$ is isomorphic to the polynomial ring

$$\mathbf{Z}[\Lambda_1, \ldots, \Lambda_{n-1}].$$

Let us now introduce the fusion algebra of type $A_{n-1}$ in a combinatorial manner. The set of dominant integral weights at level $k$ is by definition

$$P_+(n,k) = \{\sum_{i=1}^{n-1} a_i \Lambda_i \,;\, a_i \in \mathbf{Z},\, a_i \geq 0,\, \sum_{i=1}^{n-1} a_i \leq k\}.$$

Let us consider the natural inclusion $j : P_+(n,k) \to P_+(n,k+1)$ and we put $\partial P(n,k) = P_+(n,k+1) \setminus P_+(n,k)$. Let $I_{n,k}$ be the ideal of $R_n$ generated by the elements of $\partial P(n,k)$. We define the fusion algebra $R_{n,k}$ as the truncated representation ring $R_n / I_{n,k}$. It is known by [GW] that the fusion algebra $R_{n,k}$ is a free $\mathbf{Z}$ module whose basis is in one-to-one correspondence with the set $P_+(n,k)$. Let us denote the basis by the same letter $\lambda \in P_+(n,k)$. We define the fusion rule $N_{\lambda\mu}^\nu$ to be the structure constant defined by

$$\lambda \cdot \mu = \sum_\nu N_{\lambda\mu}^\nu \nu \quad \text{in} \quad R_{n,k}.$$

**Example.** In the case of $sl(2,\mathbf{C})$, the representation ring is the polynomial ring $\mathbf{Z}[\Lambda_1]$. The representation with highest weight $m\Lambda_1$ can be written as the polynomial

$$P_m(\Lambda_1) = \sum_{i=0}^{[m/2]} (-1)^i \binom{m-1}{i} \Lambda_1^{m-2i}.$$

The fusion algebra $R_{2,k}$ is by definition $\mathbf{Z}[\Lambda_1]/(P_{k+1}(\Lambda_1))$. Let us denote by $v_j$ the element corresponding to the representation with highest weight $2j\Lambda_1$. Then the structure constant $N_{j_1 j_2}^{j_3}$ is 1 if the condition

$$|j_1 - j_2| \le j_3 \le j_1 + j_2,\ j_1 + j_2 + j_3 \in \mathbf{Z},\ j_1 + j_2 + j_3 \le k$$

is satisfied and is 0 otherwise.

We have an involution on $P_+(n,k)$ defined by $\lambda^* = -w(\lambda)$ where $w$ denotes the longest element in the Weyl group. We put $N_{\lambda\mu\nu} = N_{\lambda\mu}^{\nu^*}$. We have the following basic properties of the structure constant $N_{\lambda\mu}^{\nu}$.

(1) $0 \le N_{\lambda\mu}^{\nu} \le n_{\lambda\mu}^{\nu}$
(2) $N_{\lambda\mu}^{\nu}$ is symmetric with respect to $\lambda$, $\mu$ and $\nu$.
(3) $N_{0\mu}^{\nu} = \delta_{\mu\nu}$

### 2.2. Dynkin diagram automorphism

Let $\widehat{\Lambda}_i, 0 \le i \le n-1$, denote the fundamental weights of the affine Lie algebra $\widehat{sl(n,\mathbf{C})}$. We define the set of dominant integral weights at level $k$ by

$$\widehat{P}_+(n,k) = \{\sum_{i=0}^{n-1} a_i\widehat{\Lambda}_i\,;\ a_i \in \mathbf{Z},\ a_i \ge 0,\ \sum_{i=0}^{n-1} a_i = k\}.$$

We have a natural injection $j : \widehat{P}_+(n,k) \to P_+(n)$ defined by $j(\lambda) = \sum_{i=1}^{n-1} a_i\Lambda_i$ for $\lambda = \sum_{i=0}^{n-1} a_i\widehat{\Lambda}_i$ and its image is equal to $P_+(n,k)$. It is known that $P_+(n,k)$ is in one to one correspondence with the set of integrable highest weight representations of the affine Lie algebra $\widehat{sl(n,\mathbf{C})}$ (see [K]).

The cyclic group $\mathbf{Z}_n$ acts on the set $\widehat{P}_+(n,k)$ by the Dynkin diagram automorphism

$$\sigma(\widehat{\Lambda}_i) = \widehat{\Lambda}_{i+1},$$

where the suffix is taken modulo $n$. By means of the identification using $j$, the Dynkin diagram automorphism induces a $\mathbf{Z}_n$ action on $P_+(n,k)$. More explicitly, this action is defined by

$$\sigma(\sum_{i=1}^{n-1} a_i\Lambda_i) = (k - \sum_{i=1}^{n-1} a_i)\Lambda_1 + \sum_{j=2}^{n-1} a_{j-1}\Lambda_j.$$

**Lemma 2.2.1.** *We have a bijection*

$$P_+(n,k)/\mathbf{Z}_n \to P_+(k,n)/\mathbf{Z}_k.$$

*Proof.* To see this correspondence it is convenient to consider for $\hat{\lambda} \in \widehat{P}_+(n,k)$ the sum $\hat{\lambda} + \sum_{i=0}^{n-1} \hat{\Lambda}_i$, which is expressed as $\alpha_0\hat{\Lambda}_0 + \cdots + \alpha_{n-1}\hat{\Lambda}_{n-1}$ with $\sum_{i=0}^{n-1} \alpha_i = k + n$. Let us consider a circle of the circumference $n + k$ and divide the circle with $n + k$ points in such a way that the length of each divided arc is equal to 1. We take $n$ of the above $n+k$ points so that the lengths of the arcs are $\alpha_0, \ldots, \alpha_{n-1}$. Considering up to rotation, the set of ways of dividing circles in the above manner are in one to one correspondence with $P_+(n,k)/\mathbf{Z}_n$. Now let us take the complementary $k$ points on the circle, which defines an element of $P_+(k,n)/\mathbf{Z}_k$. It is clear that this gives a bijection between $P_+(n,k)/\mathbf{Z}_n$ and $P_+(k,n)/\mathbf{Z}_k$.

