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The Character Table of $^2E_6(2)$ Acting on the Cosets of Fi_{22}

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Abstract.

We consider the permutation action of $E \cong {}^2E_6(2)$ on the cosets of its maximal subgroup $F \cong Fi_{22}$. We calculate the intersection matrices and character table of the centralizer algebra corresponding to this action. There are three reasons for the interest in this particular representation. Firstly, it is a sporadic multiplicity-free action of a simple group of exceptional Lie type. Secondly, E and F are Y-groups Y_{333} and Y_{332}, respectively, factorized over their centers. We believe that the intersection matrices we have calculated might be useful for a computer-free identification of Y_{333} with $2^3 \cdot {}^2E_6(2)$. Thirdly, the permutation group considered is the one induced by the involution centralizer on the set of points fixed by an involution in the action of the Baby Monster F_2 on the cosets of the Fischer group Fi_{23}. The latter action has the largest rank (namely 23) among the primitive multiplicity-free actions of the sporadic simple groups and the calculation of its character table is an open problem.

§1. Introduction

Let us recall some basic facts concerning permutation groups and their centralizer algebras from [BI] and [BCN]. Let X be the set of (right) cosets of a subgroup H in a finite group G. Then G induces a transitive action on X by translations and H coincides with the stabilizer $G(x_0)$ of the coset $x_0 \in X$ containing the identity (that is, of H itself). We assume that the action is faithful, that is H does not contain a non-trivial normal subgroup of G. Let χ be the permutation character of G acting on X, that is, $\chi(g) = \#\{x \mid x \in X, x^g = x\}$ for $g \in G$.

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Let V be the space of complex valued functions defined on X and let $v(x)$ denote the characteristic function of $x \in X$. Then G acts naturally on V, preserving on it the inner product $\langle . \mid . \rangle$ with respect to which $\mathcal{B} = \{v(x) \mid x \in X\}$ is an orthonormal basis. In this basis the linear transformation of V induced by $g \in G$ is given by the matrix $M(g)$ whose (x, y)-entry is 1 if $y^g = x$ and 0 otherwise. Clearly $\chi(g) = tr(M(g))$.

Let $\Gamma_0, \Gamma_1, \ldots, \Gamma_{r-1}$ be the orbits of G on the set of ordered pairs of elements of X. Then the Γ_i are called the orbitals or 2-orbits and r is known as the rank of the permutation group (G, X). It is standard to assume that $\Gamma_0 = \{(x, x) \mid x \in X\}$ is the diagonal orbital. In what follows we assume that all Γ_i are symmetrical, that is, for $x, y \in X$, $(x, y) \in \Gamma_i$ if and only if $(y, x) \in \Gamma_i$. Let $\chi = \psi_0 + \psi_1 + \cdots + \psi_{s-1}$ be the decomposition of χ into a sum of G-irreducibles. The condition assumed is equivalent to the following: $r = s$, the irreducibles ψ_i are pairwise different (that is, χ is multiplicity-free) and the Frobenius – Schur indicator of every ψ_i is 1. It is also standard to take ψ_0 to be the principal character of G.

Let $\Gamma_i(x) = \{y \mid (x, y) \in \Gamma_i\}$, $0 \le i \le r - 1$ and let $k_i = |\Gamma_i(x)|$. One can consider Γ_i as the set of edges of an undirected graph on X and we will identify Γ_i with this graph. Then $\Gamma_i(x)$ is the set of vertices adjacent to x in Γ_i and k_i is the valency of Γ_i. Let A_i be the adjacency matrix of Γ_i, that is, a matrix, whose rows and columns are indexed by the elements of X and the (x, y)-entry is 1 if $(x, y) \in \Gamma_i$ and 0 otherwise. We can consider A_i as a linear transformation of V written in the basis $\mathcal{B}$. The matrices A_i for $1 \le i \le r-1$ form a linear basis of the algebra $\mathcal{C}$ (the centralizer algebra) consisting of the matrices which commute with $M(g)$ for every $g \in G$. In particular

$$A_i \cdot A_j = \sum_{k=0}^{r-1} p_{ij}^k A_k.$$

The structure constant p_{ij}^k is equal to the number of vertices in $\Gamma_j(x)$ adjacent to a fixed vertex from $\Gamma_k(x)$ in the graph (determined by) Γ_i. Let B_i denote the $r \times r$ matrix whose (j, k)-entry is equal to p_{ij}^k, $0 \le i, j, k \le r - 1$. Then B_i is called the intersection matrix of the graph Γ_i. We will always append such a matrix by the column $(k_0, k_1, \ldots, k_{r-1})^t$. The mapping $A_i \mapsto B_i$ $(0 \le i \le r - 1)$ induces a faithful linear representation of $\mathcal{C}$; in particular A_i and B_i have the same minimal polynomial.

Let us consider the centralizer algebra from a different point of view. The G-module V possesses a decomposition $V = V_0 \oplus V_1 \oplus \cdots \oplus V_{r-1}$ into the direct sum of minimal G-invariant subspaces. These subspaces are pairwise orthogonal with respect to the inner product $\langle . \mid . \rangle$ and they support pairwise non-isomorphic irreducible representations of G. We can assume that the irreducible constituent ψ_j of χ is the character of G acting on V_j, $0 \leq j \leq r-1$. Let E_j be the linear transformation of V which acts as the identity on V_j and maps every $w \in V_k$ for $k \neq j$ to the zero vector. Then E_j belongs to $\mathcal{C}$, moreover $E_0, E_1, \ldots, E_{r-1}$ is a linear basis of $\mathcal{C}$ consisting of primitive idempotents (that is, $E_i \cdot E_j = \delta_{ij} E_i$).

Since A_i commutes with the action of G on V, by Schur's lemma it preserves every V_j as a whole and multiplies every vector of V_j by a scalar, which we denote by $p_i(j)$. This number is an entry of the transformation matrix between the two bases of $\mathcal{C}$:

$$A_i = \sum_{j=0}^{r-1} p_i(j) E_j.$$

So $p_i(0), \ldots, p_i(r-1)$ are the eigenvalues (non-necessarily distinct) of A_i and hence of the intersection matrix B_i as well. It is standard to write the inverse transformation as follows:

$$E_j = (1/n) \sum_{i=0}^{r-1} q_j(i) A_i,$$

where $n = |X|$. It is known that $q_j(i) = p_i(j) \cdot m_j / k_i$. As above, k_i is the valency of Γ_i and m_j is the dimension of V_j that is the rank of E_j and can be computed by the formulae:

$$m_j = n \cdot (\sum_{i=0}^{r-1} p_i(j)^2 / k_i)^{-1}.$$

If $(x, y) \in \Gamma_i$ then $(1/n) q_j(i)$ is the (x, y)-entry of E_j written in the basis $\mathcal{B}$. This implies an important geometrical interpretation of the $q_j(i)$. As above, let $v(x) \in \mathcal{B}$ be the characteristic function of $x \in X$ and let $v_j(x)$ be the projection of $v(x)$ into V_j, that is $v_j(x) = E_j v(x)$. If $(x, y) \in \Gamma_i$ then the inner product $\langle v_j(x) \mid v_j(y) \rangle$ equals to $(1/n) q_j(i)$. So after rescaling we obtain a realization of the elements of X as unit vectors $w_j(x) = (n/m_j) v_j(x)$ in V_j such that for $(x, y) \in \Gamma_i$ the inner product $\langle w_j(x) \mid w_j(y) \rangle$ equals to $q_j(i)/m_j = p_i(j)/k_i$ for $0 \leq i, j \leq r-1$.

The vector $w_j(x)$ is fixed by $G(x)$. Since ψ_j appears in χ with multiplicity 1, the Frobenius reciprocity rule implies that the subspace of V_j fixed by $G(x)$ is 1-dimensional. This determines $w_j(x)$ up to multiplication by a scalar. Since the action of G on every V_i can be realized by real matrices, the scalar is plus or minus one.

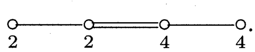
The matrix, whose (i, j)-entry is $p_i(j)$ is known as the character table of the centralizer algebra $\mathcal{C}$. We will append such a matrix by the column $(m_0, \dots, m_{r-1})^t$.

In the present paper we compute the intersection matrices and character table of the centralizer algebra corresponding to the action of ${}^2E_6(2)$ on the cosets of Fi_{22}. We use a considerable amount of unpublished information on this action and on smaller configurations. The permutation character and 2-point stabilizers were determined by S. P. Norton using the fact that ${}^2E_6(2)$ is a section in the Monster. Later the character was independently computed by T. Breuer and K. Lux. In our work we rely on the information on the permutation character. At the same time we present a self-contained identification of the 2-point stabilizers. To meet the needs of the present project we asked L. H. Soicher to compute the intersection matrices of the primitive action of $O_8^+(2) : S_3$ of degree 11 200. Also, at our request S. A. Linton has computed the sizes of double cosets in Fi_{22} of a particular 2-point stabilizer, isomorphic to $2^{10} : M_{22}$, and all other such stabilizers. This is a very delicate information which has played a crucial role in our arguments. We have also used a computer program by D. V. Pasechnik which calculates the complete set of intersection matrices and the character table of a centralizer algebra from a single intersection matrix (having pairwise distinct eigenvalues). Finally, S. V. Shpectorov has suggested many improvements of the exposition of the paper. We are very grateful to all these people for their helpful cooperation.

Throughout the paper, given a group G we write $\bar{G}$ to denote $G/O_2(G)$.

§2. Preliminaries

The group $E \cong {}^2E_6(2)$ is a flag-transitive automorphism group of a Tits building $\mathcal{E}$ with the diagram



The elements of $\mathcal{E}$ will be called points, lines, planes and symplecta, respectively (nodes from the left to the right in the diagram). There is a natural bijection between the point set Δ of $\mathcal{E}$ and the conjugacy class of central involutions in E. We will not distinguish between these two sets. The following lemma describes the action of E on Δ (see for instance [Ivn]).

Lemma 2.1. *The group E acts transitively on Δ. Let $u \in \Delta$ and let $E(u)$ be the stabilizer of u in E. Then $E(u) \cong 2_+^{1+20} : U_6(2)$ is the* *centralizer of u as a central involution in E. $E(u)$ has five orbits $\Sigma_1(u) = \{u\}$, $\Sigma_2(u)$, $\Sigma_2'(u)$, $\Sigma_4(u)$ and $\Sigma_3(u)$ on Δ with lengths 1, 1 782, 44 352, 1 824 768 and 2 097 152, respectively. If $v \in \Sigma_i^{(')}(u)$ then the product of u and v (as involutions in E) has order i. The permutation character $\mathbf{1}_{E(u)}^E$ of E acting on Δ is $1a+1\,938a+48\,620a+1\,828\,332a+2\,089\,164a$.*

The subdegree 1 782 of E acting on Δ corresponds to the collinearity graph of $\mathcal{E}$ i.e., to the graph where two points are adjacent if they are incident to a common line. The intersection matrix of this graph is the following:

0	1	0	0	0	1
1782	85	27	1	0	1782
0	672	27	42	0	44352
0	1024	1728	715	891	1824768
0	0	0	1024	891	2097152

In the above notation let $L = O_2(E(u))$, $U = E(u)/L \cong U_6(2)$ and Π be the residue of u in $\mathcal{E}$ on which U acts flag-transitively. Then Π is a rank 3 dual polar space of unitary type. The orbits of L on $\Sigma_2(u)$ are of length 2 and they correspond to 891 lines of $\mathcal{E}$ incident to u. The orbits of L on $\Sigma_2'(u)$ have length 64 and they correspond to the 693 symplecta incident to u. Every symplecton incident to u is also incident to 54 points from $\Sigma_2(u)$ and to 64 points from $\Sigma_2'(u)$. The subgraph in the collinearity graph induced by the points incident to a symplecton is strongly regular with parameters $v = 119$, $k = 54$, $\lambda = 21$, $\mu = 27$.

