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Eigenvalue Properties of Schrödinger Operators

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Abstract.

In Evans-Lewis [5] and Evans-Lewis-Saitō [6], [7], [8], [9] we have been discussing conditions for the finiteness and for the infiniteness of bound states of Schrödinger-type operators using geometric methods. Here the ideas and results obtained so far are summarized and presented in an expository manner. These bound states correspond to eigenvalues below the essential spectrum of the operator. After basic results are presented, Schrödinger operators of atomic type will be discussed to show how these basic results can be applied to various types of N-body Schrödinger operators.

Introduction

In [5], [6], [7], [8] and [9] we have been considering criteria for the bound states of Schrödinger-type operators

$$(0.1) \qquad P = -\sum_{j,k=1}^{n} \partial_j a^{jk}(x)\partial_k + q(x) \qquad x \in \mathbf{R}^n,\ \partial_j = \frac{\partial}{\partial x_j}$$

to be finite or infinite (see Assumption 1.1 for the properties satisfied by the coefficients $a^{jk}(x)$ and $q(x)$). These bound states correspond to eigenvalues below the essential spectrum of the operator. The goal of this paper is twofold:

(1) *In* §1 *the basic results for the operator* (0.1) *will be presented in a more self-contained and unified way, which we hope makes these basic results easier to be understood. Our arguments are based on the geometric method using the Agmon spectral function which was introduced in Agmon* [1]. *We are going to show that our arguments become smoother and more streamlined by restricting the operator P using only smooth cut-off functions. This was introduced in* [9]. *Here we have an opportunity to modify our way of deriving the basic results obtained in* [5] *and*

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[6]. *Other important ingredients are the results of Glazman* [10, *Chapter* 1] *on counting the eigenvalues of an abstract selfadjoint operator in a given interval. Since the proofs of some of his theorems in* [10] *are too succinct, we are proving these theorems in a more self-contained way so that our main theorems will be understood more easily.*

(2) *In* §2 *we shall discuss the Schrödinger operator of atomic type*

$$P = P_N = \sum_{i=1}^{N}\left(-\frac{1}{2m_i}\Delta_i + v_{0i}(x^i)\right) + \sum_{1\le i<j\le N} v_{ij}(x^i - x^j), \tag{0.2}$$

where

$$x^i = (x_1^i, x_2^i, \cdots, x_\nu^i) \in \mathbf{R}^\nu, \tag{0.3}$$

(*see Assumption* 2.1). *We chose the operator* (0.2) *as an example to give an idea how the general results obtained in* §1 *can be applied to various types of* N*-body Schrödinger operators since it is easier to be treated without being bothered by technical troubles. We are going to compare our results to the celebrated results by Zhislin* ([22], [23], [24], [25]), Yafaev ([20], [21]) *and others for the atomic Hamiltonian given by*

$$P = P(N, Z) = \sum_{i=1}^{N}(-\frac{1}{2m}\Delta_i - \frac{Z}{|x^i|}) + \sum_{1\le i<j\le N}\frac{1}{|x^i - x^j|}. \tag{0.4}$$

We are also giving an another proof for the finiteness of the bound states of the operator (0.2) *with "short-range" potentials* v_{jk}, $0 \le j < k \le N$, *i.e.,*

$$v_{jk} \in L_2^{\nu/2}(\mathbf{R}^\nu) \qquad (0 \le j < k \le N). \tag{0.5}$$

The results given in §1 can be applied to other types of N-body Schrödinger operators. In [8] we discussed N-body Schrödinger operators with their center of mass removed. Then the operator becomes unitarily equivalent to the operator in §1, and hence we can develop essentially the same theory as in §1 and §2. Thus we are able to treat molecular Hamiltonians. We also discussed molecular Hamiltonians with symmetry restrictions in [9]. The N-body Schrödinger operator with its center of mass removed is considered in the L_2 space whose elements are square integrable functions over

$$X = \{x \in \mathbf{R}^{\nu N} : m_1x^1 + m_2x^2 + \cdots + m_Nx^N = 0\} \tag{0.6}$$

satisfying specified symmetry conditions. Again we found that we can construct a parallel theory to those in §1 and §2. For these details see [8] and [9].

While we try to make this work self-contained, we refer to our works [5], [6], [7] and [8] when we use the exactly same propositions given in the above papers. Some technical lemmas and theorems are proved in the Appendices.

§1. The bound states of Schrödinger-type operators

Consider the Schrödinger-type operator

$$P = -\sum_{j,k=1}^{n} \partial_j a^{jk}(x)\partial_k + q(x) \qquad x \in \mathbf{R}^n,\ \partial_j = \frac{\partial}{\partial x_j}. \tag{1.1}$$

Assumption 1.1. The coefficients a^{jk} and q of the operator P is assumed to satisfy the following (i) $\sim$ (iii):

(i) Each a^{jk} is a bounded, continuous, real-valued function on $\mathbf{R}^n$.

(ii) The matrix $A(x) = \left(a^{jk}(x)\right)$ is uniformly positive definite on $\mathbf{R}^n$, i.e., there exists a constant $c_0 > 0$ such that

$$\sum_{j,k=1}^{n} a^{jk}(x)\xi_j\overline{\xi_k} \geq c_0 \sum_{j=1}^{n} |\xi|^2 \tag{1.2}$$

for all $x \in \mathbf{R}^n$ and $(\xi_1, \xi_2, \cdots, \xi_n) \in \mathbf{C}^n$.

(iii) $q \in L_1(\mathbf{R}^n)_{\text{loc}}$.

We start with the following definition.

Definition 1.2. (i) Let η be a nonnegative, bounded C^∞ function on $\mathbf{R}^n$. Let the sesquilinear form on $C_0^\infty(\mathbf{R}^n) \times C_0^\infty(\mathbf{R}^n)$ be defined by

$$\rho_\eta[\phi, \varphi] = \int_{\mathbf{R}^n} \{< \nabla(\eta\phi), \nabla(\eta\varphi) >_A + q\eta^2\phi\overline{\varphi}\}\, dx, \tag{1.3}$$

where

$$< \xi, \zeta >_A = \sum_{j,k=1}^{n} a^{jk}\xi_j\overline{\zeta_k} \tag{1.4}$$

$$(\xi = (\xi_1, \xi_2, \cdots, \xi_n), \zeta = (\zeta_1, \zeta_2, \cdots, \zeta_n) \in \mathbf{C}^n).$$

We set $\rho_\eta[\phi] := \rho_\eta[\phi, \phi]$. For $\eta \equiv 1$ we denote $\rho_1[\ ,\ ]$ simply by $\rho[\ ,\ ]$.

(ii) Define the Hilbert space $L_{2,\eta}(\mathbf{R}^n)$ by

$$L_{2,\eta}(\mathbf{R}^n) = L_2(\mathbf{R}^n, \eta^2 dx). \tag{1.5}$$

The inner product and norm of $L_{2,\eta}(\mathbf{R}^n)$ are denoted by $(\ ,\ )_\eta$ and $\|\ ,\ \|_\eta$, respectively. For $\eta \equiv 1$ we simply write $L_2(\mathbf{R}^n)$, $(\ ,\ )$, and $\|\ \|$.

The following assumption guarantees that ρ_η is closable on $L_{2,\eta}(\mathbf{R}^n)$.

Assumption 1.3. For every $\epsilon \in (0,1)$ there is a $C(\epsilon) > 0$ such that

(1.6)
$$\int_{\mathbf{R}^n} q_-|\phi|^2 dx \leq \epsilon \int_{\mathbf{R}^n} |\nabla\phi|^2 dx + C(\epsilon)\int_{\mathbf{R}^n} |\phi|^2 dx, \qquad \phi \in C_0^\infty(\mathbf{R}^n),$$

where $q_-(x) = \max(-q(x), 0)$.

It is known (Schechter [16, Theorem 7.3, p.138]) that (1.6) holds if q_- belongs to the Kato class, i.e.,

$$\lim_{r\to 0} \int_{|x-y|<r} g(x,y)|q(y)|dy = 0, \tag{1.7}$$

where

$$g(x,y) = \begin{cases} |x-y|^{2-n} & \text{if } n \geq 3, \\ |\ln|x-y|| & \text{if } n = 2, \text{ and} \\ 1 & \text{if } n = 1. \end{cases} \tag{1.8}$$

Remark 1.4. Assumptions 1.1 and 1.3 are slightly more strict than those given in [6], [7], [8], [9] although usual N-body Schrödinger operators satisfy our assumptions. Since we assume that the matrix $A(x)$ is uniformly positive, the condition on q_- seems to be easier to check (cf. the condition $\mathcal{H}(1)$ in [6, p.383]).

Proposition 1.5. *Let Assumptions* 1.1 *and* 1.3 *be satisfied. Let* ρ_η *be as in Definition* 1.2. *Then* ρ_η *is densely defined, symmetric, bounded below, and closable in* $L_{2,\eta}(\mathbf{R}^n)$.

Proof. (1) Since it is easy to see that ρ_η is densely defined, symmetric, and bounded below, we are going to give the proof that ρ_η is closable. Let $\{\phi_j\}$ be a sequence in $C_0^\infty(\mathbf{R}^n)$ such that

$$\begin{cases} \|\phi_j\|_\eta \to 0 & (j \to \infty), \\ \rho_\eta[\phi_j - \phi_k] \to 0 & (j,k \to \infty). \end{cases} \tag{1.9}$$

We have only to prove that $\rho_\eta[\phi_j] \to 0$ as $j \to \infty$.