Let us suppose that $n$ and $k$ are relatively prime. Since we have $|\sigma(\lambda)| \equiv |\lambda| \bmod n$, each orbit of the $\mathbf{Z}_n$ action on $P_+(n,k)$ contains a unique dominant integral weight $\lambda$ of level $k$ with $|\lambda| \equiv 0 \bmod n$. The orbit space $P_+(n,k)/\mathbf{Z}_n$ is identified with the set

$$\Omega_{n,k} = \{\lambda \in P_+(n,k)\ ;\ |\lambda| \equiv 0 \bmod n\}.$$

For $\lambda = \sum_{i=1}^{n-1} a_i\Lambda_i \in P_+(n,k)$ we associate the Young diagram of type $(a_1 + \cdots + a_{n-1}, a_2 + \cdots + a_{n-1}, \ldots, a_{n-1})$. We express $\lambda$ as

$$\lambda = a_{i_1}\Lambda_{i_1} + \cdots + a_{i_s}\Lambda_{i_s}, \quad a_{i_1}, \ldots, a_{i_s} \neq 0.$$

Let us consider the transposed Young diagram and the associated weight ${}^t\lambda \in P_+(k,n)$ given by

$${}^t\lambda = (i_s - i_{s-1})\Lambda_{\alpha_{i_s}} + (i_{s-1} - i_{s-2})\Lambda_{\alpha_{i_s}+\alpha_{i_{s-1}}} + \cdots + i_1\Lambda_{\alpha_{i_1}+\cdots+\alpha_{i_s}}.$$

We can easily check the following lemma.

**Lemma 2.2.2.** *In the case $n$ and $k$ are relatively prime, we have a bijection*

$$\tau : \Omega_{n,k} \to \Omega_{k,n}$$

*given by*

$$\tau(\lambda) = \sigma^{k-|\lambda|/n}({}^t\lambda).$$

## 2.3. Verlinde formula

In this paragraph, we recall the Verlinde formula which relates the fusion rule and the modular transformation $S$ matrix. For $\lambda$, $\mu \in P_+(n,k)$, we set

$$S_{\lambda\mu} = \frac{(\sqrt{-1})^{n(n-1)/2}}{\sqrt{n(k+n)^{n-1}}} \sum_{w \in W} \det w \exp\left(-\frac{2\pi\sqrt{-1}}{k+n}\langle w(\lambda+\rho), \mu+\rho\rangle\right),$$

$$T_{\lambda\mu} = \delta_{\lambda\mu} \exp 2\pi\sqrt{-1}\left(\Delta_\lambda - \frac{c_{n,k}}{24}\right),$$

where $c_{n,k}$ is the so-called central charge

$$c_{n,k} = \frac{k \dim sl(n,\mathbf{C})}{k+n}.$$

For $\lambda \in P_+(n,k)$ we denote by $\mathcal{H}_\lambda$ the integrable highest weight module of the affine Lie algebra $\widehat{sl(n,\mathbf{C})}$ at level $k$. The character $\chi_\lambda$ is by definition

$$\chi_\lambda(\tau) = Tr_{\mathcal{H}_\lambda} q^{L_0 - \frac{c_{n,k}}{24}}$$

where $L_0$ is the 0-th Sugawara operator and $q = e^{2\pi\sqrt{-1}\tau}$ with $\operatorname{Im} q > 0$. Let us recall the following fundamental result due to Kac and Peterson.

**Theorem 2.3.1 [KP].** *The set of characters $\chi_\lambda$, $\lambda \in P_+(n,k)$, are invariant under the modular transformation and they satisfy*

$$\chi_\lambda(-1/\tau) = \sum_{\mu \in P_+(n,k)} S_{\lambda\mu}\, \chi_\mu(\tau),$$

$$\chi_\lambda(\tau+1) = \exp 2\pi\sqrt{-1}\left(\Delta_\lambda - \frac{c_{n,k}}{24}\right) \chi_\lambda(\tau).$$

The matrices $S = (S_{\lambda\mu})$ and $T = (T_{\lambda\mu})$ are unitary and symmetric and satisfy the relation

$$(ST)^3 = S^2 = (\delta_{\lambda\mu^*}),$$

where $\lambda^* = -w(\lambda)$ with the longest element $w \in W$ and is the highest weight for the dual representation $V_\lambda^*$.

Let us observe that $S_{0\lambda}$ is a real number and is written as

$$S_{0\lambda} = \frac{1}{\sqrt{n(k+n)^{n-1}}} \prod_{\alpha \in \Delta_+} 2 \sin \frac{\pi \langle \lambda + \rho, \alpha \rangle}{k+n}.$$

For our purpose the following expression for the modular transformation $S$ matrix is also useful. We put

$$\lambda + \rho = \sum_{i=1}^{n} x_i \varepsilon_i, \quad \mu + \rho = \sum_{i=1}^{n} y_i \varepsilon_i.$$

Then we have

$$S_{\lambda\mu} = \frac{(\sqrt{-1})^{n(n-1)/2}}{\sqrt{n(k+n)^{n-1}}} \sum_{\sigma \in S_n} \det \sigma \ \zeta^{x_{\sigma(1)} y_1} \cdots \zeta^{x_{\sigma(n)} y_n}$$

with $\zeta = \exp \frac{2\pi\sqrt{-1}}{k+n}$, which is a minor of the Vandermonde determinant.

