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Blowing-up Behavior for Solutions of Nonlinear Elliptic Equations

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Dedicated to Prof. S.T. Kuroda on his 60th birthday

Abstract.

We consider the following nonlinear elliptic equations with real parameter λ:

$$(P_\lambda) \qquad \Delta u + f(u,\lambda) = 0, \quad u > 0 \text{ in } \Omega; \quad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a smooth bounded domain in R^n $(n \geq 2)$ and $f \geq 0$ satisfies an inequality:

$$f(u,\lambda) \leq c_1 + c_2 u^p$$

$$(c_1, c_2 > 0,\ p > 1 \text{ constants}).$$

We suppose the existence of a family of solutions $\{(u_s, \lambda_s)\}_{0<s\leq 1}$ of (P_λ) with the following properties: $(u_s, \lambda_s) \in C(\overline{\Omega}) \times R$ is continuous in s, λ_s is bounded, and $\max u_s \to \infty$ $(s \downarrow 0)$.
We investigate the asymptotic behavior of solutions near blowing-up points.

§1. Introduction

In this paper we consider the following nonlinear elliptic equations with real parameter λ:

$$(P_\lambda) \qquad \begin{cases} \Delta u + f(u,\lambda) = 0, \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here Ω is a smooth bounded domain in R^n $(n \geq 2)$ and a smooth function f satisfies the following inequality:

$$0 \leq f(u,\lambda) \leq c_1 + c_2 u^p \qquad (u \geq 0)$$

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where $c_1, c_2 > 0$, and $p > 1$ are constants. Recently many works have been done in the case that (P_λ) is Yamabe type problem, i.e., when $n \geq 3$ and f has (nearly) critical Sobolev exponents such as

$$\text{(i)} \qquad f = u^{\frac{n+2}{n-2}} + \lambda u,$$

$$\text{(ii)} \qquad f = u^{\frac{n+2}{n-2} - \lambda} \qquad (\lambda > 0).$$

See, e.g., [1, 3, 4, 6, 8, 10, and references therein]. We recall the results on the asymptotic behavior of solutions of (P_λ) when f is (i) or (ii). There are two types of results. The first one is on the behavior of solutions when Ω is a ball with center 0. In this case it is known that a family of solutions $\{(u_s, \lambda_s)\}_{s \in (0,1]} \left(\subset C^2(\Omega) \times R\right)$ exists with the following properties:

(A1) $(u_s, \lambda_s) \left(\subset C(\overline{\Omega}) \times R\right)$ is continuous in s;

(A2) λ_s is bounded;

(A3) $\max u_s \to \infty$;

(A4) $u_s(0) \to \infty, \quad u_s(x) \to 0 \quad (x \in \Omega, x \neq 0)$ as $s \downarrow 0$.

(We call such a point as $x = 0$ a blowing-up point.) For more detailed behavior see [1, 3, 4, 10].

The second one is on the behavior of solutions of (P_λ) which satisfy a minimizing sequence property for the (Sobolev) inequality:

$$\frac{\int_\Omega | \nabla u_s |^2 \, dx}{\| u_s \|_{p+1}^2} \to S_n \qquad \text{as} \quad s \downarrow 0,$$

where $p = \frac{n+2}{n-2}$ or $p = \frac{n+2}{n-2} - \lambda$ respectively, and S_n is the best Sobolev constant in R^n. Under appropriate assumptions it is proved that a blowing-up point is unique and that (A3) and similar behavior to (A4) hold ([3, 4, 6, 8, 10]).

We would like to investigate the asymptotic behavior in a neighborhood of a blowing-up point for more general domains and for more general functions.

Throughout the paper we assume that there exists a family of solutions $\{(u_s, \lambda_s)\}_{0<s\leq 1}$ of (P_λ) with the properties (A1)–(A3).