(2) It follows from Assumption 1.1, (ii) and Assumption 1.3 that there exists a positive constant C_1 such that

$$(1.10)\quad \|\phi\|^2_{\rho_\eta} \equiv \rho_\eta[\phi] + C_1\|\phi\|^2_\eta \geq \int_{\mathbf{R}^n} \{\frac{c_0}{2}|\nabla(\eta\phi)|^2 + (q_+ + 1)\eta^2|\phi|^2\}\, dx$$

for $\phi \in C_0^\infty(\mathbf{R}^n)$, where $q_+(x) = \max\{q(x), 0\}$, and hence $\{\eta\phi_j\}$ is a Cauchy sequence in both $H^1(\mathbf{R}^n)$ and $L_2(\mathbf{R}^n, q_+dx)$. Further, since $\eta\phi_j \to 0$ in $L_2(\mathbf{R}^n)$ as $j \to \infty$, it follows that

$$(1.11)\qquad \mathrm{s} - \lim_{j\to\infty} \eta\phi_j = 0 \qquad (j \to \infty)$$

in both $H^1(\mathbf{R}^n)$ and $L_2(\mathbf{R}^n, q_+dx)$ which implies that $\rho_\eta[\phi_j] \to 0$.

Q.E.D.

Definition 1.6. Let ρ_η be as above. Denote the closure of ρ_η by $\tilde{\rho}_\eta$. Let H_η be the selfadjoint operator in $L_{2,\eta}(\mathbf{R}^n)$ associated with $\tilde{\rho}_\eta$ (see, e.g., Kato [13, Chapter VI]). For $\eta \equiv 1$ H_1 will be denoted simply by H. Define $\Sigma(H_\eta)$ by

$$(1.12)\qquad \Sigma(H_\eta) = \inf \sigma_e(H_\eta),$$

where $\sigma_e(H_\eta)$ is the essential spectrum of H_η.

Now we are in a position to introduce the Agmon spectral function.

Definition 1.7. Let S^{n-1} be the unit sphere. For any set $U \subset S^{n-1}$ and for positive numbers R and δ define

$$\begin{aligned} U_\delta &:= \{\omega \in S^{n-1} : \mathrm{dist}(\omega : U) < \delta\};\\ \Gamma(U_\delta, R) &:= \{x \in \mathbf{R}^n : x = t\omega \text{ for } \omega \in U_\delta \text{ and } t > R\}\\ K(U_\delta, R; P) &:= \inf\{\rho[\varphi] :\ \varphi \in C_0^\infty(\Gamma(U_\delta, R)), \|\varphi\| = 1\};\\ (1.13)\qquad K(U : P) &:= \lim_{\delta\downarrow 0}\lim_{R\to\infty} K(U_\delta, R; P); \end{aligned}$$

and

$$\mathcal{M} := \{\omega \in S^{n-1} : K(\omega : P) = \inf_{\omega\in S^{n-1}} K(\omega : P)\},$$

where we write $K(\omega : P)$ instead of $K(\{\omega\} : P)$, and the set $\mathcal{M} \subset S^{n-1}$ is called the minimizing set.

The following properties of the Agmon spectral function are important.

Proposition 1.8. *Suppose that Assumptions* 1.1 *and* 1.3 *hold. Let H be as in Definition* 1.6.

(i) *Then $K(\omega : P)$ is a lower semicontinuous function on S^{n-1} and we have*

$$\Sigma(H) := \min_{\omega \in S^{n-1}} K(\omega : P), \tag{1.14}$$

and the minimizing set $\mathcal{M}$ is a compact set in S^{n-1} with

$$\mathcal{M} = \{\omega \in S^{n-1} : K(\omega : P) = \min_{\omega \in S^{n-1}} K(\omega : P)\}. \tag{1.15}$$

(ii) *For any $U \subset S^{n-1}$,*

$$K(U : P) = K(\overline{U} : P) = \inf_{\omega \in \overline{U}} K(\omega : P) \tag{1.16}$$

The first part of the above proposition is due to Agmon [1, Lemma 2.7, p.38]. For the proof of (ii) see [6, Lemma 5, p.380].

Let us give a necessary condition for the bound states to be finite.

Theorem 1.9 ([6, Theorem 8]). *Let Assumptions* 1.1 *and* 1.3 *hold. Let H be the selfadjoint operator associated with the closure $\tilde{\rho}$ of ρ in $L^2(\mathbf{R}^n)$. A necessary condition for the finiteness of the number of eigenvalues of H below $\Sigma(H)$ is that for some $\delta_0 > 0$ and some $R_0 > 0$*

$$K(\mathcal{M}_\delta, R; P) = K(M : P) = \Sigma(H) \qquad \text{for all } \delta \geq \delta_0 \text{ and } R \geq R_0. \tag{1.17}$$

Before proving the theorem we mention a simple fact on a linear space.

Lemma 1.10. *Let Y be a vector space over $\mathbf{C}$. Let Y_1 and Y_2 be linear subspaces of Y such that $\dim Y_2 < \infty$ and Y is the direct sum of Y_1 and Y_2 (i.e., $Y_1 \cap Y_2 = \{0\}$, and $Y = Y_1 + Y_2$). Let Y_0 be another linear subspace of Y such that $\dim Y_0 > \dim Y_2$. Then we have* $\dim(Y_0 \cap Y_1) \geq 1$.

Proof. Let $\dim Y_2 = m$ and let $\phi_1, \phi_2, \cdots, \phi_m$ be a base of Y_2. Let $\{f_j\}$, $j = 1, 2, \cdots, m+1$, be a set of $m+1$ independent vectors in Y_0. Since Y is the direct sum of Y_1 and Y_2, there exist $u_j \in Y_1$,

$j = 1, 2, \cdots, m+1$ and $a_{jk} \in \mathbf{C}$, $j = 1, 2, \cdots, m+1$, $k = 1, 2, \cdots, m$ such that

$$f_j = u_j + \sum_{k=1}^{m} a_{jk}\phi_k \qquad (j = 1, 2, \cdots, m+1). \tag{1.18}$$

Note that the system of linear equations

$$\sum_{j=1}^{m+1} c_j a_{jk} = 0 \qquad k = 1, 2, \cdots, m \tag{1.19}$$

has a nontrivial solution $(c_1, c_2, \cdots, c_{m+1})$. Then we have

$$f_0 := \sum_{j=1}^{m+1} c_j f_j = \sum_{j=1}^{m+1} c_j u_j \tag{1.20}$$

is nontrivial and belongs to $Y_0 \cap Y_1$, which completes the proof. Q.E.D.

Proof of Theorem 1.9. We are going to prove that the number of eigenvalues of H below $\Sigma(H)$ is infinite if

$$K(\mathcal{M}_\delta, R; P) < K(M : P) = \Sigma(H) \qquad (\delta > 0 \text{ and } R > 0). \tag{1.21}$$

The proof is divided into several steps.

(1) It follows from (1.21) that for each $j = 1, 2, \cdots$ there exist a positive number R_j and $\phi_j \in C_0^\infty(\Gamma(\mathcal{M}_{(1/j)}, R_j))$ such that

$$\begin{cases} \text{(a) } R_1 < R_2 < \cdots < R_j < \cdots \to \infty, \\ \text{(b) } \|\phi_j\|^2 = 1 \qquad (j = 1, 2, \cdots), \\ \text{(c) } \mathrm{supp}(\phi_j) \cap \mathrm{supp}(\phi_k) = \emptyset \qquad (j \neq k), \\ \text{(d) } \rho[\phi_j] < \Sigma(H) \qquad (j = 1, 2, \cdots). \end{cases} \tag{1.22}$$

Let X_0 be the linear subspace spanned by $\{\phi_j\}_{j=1}^{\infty}$. Note that it follows from (b) and (c) of (1.22) that

$$\rho[f] < \Sigma(H)\|f\|^2 \tag{1.23}$$

for any $f \in X_0$.

(2) Let s be a positive number such that

$$\rho[\phi] + s\|\phi\|^2 \geq 0 \qquad (\phi \in C_0^\infty(\mathbf{R}^n)). \tag{1.24}$$

Define the sesquilinear form $\rho^{(s)}$ on $C_0^\infty(\mathbf{R}^n) \times C_0^\infty(\mathbf{R}^n)$ by

$$\rho^{(s)}[\phi, \varphi] = \rho[\phi, \varphi] + s(\phi, \varphi) \qquad (\phi, \varphi \in C_0^\infty(\mathbf{R}^n)). \tag{1.25}$$

Since the potential $q^{(s)} = q(x) + s$ satisfies Assumptions 1.1 and 1.3, $\rho^{(s)}$ is closable with its closure $\tilde{\rho}^{(s)}$. Let $H^{(s)}$ be the nonnegative selfadjoint operator determined through $\tilde{\rho}^{(s)}$. Obviously we have $D(\tilde{\rho}^{(s)}) = D(\tilde{\rho})$. It follows from the uniqueness of the selfadjoint operator determined by a symmetric closed sesquilinear form (Kato [13, Chapter VI, Theorem 2.1 and Corollary 2.4, pp.322–323]) that $H^{(s)} = H + sI$, where I is the identity operator on $L_2(\mathbf{R}^n)$. Let $E^{(s)}(\cdot)$ be the spectral measure associated with $H^{(s)}$. Applying the second representation theorem (Kato [13, Chapter VI, Theorem 2.23, p.331]) to the nonnegative closed sesquilinear form $\tilde{\rho}^{(s)}$, we see that

$$\tilde{\rho}^{(s)}[f] = \int_{\mathbf{R}} \lambda \, d\|E^{(s)}(\lambda) f\|^2 \qquad (f \in D(\tilde{\rho}^{(s)}) = D(\tilde{\rho})). \tag{1.26}$$

Therefore we have

$$\begin{cases} E^{(s)}((-\infty, \lambda)) = E((-\infty, \lambda - s)) \qquad (\lambda \in \mathbf{R}) \\ \tilde{\rho}[f] = \int_{\mathbf{R}} \lambda \, d\|E(\lambda) f\|^2 \qquad (f \in D(\tilde{\rho})), \end{cases} \tag{1.27}$$

where $E(\cdot)$ is the spectral measure associated with H.