§3. The permutation character

The fact that Fi_{22} is a subgroup of ${}^2E_6(2)$ was first established by B. Fischer and his arguments were published in [Coo], Section 6. In terms of Y-groups the Fi_{22}-subgroups in ${}^2E_6(2)$ where classified by S. P. Norton in [Nor]. The permutation character of E acting on the cosets of F was computed by S. P. Norton and independently by T. Breuer and K. Lux at Aachen.

Lemma 3.1. *The group $E \cong {}^2E_6(2)$ contains three classes of maximal subgroups $F \cong Fi_{22}$ which are permuted transitively by outer automorphisms of E. The permutation character $\mathbf{1}_F^E$ is the following: $1a + 1\,938a + 48\,620a + 1\,828\,332a + 2\,909\,907x + 29\,099\,070a + 278\,555\,200a + 872\,972\,100x$, where $x = a$, b or c depending on the choice of the E-conjugacy class containing F.*

In the above lemma the permutation character is given in its decomposition into irreducibles and each irreducible is presented by its degree. In case there are several characters with the same degree we use letters a, b, etc. according to the ordering of the characters in [Atlas].

In order to simplify the references we present below the values of $\mathbf{1}_F^E$ on elements of certain classes. We use the upper case letters to name E-conjugacy classes and the lower case letters to name F-classes.

E-classes	E-normalizers	F-classes	F-normalizers	the values of $\mathbf{1}_F^E$
$1A$	${}^2E_6(2)$	$1a$	Fi_{22}	1 185 415 168
$2A$	$2^{1+20} : U_6(2)$	$2a$	$2 \cdot U_6(2)$	1 048 576
$3B$	$(3 \times O_8^+(2) : 3)2$	$3c$	$3.3^4 : 2^3.S_4.S_3$	11 200
$3C$	$3^{1+6}.2^{3+6}.(S_3 \times 3)$	$3b, 3d$		$4 + 576$
$5A$	$(D_{10} \times A_8).2$	$5a$	$F_{20} \times S_5$	168
$7B$	$(F_{21} \times L_2(7)).2$	$7a$	$F_{42} \times S_3$	28
$9A$		$9a$		1
$11A$		$11a$		3

It is straightforward to reconstruct the fusion of F-classes into E-classes (at least for elements of small order) directly from the permutation character $\mathbf{1}_F^E$. The structure of E- and F-normalizers in the above table is taken from [Atlas] for elements of order 1, 2 and 3. For the elements of order 5 and 7 the relevant information is obtained below, in Lemmas 6.4 and 6.3, respectively.

§4. Some properties of Fi_{22}

We will make use of the description of maximal subgroups in the Fischer group Fi_{22} obtained in [KW] (see also [Atlas]).

Lemma 4.1. *The Fischer group $F \cong Fi_{22}$ contains (up to conjugacy in its automorphism group) 12 classes of maximal subgroups with representatives H_i, $1 \le i \le 12$ as given on the next page. The $Aut(F)$-conjugacy classes containing H_2 and H_{11} split into two F-classes each.*

Let Ξ be the transposition graph of F. The vertices of Ξ are the Fischer transpositions of F ($2a$-involutions); two of them are adjacent if they commute. H_1 is the stabilizer of a transposition $\alpha \in \Xi$. The action induced by H_1 on the set $\Xi(\alpha)$ of transpositions adjacent to α is similar to the action of $\bar{H}_1 \cong U_6(2)$ on the set of planes of the dual polar space Π. Two transpositions from $\Xi(\alpha)$ are adjacent if and only if the corresponding planes in Π are incident to a common line. This implies that every maximal clique of Ξ has size 22. H_4 is the stabilizer of such

i	$[F : H_i]$	H_i	Remarks
1	3 510	$2 \cdot U_6(2)$	$N(2a)$
2	14 080	$O_7(3)$	
3	61 776	$O_8^+(2) : S_3$	
4	142 155	$2^{10} : M_{22}$	$N(2a_{22}2b_{231}2c_{770})$
5	694 980	$2^6 : Sp_6(2)$	$N(2b^6)$
6	1 216 215	$(2 \times 2_+^{1+8} : U_4(2)) : 2$	$N(2b)$
7	1 647 360	$S_3 \times U_4(3) : 2_2$	$N(3a)$
8	3 592 512	${}^2F_4(2)'$	
9	3 648 645	$2^{5+8} : (S_3 \times A_6)$	$N(2b^4)$
10	12 812 800	$3_+^{1+6} : 2^{3+4} : 3^2 : 2$	$N(3b)$
11	17 791 488	S_{10}	
12	679 311 360	M_{12}	

a clique and it is known to contain exactly 22 transpositions (the ones in the clique). In what follows we will denote this subgroup simply by H.

Let Ω denote the set of maximal cliques in the transposition graph Ξ naturally identified with the cosets of $H \cong 2^{10} : M_{22}$ in $F \cong Fi_{22}$. The following result was proved in [RW].

Lemma 4.2. *The subgroup $H \cong 2^{10} : M_{22}$ acting on Ω has* 8 *orbits $\Omega_0, \dots, \Omega_7$ with lengths* 1, 154, 1 024, 3 696, 4 928, 11 264, 42 240 *and* 78 848 *and stabilizers $2^{10} : (M_{22})$, $2^9.(2^4 : A_6)$, $1.(M_{22})$, $2^6.(2^4 : S_5)$, $2^4.(2^4 : A_6)$, $2.(L_3(4))$, $2^3.(2^3 : L_3(2))$ and $1.(2^4 : A_6)$, respectively. Here when a stabilizer Z is written as $X.(Y)$, we mean that $X = Z \cap O_2(H)$ and Y is the image of Z in $\bar{H} \cong M_{22}$. The permutation character $\mathbf{1}_H^F$ is the following: $1a + 78a + 429a + 1\,430a + 3\,080a + 30\,030a + 32\,032a + 75\,075a$.*

Let $z_i \in \Omega_i$, $0 \le i \le 7$, so that $y = z_0$ is the clique stabilized by H. The intersection $y \cap z_i$ has size 22, 6, 2 and 1 for i =0, 1, 4 and 5, respectively and is empty in the remaining cases. As above, let α be the transposition centralized by H_1. Then α stabilizes $z \in \Omega$ if and only if $\alpha \in z$. This implies the following.

Lemma 4.3. *The set $\Omega(\alpha)$ of elements in Ω fixed by α has size* 891*; H_1 induces on $\Omega(\alpha)$ the action of $\bar{H}_1$ as on the points of Π. Assume that $\omega_0 \in \Omega(\alpha)$. Then $\Omega(\alpha)$ intersects Ω_i in* 1, 42, 336 *and* 512 *elements for $i = 0$,* 1, 4 *and* 5, *respectively.*

Let $\mathcal{S}$ be the Steiner system $S(3,6,22)$ defined on z_0 and acted on naturally by $\bar{H} \cong M_{22}$. We see from Lemma 4.2 that the orbits of $O_2(H)$ on Ω_i have lengths 1, 2, 2^{10}, 2^4, 2^6, 2^9, 2^7 and 2^{10} for $i = 0$ to 7, respectively. Moreover, the action of $\bar{H} \cong M_{22}$ on the set of $O_2(H)$-orbits on Ω_i is trivial for $i = 0$ and 2; as on the points of $\mathcal{S}$ for $i = 5$; as on the blocks of $\mathcal{S}$ for $i = 1$, 4 and 7; as on the duads for $i = 3$ and as on the special octets for $i = 6$. The intersection matrix of F acting on the cosets of H which correspond to the subdegree 154 and the character table of the centralizer algebra are given below (cf. [ILLSS]).

0	1	0	0	0	0	0	0	1
154	1	5	1	0	0	0	0	154
0	120	5	0	21	7	0	0	3696
0	32	0	1	0	14	77	1	4928
0	0	64	0	21	0	0	16	11264
0	0	80	120	0	21	0	60	42240
0	0	0	16	0	0	0	1	1024
0	0	0	16	112	112	77	76	78848

1	154	3696	4928	11264	42240	1024	78848	1
1	−77	924	1232	−1408	−5280	−320	4928	78
1	49	546	−532	1184	−960	−176	−112	429
1	−35	294	−364	−400	120	160	224	1430
1	37	186	248	32	120	88	−712	3080
1	13	6	−28	−112	120	−32	32	30030
1	−17	24	32	32	120	−20	−172	32032
1	1	−30	−4	32	−96	16	80	75075

We will make use of the following two lemmas. The former comes from [ILLSS] and direct calculations with the permutation character, while the latter follows from calculations performed by S. A. Linton (cf. remark before Lemma 10.1).

Lemma 4.4. *The group $F \cong Fi_{22}$ acting on the cosets of $H_5 \cong 2^6 : Sp_6(2)$ has rank* 10 *with subdegrees* 1, 135, 1 260, 2 304, 8 640, 10 080, 45 360, 143 360 *and* 241 920 (*twice*). *An element of type* 5a *from F fixes exactly five cosets of H_5 in F.*

Lemma 4.5. *The subgroup $H_5 \cong 2^6 : Sp_6(2)$ acting on Ω has* 6 *orbits with lengths* 135, 756, 8 640, 15 120, 48 384 *and* 69 120.

§5. The action of F on the central involutions of E

Lemma 5.1. *Let Δ be the set of central involutions in E. Then F acting on Δ has 4 orbits Δ_i, $1 \leq i \leq 4$ with lengths* 3 510, 142 155, 3 127 410 *and* 694 980. *The stabilizer in F of an involution from Δ_i is isomorphic to $2 \cdot U_6(2)$, $2^{10} : M_{22}$, $2^{10}.L_3(4)$ and $2^6 : Sp_6(2)$, for $i = 1$, 2, 3 and 4, respectively.*

Proof. It follows from the table, given in Section 3, that Δ_1 defined as $\Delta \cap F$ is the class of $2a$-involutions in F and hence it forms an F-orbit with stabilizer $H_1 \cong 2 \cdot U_6(2)$. We will use notation introduced in and after Lemma 2.1 and assume that $u \in \Delta_1$. Then $F(u) \cong 2 \cdot U_6(2)$ intersects L in a subgroup of order 2 and $F(u)L/L = U \cong U_6(2)$. The subgroup $F(u)$ has two orbits, say $\Delta_1^2(u)$ and $\Delta_1^3(u)$ on $\Delta_1 - \{u\}$ consisting of transpositions whose products with u have order 2 and 3, respectively. By Lemma 2.1 we see that $\Delta_1^3(u) \subseteq \Sigma_3(u)$ and either $\Delta_1^2(u) \subseteq \Sigma_2(u)$ or $\Delta_1^2(u) \subseteq \Sigma_2'(u)$. The intersection matrix given after Lemma 2.1 shows that in the collinearity graph of $\mathcal{E}$ (which corresponds to $\Sigma_2(u)$) every vertex $v \in \Sigma_3(u)$ is at distance 3 from u and hence $\Delta_1^2(u) \subseteq \Sigma_2'(u)$. Since $F(u)/\langle u \rangle \cong U_6(2)$ acts on $\Delta_1^2(u)$ as it acts on the symplecta incident to u and for every $u' \in \Sigma_2(u)$ there is a unique symplecton incident to both u and u', we conclude that for every symplecton of $\mathcal{E}$ incident to u there is exactly one point in $\Delta_1^2(u)$ which is also incident to this symplecton. Consider the action of $F(u)$ on $\Sigma_2(u)$. Clearly $\langle u \rangle$ is the kernel of the action and since the stabilizer in $U_6(2)$ of a point from Π (isomorphic to $2^9.L_3(4)$) does not contain subgroups of index 2, $F(u)$ has two orbits, say $\Sigma_2^1(u)$ and $\Sigma_2^2(u)$ on $\Sigma_2(u)$. Every line of $\mathcal{E}$ incident to u is also incident to one point from $\Sigma_2^1(u)$ and to one point from $\Sigma_2^2(u)$.