The Verlinde formula relates the fusion rule and the modular transformation $S$ matrix as

$$N_{\lambda\mu}^{\nu} = \sum_{\alpha \in P_+(n,k)} \frac{S_{\lambda\alpha} S_{\mu\alpha} \overline{S_{\nu\alpha}}}{S_{0\alpha}} \tag{2.3.2}$$

(see [K] for details).

### 2.4. Behaviour of $S$, $N$ and $\Delta$ under $\sigma$ and $\tau$

We put

$$\Delta_\lambda = \frac{\langle \lambda, \lambda + 2\rho \rangle}{2(k+n)},$$

which is the value of the operator $L_0$ on the highest weight representation $V_\lambda$.

**Lemma 2.4.1 [KT].** *With respect to the action $\sigma$, the conformal weight $\Delta$ and the modular transformation matrix $S$ satisfy*

$$(1) \quad \Delta_{\sigma(\lambda)} - \Delta_\lambda = \frac{1}{n} \left( \frac{(n-1)k}{2} - |\lambda| \right),$$

$$(2) \quad S_{\sigma(\lambda)\mu} = \exp \left( \frac{2\pi\sqrt{-1}|\mu|}{n} \right) S_{\lambda\mu}.$$

Now we describe the action of $\tau$ on the modular transformation matrix $S_{\lambda\mu}$, the structure constant of the fusion algebra $N_{\lambda\mu}^{\nu}$ and the conformal weight $\Delta_\lambda$. To specify the rank and the level, we denote them by $S_{\lambda\mu}[n,k]$, $N_{\lambda\mu}^{\nu}[n,k]$ and $\Delta_\lambda[n,k]$ for the highest weights $\lambda, \mu, \nu \in P_+(n,k)$.

**Lemma 2.4.2.** *For $\tau : \Omega_{n,k} \to \Omega_{k,n}$, we have the following properties.*

$$\begin{aligned} &(1) \quad S_{\lambda\mu}[n,k] = \sqrt{\frac{k}{n}}\, \overline{S_{\tau(\lambda)\tau(\mu)}[k,n]} \\ &(2) \quad N_{\lambda\mu}^{\nu}[n,k] = N_{\tau(\lambda)\tau(\mu)}^{\tau(\nu)}[k,n] \\ &(3) \quad \Delta_\lambda[n,k] + \Delta_{\tau(\lambda)}[k,n] = 0 \mod \mathbf{Z} \end{aligned}$$

*Proof.* (1) By Lemma 4 in [KN], we have

$$S_{\lambda\mu}[n,k] = \sqrt{\frac{k}{n}} \exp \frac{2\pi\sqrt{-1}}{nk} |\lambda||\mu| \overline{S_{{}^t\lambda {}^t\mu}[k,n]}$$

for $\lambda, \mu \in P_+(n,k)$. On the other hand, it follows from the property of $S_{\lambda\mu}$ in Lemma 2.4.1 that

$$\begin{aligned} S_{\tau(\lambda)\tau(\mu)}[k,n] &= \exp 2\pi\sqrt{-1}\left(k - \frac{|\mu|}{n}\right)\frac{|\lambda|}{k} S_{{}^t\lambda {}^t\mu}[k,n] \\ &= \exp 2\pi\sqrt{-1}\left(-\frac{|\lambda||\mu|}{nk}\right) S_{{}^t\lambda {}^t\mu}[k,n]. \end{aligned}$$

Thus we obtain the assertion (1).

(2) Using the assertion (1), we obtain

$$\begin{aligned} &n\, \frac{S_{\lambda\varepsilon}[n,k] S_{\mu\varepsilon}[n,k] \overline{S_{\nu\varepsilon}[n,k]}}{S_{0\varepsilon}[n,k]} \\ &\quad = k\, \frac{\overline{S_{\tau(\lambda)\tau(\varepsilon)}[k,n]}\ \overline{S_{\tau(\mu)\tau(\varepsilon)}[k,n]} S_{\tau(\nu)\tau(\varepsilon)}[k,n]}{\overline{S_{\tau(0)\tau(\varepsilon)}[k,n]}} \end{aligned}$$

for $\lambda, \mu, \nu, \varepsilon \in \Omega_{n,k}$. By the Verlinde formula we have

$$\begin{aligned} N_{\lambda\mu}^{\nu}[n,k] &= \sum_{\alpha \in P_+(n,k)} \frac{S_{\lambda\alpha}[n,k] S_{\mu\alpha}[n,k] \overline{S_{\nu\alpha}[n,k]}}{S_{0\alpha}[n,k]} \\ &= \sum_{\varepsilon \in \Omega_{n,k}} \sum_{0 \le x \le n-1} \frac{S_{\lambda\sigma^x(\varepsilon)}[n,k] S_{\mu\sigma^x(\varepsilon)}[n,k] \overline{S_{\nu\sigma^x(\varepsilon)}[n,k]}}{S_{0\sigma^x(\varepsilon)}[n,k]}. \end{aligned}$$

Since $|\lambda|, |\mu|, |\nu| \equiv 0 \bmod n$, we have

$$S_{\lambda\sigma^x(\varepsilon)} = S_{\lambda\epsilon}, \quad S_{\mu\sigma^x(\varepsilon)} = S_{\mu\varepsilon}, \quad S_{\nu\sigma^x(\varepsilon)} = S_{\nu\varepsilon}.$$

This implies

$$N_{\lambda\mu}^{\nu}[n,k] = \sum_{\varepsilon\in\Omega_{n,k}} n \, \frac{S_{\lambda\varepsilon}[n,k]S_{\mu\varepsilon}[n,k]\overline{S_{\nu\varepsilon}[n,k]}}{S_{0\varepsilon}[n,k]}.$$

Combining with a similar formula for $N_{\tau(\lambda)\tau(\mu)}^{\tau(\nu)}[k,n]$ we obtain the assertion.