(3) Suppose that $\dim E(-\infty, \Sigma(H)) = m < \infty$. Then, setting

$$\begin{cases} Y_1 = E([\Sigma(H), \infty))L_2(\mathbf{R}^n), \\ Y_2 = E((-\infty, \Sigma(H))L_2(\mathbf{R}^n), \\ Y_0 = X_0 \end{cases} \tag{1.28}$$

in Lemma 1.10, we see that there exists a nonzero $f_0 \in L_2(\mathbf{R}^n)$ which belongs to both X_0 and $E([\Sigma(H), \infty))L_2(\mathbf{R}^n)$. Therefore, f_0 satisfies (1.23) with f replaced by f_0, and it follows from the second relation of (1.27) that

$$\begin{aligned} \rho[f_0] &= \int_{\mathbf{R}} \lambda \, d\|E(\lambda) f_0\|^2 \\ &= \int_{\Sigma(H)}^{\infty} \lambda \, d\|E(\lambda) f_0\|^2 \\ &\geq \Sigma(H)\|f_0\|^2. \end{aligned} \tag{1.29}$$

These two inequalities contradict each other, which completes the proof.

Q.E.D.

In order to give a sufficient condition for the bound states of H to be finite we are going to start with

Proposition 1.11 (cf. [5, Theorem 15], [6, Theorem 10]). *Suppose that Assumptions* 1.1 *and* 1.3 *hold. Let* η *be a nonnegative, bounded* C^∞ *function on* $\mathbf{R}^n$. *Let* H_η *be the selfadjoint operator given by Definition* 1.6. *For any* $R > 0$ *define*

$$\begin{cases} K_R = K_R(H_\eta) = \inf\{\rho_\eta[\phi] : \phi \in C_0^\infty(E_R),\ \|\phi\|_\eta = 1\}, \\ K_\infty = K_\infty(H_\eta) = \lim_{R\to\infty} K_R, \end{cases} \tag{1.30}$$

where

$$E_R = \{x \in \mathbf{R}^n : |x| > R\}. \tag{1.31}$$

Then, setting $K_\infty = \lim_{R\to\infty} K_R$, *we have*

$$K_\infty = \Sigma(H_\eta). \tag{1.32}$$

Since the idea of the proof is essentially the same as the proof of [5], Theorem 10 or [6], Theorem 15, we are going to give the proof in Appendix.

The following corollary will be used later.

Corollary 1.12. *Let* η *be a nonnegative, bounded* C^∞ *function on* $\mathbf{R}^n$ *such that all the first derivatives* $\partial_j\eta$, $j = 1, 2, \cdots, n$ *are also bounded on* $\mathbf{R}^n$. *Let* ρ_η *and* $\rho = \rho_1$ *be as in Definition* 1.2.

(i) *Then we have* $D(\tilde\rho) \subset D(\tilde\rho_\eta)$, *i.e., for* $u \in D(\tilde\rho)$ *and for any sequence* $\{\phi_j\} \subset C_0^\infty(\mathbf{R}^n)$ *such that* $\phi_j \to u$ *in* $D(\tilde\rho)$, *we have*

$$\begin{cases} s - \lim_{j\to\infty} \phi_j = u \qquad \text{in } D(\tilde\rho_\eta), \\ \lim_{j\to\infty} \rho_\eta[\phi_j] = \tilde\rho_\eta[u]. \end{cases} \tag{1.33}$$

(ii) *On the other hand, for* $u \in \tilde\rho_\eta$ *we have* $\eta u \in D(\tilde\rho)$

Proof. Since it follows from (1.10) that $\{\phi_j\}$ is a Cauchy sequence both in $H^1(\mathbf{R}^n)$ and $L_2(\mathbf{R}^n, q_+dx)$. Then it is easy to see that $\{\phi_j\}$ is also a Cauchy sequence in the norm $\| \ \|_{\rho_\eta}$. The second part of the corollary follows directly from the fact that $\rho_\eta[\phi] = \rho[\eta\phi]$ for any $\phi \in C_0^\infty(\mathbf{R}^n)$. Q.E.D.

Assumption 1.13. Let $\mathcal{M}$ be the minimizing set associated with the operator P give in Definition 1.7. We assumed that $\mathcal{M}$ is a *proper subset* of the unit sphere S^{n-1}.

Assumption 1.13 is introduced to exclude a phenomenon known as the Efimov effect in the case of N-body Schrödinger operators. For more detailed discussion and the references, see [6, p.381–382].

Lemma 1.14. *Let Assumption* 1.13 *be satisfied. Let δ be a sufficiently small positive number, and let R be a positive number. Then there exist $\alpha = \alpha_{\delta,R}, \beta = \beta_{\delta,R} \in C_0^\infty(\mathbf{R}^n)$ satisfying*

(i) $\alpha(x), \beta(x) \in [0,1]$ *and* $\alpha(x)^2 + \beta(x)^2 \equiv 1$ *for all* $x \in \mathbf{R}^n$*;*

(ii) $\mathrm{supp}(\alpha) \subset \Gamma(\mathcal{M}_\delta; R/2)$, *with* $\alpha \equiv 1$ *in* $\Gamma(\mathcal{M}_{\frac{\delta}{2}}; R)$*;*

(iii) $\mathrm{supp}(\beta) \subset X \setminus \Gamma(\mathcal{M}_{\frac{\delta}{2}}; R)$*;*

(iv) α *and* β *are homogeneous of degree 0 in* $\mathbf{R}^n \setminus B(R)$*; and*

(v) *given* $\epsilon > 0$ *there exists* $C(\epsilon) > 0$ *such that*

$$|\nabla\alpha(x)|^2 + |\nabla\beta(x)|^2 \le \left(\epsilon\alpha(x)^2 + C_\epsilon\beta(x)^2\right)\chi_\Delta/|x|^2 \quad (x \in \mathbf{R}^n),$$

where χ_Δ is the characteristic function of the set $\Delta := \Gamma(\mathcal{M}_\delta; R/2) \setminus \Gamma(\mathcal{M}_{\frac{\delta}{2}}; R)$, and $\mathcal{M}$ is the minimizing set.

For the proof see [6, Lemmas 9, 10, and Definition 11]. See also [9, Lemma 3.1]. We can take β as w in (i) of Definition 11 of [6].

Proposition 1.15. *Let Assumptions* 1.1, 1.3 *and* 1.13 *be satisfied. Let $\beta = \beta_{\delta,R}$, $\delta, R > 0$, be as in Lemma* 1.14. *Let ρ_β and H_β be as in Definitions* 1.2 *and* 1.6 *with η replaced by β, respectively. Then we have*

$$\Sigma(H_\beta) > \Sigma(H). \tag{1.34}$$

Proof. Set $\mathcal{N}(\delta) = S^{n-1} \setminus \mathcal{M}_\delta$, and let γ be a (sufficiently small) positive number. Set

$$\mathcal{N}(\delta)_\gamma = \{\omega \in S^{n-1} : \mathrm{dist}(\omega : \mathcal{N}(\delta)) < \gamma\}. \tag{1.35}$$

Let $T > R$. Then it follows that

$$K_T(H_\beta) \ge K(\mathcal{N}(\delta)_\gamma, T : P), \tag{1.36}$$

where $K_T(H_\beta)$ is as in Proposition 1.11, $K(\mathcal{N}(\delta)_\gamma, T : P)$ is as in (1.13), and we should note that $\rho_\beta[\phi] = \rho[\beta\phi]$, $\|\phi\|_\beta = \|\beta\phi\|$, and the cone

$\Gamma(\mathcal{N}(\delta)_\gamma, T)$ contains $\Gamma(\mathcal{N}(\delta), T)$. Letting $T \to \infty$ first and letting $\gamma \to 0$ next, we obtain

$$K_\infty(H_\beta) \geq K(\mathcal{N}(\delta) : P), \tag{1.37}$$

which implies by Proposition 1.11 that

$$\Sigma(H_\beta) \geq K(\mathcal{N}(\delta) : P). \tag{1.38}$$

Since $\mathrm{dist}(\mathcal{N}(\delta), \mathcal{M}) > 0$, Proposition 1.8 can be applied to get

$$\Sigma(H_\beta) \geq K(\mathcal{N}(\delta) : P) > K(\mathcal{M} : P) = \Sigma(H), \tag{1.39}$$

which completes the proof Q.E.D.

Theorem 1.17, which is one of our main results in this section, is the application of an abstract result by Glazman [10] to the operator H. Here we are going to give his result as follows:

Proposition 1.16 (Glazman, [10, p.13–15]). *Let A be a selfadjoint operator defined in a Hilbert space $\mathcal{H}$. Let λ_0 be a fixed real number. Let $E(\cdot)$ be the spectral measure associated with A. Then the dimension of $E((-\infty, \lambda_0))\mathcal{H}$ is finite if and only if there exists a linear subspaces F and G of $\mathcal{H}$ such that $\dim G < \infty$, $\mathcal{H}$ is the direct sum of F and G, and*

$$(Af - \lambda_0 f, f) \geq 0 \qquad (f \in F \cap D(A)), \tag{1.40}$$

where $(\ ,\ )$ denotes the inner product of $\mathcal{H}$, and $D(A)$ denotes the domain of A. Then the number of eigenvalues λ of A such that $\lambda < \lambda_0$ does not exceed the dimension of G.