Let $w \in \Delta_1^2(u)$ and let $\{u, x_1, x_2\}$ be a line incident to the symplecton containing the pair $\{u, w\}$, where $x_i \in \Sigma_2^i(u)$, $i = 1, 2$. Then, in the collinearity graph of $\mathcal{E}$, the point w is incident to exactly one of x_1 and x_2, say to x_1. Now $(F(u) \cap F(x_1))L/L$ is the stabilizer in U of a point in Π and hence it acts (doubly) transitively on the 21 planes of Π incident to this point. Hence in the collinearity graph of $\mathcal{E}$, the vertex x_1 is adjacent to exactly 22 vertices from Δ_1 while x_2 is adjacent to only one such vertex, namely to u. Let Δ_2 and Δ_3 be the orbits of F on Δ which contain $\Sigma_2^1(u)$ and $\Sigma_2^2(u)$, respectively. Then the above arguments show that $F(x_2) \cong 2.2^9.L_3(4)$ is contained in $F(u)$ while $F(x_1)$ contains $F(x_2)$ as a subgroup of index 22 and it is straightforward to see that $F(x_1) \cong 2^{10}.M_{22}$ (a conjugate of H). Finally, comparison of the characters $\mathbf{1}_{E(u)}^E$ and $\mathbf{1}_F^E$ shows that F has four orbits on Δ. Hence Δ_4 defined as $\Delta - \Delta_1 - \Delta_2 - \Delta_3$ is an F-orbit of length 694 980 and if follows from Lemma 4.1 that for $v \in \Delta_4$ we have $F(v) \cong 2^6 : Sp_6(2)$ (a conjugate of H_5). Q.E.D.

Let X be the set of cosets of F in E and let x be the coset fixed by F (that is F itself). For a subset Y of X let $E(Y)$ denote the elementwise stabilizer of Y in E (we write $E(a, b, \dots)$ instead of $E(\{a, b, \dots\})$). By Lemma 3.10 the character $\mathbf{1}_F^E$ has 8 irreducible components, so F has 7 orbits on $X - \{x\}$. We will use Lemma 5.1 to identify some of them.

Lemma 5.2. *Let M be a maximal subgroups in $F = E(x)$ conjugate to H or H_5. Then there is an element $z \in X - \{x\}$ such that $E(x, z) = M$. Moreover, the setwise stabilizer of $\{x, z\}$ in E is $M \times \langle \tau \rangle$ where τ is a central involution in E.*

Proof. By Lemma 5.1 there is a unique involution $\tau \in \Delta - F$ which commutes with M. This implies that $E(x, z) \geq M$ for $z = x^\tau$. Since M is maximal in F and F does not fix cosets in X other than x, we obtain $E(x, z) = H$. Q.E.D.

Notice that the orbit Δ_3 of F on Δ can not be used in similar way to produce a new F-orbit on X. In fact, an involution from Δ_3 (say x_2 as in Lemma 5.1) conjugates the coset x into the orbit with the stabilizer $2^{10} : M_{22}$. Indeed, $x_2 = ux_1$ and since u is contained in F, it stabilizes x.

The above lemma shows that H and H_5 are 2-point stabilizers in the action of E on X. From the properties of E as a Lie type group we can deduce another 2-point stabilizer.

Lemma 5.3. *Let $H_8 \cong {}^2F_4(2)'$ be a maximal subgroup in $F \cong E(x)$, isomorphic to the Tits group. Then H_8 stabilizes a vertex from $X - \{x\}$.*

Proof. Let V be the natural 27-dimensional $GF(4)$-module for E. It follows from [JLPW] or Proposition 5.4.12 in [KL], that V, restricted to H_8, has two composition factors V_1 and V_2 with dimensions 1 and 26, respectively. Substituting V by its dual, if necessary, we assume that H_8 fixes a 1-dimensional subspace V_1 in V. Comparing the ordinary and modular character tables of H_8, we conclude that V_2 is the reduction modulo 2 of a real, irreducible 26-dimensional representations of H_8. Let $N \cong 13.3$ be the normalizer of a Sylow 13-subgroup in H_8. Computing the inner product of the principal character of N and its character on V_2 (or, rather, on the real version of V_2) we obtain 0. Hence N fixes in V no 1-dimensional subspaces besides V_1. On the other hand a maximal subgroup $F_4(2)$ in E fixes a 1-dimensional subspace in V and contains

N. Hence H_8 is contained in the full stabilizer of V_1 in E, isomorphic to $F_4(2)$. The list of maximal subgroups in $F_4(2)$ obtained in [NW] shows that the normalizer in $F_4(2)$ of every subgroup ${}^2F_4(2)'$ is isomorphic to ${}^2F_4(2)$. This means that $N_E(H_8)$ contains H_8 properly and the result follows. Q.E.D.

§6. Fixed points subgraphs

Let τ be the unique involution from Δ which commutes with H (compare Lemma 5.2) and let $y = x^\tau$. Let Γ be the graph on X with the edge set $\{(x,y)^g \mid g \in E\}$. For $z \in X$ let $\Gamma(z)$ denote the set of vertices of Γ adjacent to z. Then $F = E(x)$ acts on $\Gamma(x)$ as on the cosets of $H = E(x,y)$. For an ordered pair (a,b) of adjacent vertices in Γ let $\Gamma(a,b;n)$ denote the orbit of length n of $E(a,b) \cong 2^{10} : M_{22}$ on $\Gamma(a)$. The possible values of n are listed in Lemma 4.2.

Let $\Gamma_2(x)$ be the orbit of F on $X - \{x\}$ which contains the image of x under the unique involution from Δ which commutes with H_5 (compare Lemma 5.2). Then the action of F on $\Gamma_2(x)$ is similar to its action on the cosets of H_5.

We are going to determine the structure of subgraphs in Γ induced by vertices fixed by certain (prime order) elements $d \in E$. Since we are interested in non-trivial subgraphs, we only take d from $E(x,y)$. This leaves us with a number of possibilities, among which we find the classes $11A$, $7B$, $5A$, $3B$ and $2A$ of E. Let cl be the conjugacy class of E containing d; let $D = \langle d \rangle$ be the cyclic subgroup generated by d; $M(cl) = N_E(D)$ and $\Phi(cl)$ be the set of elements from X fixed by d. Finally, let $\Gamma(cl)$ be the subgraph of Γ induced by $\Phi(cl)$. Notice that the size of $\Phi(cl)$ (i.e., the number of cosets from X fixed by d) is equal to the value of $\mathbf{1}_F^E$ on d; so we can use the values from the table in Section 3. This table shows that in each of the five cases we consider, $M(cl)$ acts transitively on $\Phi(cl)$ since only one conjugacy class of F fuses to cl. In addition, if $cl = 11A$, $7B$ or $5A$, then D is a Sylow subgroup in F and hence $N_F(D)$ acts transitively on $\Phi(cl) \cap \Theta$ for every orbit Θ of F on X.

Lemma 6.1. $\Gamma(11A)$ *is the complete graph on* 3 *vertices and* $M(11A)$ *induces on it the group* S_3.

Proof. The value of $\mathbf{1}_F^E$ on elements of type $11A$ is 3, hence $|\Phi(11A)| = 3$. Since $|\Gamma(x)| = 2 \bmod 11$, we have $\Phi(11A) \subseteq \{x\} \cup \Gamma(x)$. By the paragraph before the lemma $M(11A)$ is transitive on $\Phi(11A)$ and $N_F(D)$ is transitive on $\Phi(11A) \cap \Gamma(x)$, so the result follows. Q.E.D.

By our choice of d we have $x, y \in \Gamma(11A)$. Since $\Gamma(x, y; 1024)$ is an orbit of $E(x, y)$ on $\Gamma(x) - \{y\}$ whose length is not divisible by 11, the third vertex of $\Gamma(11A)$ must be in this orbit and we have the following.

Corollary 6.2. $\Gamma(x, y; 1024) \subseteq \Gamma(y)$.

Lemma 6.3. *$M(7B)$ induces on $\Phi(7B)$ a primitive action of degree 28 which is similar to the action of $PGL_2(7)$ on the cosets of $2 \times S_3$; the action has rank 5 with subdegrees 1, 3, 6 (twice) and 12. The intersection matrix of $\Gamma(7B)$ is the following.*

$$\begin{array}{cccccc}
0 & 1 & 0 & 0 & 0 & 1 \\
6 & 2 & 2 & 1 & 0 & 6 \\
0 & 1 & 0 & 1 & 0 & 3 \\
0 & 2 & 4 & 2 & 4 & 12 \\
0 & 0 & 0 & 2 & 2 & 6
\end{array}$$

Proof. The value of $\mathbf{1}_F^E$ on elements of type $7B$ is 28. By the paragraph before Lemma 6.1 $M(7B)$ induces on $\Phi(7B)$ a transitive action (of degree 28) and $N_F(D)$ acts transitively on $\Phi(7B) \cap \Theta$ for every orbit Θ of F on X. From the order we observe that $N_F(D)$ is contained in a maximal subgroup $H_{11} \cong S_{10}$ of F and hence $N_F(D) \cong F_{42} \times S_3$ and $C_F(D) \cong 7 \times S_3$. From the permutation character $\mathbf{1}_H^F$ given in Lemma 4.2, we get $|\Phi(7B) \cap \Gamma(x)| = 6$. Since $\bar{H} \cong M_{22}$ acting on the non-trivial elements of $O_2(H)$ has three orbits with lengths 22, 231 and 770, one can see (compare [Atlas]) that $N_H(D) \cong F_{21} \times 2$. Comparing the structures of $N_F(D)$, $C_F(D)$ and $N_H(D)$, we conclude: (a) if K is the elementwise stabilizer in $M(7B)$ of the connected component of $\Gamma(7B)$ containing x, then $K \cong F_{21}$; (b) $N_F(D)/K \cong 2 \times S_3$; (c) $C_F(D)$ acting on $\Phi(7B) \cap \Gamma(x)$ has two orbits of length 3 each; (d) there is a subgroup T of order 3 in $N_F(D) - K$ which commutes with K. Let S be the setwise stabilizer in $M(7B)$ of the connected component of $\Gamma(7B)$ which contains x and let $\hat{S} = S/K$. The connected component has 7, 14 or 28 vertices and hence $|\hat{S}| = 2^a \cdot 3 \cdot 7$ for $a = 2$, 3 or 4, respectively. By Sylow theorem $\hat{S}$ contains a subgroup of order 21. Suppose this subgroup is cyclic. Then by (d) F contains a subgroup T of order 3 such that $N_E(T)$ has order divisible by 49. Since there are no such subgroups, $\hat{S}$ contains F_{21}. Since $\hat{S}$ also contains $N_F(D)/K \cong 2 \times S_3$, it is easy to show that $\hat{S}$ is non-solvable. The non-abelian composition factor of $\hat{S}$ must be $PSL_2(7)$ and since $N_F(D)/K \cong 2 \times S_3$ contains no normal subgroups of $\hat{S}$, we obtain $\hat{S} \cong PGL_2(7)$. This means that $M(7B)$ acts on $\Phi(7B)$ as $PGL_2(7)$ acts on the antiflags of the projective plane of order 2. There are two orbitals of valency 6 with respect to this action and $\Gamma(7B)$ is characterized by the property that it splits under the restriction to $PSL_2(7)$. Q.E.D.