Before proceeding to state our result, we give the definition of blowing up points. From our assumption it follows that there exist a family of points $\{x_j\} (\subset \Omega)$, a point $x_0 \in \overline{\Omega}$, $s_j \in (0, 1]$, and λ_0 such that $x_j \to x_0, \lambda_{s_j} \to \lambda_0, u_{s_j}(x_j) \to \infty$ as $j \uparrow \infty$. We call (x_0, λ_0) or simply x_0 a *blowing-up point* with respect to $\{(u_{s_j}, \lambda_{s_j})\}_{j=1}^\infty$.

Our result is

Theorem. *Under above hypotheses the following statement holds. For each blowing-up point $x_0 \in \Omega$ there exists $r_0 > 0$ such that for each fixed r $(0 < r \leq r_0)$ there exists s $(0 < s < 1)$ such that*

$$k_1 r^{-2/(p-1)} \leq u_s(x) \leq k_2 r^{-2/(p-1)}$$

$$(| x - x_0 | \leq r).$$

Here $k_1, k_2 > 0$ are constants depending only on Ω,c_1,c_2, and p.

As a direct consequence of Theorem we have

Corollary. *Let $n \geq 3$ and let $p = \frac{n+2}{n-2}$. Then for each blowing-up point $x_0 \in \Omega$ there exists $r_0 > 0$ such that for each fixed r $(0 < r \leq r_0)$ there exists s $(0 < s < 1)$ such that*

$$\int_{|x-x_0|\leq r} u_s(x)^{\frac{2n}{n-2}}\, dx \geq k_3.$$

Here $k_3 > 0$ is a constant depending only on Ω,c_1,c_2, and p.

In Section 2 we give the proof of Theorem in the case $n = 2$. In Section 4 we sketch the proof of it in the case $n \geq 3$.

§2. Proof of Theorem $(n = 2)$

In this section we prove Theorem in the case $n = 2$. For the proof of it we need the following two lemmas.

Lemma 1. *Let $n = 2$. Suppose that a family of functions $\{v_s\}_{0<s\leq 1} \subset C^2(\Omega) \cap C(\overline{\Omega})$ satisfies the following hypotheses :*
(i) *v_s satisfies the following differential inequality*

$$\Delta v_s + k e^{v_s} \geq 0 \qquad in \quad \Omega$$

where $k > 0$ is a constant.
(ii) *$v_s \in \left(C(\overline{\Omega})\right)$ is continuous in s .*
Let $r > 0$ be such that $B(x_0, r) \equiv \{x :| x - x_0 |\leq r\} \subset \Omega$, and

$$\int_{B(x_0,r)} e^{v_1(x)} dx < \frac{4\pi}{k}.$$

Assume that for some $0 < s_1 < 1$ the following inequality holds for all $s_1 \leq s \leq 1$,

$$[e^{v_s}]_r \equiv \left\{ \frac{1}{2\pi} \int_0^{2\pi} e^{v_s(x_0+x(\theta))/2} d\theta \right\}^2$$
$$< \frac{2}{kr^2}, \qquad x(\theta) \equiv r(\cos\theta, \sin\theta).$$

Then for all $s_1 \le s \le 1$

(i) $$e^{v_s(x_0)} < 4[e^{v_s}]_r,$$

(ii) $$\int_{B(x_0,r)} e^{v_s(x)} dx < \frac{4\pi}{k}.$$

Lemma 2. *Let* $n = 2$. *Let* $x_0 \in \Omega$ *be a blowing-up point. Let* r *be such that* $B(x_0, r) \subset \Omega$. *Suppose that for some* $0 < s_1 < 1$ *a solution* u_s *of* (P_λ) *with* $\lambda = \lambda_s$ *satisfies*

$$u_s(x) < \left(\frac{2}{k}\right)^{1/(p-1)} | x - x_0 |^{-2/(p-1)} \qquad (x \in B(x_0, r))$$

for all $s \in [s_1, 1]$. *Then* $v_s \equiv (p-1)\ln u_s$ *satisfies a differential inequality:*

$$\Delta v_s + ke^{v_s} \ge 0 \qquad (x \in B(x_0, r))$$

for all $s \in [s_1, 1]$, *where* k *is a constant independent of* x_0,r,s_1.