Since the proof is given rather implicitly in Glazman [10], we shall give a proof in Appendix.

Let $\epsilon > 0$. In order to give a sufficient condition for the finiteness of the bound states of H, we are going to introduce an operator P_ϵ defined by

$$\begin{cases} P_\epsilon = -\displaystyle\sum_{j,k=1}^{n} \partial_j a^{jk}(x)\partial_k + q_\epsilon(x), \\ q_\epsilon(x) = q(x) - \dfrac{\epsilon}{|x|^2}\chi_\Delta, \end{cases} \tag{1.41}$$

where χ_Δ is as in Lemma 1.14. Since the behavior of q_ϵ at infinity is the same as q, we have

$$\Sigma(H_\epsilon) = K(M : P_\epsilon) = K(M : P) = \Sigma(H). \tag{1.42}$$

Theorem 1.17 ([7, Theorem 13]). *Let Assumptions* 1.1, 1.3, *and* 1.13 *hold. Suppose that there exist* $\delta_0 > 0$, $\epsilon > 0$, *and* $R_0 > 0$ *such that*

$$K(M_\delta, R; P_\epsilon) = \Sigma(H) \qquad \text{for all } \delta \le \delta_0, \text{ and } R \ge R_0. \tag{1.43}$$

Then H has no more than a finite number of eigenvalues in $(-\infty, \Sigma(H))$.

Proof. (1) Let $\alpha = \alpha_{\delta_0,R_0}$, $\beta = \beta_{\delta_0,R_0}$ be as in Lemma 1.14. Let $\phi \in C_0^\infty(\mathbf{R}^n)$. Then, using the IMS localization formula (Ismagilov [12], Morgan [14], Morgan and Simon [15]), and (v) of Lemma 1.14, we have

(1.44)
$$\begin{aligned}\rho[\phi] &= \int_{\mathbf{R}^n} \{|\nabla(\alpha\phi)|_A^2 + q|\alpha\phi|^2 - (|\alpha|_A^2 + |\beta|_A^2)|\phi|^2\}\,dx + \rho_\beta[\phi] \\ &\ge \int_{\mathbf{R}^n} \{|\nabla(\alpha\phi)|_A^2 + q_\epsilon|\alpha\phi|^2\}\,dx + \rho_\beta[\phi] - \int_{\mathbf{R}^n} \frac{C_\epsilon}{|x|^2}\chi_\Delta|\beta\phi|^2\,dx,\end{aligned}$$

where C_ϵ is a positive constant depending only on ϵ and χ_Δ is as in Lemma 1.14 with R and δ replaced by R_0 and δ_0. Then (1.44) is combined with (1.43) to give

(1.45)
$$\rho[\phi] \ge \Sigma(H)\|\alpha\phi\|^2 + \rho_\beta[\phi] - \int_{\mathbf{R}^n} \frac{C_\epsilon}{|x|^2}\chi_\Delta|\beta\phi|^2\,dx \qquad (\phi \in C_0^\infty(\mathbf{R}^n)).$$

(2) Define the linear form ρ'_β on $C_0^\infty(\mathbf{R}^n) \times C_0^\infty(\mathbf{R}^n)$ by

$$\rho'_\beta[\phi, \varphi] = \rho_\beta[\phi, \varphi] - \int_{\mathbf{R}^n} \frac{C_\epsilon}{|x|^2}\chi_\Delta\beta\phi\beta\overline{\varphi}\,dx. \tag{1.46}$$

Then, since the potential

$$q'(x) = q(x) - \frac{C_\epsilon}{|x|^2}\chi_\Delta(x) \tag{1.47}$$

satisfies Assumptions 1.1 and 1.3, the linear form ρ'_β is closable with its closure $\tilde{\rho}'_\beta$. Let H'_β be the selfadjoint operator in $L_{2,\beta}(\mathbf{R}^n)$ determined through $\tilde{\rho}'_\beta$. Thus, using the denseness of $C_0^\infty(\mathbf{R}^n)$ in $D(\tilde{\rho})$ and Corollary 1.12, we obtain from (1.45)

$$\tilde{\rho}[u] \ge \Sigma(H)\|\alpha u\|^2 + \tilde{\rho}'_\beta[u] \qquad (u \in D(\tilde{\rho})). \tag{1.48}$$

(3) By noting that $q'(x) - q(x) \to 0$ uniformly as $|x| \to \infty$, it follows from Propositions 1.11 and 1.15 that

$$\Sigma(H'_\beta) = \Sigma(H_\beta) > \Sigma(H). \tag{1.49}$$

Therefore, the spectrum of H'_β in $(-\infty, \Sigma(H))$ is only a finite number of eigenvalues with finite multiplicity. Let $\varphi_1, \varphi_2, \cdots, \varphi_m$ be the eigenfunctions corresponding to these eigenvalues. Set

$$F = \{u \in L_2(\mathbf{R}^n) : (u, \beta^2 \varphi_j) = 0, \ j = 1, 2, \cdots, m\}. \tag{1.50}$$

Then $F^\perp$ is the linear m-dimensional subspace spanned by φ_j, $j = 1, 2, \cdots, m$. Let $u \in D(\tilde{\rho}) \cap F$. Then it follows from the second representation theorem of the closed symmetric linear form (e.g., Kato [13, Chapter IV, Theorem 2.23]) that

$$\begin{aligned} \tilde{\rho}'_\beta[u] &= \int_{\mathbf{R}} \lambda \, d\|E'_\beta(\lambda)u\|^2_\beta \\ &= \int_{\Sigma(H)}^{\infty} \lambda \, d\|E'_\beta(\lambda)u\|^2_\beta \\ &\geq \Sigma(H)\|\beta u\|^2, \end{aligned} \tag{1.51}$$

where $E'_\beta(\cdot)$ is the spectral measure associated with H'_β. This, together with (1.48), gives

$$\tilde{\rho}[u] \geq \Sigma(H)\{\|\alpha u\|^2 + \|\beta u\|^2\} = \Sigma(H)\|u\|^2 \tag{1.52}$$

for any $u \in D(\tilde{\rho}) \cap F$, and hence we have

$$(Hu - \Sigma(H)u, u) \geq 0 \qquad (u \in D(H) \cap F). \tag{1.53}$$

Thus, the condition (1.40) in Proposition 1.16 was verified, which completes the proof. Q.E.D.

Corollary 1.18. *The number of eigenvalues of H below $\Sigma(H)$ is less than the number of eigenvalues of H'_β below $\Sigma(H)$ for H'_β given above.*

Remark 1.19. Notice the gap between the conditions (1.17) in Theorem 1.9 and (1.43) in Theorem 1.17. We are led to the following question:

Under Assumptions 1.1, 1.3 *and* 1.13, *are there conditions which can be imposed upon M that will insure that* (1.17) *is a necessary and sufficient condition for the finiteness of $\sigma(H) \cap (-\infty, \Sigma(H))$?*

When stronger conditions are imposed on q, then (1.17) (with S^{n-1} substituted for M_δ and the location of M left unspecified) is known to be a sufficient condition for the finiteness of $\sigma(H) \cap (-\infty, \Sigma(H))$, see Simon [18, pp.517–518], and the related "open question" on p.518 of that article. However, these stronger conditions do not include N-body systems for $N \geq 3$.

Recently Donig [3] answered the open question in the affirmative. While the conditions imposed on his potential is slightly more strict than ours, Coulomb potentials satisfy his condition.

§2. Schrödinger operators of atomic type

In this section we consider the $(N+1)$-body Schrödinger operator of atomic-type

$$(2.1) \qquad P = P_N = \sum_{i=1}^{N} \left(-\frac{1}{2m_i}\Delta_i + v_{0i}(x^i) \right) + \sum_{1 \leq i < j \leq N} v_{ij}(x^i - x^j),$$

in $\mathbf{R}^{\nu N}$, where $N \geq 3$,

$$(2.2) \qquad \begin{cases} x^i = (x_1^i, x_2^i, \cdots, x_\nu^i) \in \mathbf{R}^\nu & (i = 1, 2, \cdots, N), \\ m_i > 0 \qquad (i = 1, 2, \cdots, N), \\ x = (x^1, x^2, \cdots, x^N) \in \mathbf{R}^{\nu N}, \end{cases}$$

and Δ_i is the Laplacian in $\mathbf{R}^\nu$ with respect to the variables $x^i = (x_1^i, x_2^i, \cdots, x_\nu^i)$ with $\nu \geq 3$. The atomic Hamiltonian is given by

$$(2.3) \qquad P = P(N, Z) = \sum_{i=1}^{N} (-\frac{1}{2m}\Delta_i - \frac{Z}{|x^i|}) + \sum_{1 \leq i < j \leq N} \frac{1}{|x^i - x^j|},$$

where N, ν are as above, and m and Z are positive numbers corresponding to the mass and charge of the nucleus, respectively.