(3) For $\lambda = \sum_{i=1}^{n-1} a_i\Lambda_i \in P_+(n,k)$, we associate the Young diagram of type $(x_1, \dots, x_n)$ with $x_i = \sum_{j=i}^{n-1} a_j, 1 \le i \le n-1$, and $x_n = 0$. We put $|\lambda| = \sum_{i=1}^{n} x_i$. In a similar way, we denote by $(y_1, \dots, y_k)$ the Young diagram for ${}^t\lambda$. We have

$$\begin{aligned}
\langle \lambda, \lambda + 2\rho\rangle &= \langle \lambda+\rho, \lambda+\rho\rangle - \langle\rho,\rho\rangle \\
&= \sum_{i=1}^{n}\left(x_i - i - \frac{|\lambda|}{n} + \frac{n+1}{2}\right)^2 - \frac{n(n^2-1)}{12} \\
&= \sum_{i=1}^{n} x_i^2 - 2\sum_{i=1}^{n} i x_i - \frac{|\lambda|^2}{n} + (n+1)|\lambda|.
\end{aligned}$$

This implies

$$\begin{aligned}
\langle \lambda, \lambda+2\rho\rangle + \langle {}^t\lambda, {}^t\lambda + 2\rho\rangle = &\sum_{i=1}^{n} x_i^2 - 2\sum_{i=1}^{n} i x_i + \sum_{j=1}^{k} y_j^2 - 2\sum_{j=1}^{k} j y_j \\
&- \frac{|\lambda|^2}{nk}(n+k) + (n+k)|\lambda| + 2|\lambda|.
\end{aligned}$$

According to Lemma 2 in [KN], the $n+k$ numbers $x_i + n - i$ $(1 \le i \le n)$, and $n-1+j-y_j$ $(1 \le j \le k)$ are both obtained as a permutation of $\{0, 1, \dots, n+k-1\}$. Since

$$\sum_{i=1}^{n} x_i^2 - 2\sum_{i=1}^{n} i x_i + \sum_{j=1}^{k} y_j^2 - 2\sum_{j=1}^{k} j y_j = -2|\lambda|,$$

we obtain the equation

$$\Delta_\lambda + \Delta_{{}^t\lambda} = \frac{|\lambda|}{2}\left(1 - \frac{|\lambda|}{nk}\right).$$

Applying Lemma 2.4.1, one has

$$\Delta_{\tau(\lambda)} - \Delta_{{}^t\lambda} = \frac{|\lambda|}{2}\left(\frac{|\lambda|}{nk} - 1\right) \mod \mathbf{Z}.$$

Thus we obtain the assertion $\Delta_\lambda + \Delta_{\tau(\lambda)} = 0 \mod \mathbf{Z}$.

### 2.5. Fusion algebras in conformal field theory

The fusion algebra was originally defined in the context of conformal field theory (see [V], [TUY] and [B]). Let us describe briefly the fusion algebra associated with the $SU(n)$ Wess-Zumino-Witten model at level $k$ on the Riemann sphere. We fix a coordinate function $t$ on the Riemann sphere $\mathbf{C}P^1$. Let $P_1, \ldots, P_m$ be distinct points of $\mathbf{C}P^1$ with $t(P_j) = \xi_j, \xi_j \neq 0, 1 \leq j \leq m$ and to each point we associate $\lambda_1, \ldots, \lambda_m \in P_+(n,k)$.

Let $T$ denote the endomorphism on $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_m}$ defined by

$$T(v_1 \otimes \cdots \otimes v_m) = \sum_{i=1}^{m} \xi_i v_1 \otimes \cdots \otimes X_\theta v_i \otimes \cdots \otimes v_m$$

where $X_\theta$ is associated with the longest root $\theta$. More explicitly $X_\theta \in sl(n, \mathbf{C})$ is written as $E_{1n}$ where $E_{ij}$ is the matrix unit such that $ij$ component is 1 and the other components are 0. We define

$$V^\dagger_{\mathbf{C}P^1}(P_1, \ldots, P_m; \lambda_1, \ldots, \lambda_m)$$

to be the space of $sl(n, \mathbf{C})$ invariant $m$-linear form

$$\varphi : V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_m} \to \mathbf{C}$$

satisfying $\varphi \circ T^{k+1} = 0$. It turns out that the above vector space is isomorphic to the dual of the space of conformal blocks (see [TUY] and [B]). We put

$$N(\lambda_1, \ldots, \lambda_m) = \dim V^\dagger_{\mathbf{C}P^1}(P_1, \ldots, P_m; \lambda_1, \ldots, \lambda_m),$$

which is related to the structure constant of the fusion algebra by

$$N^\nu_{\lambda\mu} = N(\lambda, \mu, \nu^*).$$

## §3. Level-rank duality of link invariants

### 3.1. Link invariants via face Boltzmann weights

Let us recall some basic properties of the link invariants for oriented framed links associated with $sl(n, \mathbf{C})$. For a more detailed description we refer the readers to [TW] and [KT]. Let $L$ be an oriented framed link in $S^3$ with $m$ components. To each component $L_i$, $1 \le i \le m$, we assign a highest weight $\lambda_i \in P_+(n,k)$, and we denote by $J(L, \lambda_1, \ldots, \lambda_m)$ the associated invariant. We put

$$q = \exp\left(\frac{2\pi\sqrt{-1}}{k+n}\right), \quad t = \exp\left(\frac{\pi\sqrt{-1}}{n(k+n)}\right).$$