Lemma 6.4. *$M(5A)$ induces on $\Phi(5A)$ an action of degree 168 which is similar to the action of S_8 on the cosets of $S_5 \times 2$ (having two orbits with lengths 2 and 6 in the natural action of S_8); the action has rank 6 with subdegrees 1, 5, 12, 30 and 60 (twice). The intersection matrix corresponding to $\Gamma(5A)$ is as given below. The 2-point stabilizers of the action (ordered in accordance with the rows of the intersection matrix) are $S_5 \times 2$, $D_8 \times 2$, Z_4, $S_4 \times 2$, 2^2 and F_{20}.*

$$\begin{array}{rrrrrrr}
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
30 & 8 & 8 & 6 & 2 & 0 & 30 \\
0 & 16 & 9 & 0 & 12 & 5 & 60 \\
0 & 1 & 0 & 0 & 2 & 0 & 5 \\
0 & 4 & 12 & 24 & 10 & 20 & 60 \\
0 & 0 & 1 & 0 & 4 & 5 & 12
\end{array}$$

Proof. We know from [Atlas] the orders of the normalizers of D in F and E. This tells us that $N_F(D) < S_{10} < F$ and $N_E(D) < (A_5 \times A_8).2 < O_{10}^-(2) < {}^2E_6(2)$. In particular $N_F(D) \cong F_{20} \times S_5$. On the other hand we can check that $N_F(D)$ induces on $\Phi(5A) \cap \Gamma(x)$ an action of $S_5 \times 2$ of degree 30. Hence the kernel of $N_E(D)$ acting on $\Phi(5A)$ is D_{10} and the action must be isomorphic to S_8. There are two conjugacy classes of subgroups in S_8 isomorphic to $S_5 \times 2$. Let R_1 and R_2 be their representatives such that R_1 has two orbits with lengths 2 and 6 on the 8-element set Y naturally permuted by S_8 and R_2 has three orbits on Y with lengths 1, 2 and 5. Then the action of S_8 on the cosets of R_1 preserves an imprimitivity system with blocks of size 6 and the action on the blocks is similar to the action on 2-element subsets of Y with subdegrees 1, 12 and 15. The action on the cosets of R_2 preserves an imprimitivity system with blocks of size three and the action on the blocks is similar to the action on 3-element subsets of Y with subdegrees 1, 10, 15 and 30. Let $z \in X - \{x\}$ be a point stabilized by the subgroup $H_8 \cong {}^2F_4(2)'$ in $F = E(x)$ (cf. Lemma 5.3). Then F acts on the suborbit $\{z^f \mid f \in F\}$ as it acts on the cosets of H_8. The permutation character of this action is given in [ILLSS] and it has value 12 on $5A$-elements. This means that 12 is a subdegree of $M(5A)$ acting on $\Phi(5A)$ and from above we conclude that the action is of S_8 on the cosets of R_1. Using the information about the action on the imprimitivity blocks it is easy to calculate the intersection matrix and 2-point stabilizers. Q.E.D.

0	1	0	0	0	0	0	0	1
243	26	18	14	3	2	0	0	243
0	24	9	6	9	6	0	0	324
0	112	36	43	42	46	162	0	1944
0	48	108	84	81	84	0	108	3888
0	32	72	92	84	80	81	108	3888
0	0	0	4	0	1	0	0	48
0	0	0	0	24	24	0	27	864

Lemma 6.5. *$M(3B)$ induces on $\Phi(3B)$ a primitive action of $O_8^+(2).S_3$ of degree 11 200; the action has rank 8 with subdegrees 1, 48, 243, 324, 864, 1 944 and 3 888 (twice). The intersection matrix of $\Gamma(3B)$ is as given above.*

Proof. All elements of order 3 in $E(x,y)$ are conjugate and the character $\mathbf{1}_H^F$ (cf. Lemma 4.2) shows that such an element d belongs to the class $3c$ in F and hence to the class $3B$ in E. It is easy to calculate that d fixes 243 points in $\Gamma(x)$ and $N_F(D)$ acts transitively on these points. The intersection matrices of $O_8^+(2) : S_3 \cong N_E(D)/D$ in its primitive action of degree 11 200 were computed by L. H. Soicher. There is only one orbital of valency 243 and we identify it with $\Gamma(3B)$. Q.E.D.

Lemma 6.6. *$M(2A)$ induces on $\Phi(2A)$ a primitive action of degree 1 048 576, similar to the action of $2^{20} : U_6(2)$ on the cosets of $U_6(2)$; the action has rank 6 with subdegrees 1, 891, 24 948, 228 096, 295 680 and 498 960. The intersection matrix of $\Gamma(2A)$ and the character table of the corresponding centralizer algebra are as given below.*

0	1	0	0	0	0	1
891	42	12	2	0	0	891
0	336	15	42	24	0	24948
0	512	384	147	192	216	228096
0	0	480	420	419	432	498960
0	0	0	280	256	243	295680

1	891	24948	228096	498960	295680	1
1	−133	3444	−9472	−2800	8960	891
1	123	628	768	−240	−1280	24948
1	−37	84	512	−560	0	228096
1	27	−108	0	−432	512	295680
1	−5	−12	−256	528	−256	498960

Proof. A transposition from F is a central involution in E. By Lemma 4.3 $\Gamma(2A)$ has valency 891, so the corresponding orbital is uniquely determined. Q.E.D.

The set $\Phi(2A) \cap \Gamma(x)$ coincides with the set $\Omega(\alpha)$ from Lemma 4.3. The entry 42 in the second row and second column of the intersection matrix of $\Gamma(2A)$ shows that y is adjacent to 42 vertices from $\Phi(2A) \cap \Gamma(x)$. By Lemma 4.3 these 42 vertices are contained in Ω_1 which is the same as $\Gamma(x, y; 154)$ and we have the following.

Corollary 6.7. $\Gamma(x, y; 154) \subseteq \Gamma(y)$.

Let us look more closely at the subgraph $\Gamma(5A)$.

Lemma 6.8. *Let d be an element of type $5A$ from $E(x, y)$. Then the number of vertices fixed by d in $\Gamma(x, y; n)$ is 1 for $n = 1$ and 3696; 4 for $n = 154$, 1 024 and 11 264; 8 for $n = 4\,928$ and 78 848.*

Proof. By Lemma 6.4 and its proof, $N_{E(x,y)}(D)$ induces on $\Phi(5A) \cap \Gamma(x)$ an action of degree 30 and order 8. By the Frattini argument $N_{E(x,y)}(D)$ is transitive on $\Phi(5A) \cap \Gamma(x, y; n)$ for every n and the result follows from elementary congruences. Q.E.D.

As above let d be an element of type $5A$ in $E(x, y)$ and $D = \langle d \rangle$. Because of the obvious symmetry between x and y, Lemma 6.8 describes the orbits of $N_{E(x,y)}(D)$ on $\Phi(5A) \cap \Gamma(y)$ and we can locate them in $\Gamma(y, x; n)$ for suitable values of n. The intersection matrix of $\Gamma(5A)$, given in Lemma 6.4 shows that y is adjacent to 8 vertices from $\Phi(5A) \cap \Gamma(x)$; we have 4 of them in $\Gamma(y, x; 154) = \Gamma(x, y; 154)$ (compare Corollary 6.7) and 4 in $\Gamma(y, x; 1024) = \Gamma(x, y; 1024)$ (compare Corollary 6.2). From the intersection matrix of $\Gamma(5A)$ we see that y is adjacent to a single vertex in the orbit of length 5 of $N_F(D)$ on $\Phi(5A)$. By Lemma 6.8 this vertex is in $\Gamma(y, x; 3696)$ (since it can not be $\{x\}$). We observed in Lemma 4.4 that $\Phi(5A) \cap \Gamma_2(x)$ has size 5 and clearly $N_F(D)$ acts on it transitively. This gives the following.

Corollary 6.9. $\Gamma(y, x; 3696) \subseteq \Gamma_2(x)$.

Let 60_1 and 60_2 be the orbits of length 60 of $N_F(D)$ on $\Phi(5A)$. We assume that y is adjacent to 4 vertices from 60_1. Then these 4 vertices form an orbit of $N_{E(x,y)}(D)$ and are contained in $\Gamma(y, x; 11264)$. There are 16 vertices in 60_2 adjacent to y. By what we already know, Lemma 6.8 imply that these 16 vertices split under the action of $N_{E(x,y)}(D)$ into two orbits of length 8 each and these two orbits are contained in $\Gamma(y, x; 4928)$ and in $\Gamma(y, x; 78848)$. This gives us the following important conclusion.

Lemma 6.10. *$\Gamma(y,x;4928)$ and $\Gamma(y,x;78848)$ are in the same orbit of $F = E(x)$ on X.*

Notice that if $z \in 60_2$ then the stabilizer of z in $N_F(D)$ acting on $\Gamma(x) \cap \Gamma(z) \cap \Gamma(5A)$ has two orbits of length 4 each.

§7. The eigenspace V_1

The information on the local structure of Γ established in the previous section turns out to be sufficient to calculate a non-trivial eigenvalue of the intersection matrix of Γ (without knowing the matrix itself).

Let $\Gamma_0, \Gamma_1, \dots, \Gamma_7$ be the orbitals of E acting on X, so that Γ_0 is the diagonal orbital and $\Gamma_1 = \Gamma$. The valency of Γ_i will be denoted by k_i and we will write k instead of k_1. Let V be the space of complex valued functions defined on X and let $V = V_0 \oplus V_1 \oplus \cdots \oplus V_7$ be the decomposition of V into minimal E-invariant subspaces. We denote by m_i the dimension of V_i and assume that $m_i > m_j$ for $i > j$. The values of m_j can be read from Lemma 3.1. The subspace V_0 supports the trivial representation of E and V_1 is 1938-dimensional. For $a \in X$ let $v(a)$ be the characteristic function of a, $v_1(a)$ be the projection of $v(a)$ into V_1 and $w_1(a) = (n/m_1)v_1(a)$ where $n = |X|$. Then for $(a,b) \in \Gamma_i$ the inner product $\langle w_1(a) \mid w_1(b)\rangle$ equals to $p_i(1)/k_i$ and $(p_0(1)/k_0, p_1(1)/k_1, \dots, p_7(1)/k_7)$ is a left eigenvector of the intersection matrix of Γ. We emphasize that the $w_1(a)$ are unit vectors.