The sesquilinear form ρ associated with the operator (2.1) is given by

$$(2.4) \qquad \begin{aligned} \rho[\phi, \varphi] = & \sum_{i=1}^{N} \frac{1}{2m_i} \int_{\mathbf{R}^{\nu N}} \nabla^i \phi(x) \cdot \overline{\nabla^i \varphi} \, dx \\ & + \sum_{i=1}^{N} \int_{\mathbf{R}^{\nu N}} v_{0i}(x^i) \phi(x) \overline{\varphi(x)} \, dx \\ & + \sum_{1 \leq i < j \leq N} \int_{\mathbf{R}^{\nu N}} v_{ij}(x^i - x^j) \phi(x) \overline{\varphi(x)} \, dx \end{aligned}$$

for $\phi, \varphi \in C_0^\infty(\mathbf{R}^{\nu N})$, where

$$\nabla^i = (\frac{\partial}{\partial x_1^i}, \frac{\partial}{\partial x_2^i}, \cdots, \frac{\partial}{\partial x_\nu^i}) \tag{2.5}$$

for $i = 1, 2, \cdots, N$. For the potentials v_{ij}, we assume the following

Assumption 2.1. For $0 \le i < j \le N$, v_{ij} is a real-valued function satisfying

(i) $v_{ij} \in L^1_{\text{loc}}(\mathbf{R}^\nu)$,
(ii) $\lim_{|y|\to\infty} v_{ij}(y) = 0$, and
(iii) $(v_{ij})_- \in M(\mathbf{R}^\nu)$.

Then, setting

$$V_{ij}(x) = \begin{cases} v_{0j}(x^j) & (i = 0, j = 1, 2, \cdots, N), \\ v_{ij}(x^i - x^j) & (1 \le i < j \le N), \end{cases} \tag{2.6}$$

where $x = (x^1, x^2, \cdots, x^N) \in \mathbf{R}^{\nu N}$ as in (2.2), and

$$q(x) = \sum_{j=1}^N V_{0j}(x) + \sum_{1\le i<j\le N} V_{ij}(x), \tag{2.7}$$

we easily see that $q(x)$ satisfies Assumptions 1.1 and 1.3 (see Agmon [1, Lemma 4.7] for the proof that $q_- \in M(\mathbf{R}^{\nu N})$. Thus, the corresponding sesquilinear form ρ (or, more exactly, the closure $\tilde{\rho}$ of ρ) determines a selfadjoint operator in $L_2(\mathbf{R}^{\nu N})$. Henceforth, the selfadjoint realization will be denoted by P again.

We are now introducing the subsystems of the operator P.

Definition 2.2 (Subsystems of P).

Let $S^{\nu N-1}$ be the unit sphere of $\mathbf{R}^{\nu N}$. For $\omega \in S^{\nu N-1}$ define the subsystem P_ω of P by

$$P_\omega = -\sum_{j=1}^N \frac{1}{2m_j}\Delta_j + \sum_{\omega^i=0} v_{0i}(x^i) + \sum_{\omega^i=\omega^j} v_{ij}(x^i - x^j), \tag{2.8}$$

where $\omega = (\omega^1, \omega^2, \cdots, \omega^N)$ and $\sum_{\omega^i=0}$ [or $\sum_{\omega^i=\omega^j}$] means summation over those indices i for which $\omega^i = 0$ [or those pair of indices (i, j), $1 \le i < j \le N$, for which $\omega^i = \omega^j$]. The selfadjoint realization of P_ω in $L^2(\mathbf{R}^{\nu N})$ will continue to be denoted by P_ω.

The following fact given by Agmon [1, Lemma 4.8, p.66] will play an important role:

Proposition 2.3 ($K(\omega)$ and subsystems (Agmon [1, Lemma 4.8])). *Let P be as in* (2.2) *and satisfy Assumption* 2.1. *Let P_ω be the subsystem of P defined above. Then, for any $\omega \in S^{\nu N-1}$*

$$K(\omega; P) = K(\omega; P_\omega) = \Sigma(P_\omega) = \Lambda(P_\omega), \tag{2.9}$$

where $\Lambda(A)$ and $\Sigma(A)$ denote the infimum of the spectrum and essential spectrum of A, respectively.

Let $\mathcal{M}$ be the minimizing set for the Schrödinger operator P of atomic type (see Definition 1.7).

Definition 2.4 (Sets $\mathcal{M}_i$ and subsystems P_i). For $i = 1, 2, \cdots, N$, define

(2.10)
$$\mathcal{M}_i = \{\omega = (\omega^1, \omega^2, \cdots, \omega^N) \ : \omega^j = \delta_{ij}\eta \text{ for } \eta \in S^{\nu-1}, j = 1, 2, \cdots, N\},$$

where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $j \neq i$. The set $\mathcal{M}_i$ is a closed subset of $S^{\nu N-1}$. Let P_ω be given by (2.8). Since for any $\omega \in \mathcal{M}_i$ the subsystem P_ω has the same form, we set $P_\omega = P_i$ for $\omega \in \mathcal{M}_i$, i.e.,

$$P_i = -\sum_{j=1}^{N} \frac{1}{2m_j}\Delta_j + \sum_{j\neq i} v_{oj}(x^j) + \sum_{\substack{1\leq j<k\leq N \\ j\neq i \text{ and } k\neq i}} v_{jk}(x^j - x^k). \tag{2.11}$$

The subsystem P_i is the subsystem of $(N-1)$ electrons $x^1, \cdots, x^{i-1}, x^{i+1}, \cdots, x^N$.

In this section we assume that the lower bound $\Sigma(P)$ of the essential spectrum of P is determined only by subsystems of $N-1$ electrons.

Assumption 2.5. Let P be the atomic-type Hamiltonian (2.1). Let $\mathcal{M}$ be the minimizing set of P. Assume that

$$\mathcal{M} \subset \bigcup_{i=1}^{N} \mathcal{M}_i. \tag{2.12}$$

Assumption 2.5 implies that the minimizing set $\mathcal{M}$ is not only a closed set of $S^{\nu N-1}$, but also a proper subset of $S^{\nu N-1}$. Thus, this assumption implies Assumption 1.12 for our operator P.

Definition 2.6 (Operators P'_i and L_i). Let P be as above and for each $i = 1, 2, \cdots, N$ define

$$(2.13) \qquad P'_i = -\sum_{j\neq i} \frac{1}{2m_j}\Delta_j + \sum_{j\neq i} v_{0j}(x^j) + \sum_{\substack{1\leq j<k\leq N \\ j\neq i \text{ and } k\neq i}} v_{jk}(x^j - x^k),$$

The selfadjoint realization of P'_i in $L_2(\mathbf{R}^{\nu(N-1)})$ is also denoted by P'_i. We also set

$$(2.14) \qquad L_i = P - P'_i = -\frac{1}{2m_i}\Delta_i + v_{0i}(x^i) + \sum_{\substack{1\leq j<k\leq N \\ j=i \text{ or } k=i}} v_{jk}(x^j - x^k).$$

Now we are in a position to give a criterion for the finiteness of the bound states of the atomic-type Hamiltonian P.

Theorem 2.7 (Finiteness of bound states ([7, Theorem 3.4)).
Let P be given by (2.1) *and let Assumptions* 2.1 *and* 2.5 *be satisfied. Let P'_i and L_i be as above. Suppose there exist positive numbers δ_0, R_0, and ϵ such that*

$$(2.15) \qquad (L_i\phi, \phi)_{L^2(\mathbf{R}^{\nu N})} \geq \int_{\mathbf{R}^{\nu N}} \frac{\epsilon}{|x|^2}|\phi|^2\, dx$$

for each $i = 1, 2, \cdots, N$ such that $\mathcal{M}_i \subset \mathcal{M}$ and for every $\phi \in C_0^\infty(\Gamma((\mathcal{M}_i)_{\delta_0}; R_0))$. Then P has at most a finite number of bound states.

For the proof see the proof of Theorem 3.4 in [7].

Let us next discuss the infiniteness of the bound states. It follows from Assumption 2.5 that there exist some $i \subset \{1, 2, \cdots, N\}$ such that $\mathcal{M}_i \subset \mathcal{M}$. In view of Theorem 1.9 we are looking for a condition which guarantees the existence of a sequence of functions $\{F_n\}$ such that

$$(2.16) \qquad F_n \in C_0^\infty(\Gamma(\mathcal{M}_i)_{\delta_n}; R_n))$$

with $\delta_n \downarrow 0$ and $R_n \uparrow \infty$ as $n \to \infty$, and

$$(2.17) \qquad \rho[F_n] = (PF_n, F_n)_{L^2(\mathbf{R}^{\nu N})} < \Sigma(P) \qquad (n = 1, 2, \cdots, N),$$

which gives the inequality (1.21) immediately. Write $x \in \mathbf{R}^{\nu N}$ as

$$(2.18) \qquad x = (x^i, x') \qquad (x' = (x^1, \cdots, x^{i-1}, x^{i+1}, \cdots, x^N)).$$

We are going to find F_n with the form

$$F_n(x^i, x') = \theta_n(x^i)\phi_n(x') \qquad (n = N_0, N_0 + 1, \cdots), \tag{2.19}$$

where N_0 is a positive integer determined later. As for θ_n, we have the following

Proposition 2.8 [7, Proposition 4.4]. *Let $q > 1$. Then there exists a sequence $\{\theta_n\} = \{\theta_{n,q}\}$ of functions on $\mathbf{R}^\nu$ such that, for $n = 1, 2, \cdots$,*
1) $\theta_n \in C_0^\infty(\mathbf{R}^\nu)$,
2) $\|\theta_n\|_{L^2(\mathbf{R}^\nu)} = 1$,
3) $\operatorname{supp} \theta_n \subset \{x^i \in \mathbf{R}^\nu : n^q \leq |x^i| \leq 5n^q\}$,
4) *there exists a constant $C_2 = C_2(q)$, independent of $n = 1, 2, \cdots$, satisfying*

$$0 \leq (-\frac{1}{2m_i}\Delta_i\theta_n, \theta_n)_{L^2(\mathbf{R}^\nu)} \leq \frac{C_2}{n^{2q}}.$$

The construction of θ_n, $n = 1, 2, \cdots$, is easy and direct. See the proof of Proposition 4.4 of [7].