In the case $\lambda_i = \Lambda_1$, $1 \le i \le m$, the invariant $J_L = J(L, \Lambda_1, \ldots, \Lambda_1)$ is characterized by the skein relation

$$tJ_{L_+} - t^{-1}J_{L_-} = (q^{1/2} - q^{-1/2})J_{L_0}$$

and the condition

$$J_{\bigcirc} = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}.$$

With respect to the following local modifications of the framed link we have

$$J \;\lambda = \frac{S_{0\lambda}}{S_{00}} J$$

$$J \;\lambda = \exp 2\pi\sqrt{-1}\Delta_\lambda \, J \;\lambda,$$

$$J \;\mu = \frac{S_{\lambda\mu}}{S_{0\lambda}} J \;\lambda.$$

The invariant $J(L, \lambda_1, \ldots, \lambda_m)$ is related to the fusion algebra $R_{n,k}$ in the following way. We have a multi-linear map

$$J(L, \cdot) : R_{n,k}^{\otimes m} \to \mathbf{C}$$

by associating to $\lambda_1 \otimes \cdots \otimes \lambda_m$ the invariant $J(L, \lambda_1, \ldots, \lambda_m)$. This map is compatible with the product structure of the fusion algebra (see [Ko2]).

In view of the above multi-linear map, we first deal with the case when $\lambda_i$, $1 \le i \le m$, is one of the fundamental weights $\Lambda_1, \ldots, \Lambda_{n-1}$. For the purpose of describing the symmetry, it is useful to introduce an expression of the link invariants using the Boltzmann weights for face models.

Let us denote by $\Phi_n$ the set of fundamental weights $\{\Lambda_1, \ldots, \Lambda_{n-1}\}$. We consider a projection diagram of the oriented framed link $L$ with the blackboard framing. We fix $\lambda_1, \ldots, \lambda_m \in \Phi_n$ and to each component $L_i$ we associate $\lambda_i$. A state is a map $s$ from the set of regions of the projection diagram to $P_+(n,k)$ such that the following admissibility conditions (1) and (2) are satisfied.

(1) If $D_i$ and $D_j$ are adjacent regions as shown in Figure 1, then for $\mu = s(D_i)$ and $\nu = s(D_j)$ we have $N^{\nu}_{\lambda_i \mu} \neq 0$ where $\lambda_i$ is the highest weight associated to the edge incident to both $D_i$ and $D_j$.

(2) For the non-compact region $D_0$ one has $s(D_0) = 0$.

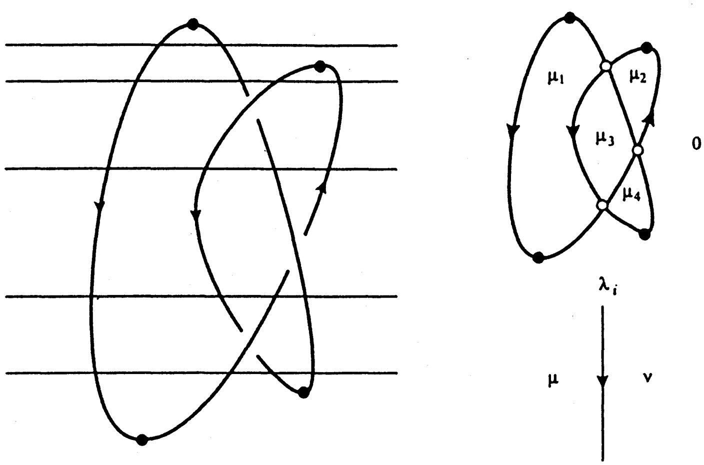


Fig. 1.

We observe that in the above situation the structure constant $N^{\nu}_{\lambda_i \mu}$ is equal to 1. This follows from the Littlewood-Richardson rule and the inequality (1) in 2.1. We have 4 kinds of vertices corresponding to overcrossing, undercrossing, creation and annihilation. For each vertex $v$ and a state $s$ we have a way to associate the face Boltzmann weight $W_v(s)$ such that the link invariant $J(V, \lambda_1, \dots, \lambda_m)$ is expressed as

$$J(L, \lambda_1, \dots, \lambda_m) = \sum_{s:\text{state}} \prod_{v:\text{vertex}} W_v(s).$$

An explicit form of such Boltzmann weights might be found in [JMO] up to some normalization.

### 3.2. $\mathbf{Z}_n$ symmetry for link invariants

Let us review the behaviour of the invariant $J(L, \lambda_1, \dots, \lambda_m)$ under the action of the Dynkin diagram automorphism. A detailed account of the subject is given in [KT]. For $\lambda = (\lambda_1, \dots, \lambda_m)$ we define a map $\phi_\lambda : (\mathbf{Z}_n)^{\oplus m} \to \mathbf{Z}_{2n}$ by

$$\phi_\lambda(x_1, \dots, x_m) = nk \sum_{i=1}^{m} l_{ii} x_i - k \sum_{i,j} l_{ij} x_i x_j - 2 \sum_{i,j} |\lambda(j)| l_{ij} x_i .$$

**Proposition 3.2.1 [KT].** *Using the above notation, we have*

$$\frac{J(L, \sigma^{x_1}(\lambda_1), \dots, \sigma^{x_m}(\lambda_m))}{J(L, \lambda_1, \dots, \lambda_m)} = \exp\left( \frac{\pi\sqrt{-1}}{n} \phi_\lambda(x_1, \dots, x_m) \right).$$

If $\lambda_1, \dots, \lambda_m \in \Omega_{n,k}$, then the above formula is simplified as
(3.2.2)

$$\frac{J(L, \sigma^{x_1}(\lambda_1), \dots, \sigma^{x_m}(\lambda_m))}{J(L, \lambda_1, \dots, \lambda_m)} = \exp\left( \frac{\pi\sqrt{-1}}{n} ((n-1)k \sum_{i,j} l_{ij} x_i x_j) \right).$$

### 3.3. Level-rank duality of link invariants

As in the previous section we denote by $\Phi_n$ the set of fundamental weights for $sl(n, \mathbf{C})$. We define $\Phi_{n,k}$ to be the set of weights $\lambda \in P_+(n, k)$ such that there exists $g \in \mathbf{Z}_n$ with $g(\lambda) \in \Phi_n$. Namely, $\Phi_{n,k}$ consists of the elements which are $\mathbf{Z}_n$ equivalent to fundamental weights with respect to the Dynkin diagram automorphism.