For the proof of Lemma 1 see [7; Proposition] or [2]. In Section 3 we prove Lemma 2.

Proof of Theorem. We set $v_s \equiv (p-1)\ln u_s$. Let $k > 0$ be a constant as in Lemma 2. Let r_0 be so small that $B(x_0, r_0) \subset \Omega$,

(1) $$\int_{B(x_0,r_0)} e^{v_1(x)} dx < \frac{4\pi}{k},$$
$$e^{v_1(x)} < \frac{2}{k} | x - x_0 |^{-2} \qquad (x \in B(x_0, r_0)).$$

Let $0 < r \le r_0$ be fixed. Suppose that for some $s_1 > 0$, v_s satisfies

(2) $$e^{v_s(x)} \equiv u_s^{p-1}(x) < \frac{2}{k} | x - x_0 |^{-2} \qquad (x \in B(x_0, r))$$

for all $s \in [s_1, 1]$. Then by lemma 2, v_s satisfies

(3) $$\Delta v_s + ke^{v_s} \ge 0 \qquad (x \in B(x_0, r)).$$

Let $x_s \in B(x_0, r)$ be a maximal point of u_s in $B(x_0, r)$:

$$u_s(x_s) = \max_{B(x_0,r)} u_s(x) .$$

Then by (2)

$$e^{v_s(x_s)} < \frac{2}{k} \mid x_s - x_0 \mid^{-2} .$$

We consider u_s a solution of the following linear elliptic equation

$$\Delta u_s + c_s(x) u_s = 0 ; \quad c_s(x) \equiv \frac{f(u_s(x), \lambda_s)}{u_s(x)} .$$

Since x_0 is a blowing-up point, we may assume that $u_s(x) \geq 1$ for $x \in B(x_0, r)$. Then $c_s(x)$ satisfies

$$c_s(x) \leq c_1 + c_2 u_s^{p-1}(x) \leq c_1 + \frac{2c_2}{k} \mid x_s - x_0 \mid^{-2} .$$

Hence by Harnack's theorem there is a constant c' such that

$$u_s(x_s) \leq c' \min u_s(x) \tag{4}$$

for all x with $\mid x - x_0 \mid \leq \mid x_s - x_0 \mid$. Here c' depends only on p,c_1,c_2. On the other hand, since (1),(2), and (3) hold, we have by Lemma 1

$$u_s^{p-1}(x_0) \equiv e^{v_s(x_0)} < 4[e^{v_s(x)}]_r \qquad s \in [s_1, 1].$$

Hence, by (2), (4)

$$\begin{aligned} u_s(x_s) &\leq c' u_s(x_0) \\ &\leq 2^{3/(p-1)} c' k^{-1/(p-1)} r^{-2/(p-1)}. \end{aligned}$$

Applying Harnack's theorem again we get an inequality:

$$u_s(x_s) \leq c \min_{B(x_0,r)} u_s(x). \tag{5}$$

Here c is a constant depending only on p,c_1,c_2. Since x_0 is a blowing-up point, this implies that (2) does not hold for all $s \in (0, 1]$.
Set

$$s_2 \equiv \inf\{s' : (2) \text{ holds for all} s \in [s', 1]\}.$$

Then $s_2 > 0$, and (2) does not hold for $s = s_2$, i.e., there exists $x' \in B(x_0, r)$ such that

$$u_{s_2}^{p-1}(x') \equiv e^{v_{s_2}(x')} = \frac{2}{k} \mid x' - x_0 \mid^{-2} .$$

On the other hand, by Harnack's inequality (5) we have

$$c^{-1}u_{s_2}(x) \le u_{s_2}(x') \le c u_{s_2}(x) \qquad (x \in B(x_0, r))\,.$$

Hence we have

$$\begin{aligned}\frac{2}{k}r^{-2} &\le u_{s_2}^{p-1}(x') \le \max_{B(x_0,r)} u_{s_2}^{p-1}(x) \\ &\le c^{p-1} \min_{B(x_0,r)} u_{s_2}^{p-1}(x) \le \frac{2c^{p-1}}{k} r^{-2}.\end{aligned}$$

Thus we obtain

$$k_1 r^{-2/(p-1)} \le u_{s_2}(x) \le k_2 r^{-2/(p-1)},$$

$$k_1 \equiv c^{-1}\left(\frac{2}{k}\right)^{\frac{1}{p-1}}, \qquad k_2 \equiv c\left(\frac{2}{k}\right)^{\frac{1}{p-1}}.$$

Q.E.D.