In order to discuss the construction of $\phi_n(x')$, we need the next

Assumption 2.9. The potentials v_{ij}, $0 \leq i < j \leq N$, satisfy

$$v_{ij} \in M_{\mathrm{loc}}(\mathbf{R}^\nu). \tag{2.20}$$

Let i be as above. Then it follows from the HVZ theorem (see [11], [19], [22]) combined with Assumption 2.5 that $\Sigma(P) = \Lambda(P_i') < 0$ and $\Sigma(P_i') > \Lambda(P_i')$, and hence $\Lambda(P_i')$ is the lowest eigenvalue (ground state) of P_i' with the eigenfunction $\Phi_i(x')$. In fact, suppose that $\Sigma(P_i') = \Lambda(P_i')$. Then we see from the HVZ theorem that there should exist a subsystem P_i'' of P_i' such that

$$\Lambda(P_i'') = \Sigma(P_i') = \Lambda(P_i') = \Sigma(P). \tag{2.21}$$

This contradicts Assumption 2.5 since the lower bound $\Lambda(P_i'')$ of the subsystem P_i'', which is different from any P_j', $j = 1, 2, \cdots, N$ coincides with $\Sigma(P)$. It follows from [1, Theorem 5.9] that the eigenfunction $\Phi_i(x')$ decays exponentially. Similarly, using Assumption 2.9, too, we can prove that any first derivatives of $\Phi_i(x')$ decay exponentially ([7, Proposition 4.2]). Now we shall prove that $\phi_n(x')$ in (2.19) can be constructed by truncating Φ_i using a smooth function, and then approximating with functions in $C_0^\infty(\mathbf{R}^{\nu(N-1)})$.

Proposition 2.10. *Let Assumptions* 2.1, 2.5, *and* 2.9 *hold. Then, for some positive integer* N_0 *and each integer* $n \geq N_0$, *there exists* $\phi_n \in C_0^\infty(\mathbf{R}^{\nu(N-1)})$ *satisfying*

$$\begin{cases} \|\phi_n\|_{L^2(\mathbf{R}^{\nu(N-1)})} = 1, \\ \operatorname{supp}\phi_n \subset \{x' \in \mathbf{R}^{\nu(N-1)} \ : |x'| \leq 2n\}, \\ (P_i'\phi_n, \phi_n)_{L^2(\mathbf{R}^{\nu(N-1)})} \leq \Sigma(P) + C_1 \dfrac{e^{-nc_0}}{n} \end{cases} \tag{2.22}$$

with positive constants c_0 *and* C_1.

For an integer $1 \leq i \leq N$ set

$$I_i(x) = v_{0i}(x^i) + \sum_{\substack{1 \leq j < k \leq N \\ j=i \text{ or } k=i}} v_{jk}(x^j - x^k). \tag{2.23}$$

We have $P = P_i + I_i$.

Theorem 2.11 (Infiniteness of bound states, [7, Theorem 4.7]).
Let Assumptions 2.1, 2.5 *and* 2.9 *be satisfied. Suppose that there exists an integer* $1 \leq i \leq N$, $\mathcal{M}_i \subset \mathcal{M}$, *positive numbers* δ_0, R_0, c_*, *and* $s \in (0, 2)$ *such that*

$$I_i(x) \leq -c_*|x^i|^{-s} \qquad (x \in \Gamma((\mathcal{M}_i)_{\delta_0}; R_0)). \tag{2.24}$$

Then P *has infinitely many bound states.*

For the proof see the proof of [7, Theorem 4.7] and [7, Proposition 4.5]. We have only to show that the sequence $\{F_n\}$ above satisfies (2.17).

The following theorem on the finiteness and infiniteness of the bound states for the atomic Hamiltonian is well-known: Zhislin ([22], [23], [24], [25]), Yafaev ([20], [21]), and others.

Theorem 2.12 (Zhislin ([22], [23], [24], [25]), Yafaev ([20], [21]), and others). *Let* $N, \nu \geq 3$ *be integers. Suppose that Assumption* 3.2 *is satisfied for* $P = P(N, Z)$ *given by* (2.2).

(i) *Suppose that*

$$Z \leq N - 1. \tag{2.25}$$

Then $P = P(N, Z)$ *has at most a finite number of bound states.*

(ii) *Suppose that*

$$Z > N - 1. \tag{2.26}$$

Then $P = P(N, Z)$ *has infinitely many bound states.*

Using Theorems 2.7 and 2.11 we can give a proof of the above celebrated theorem except the case $Z = N - 1$. Let $Z < N - 1$. Since it is easy to see that, for $x \in \Gamma((\mathcal{M}_i)_\delta; R)$ with $0 < 2\delta < 1$, we have

$$\begin{cases} |x^i| > (1-\delta)|x|, \\ |x^i - x^j| < (1+2\delta)|x|, \end{cases} \tag{2.27}$$

it follows that, for $\phi \in C_0^\infty(\Gamma((\mathcal{M}_i)_\delta; R))$,

$$\begin{aligned}(L_i\phi, \phi)_{L^2(\mathbf{R}^{\nu N})} &\geq \int_{\mathbf{R}^{\nu N}} \Big[-\frac{Z}{|x^i|} + \sum_{j \neq i} |x^i - x^j|\Big] |\phi|^2 \, dx \\ &\geq \int_{\mathbf{R}^{\nu N}} \Big[\frac{N-1}{(1+2\delta)|x|} - \frac{Z}{(1-\delta)|x|}\Big] |\phi|^2 \, dx \\ &\geq \int_{\mathbf{R}^{\nu N}} \frac{\epsilon}{|x|^2} |\phi|^2 \, dx \end{aligned} \tag{2.28}$$

if $\delta > 0$ is sufficiently small and $R > 1$, where

$$\epsilon = \frac{N-1}{1+2\delta} - \frac{Z}{1-\delta} > 0. \tag{2.29}$$

Thus we see that the condition (2.15) in Theorem 2.7 is satisfied for every $i = 1, 2, \cdots, N$. In the case that $Z > N - 1$, see the proof of Theorem 4.8 of [7].

Concerning Assumption 2.5, [7] gave a proof of the following theorem (Theorem 5.2):

Theorem 2.13. *Let* $N, \nu \geq 3$ *be integers and* $P = P(N, Z)$ *be as in* (2.2). *Suppose that*

$$Z > N - 2. \tag{2.30}$$

Then the operator $P = P(N, Z)$ *satisfies Assumption* 2.5, *i.e., the lower bound of* P *is determined only by subsystems of* $N - 1$ *electrons.*

Finally consider the case where the potentials are "short-range", i.e., $v_{ij} \in L_{\nu/2}(\mathbf{R}^\nu)$. It is known that the bound states are finite in this case (Sigal [17]). We are going to give another simple proof for the slightly more general version.

Theorem 2.14. *Let Assumptions* 2.1 *and* 2.5 *hold. Suppose that*

$$(2.31)\qquad (v_{jk})_-(\cdot) \in L_{\nu/2}(\mathbf{R}^\nu) \qquad (0 \le j < k \le N,\ j = i \ \textit{or}\ k = i)$$

for any i such that $\mathcal{M}_i \subset \mathcal{M}$, where $(v_{jk})_-$ is the negative part of v_{jk}. Then the operator P given by (2.1) *has at most finite bound states.*

Proof. Let $\epsilon > 0$. Let δ_0 be a positive number such that $1-2\delta_0 > 0$. Then there exists $R_0 > 0$ satisfying

(2.32)
$$\Big[\int_{|y|>cR_0} \{(v_{jk})_-(y)\}^{\nu/2}\,dy\Big]^{2/\nu} < \epsilon \qquad \le j < k \le N,\ j = i \ \text{or}\ k = i),$$

where $c = 1 - 2\delta_0$. Let $\phi \in C_0^\infty(\Gamma((\mathcal{M}_i)_{\delta_0}; R_0))$. Since we have

$$(2.33)\qquad x \in \Gamma((\mathcal{M}_i)_{\delta_0}; R_0) \implies \begin{cases} |x^i| > (1-\delta_0)R_0, \\ |x^i - x^j| > (1-2\delta_0)R_0, \end{cases}$$

it follows from the Hölder inequality that

(2.34)
$$\begin{cases} \displaystyle\int_{\mathbf{R}^\nu} (v_{0i})_-(x^i)|\phi|^2\,dx^i \\ \displaystyle\qquad \le \Big[\int_{|y|>cR_0} \{(v_{0i})_-\}^{\nu/2}\,dx^i\Big]^{2/\nu}\Big[\int_{\mathbf{R}^\nu} |\phi|^{2\nu/(\nu-2)}\,dx^i\Big]^{(\nu-2)/\nu}, \\ \displaystyle\int_{\mathbf{R}^\nu} (v_{jk})_-(x^j - x^k)|\phi|^2\,dx^i \\ \displaystyle\qquad \le \Big[\int_{|y|>cR_0} \{(v_{jk})_-\}^{\nu/2}\,dx^i\Big]^{2/\nu}\Big[\int_{\mathbf{R}^\nu} |\phi|^{2\nu/(\nu-2)}\,dx^i\Big]^{(\nu-2)/\nu}, \end{cases}$$

where $1 \le j < k \le N$, $j = i$ or $k = i$. It follows from a Sobolev-type inequality (e.g., [4, Theorem III.3.6]) that

$$(2.35)\qquad \Big[\int_{\mathbf{R}^\nu} |\phi|^{2\nu/(\nu-2)}\,dx^i\Big]^{(\nu-2)/\nu} \le \gamma \int_{\mathbf{R}^\nu} |\nabla^i\phi|^2\,dx^i,$$