For an oriented framed link $L$ with $m$ components, we consider a similar construction as in 3.1 with $\lambda_1, \dots, \lambda_m \in \Omega_{n,k}$. Let us denote by $\mathcal{S}_{n,k}$ the set of admissible states satisfying (1) and (2) in 3.1. Since $N_{\lambda\mu}^{\nu} = N_{\sigma^p(\lambda)\sigma^q(\mu)}^{\sigma^{p+q}(\nu)}$ the multiplicity appearing in this construction is also at most 1. The local Boltzmann weight behaves as follows with respect to the map $\tau : \Omega_{n,k} \to \Omega_{k,n}$.

We have a one-to-one correspondence $T : \mathcal{S}_{n,k} \to \mathcal{S}_{k,n}$ defined by $(T(s))(D) = \tau(s(D))$. The following lemma is essentially due to [JMO] (see also [SA]), where the corresponding statement is shown for the fundamental representation up to some phase factor. Computing the phase factor by means of Lemma 2.4.1 and Lemma 2.4.2, we obtain the following lemma.

**Lemma 3.3.1.** *For any type of vertex $v$ in the graph associated with the link diagram we have we have*

$$W_v^{\tau(\lambda)}(s) = \overline{W_v^{\lambda}(T(s))}$$

*for any admissible state $s \in \mathcal{S}_{n,k}$.*

**Proposition 3.3.2.** *Let $L$ be an oriented framed link with $m$ components. For $\lambda_1, \ldots, \lambda_m \in \Omega_{n,k}$ we have*

$$J(L, \tau(\lambda_1), \ldots, \tau(\lambda_m)) = \overline{J(L, \lambda_1, \ldots, \lambda_m)}.$$

*Proof.* By the behaviour of the face Boltzmann weights with respect to $\tau$ it can be shown that the assertion holds if $\lambda_1, \ldots, \lambda_m \in \Phi_{n,k}$. For $\lambda, \mu \in \Omega_{n,k}$, we observe by means of the Littlewood-Richardson rule and the inequality (1) in 2.1 that $N_{\lambda\mu}^{\nu} \neq 0$ only if $\nu \in \Omega_{n,k}$. Moreover, it is clear that $\Omega_{n,k}$ is generated by $\Phi_{n,k}$ in the fusion algebra. Let us suppose that $\lambda_j \in \Omega_{n,k}$ is written as a polynomial

$$\lambda_j = P_j(\mu_1, \mu_2, \ldots), \quad \mu_1, \mu_2, \ldots \in \Phi_{n,k}.$$

We have

$$\tau(\lambda_j) = P_j(\tau(\mu_1), \tau(\mu_2), \ldots)$$

and it follows from the expression of the link invariant by the face Boltzmann weights and Lemma 3.3.1 together with the compatibility of $R_{n,k}^{\otimes m} \to \mathbf{C}$ with the product structure of the fusion algebra explained in 3.1 that the assertion holds for any $\lambda_1, \ldots, \lambda_m \in \Omega_{n,k}$. This completes the proof.

## §4. $PSU(n)$ invariants of 3-manifolds

### 4.1. Review of the definition of $SU(n)$ invariants

Let $M$ be a closed oriented 3-manifold. We suppose that $M$ is obtained by the Dehn surgery on a framed link $L$ with $m$ components in $S^3$. As in [KT] we put $C_{n,k} = \exp\left(-\frac{2\pi\sqrt{-1}}{8} c_{n,k}\right)$ with the central charge $c_{n,k} = \frac{k \dim sl(n,\mathbf{C})}{k+n}$. Let $m$ be the number of components of the link $L$. We consider the sum

$$\begin{aligned} &Z_k(M, SU(n)) \\ (4.1.1) \quad &= C_{n,k}^{sign(L)} \sum_{\lambda_1, \ldots, \lambda_m \in P_+(n,k)} S_{0\lambda_1} \cdots S_{0\lambda_m} J(L, \lambda_1, \ldots, \lambda_m) \end{aligned}$$

where $sign(L)$ denotes the signature of the linking matrix of $L$. It was shown in [KT] that $Z_k(M, SU(n))$ is a topological invariant of $M$ (see also [TW]).

### 4.2. $PSU(n)$ invariants and level-rank duality

As is shown in 2.2, the set of weights $P_+(n,k)$ admits a $\mathbf{Z}_n$ action derived from the Dynkin diagram automorphism. Let us suppose that $n$ and $k$ are relatively prime. In this case the $\mathbf{Z}_n$ action is fixed point free and moreover, the invariant $Z_k(M, SU(n))$ admits the following factorization.