We are going to write down certain expressions for the inner products $\langle w_1(y) \mid w_1(z)\rangle$ for various vertices $y, z \in \Gamma(x)$ but before doing so we introduce some notation concerning the action of $F = E(x)$ on $\Gamma(x)$.

Let Ω be the set of cosets of $H \cong 2^{10} : M_{22}$ in F, which can be identified with $\Gamma(x)$. Let $\Omega_0, \dots, \Omega_7$ be the orbits of F on Ω as in Lemma 4.2, in particular $\Omega_0 = \{y\}$. Let U denote the space of complex valued functions on Ω and $U = U_0 \oplus U_1 \oplus \cdots \oplus U_7$ be its decomposition into minimal F-invariant subspaces. The dimension of U_i will be denoted by d_i and we assume that $d_i > d_j$ for $i > j$. The values of d_i one can get from Lemma 4.2. For $z \in \Omega$ let $u(z)$ be the characteristic function of z, $u_j(z)$ be its projection into U_j and $t_j(z) = (k/d_j)u_j(z)$ (where $k = |\Omega|$). For $z \in \Omega_i$ put $\pi_i(j) = \langle t_j(y) \mid t_j(z)\rangle$. These numbers can be calculated from the character table corresponding to the action of F on the cosets of H and given after Lemma 4.3.

The following result can be checked by straightforward calculations using [Atlas].

Lemma 7.1. *The* 1938*-dimensional* E*-irreducible module* V_1 *when restricted to* F *decomposes as* $U_0 \oplus U_1 \oplus U_2 \oplus U_3$ *where the irreducibles* U_j, $j = 0, 1, 2$ *and* 3 *are those involved in the permutation character* $\mathbf{1}_H^F$.

Let us consider V_1 as a module for $F = E(x)$, possessing the decomposition into F-irreducibles from Lemma 7.1. Notice that the U_j are pairwise orthogonal with respect to an inner product on V_1 preserved by E and that the restriction of this inner product to U_j is the unique one (up to a scalar) preserved by F.

It is clear that $w_1(x) \in U_0$. Let us locate the vectors $w_1(z)$ for $z \in \Gamma(x)$ in the above decomposition of V_1. Since U_j appears in the permutation character of F acting on the cosets of $F(z)$ with multiplicity 1, we conclude that the vectors in U_j fixed by $F(z)$ form a 1-dimensional subspace. Since both $t_j(z)$ and the projection of $w_1(z)$ to U_j are fixed by $F(z)$, they must differ by a scalar multiple and hence

$$w_1(z) = \sum_{j=0}^{3} t_j(z) \cdot \alpha_j,$$

for some scalars α_j, which are independent of the choice of $z \in \Gamma(x)$. Since the decomposition of V_1 we are dealing with is orthogonal, for $z \in \Omega_i$ we have

$$\langle w_1(y) \mid w_1(z) \rangle = \sum_{j=0}^{3} \langle t_j(y) \mid t_j(z) \rangle \cdot \alpha_j^2 = \sum_{j=0}^{3} \pi_j(i) \cdot \alpha_j^2.$$

In addition the vector $t_0(z)$ is a unit vector in U_0 independent of the choice of $z \in \Gamma(x)$, so we can assume that they all coincide with $w_1(x)$. In this case $\langle w_1(x) \mid w_1(z) \rangle = \alpha_0$ for every $z \in \Gamma(x)$.

Thus we have four unknowns α_j, $0 \leq j \leq 3$ which determine the inner products $\langle w_1(y) \mid w_1(z) \rangle$ (depending on i such that $z \in \Omega_i$). Now we are going to turn the structural results on Γ proved in the previous section into equations on the α_j.

Let $z_0, \ldots, z_7 \in \Gamma(x)$ be such that $z_i \in \Omega_i$. In these terms Lemmas 6.2 and 6.7 mean that $z_1, z_2 \in \Gamma(y)$. Hence $\langle w_1(y) \mid w_1(z_1) \rangle = \langle w_1(y) \mid w_1(z_2) \rangle = \langle w_1(y) \mid w_1(x) \rangle = \alpha_0$ and we come to the following two equations:

$$\sum_{j=0}^{3} \pi_j(1) \cdot \alpha_j^2 = \alpha_0, \tag{7.1}$$

$$\sum_{j=0}^{3} \pi_j(2) \cdot \alpha_j^2 = \alpha_0. \tag{7.2}$$

Lemma 6.10 means that (y, z_4) and (y, z_7) are in the same E-orbit. Hence $\langle w_1(y) \mid w_1(z_4)\rangle = \langle w_1(y) \mid w_1(z_7)\rangle$ and we have

$$\sum_{j=0}^{3} (\pi_j(4) - \pi_j(7)) \cdot \alpha_j^2 = 0. \tag{7.3}$$

Finally $w_1(y)$ is a unit vector and hence

$$\sum_{j=0}^{3} \alpha_j^2 = 1. \tag{7.4}$$

Thus we have obtained a system of four equations in four unknowns which turns out to have a unique meaningful solution. Let us substitute in the equations the values of $\pi_j(i)$ computed from the character table given after Lemma 4.3.

$$\alpha_0^2 - (1/2)\alpha_1^2 + (7/22)\alpha_2^2 - (5/22)\alpha_3^2 = \alpha_0, \tag{7.5}$$

$$\alpha_0^2 - (5/16)\alpha_1^2 - (11/64)\alpha_2^2 + (5/32)\alpha_3^2 = \alpha_0, \tag{7.6}$$

$$(3/16)\alpha_1^2 - (75/704)\alpha_2^2 - (27/352)\alpha_3^2 = 0, \tag{7.7}$$

$$\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1. \tag{7.8}$$

Comparing the left sides of (7.5) and (7.6), and using (7.7) and (7.8), we obtain a system of three linear equations in four unknowns α_j^2, $0 \le j \le 3$, from which we deduce the following expressions:

$$\alpha_1^2 = (1 - \alpha_0^2)(60/181), \tag{7.9}$$

$$\alpha_2^2 = (1 - \alpha_0^2)(66/181), \tag{7.10}$$

$$\alpha_3^2 = (1 - \alpha_0^2)(55/181). \tag{7.11}$$

Substituting these expressions in (7.5) we obtain a quadratic equation on α_0. There is an obvious and meaningless solution of this equation, namely $\alpha_0 = 1$ (which imply $w_1(z) = w_1(x)$ for all $z \in \Gamma(x)$, definitely not the case). The second solution is $\alpha_0 = -43/405$.

As soon as we know α_0, the expressions (7.9)–(7.11) give us α_1^2, α_2^2 and α_3^2 and using the values $\pi_j(i)$ computed from the character table of F acting on $\Gamma(x)$, we determine the inner products $\langle w_1(y) \mid w_1(z_i)\rangle$ for all $0 \leq i \leq 7$.

Lemma 7.2. *Let $z_i \in \Omega_i$, $0 \leq i \leq 7$. Then the inner product $\langle w_1(y) \mid w_1(z_i)\rangle$ equals to* 1, $-43/405$, $-43/405$, $69/405$, $13/405$, $-1/405$, $-15/405$ *and* $13/405$, *respectively.*

§8. The subdegrees

In this section we determine the subdegrees and 2-point stabilizers of E acting on X. We follow the notation introduced above.

By Lemma 4.2 there are 7 orbits of E on the paths of length 2 in Γ with representatives (y, x, z_i) for $1 \leq i \leq 7$. By Corollaries 6.2 and 6.7 z_1 and z_2 are adjacent to y. By Lemma 6.10 z_4 and z_7 are in the same $E(y)$-orbit. On the other hand by Lemma 7.2 for $i = 3$, 4, 5 and 6 the inner products $\langle w_1(y) \mid w_1(z_i)\rangle$ are pairwise different and none of then is equal to the inner product $-43/405$ of vectors representing adjacent vertices. So we have the following.

Lemma 8.1. *The subgroup $F = E(x)$ has four orbits on vertices at distance* 2 *from x in Γ.*

Let $\Gamma_0(x) = \{x\}$, $\Gamma_1(x) = \Gamma(x)$ and let $\Gamma_2(x)$, $\Gamma_3(x)$, $\Gamma_4(x)$ and $\Gamma_5(x)$ be the orbits of F on vertices at distance 2 from x in Γ. We will assume that $\Gamma(y, x; 3696) \subseteq \Gamma_2(x)$; $\Gamma(y, x; 4928) \cup \Gamma(y, x; 78848) \subseteq \Gamma_3(x)$; $\Gamma(y, x; 42240) \subseteq \Gamma_4(x)$ and $\Gamma(y, x; 11264) \subseteq \Gamma_5(x)$. Then by Corollary 6.9 $\Gamma_2(x)$ is as defined in Section 6, in particular $|\Gamma_2(x)| = 694\,980$. Since the rank of E on X is 8, there are 2 orbits of F on the set of vertices at distance more than 2 from x in Γ. We denote these orbits by $\Gamma_6(x)$ and $\Gamma_7(x)$.

Let us choose a family of representatives $y_i \in \Gamma_i(x)$, $0 \leq i \leq 7$. We are going to introduce for every i from 0 to 7 a subgroup K_i in $E(x, y_i)$. In some cases it will be clear from the very beginning that $K_i = E(x, y_i)$. For the remaining cases this equality will come at the end because of the equality

$$[E : E(x)] = \sum_{i=0}^{7} [E(x) : K_i]. \tag{8.1}$$

Clearly, we take $K_0 = E(x) \cong Fi_{22}$. Because of Lemma 5.2 we can take $K_1 = E(x, y_1) \cong 2^{10} : M_{22}$ and $K_2 = E(x, y_2) \cong 2^6 : Sp_6(2)$.

Consider $E(x, y_3)$. We assume that $y_3 \in \Gamma(y, x; 4928)$ and hence $E(x, y, y_3) \cong 2^4.2^4.A_6$ (compare Lemma 4.2). Let d be an element of type $5A$ from $E(x, y, y_3)$ and $D = \langle d \rangle$. This means that $x, y, y_3 \in \Gamma(5A)$. Let Θ be the orbit of y under $E(x, y_3)$. Clearly $|E(x, y_3)| = |\Theta| \times |E(x, y, y_3)|$. By the remark after Lemma 6.10, the intersection Θ' of Θ and $\Gamma(5A)$ is of size 4 and $N_{E(x,y_3)}(D)$ acts transitively on Θ'. We define K_3 to be the setwise stabilizer of Θ' in $E(x, y_3)$. We can write that $K_3 \cong 2^4.2^4.A_6.2.2$. The precise structure of K_3 will be established later.

Let us turn to $E(x, y_4)$ and start by calculating the normalizer in E of $Q = O_2(E(x, y_2)) \cong 2^6$.

Lemma 8.2. *Let $P = N_E(Q)$. Then $P/Q \cong [2^9].Sp_6(2)$.*

Proof. We claim that there is a unique central involution τ in E such that $Q \leq O_2(C_E(\tau))$ (the uniqueness will immediately imply that $P \leq C_E(\tau)$). Consider the action of F on the set of central involutions in E. We will follow the notation of Lemma 5.1. Let F_i denote the centralizer in F of $\tau_i \in \Delta_i$ for $1 \leq i \leq 4$. Suppose that $Q \leq O_2(F_i)$. Certainly $i \neq 1$. Since $O_2(F_2)$ and $O_2(F_3)$ are abelian and Q is self-centralized in F, $i \neq 2, 3$. Of course we can make $Q = O_2(F_4)$, but then τ_4 becomes uniquely determined since $F_4 = N_F(Q)$. Thus $Q \leq C_E(\tau_4)$. Since $U_6(2) \cong C_E(\tau_4)/O_2(C_E(\tau_4))$ does not contain a 2-local subgroup with a section $Sp_6(2)$, we conclude that $Q \leq O_2(C_E(\tau_4))$ and the claim follows.