§3. Proof of Lemma 2

Proof of Lemma 2. Since u_s is a solution of (P_λ) with $\lambda = \lambda_s$, $v_s(x)$ satisfies

$$\Delta v_s + \frac{1}{p-1} \mid \nabla v_s \mid^2 + (p-1)\frac{f(u_s, \lambda_s)}{u_s} = 0.$$

On the other hand, by our assumption on f

$$\frac{f(u, \lambda)}{u} \le c_1 + c_2 u^{p-1} \qquad (u \ge 1).$$

Hence we get a differential inequality

$$\Delta v_s + \frac{1}{p-1} \mid \nabla v_s \mid^2 + (p-1)c_3 e^{v_s} \ge 0$$

$$(c_3 = c_1 + c_2).$$

Therefore if we can estimate the term $\mid \nabla v_s \mid^2$ by e^{v_s} , i.e.,

$$\mid \nabla v_s \mid^2 \le c_4' e^{v_s} \quad \text{or} \quad \mid \nabla u_s \mid^2 \le c_4 u_s^{(p+1)},$$

then we get a differential inequality

$$\Delta v_s + k' e_s^v \ge 0$$

$$(k' = (p-1)(c_3 + c_4)).$$

In the following we estimate the term $|\nabla u_s|^2$ by u_s^{p+1}.
Set

$$M_s \equiv \max_{B(x_0,r)} u_s(x), \quad m_s \equiv \min_{B(x_0,r)} u_s(x),$$

and choose $K_1 > \frac{M_1}{m_1}$. Then by the continuity of u_s $\left(\subset C(\overline{\Omega})\right)$ in s, we have for some $s_2 > 0$

$$M_s \le K_1 u_s(x) \qquad (x \in B(x_0, r)) \tag{6}$$

for $s_2 \le s \le 1$. On the other hand, by Sperb's lemma [9; Lemma 5.1]

$$P_s(x) \equiv \frac{|\nabla u_s(x)|^2}{2} + \int_0^{u_s(x)} f(t, \lambda_s)\, dt \qquad (x \in B(x_0, r))$$

attains its maximum at the point where $\nabla u_s = 0$ or on $\partial B(x_0, r)$. Since x_0 is a blowing-up point, we may assume that P_s attains its maximum where $\nabla u_s(x) = 0$. Hence we have

$$|\nabla u_s|^2 \le 2\left(c_1 + \frac{c_2}{p+1}\right) M_s^{p+1} \qquad (x \in B(x_0, r)) \tag{7}$$

for $s_2 \le s \le 1$. By (6) and (7)

$$\frac{|\nabla u_s|^2}{u_s^2} \le 2K_1^{p+1}\left(c_1 + \frac{c_2}{p+1}\right) u_s^{p-1}.$$

Therefore we get a differential inequality

$$\Delta v_s + K_2 e^{v_s} \ge 0 \quad ; \qquad K_2 \equiv \left(2K_1^{p+1}\left(c_1 + \frac{c_2}{p+1}\right) + c_3\right)(p-1).$$