We consider the Gauss sum

$$G_{n,k} = \sum_{j=0}^{n-1} e((n-1)kj^2)$$

where $e(x)$ denotes $\exp(\pi\sqrt{-1}x/n)$. For $\lambda \in \Omega_{n,k}$ we have

$$\sum_{j=0}^{n-1} \exp 2\pi\sqrt{-1}\Delta_{\sigma^j(\lambda)} = G_{n,k} \exp 2\pi\sqrt{-1}\Delta_\lambda.$$

As in the previous paragraph, we suppose that $M$ is a closed oriented 3-manifold obtained as the Dehn surgery on a framed link $L$ in $S^3$. Let $(l_{ij})_{1\le i,j\le m}$ be the linking matrix of $L$. It is known by [MOO] (see also [Ko1]) that

(4.2.1)

$$\xi_{n,k}(M) = \left(\frac{\sqrt{n}}{G_{n,k}}\right)^{sign(L)} \left(\frac{1}{\sqrt{n}}\right)^m \sum_{x_1,\dots,x_m\in\mathbf{Z}_m} e((n-1)k\sum_{i,j} l_{ij}x_ix_j)$$

is a topological invariant of $M$. We put

(4.2.2)

$$Z_k(M, PSU(n)) = \left(\frac{C_{n,k}G_{n,k}}{\sqrt{n}}\right)^{sign(L)} (\sqrt{n})^m \times \sum_{\lambda_1,\dots\lambda_m\in\Omega_{n,k}} S_{0\lambda_1}\dots S_{0\lambda_m} J(L,\lambda_1,\dots,\lambda_m).$$

**Proposition 4.2.3.** *Let us suppose that $n$ and $k$ are relatively prime. Then we have*

$$Z_k(M, SU(n)) = \xi_{n,k}(M) Z_k(M, PSU(n)).$$

*Proof.* By using 3.2.2, we have

$$\begin{aligned}
&\sum_{\lambda_1,\ldots,\lambda_m \in P_+(n,k)} S_{0\lambda_1}\cdots S_{0\lambda_m} J(L,\lambda_1,\ldots,\lambda_m)\\
&= \sum_{\lambda_1,\ldots,\lambda_m \in \Omega_{n,k}} \sum_{x_1,\ldots,x_m \in \mathbf{Z}_n} S_{0\lambda_1}\cdots S_{0\lambda_m} J(L,\sigma^{x_1}(\lambda_1),\ldots,\sigma^{x_m}(\lambda_m))\\
&= e((n-1)k \sum_{x_1,\ldots,x_m \in \mathbf{Z}_n} l_{ij}x_ix_j)\\
&\qquad \times \sum_{\lambda_1,\ldots,\lambda_m \in \Omega_{n,k}} S_{0\lambda_1}\cdots S_{0\lambda_m} J(L,\lambda_1,\ldots,\lambda_m),
\end{aligned}$$

which shows the assertion.

We are going to show that $Z_k(M, PSU(n))$ is actually a topological invariant of $M$. This is not obvious from the above factorization, since the invariant $\xi_{n,k}(M)$ defined by the linking matrix might be equal to zero.

**Lemma 4.2.4.** *For $\lambda, \mu, \nu \in \Omega_{n,k}$ we have*

$$C_{n,k}G_{n,k} \sum_{\mu \in \Omega_{n,k}} S_{\lambda\mu}S_{\mu\nu} \exp 2\pi\sqrt{-1}(\Delta_\lambda + \Delta_\mu + \Delta_\nu) = S_{\lambda\nu}.$$

*Proof.* Let us start with the equality

$$C_{n,k} \sum_{\mu \in P_+(n,k)} S_{\lambda\mu}S_{\mu\nu} \exp 2\pi\sqrt{-1}(\Delta_\lambda + \Delta_\mu + \Delta_\nu) = S_{\lambda\nu} \tag{4.2.5}$$

which follows from the explicit form of the matrix $S$ given in 2.3. We see that the equality (4.2.5) reveals the fact that the modular group acts on the space of characters $\chi_\lambda, \lambda \in P_+(n,k)$ as stated in Theorem 2.3.1. Since $\lambda, \nu \in \Omega_{n,k}$ we have $S_{\lambda\sigma(\mu)} = S_{\lambda\mu}$ by Lemma 2.4.1 (2). We decompose the left hand side of the equality 4.2.5 with respect to the $\mathbf{Z}_n$ action on $\mu$. Combining with Lemma 2.4.1 (1), we obtain the desired equality.

The above lemma shows that the matrices

$$\tilde{S} = (\sqrt{n}S_{\lambda\mu})_{\lambda,\mu \in \Omega_{n,k}}$$

and

$$\tilde{T} = diag\left(\exp 2\pi\sqrt{-1}\left(\Delta_\lambda - \frac{c_{n,k}}{24}\right) G_{n,k}^{-1/3}\right)_{\lambda \in \Omega_{n,k}}$$

satisfy

$$(\tilde{S}\tilde{T})^3 = (\tilde{S})^2$$

and define a linear representation of $SL(2, \mathbf{Z})$.

We are in a position to show the following proposition.

**Proposition 4.2.6.** *$Z_k(M, PSU(n))$ is a topological invariant of a closed oriented 3-manifold $M$.*

*Proof.* As is explained in [RT], we show the invariance under the Kirby moves in the sense of Fenn and Rourke [FR]. It is enough to check the equality

$$C_{n,k}G_{n,k} \sum_{\mu \in \Omega_{n,k}} S_{0\mu}\, J(L', \lambda_1, \ldots, \lambda_m, \mu) = J(L, \lambda_1, \ldots, \lambda_m)$$

for the link diagram $L'$ obtained by adding one component with framing 1 as shown in Figure 2. Here $\mu$ stands for the representation associated to the new component. Let $l$ denote the number of strands passing through the new component in Figure 2.