Let S be the image of P in Out $Q \cong L_6(2)$. Then S contains the image of $N_F(Q)$ isomorphic to $Sp_6(2)$. Since $Sp_6(2)$ is maximal in $L_6(2)$ and the latter is not involved in $C_E(\tau)$, $S \cong Sp_6(2)$. So we only have to consider $D = C_E(Q)$. Since every subgroup of $U_6(2)$ having $Sp_6(2)$ as a factor group is $Sp_6(2)$ itself, $D \leq O_2(C_E(\tau))$ and lemma follows from the basic properties of extraspecial groups. Q.E.D.

Corollary 8.3. *Let Σ be the orbit of x under $N_E(Q)$. Then $|\Sigma| =$* 512.

Clearly Q fixes Σ elementwise. Let us locate some elements of Σ. The setwise stabilizer of $\{x, y_2\}$ clearly normalizes Q and hence $y_2 \in \Sigma$. By Lemma 4.5 $E(x, y_2) = H_5$ has an orbit Ψ of length 135 on $\Gamma(x)$. Since $Q = O_2(E(x, y_2))$, Q fixes Ψ elementwise. Without loss of generality we assume that $y \in \Psi$, which means that $Q \leq E(x, y)$. Let τ be the central involution in E, such that $\langle \tau \rangle \times E(x, y)$ is the setwise stabilizer of $\{x, y\}$.

Then τ normalizes (even commutes with) Q which means that $\Psi \subseteq \Sigma$. The group $E(x, y_2)$ acts on Ψ as $Sp_6(2) \cong E(x, y_2)/Q$ acts on the set of points of the symplectic dual polar space. In particular $E(x, y, y_2) \cong 2^6.2^6.L_3(2)$ acting on Ψ has 4 orbits Ψ_0, Ψ_1, Ψ_2 and Ψ_3 with lengths 1, 14, 56 and 64, respectively. Now one can see (compare Lemma 4.2) that $\Psi_1 \subseteq \Gamma(x, y; 154)$, $\Psi_2 \subseteq \Gamma(x, y; 3696)$ and $\Psi_3 \subseteq \Gamma(x, y; 42240)$. This means that for $i = 2$ and 4 the set $\Sigma \cap \Gamma_i(x)$ contains more than one vertex. By Lemma 4.4 and Corollary 8.3 we have $|\Sigma \cap \Gamma_2(x)| \geq 136$ and hence $|\Sigma \cap \Gamma_4(x)| \leq 240$. Assuming that y_4 is in $\Sigma \cap \Gamma_4(x)$, we define K_4 to be the normalizer of Q in $E(x, y_4)$. By the above K_4 is a subgroup in $K_2 = N_{E(x)}(Q) \cong 2^6 : Sp_6(2)$, which contains Q and the index $[K_2 : K_4]$ is at most 240.

By Lemmas 4.2 and 6.6 one can see that the subgroup $E(x, y, y_5) \cong 2.L_3(4)$ contains a unique transposition τ from $E(x)$, and this transposition is in the center of $E(x, y, y_5)$. We define K_5 to be $C_{E(x,y_5)}(\tau)$. It follows from Lemmas 4.2, 6.6 and the intersection matrix of $\Gamma(2A)$ that $K_5 \cong 2.L_3(4).2$.

Now we are going to deduce some information on stabilizers in $E(x)$ of vertices at distance more than 2 from x in Γ. By Lemma 5.3 there is a point $z \in X - \{x\}$ whose stabilizer in $F = E(x)$ is $H_8 \cong {}^2F_4(2)'$. By Lemma 4.2 every pair of vertices from $\Gamma(x)$ is stabilized by an element of type $3c$ from $F = E(x)$. On the other hand the permutation character of F on the cosets of H_8 given in [ILLSS] is zero on elements of type $3c$. Hence z is at distance more than 2 from x in Γ. Without loss of generality we assume that $z = y_6$ and put $K_6 = E(x, y_6) \cong {}^2F_4(2)'$.

Let d be an element of type $3b$ in F, $D = \langle d \rangle$, so that $N_F(D) = H_{10}$. The character $\mathbf{1}_F^E$ shows that d is of type $3C$ in E and $M(3C)$ has two orbits on $\Phi(3C)$, say $\Phi_1(3C)$ and $\Phi_2(3C)$ with lengths 4 and 576, respectively. We assume that $x \in \Phi_1(3C)$. Let S be a Sylow 3-subgroup of F contained in $N_F(D)$. Since $|\Phi(9A)| = 1$, S acts non-trivially on $\Phi_1(3C)$ and hence the action of $M(3C)$ on this set is doubly transitive. Let $u \in \Phi_1(3C) - \{x\}$. Then $K = N_F(D) \cap E(x, u)$ is an index 3 subgroup in H_{10}. The list of maximal subgroups of F shows that every subgroup L, such that $K < L \leq F$ contains a Sylow 3-subgroup of F and since $|\Phi(9A)| = 1$, L can not be a 2-point stabilizer. Hence $E(x, u) = K$ and it is a $\{2, 3\}$-group having index 3 in H_{10}. Notice that for $i \leq 6$ the order of $E(x, y_i)$ is divisible by a prime greater than 3. So we can assume that $u = y_7$ and put $K_7 = K$.

Now by direct calculations we see that $\sum_{i=0}^{7} [E(x) : K_i]$ is less than $[E : F]$ unless K_4 has index exactly 240 in K_2. In the latter case the equality (8.1) holds and we conclude the following.

Lemma 8.4. *K_4 has index 240 in K_2 and $E(x, y_i) = K_i$ for every $0 \leq i \leq 7$.*

So we have calculated the subdegrees of the action of E on X and proceed to determination of the precise structure of the 2-point stabilizers $E(x, y_i)$. For $i = 0$, 1, 2, 5 and 6 everything is clear.

The group $Sp_6(2)$ contains a unique conjugacy class of subgroups of index 240 (cf. [Atlas]) and we obtain the following.

Lemma 8.5. $E(x, y_4) \cong 2^6 : U_3(3)$.

Let us turn to $E(x, y_3)$. Let $\Theta = \Theta(x, y_3) = \{z \mid z \in \Gamma(x), y_3 \in \Gamma(z, x; 4928)\}$. Clearly Θ is the same as defined before Lemma 8.2. By now we have the following information on Θ (cf. Lemmas 4.2 and 6.4).

Lemma 8.6. *The set Θ is of size 4, the normalizer in $E(x, y_3)$ of a Sylow 5-subgroup induces on Θ the regular action of Z_4 and for $z \in \Theta$ we have $E(x, y_3, z) \cong 2^4.2^4.A_6$.*

Clearly $E(x, y_3)$ is a subgroup in the setwise stabilizer of Θ in $E(x)$. Let us specify Θ and its stabilizer in $E(x)$.

Let Ω be the graph of valency 154 on $\Gamma(x)$ in which $z \in \Gamma(x)$ is adjacent to the vertices from $\Gamma(x, z; 154)$. The intersection matrix of Ω is given after Lemma 4.3. Let $z \in \Gamma(x, y; 4928)$. Then $z \in \Gamma_3(y)$ and $x \in \Theta(y, z)$. The intersection matrix of Ω shows that there is a unique vertex w, adjacent to both y and z in Ω. Since $E(y, x, z) = E(y, x, z, w) \leq E(y, z, w)$, it is clear that $w \in \Theta(y, z)$ and we obtain the following.

Lemma 8.7. *The subgraph of Ω induced by Θ is of valency* 1.

Let $\Theta = \{z_1, z_2, z_3, z_4\}$ and assume that $\{z_1, z_2\}$ and $\{z_3, z_4\}$ are edges of Ω. Let z_0 be the unique vertex adjacent in Ω to both z_1 and z_2. Put $L = E(x, z_1, z_2) = E(x, z_0, z_1, z_2)$ and let $\Lambda(z_i)$ be the orbit of z_i under L, $i = 3$ or 4. Since $L \cong 2^9.2^4.A_6$ and $E(x, y_3, z_1) \cong 2^4.2^4.A_6$ is contained in the elementwise stabilizer of Θ, we conclude that the length of $\Lambda(z_i)$ divides 2^5.

Let S be the Steiner system $S(3, 6, 22)$ naturally associated with $E(x, z_1) \cong 2^{10} : M_{22}$. Then $\{z_0, z_2\}$ is an orbit of $O_2(E(x, z_1))$ which corresponds to a block B of S. Let Λ_1 be the union of all orbits of $O_2(E(x, z_1))$ on $\Gamma(x, z_1; 154)$ which correspond to S-blocks disjoint from B. Then Λ_1 is an orbit of L of length 2^5. For $i = 0$ and 2 let Λ_i denote the similar orbit of L in $\Gamma(x, z_i; 154)$. Now from Lemma 4.2 and basic properties of the Steiner system S it is not difficult to show the following.

Lemma 8.8. *Every orbit of L on $\Gamma(x) - \{z_0, z_1, z_2\}$ whose length divides 2^5 coincides with Λ_i for $i = 0$, 1 or 2.*

The intersection matrix of Ω shows that Λ_i and Λ_j are disjoint for $i \neq j$. Hence $\Lambda(z_3) = \Lambda(z_4) = \Lambda_i$ for some $0 \leq i \leq 2$. Since Θ contains only two Ω-edges (cf. Lemma 8.7), i must be equal to 0. Remembering that the setwise stabilizer in M_{22} of a pair of disjoint blocks of S is isomorphic to M_{10}, we obtain the following.

Lemma 8.9. *There is a unique vertex $z_0 \in \Gamma(x)$ such that $\Theta \subseteq \Gamma(x, z_0; 154)$. The pairs $\{z_1, z_2\}$ and $\{z_3, z_4\}$ are orbits of $O_2(E(x, z_0))$ and correspond to disjoint blocks of the Steiner system associated with $E(x, z_0)$. The setwise stabilizer of Θ in $E(x)$ is the full preimage in $E(x, z_0)$ of an M_{10}-subgroup from $\bar{E}(x, z_0) \cong M_{22}$.*

Thus the setwise stabilizer R of Θ in $E(x)$ contains $E(x, y_3)$ as a subgroup of index 2. Since R induces D_8 on Θ while $E(x, y_3)$ induces Z_4, $E(x, y_3)$ is uniquely specified.

Lemma 8.10. *$E(x, y_3) \cong 2^4.2^4.A_6.Z_4 \cong 2^9.M_{10}$ is a subgroup in a conjugate of $E(x, y_1)$.*

In order to identify $E(x, y_7)$ we need the size of the intersection of $\Gamma(3B)$ and $\Gamma_7(x)$. We prove a more general proposition which will be used later.