γ being a positive constant depending only on ν. Then we obtain from

(2.34) and (2.35) that

(2.36)

$$\int_{\mathbf{R}^\nu} [(v_{0i})_-(x^i) + \sum_{\substack{1\le j<k\le N \\ j=i \text{ or } k=i}} (v_{jk})_-(x^j - x^k)]|\nabla^i \phi|^2\, dx^i \le (2N-1)\epsilon \int_{\mathbf{R}^\nu} |\nabla^i \phi|^2\, dx^i$$

for any $\phi \in C_0^\infty(\Gamma((\mathcal{M}_i)_{\delta_0}; R_0))$. The inequality (2.36) is combined with the Hardy inequality

$$\int_{\mathbf{R}^\nu} \frac{|\phi|^2}{|x|^2}\, dx^i \le \int_{\mathbf{R}^\nu} \frac{|\phi|^2}{|x^i|^2}\, dx^i \le \frac{4}{(\nu-2)^2} \int_{\mathbf{R}^\nu} |\nabla^i \phi|^2\, dx^i, \tag{2.37}$$

where $\phi \in C_0^\infty(\mathbf{R}^{\nu N})$, to give

(2.38)

$$(L_i\phi, \phi)_{L_2(\mathbf{R}^{\nu N})} - \int_{\mathbf{R}^{\nu N}} \frac{\epsilon|\phi|^2}{|x|^2}\, dx \ge \int_{\mathbf{R}^{\nu N}} [1 - (2N-1)\gamma\epsilon - \frac{4\epsilon}{(\nu-2)^2}]|\nabla^i\phi|^2\, dx$$

for any $\phi \in C_0^\infty(\Gamma((\mathcal{M}_i)_{\delta_0}; R_0))$. Therefore, choosing $\epsilon > 0$ sufficiently small, we see that the right-hand side of (2.37) is nonnegative for $\phi \in C_0^\infty(\Gamma((\mathcal{M}_i)_{\delta_0}; R_0))$. Thus the condition (2.15) is satisfied, which complete the proof. Q.E.D.

Appendices

A.1 The infimum of the essential spectrum of H_η

Proof of Proposition 1.11.

(1) Let $\Lambda \in \sigma_e(H_\eta)$ with a singular sequence $\{u_j\}$, i.e.,

$$\begin{cases} \text{(a) } u_j \in D(H_\eta) \qquad (j = 1, 2, \cdot), \\ \text{(b) } \|u_j\|_\eta = 1 \qquad (j = 1, 2, \cdot), \\ \text{(c) } \mathrm{w} - \lim_{j\to\infty} u_j = 0 \qquad in L_{2,\eta}(\mathbf{R}^n), \\ \text{(d) } \mathrm{s} - \lim_{j\to\infty} (H_\eta - \lambda)u_j = 0 \qquad in L_{2,\eta}(\mathbf{R}^n). \end{cases} \tag{A.1.1}$$

Introduce an inner product $(\ ,\ )_{\rho_\eta}$ and norm $\|\ \|_{\rho_\eta}$ in $C_0^\infty(\mathbf{R}^n)$ by

$$\text{(A.1.2)}\qquad \begin{cases} (\phi,\varphi)_{\rho_\eta} = \rho_\eta[\phi,\varphi] + C_1(\phi,\varphi)_\eta, \\ \|\phi\|_{\rho_\eta} = [(\phi,\phi)_{\rho_\eta}]^{1/2}, \end{cases}$$

where the positive constant C_1 is as in (1.10). Note that we obtain from (1.10)

$$\text{(A.1.3)}\qquad \begin{cases} \int_{\mathbf{R}^n} |\nabla(\eta\phi)|^2\,dx \le \dfrac{2}{c_0}\|\phi\|_{\rho_\eta}^2, \\ \|\phi\|_\eta \le \|\phi\|_{\rho_\eta} \end{cases}$$

for $\phi \in C_0^\infty(\mathbf{R}^n)$. Then $C_0^\infty(\mathbf{R}^n)$ becomes a pre-Hilbert space with the inner product $(\ ,\ )_{\rho_\eta}$ and norm $\|\ \|_{\rho_\eta}$, and the domain $D(\tilde{\rho}_\eta)$ of the closed linear form $\tilde{\rho}_\eta$ is the completion of $C_0^\infty(\mathbf{R}^n)$ by $\|\ \|_{\rho_\eta}$. The inner product and norm of $D(\tilde{\rho}_\eta)$ will be denoted again by $(\ ,\ )_{\rho_\eta}$ and norm $\|\ \|_{\rho_\eta}$. We have

$$\text{(A.1.4)}\qquad \begin{cases} (u,v)_{\rho_\eta} = \tilde{\rho}_\eta[u,v] + C_1(u,v)_\eta, \\ \|u\|_{\rho_\eta} \ge \|\phi\|_\eta^2 \end{cases}$$

for $u, v \in D(\tilde{\rho}_\eta)$.

(2) Since $C_0^\infty(\mathbf{R}^n)$ is dense in the Hilbert space $D(\tilde{\rho}_\eta)$, there exists a sequence $\{\phi_j\} \subset C_0^\infty(\mathbf{R}^n)$ such that

$$\text{(A.1.5)}\qquad \|u_j - \phi_j\|_{\rho_\eta} \to 0 \qquad (j \to \infty).$$

Then it follows that

$$\text{(A.1.6)}\qquad \begin{cases} \text{(a) } \|\phi_j\|_\eta \to 1 \qquad (j \to \infty), \\ \text{(b) } \mathrm{w} - \lim\limits_{j\to\infty} \phi_j = 0 \qquad \text{in } D(\tilde{\rho}_\eta), \\ \text{(c) } \rho_\eta[\phi_j] \to \lambda \qquad (j \to \infty). \end{cases}$$

In fact, (a) follows directly from (b) of (A.1.1) and (A.1.3). As for (b), we have for any $v \in D(\tilde{\rho}_\eta)$

(A.1.7)

$$\begin{aligned} (\phi_j, v)_{\rho_\eta} &= (\phi_j - u_j, v)_{\rho_\eta} + (u_j, v)_{\rho_\eta} \\ &= (\phi_j - u_j, v)_{\rho_\eta} + ((H_\eta - \lambda)u_j, v)_\eta + (\lambda + C_1)(u_j, v) \\ &\to 0 \end{aligned}$$

as $j \to \infty$, where we have used (c), (d) of (A.1.1), and we should note that

$$\tilde{\rho}_\eta[u_j, v] = (H_\eta u_j, v)_\eta \tag{A.1.8}$$

(see, e.g., Kato [13, Theorem VI.2.1, p.322]). Finally, since we have

$$\begin{aligned} \rho_\eta[\phi_j] &= \|\phi_j\|^2_{\rho_\eta} - C_1\|\phi_j\|^2_\eta \\ &= \|u_j\|^2_{\rho_\eta} - C_1\|u_j\|^2_\eta + \gamma_j \\ &= \tilde{\rho}_\eta[u_j] + \gamma_j \\ &= ((H_\eta - \lambda)u_j, u_j)_\eta + \lambda\|u_j\|^2_\eta + \gamma_j \\ &= \lambda + ((H_\eta - \lambda)u_j, u_j)_\eta + \gamma_j, \end{aligned} \tag{A.1.9}$$

where $\gamma_j \to 0$ and $((H_\eta - \lambda)u_j, u_j)_\eta$ converge to 0 as $j \to \infty$, we obtain (d).

(3) Let $\alpha(x)$ be a C^∞ function on $\mathbf{R}^n$ such that

$$\alpha(x) = \begin{cases} 0 & x \in B_R = \{x \in \mathbf{R}^n : |x| \le R\}, \\ 1 & x \in E_{R+1}, \end{cases} \tag{A.1.10}$$

$0 \le \alpha \le 1$, and $|\nabla\alpha|$ is bounded on $\mathbf{R}^n$. Set $|\xi|_A = [< \xi, \xi >_A]^{1/2}$ for $\xi \in \mathbf{C}^n$. Then it follows from the identity

(A.1.11)

$$|\nabla(\alpha\eta\phi)|^2_A = \alpha^2|\nabla(\eta\phi)|^2_A + |\nabla\alpha|^2_A|\eta\phi|^2 + 2\alpha\eta\Re\{\overline{\phi} < \nabla(\eta\phi), \nabla\alpha >_A\},$$

where $\phi \in C_0^\infty(\mathbf{R}^n)$, that

$$|\nabla(\alpha\eta\phi)|^2_A \le (1+\delta)|\nabla(\eta\phi)|^2_A + C_\delta\chi_{R,R+1}|\eta\phi|^2 \tag{A.1.12}$$

for $\phi \in C_0^\infty(\mathbf{R}^n)$, where δ is an arbitrary positive number, $\chi_{R,R+1}$ is the characteristic function of $\{x \in \mathbf{R}^n : R < |x| \le R+1\}$, and

$$C_\delta = (1+\delta^{-1}) \max_{x\in\mathbf{R}^n} |\nabla\alpha|^2_A. \tag{A.1.13}$$

Further we have

$$\begin{aligned} q(x)\alpha^2|\eta\phi|^2 &= q_+\alpha^2|\eta\phi|^2 - q_-\alpha^2|\eta\phi|^2 \\ &= \alpha^2 q_+|\eta\phi|^2 - q_-|\eta\phi|^2 + (1-\alpha^2)q_-|\eta\phi|^2 \\ &\le q|\eta\phi|^2 + (1-\alpha^2)q_-|\eta\phi|^2. \end{aligned} \tag{A.1.14}$$