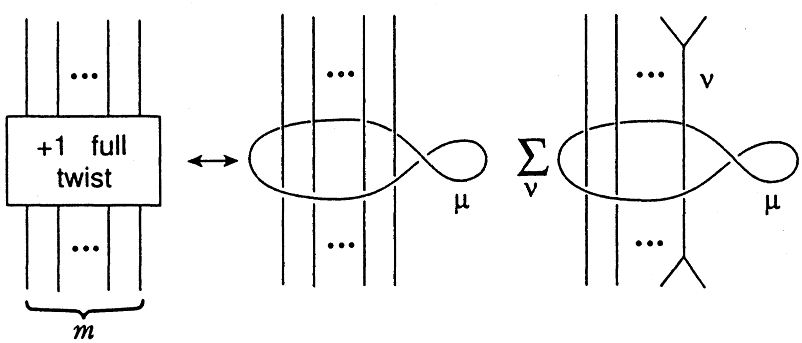


Fig. 2.

We show the assertion by induction with respect to $l$. As a special case of Lemma 4.2.4, we have

$$C_{n,k}G_{n,k} \sum_{\mu \in \Omega_{n,k}} S_{0\mu}S_{\lambda\mu} \exp 2\pi\sqrt{-1}(\Delta_\lambda + \Delta_\mu) = S_{0\lambda}.$$

This settles the case $l = 1$ and in particular, putting $\lambda = 0$, we obtain the case $l = 0$. Let us recall that $\Omega_{n,k}$ is closed under the product structure of the fusion algebra. Now we apply the fusion rule as shown in Figure 2, which implies that the linear map associated with the two tangles in Figure 2 are identical. This procedure permits us to reduce to the case of $l-1$ strands and we are done by induction.

Now we are in a position to show the following level-rank duality of $PSU(n)$ invariants.

**Theorem 4.2.7.** *Let us suppose that $n$ and $k$ are relatively prime. Then we have*

$$Z_k(M, PSU(n)) = \overline{Z_n(M, PSU(k))}.$$

*Proof.* First, we see

$$C_{n,k} = \exp(-\frac{nk-1}{4}\pi\sqrt{-1})\,\overline{C_{k,n}}$$

and

$$G_{n,k} = \exp(\frac{nk-1}{4}\pi\sqrt{-1})\sqrt{\frac{n}{k}}\,\overline{G_{k,n}}.$$

By Lemma 2.4.2 we have

$$\sqrt{n}S_{0\lambda}[n,k] = \sqrt{k}S_{0\tau(\lambda)}[k,n].$$

Here we recall that they are real numbers. Using Proposition 3.3.2, we obtain the equality

$$\begin{aligned}(\sqrt{n})^m & \sum_{\lambda_1,\dots,\lambda_m\in\Omega_{n,k}} S_{0\lambda_1}\cdots S_{0\lambda_m} J(L,\lambda_1,\dots,\lambda_m)\\ =(\sqrt{k})^m & \sum_{\tau(\lambda_1),\dots,\tau(\lambda_m)\in\Omega_{k,n}} S_{0\tau(\lambda_1)}\cdots S_{0\tau(\lambda_m)}\overline{J(L,\tau(\lambda_1),\dots,\tau(\lambda_m))}.\end{aligned}$$

This completes the proof.

Let us describe briefly the duality relation of the representations of the mapping class groups associated with $SU(n)$ Wess-Zumino-Witten model at level $k$ and $SU(k)$ model at level $n$. Let $\Sigma$ denote a closed oriented surface of genus $g$ and $\mathcal{M}_g$ its mapping class group. Using the notation of 2.5, we put

$$\mathcal{H}_\Sigma[n,k] = \oplus_{\lambda_1,\dots,\lambda_g\in P_+(n,k)} V^\dagger_{\mathbf{C}P^1}(\lambda_1,\lambda_1^*,\dots,\lambda_g,\lambda_g^*)$$

which is isomorphic to the space of conformal blocks of $SU(n)$ Wess-Zumino-Witten model at level $k$ in the sense of [TUY]. As in [RT] and

[Ko2], we have a projectively linear unitary representation of the mapping class group

$$\mathcal{M}_g \to GL(\mathcal{H}_\Sigma[n,k]).$$

Let us recall that this representation is expressed in terms of the Boltzmann weight of the face model by considering a $(2g, 2g)$ tangle associated with an element of the mapping class group.

Let us now suppose that $n$ and $k$ are relatively prime. We put

$$\mathcal{H}_\Sigma[n,k]/\mathbf{Z}_n = \oplus_{\lambda_1,\ldots,\lambda_g \in \Omega_{n,k}} V^\dagger_{\mathbf{C}P^1}(\lambda_1, \lambda_1^*, \ldots, \lambda_g, \lambda_g^*).$$

Our previous construction permits us to construct a projectively linear unitary representation

$$\rho_g[n,k] : \mathcal{M}_g \to GL(\mathcal{H}_\Sigma[n,k]/\mathbf{Z}_n).$$

In particular, in the case $g = 1$ this representation is identical to the representation considered in the remark after Lemma 4.2.4. By the Verlinde formula, we see that

$$\dim \mathcal{H}_\Sigma[n,k]/\mathbf{Z}_n = \sum_{\lambda \in \Omega_{n,k}} \left( \frac{1}{\sqrt{n} S_{0\lambda}[n,k]} \right)^{2g-2}.$$

Combining with our previous discussion, we have the following proposition.

**Proposition 4.2.8.** *We have a non degenerate bilinear pairing of the vector spaces $\mathcal{H}_\Sigma[n,k]/\mathbf{Z}_n \times \mathcal{H}_\Sigma[k,n]/\mathbf{Z}_k \to \mathbf{C}$ and the representation of the mapping class group $\rho_g[n,k]$ is the contragredient representation of $\rho_g[k,n]$.*

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