We may assume that

$$K_2 \ge 1, \qquad K_2 > k,$$

where k is the constant determined by (11) which is independent of x_0,r,s_2. Note that K_2 depends on x_0,r,s_2 . In the following we improve the above differential inequality and obtain:

$$\Delta v_s + k e^{v_s} \ge 0 \qquad (x \in B(x_0, r)).$$

If necessary, by choosing $r > 0$ sufficiently small we may assume that

$$e^{v_1(x)} < \frac{2}{K_2} r^{-2} \qquad (|x - x_0| \le r), \tag{8}$$

$$K_2 \int_{B(x_0,r)} e^{v_1(x)}\, dx < 4\pi.$$

By the continuity of v_s in s, it follows that for some $s' > 0$, (8) holds for all $s' \le s \le 1$. Hence by Lemma 1 we have

$$e^{v_s(x_0)} \le 4[e^{v_s(x)}]_r < 8r^{-2} \tag{9}$$

for all $s' \le s \le 1$. On the other hand, by Harnack's theorem there exists a constant c' such that

$$\max_{|x-x_0|\le r} u_s(x) \le c' u_s(x_0).$$

Hence by (9) we have

$$u_s(x_s) \le 2^{3/(p-1)} c' r^{-2/(p-1)}.$$

Applying Harnack's theorem again we get

$$u_s(x_s) \le c u_s(x) \qquad (x \in B(x_0, r))$$

for all $s' \le s \le 1$. Here c is a constant depending only on p,c_1,c_2. Then repeating the above arguments we get a differential inequality

$$\Delta v_s + k e^{v_s} > 0 \qquad (x \in B(x_0, r)), \tag{10}$$

$$k \equiv \left(2c^{p+1} \left(c_1 + \frac{c_2}{p+1} \right) + c_3 \right) (p-1). \tag{11}$$

Since $k < K_2$, from the continuity of $u_s(x)$ in s it follows that there exists s'' such that for all $s'' \le s \le 1$

$$u_s(x)^{p-1} \equiv e^{v_s(x)} < \frac{2}{k} r^{-2} \qquad x \in B(x_0, r), \tag{12}$$

$$\int_{B(x_0,r)} e^{v_s(x)} dx < \frac{4\pi}{k}. \tag{13}$$

Set

$$s^* \equiv \inf\{s'' : (10),(12) \text{ hold for} s'' \le s \le 1\}$$

Suppose that $s_1 < s^*$. Then repeating the above argument we conclude that a differential inequality (10) holds for all $s \in [s^*, 1]$. This contradicts the definition of s^*. Thus we have $s^* = s_1$. Q.E.D.

§4. Proof of Theorem $n \geq 3$

In this section we sketch the proof of Theorem when $n \geq 3$. We may assume that $0 \notin \Omega$ and introduce spherical coordinates:

$$x = r\omega \quad \left(r = \mid x \mid, \quad \omega \in S^{n-1}\right).$$

Let $x_0 \in \Omega$ be a blowing-up point. Let $r_0 > 0$ be such that $B(x_0, r_0) \subset \Omega$.
Suppose that

$$u_s(x) \leq \mid x - x_0 \mid^{-2/(p-1)} \qquad (x \in B(x_0, r_0)).$$

Then we have

$$\mid u_s \mid_{C^2(B(x_0,r_0))} \leq c'(c_1 + c_2 M_s^p), \qquad M_s \equiv \max_{B(x_0,r_0)} u_s(x).$$

On the other hand, by Sperb's lemma [9; Lemma 5.2] we get

$$\mid \nabla u_s \mid^2 \leq 2\left(c_1 M_s + \frac{c_2}{p+1} M_s^{p+1}\right).$$

Hence $v_s \equiv (p-1)\ln u_s$ satisfies a differential inequality

$$(v_s)_{rr} + \frac{(v_s)_r}{r} + c\frac{M_s^{p+1}}{u_s^2} \geq 0,$$

where c is a constant depending only on Ω,c_1,c_2, and p. We consider $v_s(r\omega)$ a function $w_{s,\omega}(y)$ defined in R^2 near $\mid y \mid = \mid x_0 \mid$:

$$w_{s,\omega}(y) \equiv v_s(r\omega), \qquad \mid y \mid = r, \quad y \in R^2.$$

Now we have a two-parameter family of functions $\{w_{s.\omega}\}_{s,\omega}$. Repeating similar arguments as in Sections 2 and 3 we can conclude the assertion in Theorem. Q.E.D.

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