Therefore, it follows that

(A.1.15)

$$\rho_\eta[\alpha\phi] \le (1+\delta)\rho_\eta[\phi] + C_\delta \int_{B_{R+1}} |\eta\phi|^2\, dx + \int_{\mathbf{R}^n} (1-\alpha^2)q_-|\eta\phi|^2\, dx.$$

Here it follows from Assumption 1.3 and (A.1.3) that

(A.1.16)

$$\begin{aligned}
&\int_{\mathbf{R}^n} (1-\alpha^2)q_-|\eta\phi|^2\, dx \\
&\quad \le \Big\{\int_{\mathbf{R}^n} q_-|\eta\phi|^2\, dx\Big\}^{1/2}\Big\{\int_{\mathbf{R}^n} q_-|(1-\alpha^2)\eta\phi|^2\, dx\Big\}^{1/2} \\
&\quad \le \Big\{\int_{\mathbf{R}^n} |\nabla(\eta\phi)|^2\, dx + C(1)\|\phi\|_\eta^2\Big\}^{1/2} \\
&\qquad \Big\{\delta^2 \int_{\mathbf{R}^n} |\nabla((1-\alpha^2)\eta\phi)|^2\, dx + C(\delta^2)\|(1-\alpha^2)\phi\|_\eta^2\Big\}^{1/2} \\
&\quad \le C_2\|\phi\|_{\rho_\eta}\Big\{\delta\|\phi\|_{\rho_\eta} + C'(\delta)\Big[\int_{B_{R+1}} |\eta\phi|^2\, dx\Big]^{1/2}\Big\},
\end{aligned}$$

where $C(1)$ and $C(\delta^2)$ are as in (1.6) with ϵ replaced by 1 and δ^2, respectively, C_2 is a positive constant independent of δ, and $C'(\delta)$ is a positive constant which may depend on δ. Thus, combining (A.1.14) with (A.1.15), substituting $\phi = \phi_j$, and taking note of the definition of K_R, we obtain with another constants C_2' and $C''(\delta)$

(A.1.17)

$$\begin{aligned}
&K_R\|\alpha\phi_j\|_\eta^2 \\
&\quad \le \rho_\eta[\alpha\phi_j] \\
&\quad \le (1+\delta)\rho_\eta[\phi_j] + C_2'\|\phi_j\|_{\rho_\eta}\Big\{\delta\|\phi_j\|_{\rho_\eta} + C''(\delta)\Big[\int_{B_{R+1}} |\eta\phi_j|^2\, dx\Big]^{1/2}\Big\}.
\end{aligned}$$

(4) Using (A.1.6) and the Rellich theorem, we see that, for any $0 < R < \infty$,

(A.1.18)
$$\int_{B_R} |\eta\phi_j|^2\, dx \to 0$$

as $j \to \infty$, where we should note that (c) of (A.1.1) and (A.1.5) imply that

(A.1.19)
$$\mathrm{w}-\lim_{j\to\infty} \phi_j = 0 \qquad \text{in } L_{2,\eta}(\mathbf{R}^n).$$

From (A.1.18) we see that

$$\text{(A.1.20)} \qquad \begin{aligned} &\lim_{j\to\infty} \|\alpha\phi_j\|_\eta^2 \\ &= \lim_{j\to\infty} \{\int_{\mathbf{R}^n} |\phi_j\eta|^2\,dx + \int_{\mathbf{R}^n} (1-\alpha^2)|\phi_j\eta|^2\,dx\} \\ &= \lim_{j\to\infty} \|\phi_j\|_\eta^2 \\ &= 1. \end{aligned}$$

Thus, by letting $j \to \infty$ in (A.1.17) and using (c) of (A.1.6), (A.1.18), and (A.1.20), it follows that

$$\text{(A.1.21)} \qquad K_\infty \le (1+\delta)\lambda + \delta C_2' C_3$$

with $C_3 = \sup_j \|\phi_j\|_{\rho_\eta}$. Since δ is arbitrary, we have proved that $K_\infty \le \lambda$ for any $\lambda \in \sigma_e(H_\eta)$, i.e., $K_\infty \le \Sigma(H_\eta)$.

(5) Let $\mu < \Sigma(H_\eta)$. Then in $(-\infty, \mu]$ the spectrum $\sigma(H_\eta)$ of H_η consists of a finite number (M say) of eigenvalues λ_k, $k = 1, 2, \cdots, M$, repeated according to multiplicity, with corresponding eigenfunctions $\varphi_k \in D(H_\eta) \subset D(\tilde\rho_\eta)$. Let $E_\eta(\cdot)$ be the spectral measure associated with H_η. Then note that we have

$$\text{(A.1.22)} \qquad \begin{aligned} \tilde\rho_\eta[\phi] &= (H_\eta\phi, \phi)_\eta \\ &= \sum_{k=1}^{M} \lambda_k |(\phi, \varphi_k)_\eta|^2 + \int_\mu^\infty \lambda\, d(E_\eta(\lambda)\phi, \phi)_\eta \\ &\ge \sum_{k=1}^{M} \lambda_k |(\phi, \varphi_k)_\eta|^2 + \mu \|E_\eta((\mu,\infty))\phi\|_\eta^2 \\ &= \sum_{k=1}^{M} (\lambda_k - \mu) |(\phi, \varphi_k)_\eta|^2 + \mu\|\phi\|_\eta^2 \end{aligned}$$

for $\phi \in D(H_\eta)$. Further, since $D(H_\eta)$ is dense in $D(\tilde\rho_\eta)$, the inequality (A.1.22) holds for any $\phi \in D(\tilde\rho_\eta)$. Now choose $\{\phi_j\} \subset C_0^\infty(\mathbf{R}^n)$ such that

$$\text{(A.1.23)} \qquad \begin{cases} \text{(a)} \ \lim_{j\to\infty} \rho_\eta[\phi_j] = R_\infty, \\ \text{(b)} \ \|\phi_j\|_\eta = 1 \qquad (j = 1, 2, \cdots), \\ \text{(c)} \ \text{supp}\phi_j \cap \text{supp}\phi_\ell = \emptyset \qquad (j, \ell = 1, 2, \cdots,\ j \ne \ell). \end{cases}$$

Let $\phi = \phi_j$ and make $j \to \infty$ in (A.1.22). Then it follows that

$$\text{(A.1.24)} \qquad K_\infty \ge \mu,$$

where we should note that ϕ_j converges to 0 weakly in $L_{2,\eta}(\mathbf{R}^n)$ as $j \to \infty$. Since $\mu < \Sigma(H_\eta)$ is arbitrary, we obtain $K_\infty \geq \Sigma(H_\eta)$, which completes the proof. Q.E.D.

A.2 Proof of Glazman's theorem

Proof of Proposition 1.16.

(1) Suppose that the dimension of $E((-\infty, \lambda_0))\mathcal{H}$ is finite. Then set

$$\begin{cases} F = E([\lambda_0, \infty))\mathcal{H}, \\ G = E((-\infty, \lambda_0))\mathcal{H}. \end{cases} \tag{A.2.1}$$

Then the dimension of G is finite, and $\mathcal{H}$ is the direct sum of F and G. Further, for $f \in D(A) \cap F$ we have

$$\begin{aligned} (Af, f) &= \int_{\lambda_0}^{\infty} \lambda \, d\|E(\lambda)f\|^2 \\ &\geq \lambda_0 \|E([\lambda_0, \infty))f\|^2 \\ &= \lambda_0 (f, f), \end{aligned} \tag{A.2.2}$$

where $\| \quad \|$ denotes the norm of $\mathcal{H}$, and we have used the relation $\|E([\lambda_0, \infty))f\| = \|f\|$ for $f \in F$. This implies that (1.40) is satisfied.

(2) Suppose that there exists subspaces F and G of $\mathcal{M}$ satisfying the conditions in Proposition 1.16. Set $m = \dim G$ and suppose that

$$\dim E((-\infty, \lambda_0))\mathcal{H} \geq m + 1. \tag{A.2.3}$$

Then it follows from Lemma A.1.1 that

$$E((-\infty, \lambda_0))\mathcal{H} \cap F \neq \emptyset. \tag{A.2.4}$$

In fact we can assume that there exists a nonzero element f_0 such that

$$f_0 \in E((-\infty, \lambda_0 - \mu))\mathcal{H} \cap F \cap D(A) \tag{A.2.5}$$

with $\mu > 0$ because we can choose the $m + 1$ independent elements $f_1, f_2, \cdots, f_{m+1}$ in $E((-\infty, \lambda_0)\mathcal{H}$ so that all f_j belong to $E((-\infty, \lambda_0 - \mu)\mathcal{H} \cap D(A)$, which is possible in either case where the spectrum of A in $(-\infty, \lambda_0)$ contains the essential spectrum or it consists only the discrete spectrum. Thus it follows that

$$\begin{aligned} (Af_0, f_0) &= \int_{-\infty}^{\lambda_0 - \mu} \lambda \, d\|E(\lambda)f_0\|^2 \\ &< \lambda_0 \|E((-\infty, \lambda_0 - \mu))f_0\|^2 \\ &= \lambda_0 (f_0, f_0). \end{aligned} \tag{A.2.6}$$

This contradicts (1.40). Therefore, we have shown that

(A.2.7) $$\dim E((-\infty, \lambda_0))\mathcal{H} \leq m,$$

which completes the proof. Q.E.D.

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