Lemma 8.11. *Let d be an element of type cl from E contained in $E(x) = F$, $D = \langle d \rangle$ and let $\Gamma(cl)$ be the subgraph of Γ induced by the vertices fixed by d. Then the orbits of $N_{E(x)}(D)$ on $\Gamma(cl)$ (ordered as the rows of the intersection matrix of $\Gamma(cl)$ given in Section 6) are contained in*

(i) $\Gamma_0(x)$, $\Gamma_1(x)$, $\Gamma_5(x)$, $\Gamma_4(x)$ *and* $\Gamma_2(x)$ *for* $cl = 7B$;
(ii) $\Gamma_0(x)$, $\Gamma_1(x)$, $\Gamma_3(x)$, $\Gamma_5(x)$ *and* $\Gamma_6(x)$ *for* $cl = 5A$;
(iii) $\Gamma_0(x)$, $\Gamma_1(x)$, $\Gamma_3(x)$, $\Gamma_4(x)$, $\Gamma_5(x)$, $\Gamma_7(x)$ *and* $\Gamma_7(x)$ *for* $cl = 3B$;
(iv) $\Gamma_0(x)$, $\Gamma_1(x)$, $\Gamma_2(x)$, $\Gamma_5(x)$, $\Gamma_7(x)$ *and* $\Gamma_3(x)$ *for* $cl = 2A$.

Proof. Using the permutation characters of $E(x) = F$ on the cosets of $H = E(x, y_1)$, $H_5 = H(x, y_2)$, $H_8 = H(x, y_6)$ (cf. [ILLSS]) and the embeddings $E(x, y_3) < E(x, y_1)$, $E(x, y_4) < E(x, y_2)$, we calculate the number of points in $\Gamma_i(x)$ fixed by an element of type cl contained in F for $i \leq 6$. In the case (iii) there are two orbits of length 3 888. Since there are two paths of length 2 joining x and y_5, we have a unique way to locate these orbits and the result follows. Q.E.D.

Lemma 8.12. *$E(x, y_7)$ is an index 3 subgroup in the maximal subgroup H_{10} of $F = E(x)$. In the permutation action of F on the cosets of $E(x, y_7)$ an element of type $3C$ fixes 912 points. These two properties characterize $E(x, y_7)$ up to conjugacy in F.*

Proof. The first sentence follows from the definition of $E(x, y_7)$. The second one comes from Lemma 8.11 (iii), so all we have to prove is the last sentence. The GAP system [GAP] contains the character table of H_{10} along with the fusion pattern of the H_{10}-classes into F-classes. There is one non-principal H_{10}-character of degree 1 and four characters of degree 2 (with pairwise different kernels). So we have five candidates for the permutation character of H_{10} of degree 3 (on the cosets of $E(x, y_7)$). Inducing these five characters to F we observe that the one involving irreducibles of degree 1 only, does not lead to a permutation character at all (the induced character involves negative values). Among the other four, only one (denote it by ϕ) when induced gives the character value 912 on $3C$-elements. It was checked that the kernel of ϕ is a normal subgroup of index 6 in H_{10} and the result follows. Q.E.D.

We can summarize this section by the following.

Proposition 8.13. *The 2-point stabilizers of $E \cong {}^2E_6(2)$ acting on the cosets of $F \cong Fi_{22}$ are Fi_{22}, $2^{10} : M_{22}$, $2^6 : Sp_6(2)$, $2^9.M_{10}$, $2^6 : U_3(3)$, $2.L_3(4).2$, ${}^2F_4(2)'$ and $3_+^{1+6} : 2^{3+4} : 3 : 2$.*

§9. The eigenvector

Lemma 7.2 gives six entries $p_i(1)/k_i$, $0 \leq i \leq 5$ in the left eigenvector of the intersection matrix of Γ corresponding to the idempotent of rank 1938. In this section we calculate the missing two entries. We will use for this purpose the character table of the centralizer algebra corresponding to the action of $M(2A)$ on $\Phi(2A)$ given after Lemma 6.6.

For the rest of the section we assume that $\Gamma(2A)$ contains y_i for $i = 0$, 1, 2, 3, 5 and 7 (cf. Lemma 8.11 (iv)). Let $C = C_E(d) = M(2A)$.

Lemma 9.1. *The 1938-dimensional E-module V_1 decomposes into C-irreducibles as $W_0 \oplus W_1 \oplus W_2 \oplus W_3$ where the W_j have dimensions 1, 22, 891 and 1024, respectively.*

Proof. Since $Q = O_2(C)$ is extraspecial of order 2^{21}, V_1 restricted to Q must involve the faithful component of dimension 1024 and only one such component fits. Since the central involution d of C is conjugate in E to an involution from $Q - \langle d \rangle$, there must be an irreducible component whose kernel is exactly $\langle d \rangle$. Such a component has dimension equal to the length of an orbit of C on the non-zero vectors of the module (dual to) $Q/\langle d \rangle$. Only one such component of dimension 891 fits. For the remaining components Q is in the kernel and the conclusion comes from the character table of $U_6(2) \cong C/Q$ in [Atlas]. Q.E.D.

As above for $z \in \Gamma$ let $w_1(z)$ be the unit vector in V_1 realizing z. Suppose that $z \in \Gamma(2A)$. The projection of $w_1(z)$ into W_j can be non-zero only if $C \cap E(z) \cong 2 \cdot U_6(2)$ fixes a vector in W_j. Such a vector is fixed if and only if W_j is involved in the permutation character of $C = M(2A)$ acting on the vertices of $\Gamma(2A)$. The degrees of the irreducible constituents in the permutation character can be read from the character table of the corresponding centralizer algebra given after Lemma 6.6 and we observe that only W_0 and W_2 can be involved. On the other hand W_0 supports the principal character and supports W_2 the unique faithful character of degree 891, so both W_0 and W_2 are involved.

For $j = 0$ or 2 and $z \in \Gamma(2A)$ let $s_j(z)$ be the unit vector in W_j realizing z with respect to the centralizer algebra of C acting on $\Gamma(2A)$, clearly $s_j(z)$ is fixed by $E(z) \cap C$. Then

$$w_1(z) = s_0(z) \cdot \beta_0 + s_2(z) \cdot \beta_2 \tag{9.1}$$

for some β_0 and β_2 independent of z and satisfying

$$\beta_0^2 + \beta_2^2 = 1. \tag{9.2}$$

Then Lemmas 7.2, 8.11 (iv) and the character table given after Lemma 6.6 give us the following (we assume that $x, y_1 \in \Gamma(2A)$):

$$-43/405 = \langle w_1(x) \mid w_1(y_1) \rangle = \beta_0^2 + (-133/891)\beta_2^2. \tag{9.3}$$

Now (9.2) and (9.3) imply $\beta_0^2 = 3/80$, $\beta_2^2 = 77/80$ and using the character table of C acting on $\Gamma(2A)$ we can compute the inner product $\langle w_1(u) \mid w_1(v) \rangle$ for every pair $u, v \in \Gamma(2A)$. In particular

$$p_7(1)/k_7 = \langle w_1(x) \mid w_1(y_7) \rangle = 1/15.$$

In order to compute the last unknown entry in the eigenvector, namely $p_6(1)/k_6$ we use the first orthogonality relation [BI]. This relation applied to the vector we are studying and the one corresponding to the trivial idempotent of rank 1 gives

$$\sum_{i=0}^{7} p_i(1) = 0.$$

This immediately implies $p_6(1)/k_6 = -1/27$. Thus we have proved the following.

Proposition 9.2. *The eigenvector of the intersection matrix of* Γ *corresponding to the idempotent of rank* 1938 *is the following*:
$(1, -43/405, 69/405, 13/405, -15/405, -1/405, -15/405, 27/405)$.

§10. The intersection matrix

In this section we calculate the intersection matrix B_1 of the graph Γ. By the definition the (j,i)-entry p^i_{1j} of B_1 is the number of vertices in $\Gamma_j(x)$ adjacent to y_i in Γ. Since all orbitals of E acting on Γ are symmetrical, p^i_{1j} is also the number of vertices in $\Gamma(x)$ contained in $\Gamma_j(y_i)$. So the entries of the intersection matrix are the sizes of parts in the partitions

$$\Gamma(x) = \bigcup_{j=0}^{7} (\Gamma(x) \cap \Gamma_j(y_i)), \qquad 0 \le i \le 7. \tag{10.1}$$

It is clear that for every i and j the set $\Gamma(x) \cup \Gamma_j(y_i)$ is a union of orbits of $E(x, y_i)$ on $\Gamma(x)$. So the partitions of $\Gamma(x)$ into $E(x, y_i)$-orbits are refinements of the partitions (10.1). In this context it is quite helpful (and crucial for our approach) to know the lengths of $E(x, y_i)$-orbits on $\Gamma(x)$. For $i = 1$ the information is contained in Lemma 4.2. For $i = 6$ the orbit lengths follow from the inner product of the permutation characters and elementary congruences. For the remainder i the information is much more non-trivial and was obtained by S. A. Linton using explicit calculations in the Fischer group Fi_{22}. The result is contained in Lemma 4.5 for $i = 2$ and in the following lemma for $3 \le i \le 7$.

Lemma 10.1. *The orbits lengths of* $E(x, y_i)$, $3 \le i \le 7$ *on* $\Gamma(x)$ *are the following.*

(i) *The subgroup* $E(x, y_3) \cong 2^9.M_{10}$ *acting on* $\Gamma(x)$ *has* 26 *orbits with lengths* 1, 4, 60, 64^2, 90, 480, 576, 720, $1\,024^3$, $1\,920^4$, $2\,560^2$, 2 880, 3 840, 6 144, 11 520, $15\,360^2$, 23 040, *and* 46 080.

(ii) *The subgroup* $E(x, y_4) \cong 2^6 : U_3(3)$ *acting on* $\Gamma(x)$ *has* 25 *orbits with lengths* 36^2, 63, 126^2, 288^2, 504, $2\,016^4$, $2\,304^2$, 3 024, $4\,032^3$, $8\,064^2$, $16\,128^4$ *and* 32 256.

(iii) *The subgroup* $E(x, y_5) \cong 2.L_3(4).2$ *acting on* $\Gamma(x)$ *has* 35 *orbits with lengths* 2, 42^2, 105, 112, 224^2, 280, 420, 480, 504, 840^3, 1 120, $1\,344^2$, $2\,520^2$, $3\,360^6$, 4 032, $5\,040^2$, 6 720, $10\,080^2$, $13\,440^2$ *and* $20\,160^2$.

(iv) *The subgroup* $E(x, y_6) \cong {}^2F_4(2)'$ *acting on* $\Gamma(x)$ *has* 3 *orbits with lengths* 1 755, 28 080 *and* 112 320.

(v) *The subgroup* $E(x, y_7) \cong 3_+^{1+6}.2^{3+4} : 3 : 2$ *acting on* $\Gamma(x)$ *has* 6 *orbits with lengths* 2 187, 5 832, 11 664, 17 496, 34 992 *and* 69 984.

The following lemma is obvious.

Lemma 10.2. *Suppose that* $E(x, y_i)$ *acting on* $\Gamma(x) \cap \Gamma_j(y_i)$ *has an orbit of length* l. *Then* $E(x, y_j)$ *acting on* $\Gamma(x) \cap \Gamma_i(y_j)$ *has an orbit of length* $l \cdot k_i/k_j$.

In order to simplify our terminology we present a matrix (denoted by D) and will prove below that it is equal to the intersection matrix B_1 of Γ. First of all, it is straightforward to check that the vector in Proposition 9.2 is a left eigenvector of D.

$$D = \begin{array}{ccccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
142155 & 1178 & 756 & 68 & 36 & 2 & 0 & 0 & 142155 \\
0 & 3696 & 135 & 60 & 324 & 42 & 0 & 0 & 694980 \\
0 & 83776 & 15120 & 17899 & 27720 & 20692 & 28080 & 11664 & 175134960 \\
0 & 42240 & 77760 & 26400 & 19035 & 18120 & 0 & 34992 & 166795200 \\
0 & 11264 & 48384 & 94592 & 86976 & 98315 & 112320 & 93312 & 800616960 \\
0 & 0 & 0 & 576 & 0 & 504 & 1755 & 0 & 3592512 \\
0 & 0 & 0 & 2560 & 8064 & 4480 & 0 & 2187 & 38438400
\end{array}$$

Clearly the first row and column in D are as in B_1.

Lemmas 8.1, 8.4 and straightforward calculations imply the following.

Lemma 10.3. *The second row and column in* D *are as in* B_1.

Lemma 10.4. *The third row and column of* D *are as in* B_1.

Proof. The lengths of orbits of $E(x, y_2)$ on $\Gamma(y_2)$ are given in Lemma 4.5. By Lemma 10.3 the orbit of length 756 is contained in $\Gamma(x)$. The intersection matrix of $\Gamma(7B)$ and Lemma 8.11 (i) show that $p_{12}^2 \neq 0$. Clearly the length of an orbit of $E(x, y_2)$ on $\Gamma(y_2) \cap \Gamma_2(x)$ must be equal to a subdegree of $E(x) = F$ acting on the cosets of $E(x, y_2) = H_5$. By Lemmas 4.4 and 4.5 this implies that $p_{12}^2 = 135$. Now the intersection matrix of $\Gamma(7B)$ and Lemma 8.11 (i) imply that each of the remaining orbits of $E(x, y_2)$ on $\Gamma(y_2)$ whose length is not divisible by 7 must be in $\Gamma_4(x)$. There are two such orbits with lengths 8 640 and 69 120. By now there are two orbits left and from the intersection matrix of $\Gamma(2A)$ and Lemma 8.11 (iv) it follows that both p_{13}^2 and p_{15}^2 are non-zero. The divisibility condition forces the orbit of length 15 120 to be in $\Gamma_3(x)$ and so the one of length 48 384 is in $\Gamma_5(x)$. Q.E.D.

Lemma 10.5. *The seventh row and column of D are as in B_1.*

Proof. By Lemma 10.1 (iv) there are at most three indices i for which $p^6_{1i} \neq 0$. On the other hand the intersection matrix of $\Gamma(5A)$ and Lemma 8.11 (ii) show that for $i \in \{3,5,6\}$ these parameters must be non-zero. Now the result follows from straightforward calculations using Lemmas 10.2 and 10.1 (i), (iii), (iv). Q.E.D.

Before proceeding to the next case we prove the following easy lemma.

Lemma 10.6. *Let $E(x, y_5) \cong 2.L_3(4).2$ act transitively on a set Θ of size 280, 480, 840, 3 360 or 6 720. Let σ be a subgroup of order 3 in $E(x, y_5)$. Then the number of elements fixed by σ in Θ is 1, 12, 2^a3 for $a \leq 2$, 2^b3 for $b \leq 2$ or 2^c3 for $c \leq 3$, respectively.*

Proof. The list of maximal subgroups of $L_3(4)$ [Atlas] shows that the action of degree 280 is unfaithful primitive with character value 1 on 3-elements while the one of degree 480 is on the cosets of $L_3(2)$ with the character value 12 on 3-elements. The normalizer N of σ in $E(x, y_5)$ has order 72. Let $a \in \Theta$ be an element fixed by σ. Then σ is a Sylow 3-subgroup of $E(x, y_5, a)$ and by Frattini argument N acts transitively on the set of all elements from Θ fixed by σ. Now the conclusion comes from an easy observation that in a group of order 96 or 24 the normalizer of a 3-subgroup has order at least 6. Q.E.D.

Lemma 10.7. *The eighth row and column in D are as in B_1.*

Proof. The intersection matrix of $\Gamma(3B)$ and Lemma 8.11 (iii) show that for $i \in \{3,4,5,7\}$ the number p^7_{1i} is non-zero. The lengths of orbits of $E(x, y_7)$ on $\Gamma(y_7)$ are given in Lemma 10.1 (v). Lemmas 10.2, 10.1 (iii) show that only three orbits with lengths 5 832, 17 496 and 69 984 might be contained in $\Gamma_5(x)$. These orbits then would correspond to orbits of $E(x, y_5)$ on $\Gamma_7(x)$ with lengths 280, 480 and 3 360, respectively. By Lemma 10.6 an element of order 3 from $E(x, y_5)$ fixes in the union of the latter three orbits at most 1+12+12=25 elements. On the other hand it follows from the intersection matrix of $\Gamma(3B)$ and Lemma 8.11 (iii) that such an element fixes exactly 1+24=25 vertices in $\Gamma(y_5) \cap \Gamma_7(x)$ so all the above three orbits of $E(x, y_7)$ are in $\Gamma_5(x)$. Applying Lemmas 10.1 and 10.2 to the remaining orbits we see that $p^7_{13} = 11\,664$, $p^7_{14} = 34\,992$ and finally $p^7_{17} = 2\,187$. Q.E.D.

It only remains to evaluate the submatrix in B_1 consisting of the elements p^i_{1j} with $3 \leq i, j \leq 5$. We show first that with the information available it is sufficient to determine just one entry in this submatrix, say p_{14}^5.

Lemma 10.8. *Suppose that* p_{14}^5 *is equal to the corresponding entry in* D. *Then* D *coincides with* B_1.

Proof. Let $C = B_1 - D$ and let the (i,j)-entry of C be denoted by c_{ij}. By Lemmas 10.3, 10.4, 10.5 and 10.7 $c_{ij} = 0$ unless $3 \leq i \leq 5$ and $3 \leq j \leq 5$. Since both B_1 and D have constant column sum (equal to $k = 142\,155$),

$$c_{3j} + c_{4j} + c_{5j} = 0, \qquad 3 \leq j \leq 5. \tag{10.2}$$

The vector given in Proposition 9.2 is a left eigenvector for both B_1 and D. Hence C applied to this vector gives zero and we have the following:

$$13c_{3j} - 15c_{4j} - c_{5j} = 0, \qquad 3 \leq j \leq 5. \tag{10.3}$$

Now (10.2) and (10.3) imply that $c_{3j} = c_{4j}$, $c_{5j} = -2c_{4j}$. Finally, $k_i p_{1j}^i = k_j p_{1i}^j$ and similar relations hold for the entries of D, hence $k_i c_{ji} = k_j c_{ij}$ for $0 \leq i,j \leq 7$ and the result follows. Q.E.D.

Let us proceed with determination of p_{14}^5. The center of $E(x, y_5)$ is of order 2 and it is generated by a $2A$-involution. Hence we conclude (assuming that $y_5 \in \Gamma(2A)$) that $\Gamma(y_5) \cap \Gamma_j(x) \cap \Gamma(2A)$ is a union of orbits of $E(x, y_5)$ on $\Gamma(y_5)$ for every $0 \leq j \leq 7$. This observation along with the intersection matrix of $\Gamma(2A)$ and Lemmas 10.1 (iii) and 8.11 (iv) enable us to locate some further orbits of $E(x, y_5)$ on $\Gamma(y_5)$.

Lemma 10.9. *The orbits of* $E(x, y_5)$ *on* $\Gamma(y_5)$ *with lengths* 2, 42, 42, 105, 280 *and* 420 *are contained in* $\Gamma_1(x)$, $\Gamma_2(x)$, $\Gamma_5(x)$, $\Gamma_5(x)$, $\Gamma_7(x)$ *and* $\Gamma_3(x)$, *respectively.*

The intersection matrix of $\Gamma(7B)$, Lemmas 8.11 (i) and 10.1 (iii) give us.

Lemma 10.10. *The orbit of length* 480 *of* $E(x, y_5)$ *on* $\Gamma(y_5)$ *is contained in* $\Gamma_4(x)$.

It follows from the proof of Lemma 10.7 that one orbit of length 840 of $E(x, y_5)$ on $\Gamma(y_5)$ is contained in $\Gamma_7(x)$. This together with Lemmas 10.9, 10.10, 10.1 (ii), (iii) and 10.2 give us the following.

Lemma 10.11. $E(x, y_5)$ *acting on* $\Gamma(y_5) \cap \Gamma_4(x)$ *has one orbit of length* 480, *at most two orbits of length* 840, *at most four orbits of length* 3 360 *and at most one orbit of length* 6 720.

Let σ be a subgroup of order 3 in $E(x, y_5)$. The intersection matrix of $\Gamma(3B)$ and Lemma 8.11 (iii) show that σ fixes 84 vertices in $\Gamma(y_5) \cap \Gamma_4(x)$. On the other hand Lemma 10.6 says that σ fixes 12 vertices in the orbit of length 480 and gives upper bounds on the numbers of vertices fixed by σ in orbits of lengths 840, 3360 and 6720. These bounds imply the following.

Lemma 10.12. $p^5_{14} = 480 + 840 \cdot \gamma$ *where* $\gamma = 18$, 21, 22, 24, 25 *or* 26.

The entry in D corresponds to $\gamma = 21$. We have checked using the computer program of D. V. Pasechnik, mentioned in the introduction that each of the other five possibilities for γ allowed by Lemma 10.12 leads to a matrix with non-integral spectrum. Since the multiplicities of the centralizer algebra are pairwise distinct, such a matrix can not possibly be the intersection matrix of Γ and we have.

Proposition 10.13. *D is the intersection matrix of* Γ.

We conclude the paper by presenting the intersection matrix B_2 corresponding to the second smallest non-trivial valency 694 980 and the character table of the centralizer algebra, both computed using the program of D. V. Pasechnik. Directly from the shapes of $B_1 = D$ and B_2 (given below) we see that the action is not distance-transitive (as was earlier proved in [CLS]).

$$B_2 = \begin{array}{rrrrrrrrr}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
694980 & 13644 & 2592 & 1140 & 660 & 210 & 36 & 0 & 694980 \\
0 & 143360 & 62532 & 38400 & 0 & 20160 & 8064 & 0 & 38438400 \\
0 & 287280 & 174960 & 142644 & 73920 & 97860 & 68040 & 28080 & 175134960 \\
0 & 135 & 0 & 60 & 3696 & 42 & 324 & 0 & 142155 \\
0 & 241920 & 419904 & 447360 & 236544 & 478788 & 457920 & 561600 & 800616960 \\
0 & 8640 & 34992 & 64800 & 380160 & 95400 & 158580 & 93600 & 166795200 \\
0 & 0 & 0 & 576 & 0 & 2520 & 2016 & 11700 & 3592512
\end{array}$$

The character table of ${}^2E_6(2)$ acting on the cosets of Fi_{22}:

1	142155	694980	175134960	166795200	800616960	3592512	38438400	1
1	−15093	118404	5621616	−6177600	−1976832	−133056	2562560	1938
1	10827	38340	468720	1321920	−2059776	−84672	304640	48620
1	−373	12804	−139664	4800	40448	28224	53760	1828332
1	3915	3780	123120	−14400	−69120	16704	−64000	2909907
1	−1269	3204	9072	−2880	13824	−4032	−17920	29099070
1	459	324	−8208	−5184	13824	−1728	512	278555200
1	−117	−252	2160	1728	−4608	576	512	872972100

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