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Analysis on Anticommuting Self-Adjoint Operators

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Dedicated to Professor ShigeToshi Kuroda on the occasion of his 60th birthday

I. Introduction

Two *bounded* linear operators A and B in a Hilbert space $\mathcal{H}$ are said to anticommute if $AB + BA = 0$. However, if A and B are *unbounded*, then this definition of anticommutativity does not work, because $AB + BA$ may not make sense on any vector in $\mathcal{H}$.

A proper notion of anticommutativity of (*unbounded*) self-adjoint operators was given by Vasilescu [23]. Samoilenko [21] and Pedersen [20] gave several equivalent characterizations of the anticommutativity and discussed some aspects of anticommuting self-adjoint operators.

Following [20], we say that *two self-adjoint operators A and B in a Hilbert space anticommute if*

$$e^{itA}B \subset Be^{-itA}$$

for all $t \in \mathbb{R}$. We remark that this definition is symmetric in A and B [20] and gives an extension of the notion of anticommutativity of bounded operators mentioned above.

Families of anticommuting self-adjoint operators are not only interesting in its own right (in particular, from representation theoretical points of view), but also may be important in applications (e.g., analysis of operators of Dirac's type [3, 5-8, 13, 16] and supersymmetric quantum theory [1, 2, 4, 9, 15, 17, 18]).

In [10, 11] the present author has developed analysis on anticommuting self-adjoint operators; The paper [10] is concerned with algebraic properties of the partial isometries assoicated with anticommuting self-adjoint operators and analysis of the sum of two anticommuting

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self-adjoint operators, while the paper [11] gives a characterization of the anticommutativity of self-adjoint operators in connection with Clifford algebra and discusses some consequences of it, one of which can be applied to the self-adjointness problem of some classes of operators of Dirac's type in both finite and infinite dimensions. In this paper we summarize the main results obtained in [10, 11].

II. Product of two anticommuting self-adjoint operators

In this section we describe some results on a product of two anticommuting self-adjoint operators. We denote by $D(A)$ the domain of the operator A.

For the reader's convenience, we first summarize as a lemma some known facts on anticommuting self-adjoint operators.

Let A be a self-adjoint operator in a Hilbert space with the spectral family $\{E_A(\lambda)|\lambda \in \mathbb{R}\}$. Then the polar decomposition of A is given by

$$A = U_A|A|$$

with

$$U_A = 1 - E_A(0) - E_A(-0),$$

see, e.g., [19, p.358]. We call U_A the partial isometry associated with the self-adjoint operator A.

Lemma 2.1 [20, 23]. *Let A and B be anticommuting self-adjoint operators in a Hibert space. Then the following* (i)–(vii) *hold:*

(i) $U_B A \subset -AU_B$ *and* $U_A B \subset -BU_A$.
(ii) $U_B|A| \subset |A|U_B$ *and* $U_A|B| \subset |B|U_A$.
(iii) $|A|$ *and* $|B|$ *commute.*
(iv) $U_A U_B = -U_B U_A$.
(v) A *and* $|B|$ *commute and* B *and* $|A|$ *commute.*
(vi) $D(A) \cap D(B) \cap D(AB) = D(A) \cap D(B) \cap D(BA)$ *and*

$$(AB + BA)f = 0, \quad f \in D(A) \cap D(B) \cap D(AB).$$

(vii) $A + B$ *is self-adjoint.*

For two anticommuting self-adjoint operators A and B in a Hilbert space, we consider the product

$$C_0(A, B) = iAB$$

with $D(C_0(A,B)) = D(A) \cap D(B) \cap D(AB)$. It follows from Lemma 2.1(vi) that

$$D(C_0(A,B)) = D(C_0(B,A)),$$

and

$$[C_0(A,B) + C_0(B,A)]f = 0, \quad f \in D(C_0(A,B)).$$

In particular, $C_0(A,B)$ is symmetric.

Theorem 2.2 [10]. *Let A and B be anticommuting self-adjoint operators in a Hilbert space. Then*

(i) *$C_0(A,B)$ is essentially self-adjoint.*
(ii) *Let $C(A,B)$ be the closure of $C_0(A,B)$. Then*

$$C(A,B) = -C(B,A).$$

(iii) *The operator $C(A,B)$ is essentially self-adjoint on every core for $A^2 + B^2$.*

Remark. We can find a dense domain $\mathcal{D}$ on which $C_0(A,B)^k$ is essentially self-adjoint for all $k \in \mathbb{N}$ [10, Theorem 2.3].

By Lemma 2.1 (vi) we have

$$AC_0(A,B) + C_0(A,B)A = 0, \quad BC_0(A,B) + C_0(A,B)B = 0,$$

on a suitable domain, respectively. Hence $C(A,B)$ may have a chance to anticommute with A and B. In fact, the following theorem holds.

Theorem 2.3 [10]. *The operator $C(A,B)$ anticommutes with A, B, and $A + B$.*

III. Algebraic properties of the partial isometries associated with anticommuting self-adjoint operators

Theorem 2.3 shows that, given two anticommuting self-adjoint operators A and B in a Hilbert space, we have a triple $\{A, B, C(A,B)\}$ of mutually anticommuting self-adjoint operators. It is interesting to investigate structures of this triple. We do it by analyzing the algebraic structure of the partial isometries of U_A, U_B, and $U_{C(A,B)}$. Thus our first task is to compute products of these partial isometries. A key tool for this purpose is the following formula for the partial isometry associated with a self-adjoint operator.

Lemma 3.1. *Let A be a self-adjoint operator. Then*

$$U_A = s - \lim_{\epsilon \to +0} A(A^2+\epsilon)^{-1/2}.$$

Proof. This can be proven by the functional calculus for self-adjoint operators. For the details, see [10]. Q.E.D.

Remark. Let A be a self-adjoint operator and P_A be the orthogonal projection onto $(\mathrm{Ker}\ A)^{\perp}$. Then:

$$U_A = \mathrm{sgn}(A)P_A,$$

where $\mathrm{sgn}(\lambda) = \lambda/|\lambda|, \lambda \in \mathbb{R} \setminus \{0\}$.

We also note the following fact.

Lemma 3.2 [10]. *Let A and B be anticommuting self-adjoint operators in a Hibert space. Then:*

(i) *P_A and P_B commute.*
(ii) *P_A and U_B commute, and P_B and U_A commute.*

Using Lemmas 3.1, 3.2 and some technical facts, we can obtain the following results .

Theorem 3.3 [10]. *Let A and B be anticommuting self-adjoint operators in a Hilbert space. Then:*

$$\begin{aligned} U_AU_B &= -iU_{C(A,B)}, \\ U_{C(A,B)}U_A &= -iP_AU_B = -iU_BP_A, \\ U_{C(A,B)}U_B &= iP_BU_A = iU_AP_B. \end{aligned}$$

In the rest of this section, we assume that A *and* B *are anticommuting self-adjoint operators in a Hilbert space* $\mathcal{H}$. To rewrite the formulas given in Theorem 3.3 as commutation relations, we introduce

$$X_1 = i\frac{U_A}{2},\ X_2 = i\frac{U_B}{2},\ X_3 = i\frac{U_{C(A,B)}}{2},$$
$$Y_1 = 1,\ Y_2 = P_B,\ Y_3 = P_A,\ Y_4 = P_AP_B.$$

For bounded linear operators X, Y on $\mathcal{H}$, we define

$$[X,Y] = XY - YX.$$

Theorem 3.4 [10]. *The following commutation relations hold:*

$$[X_j, X_k] = \sum_{\ell=1}^{3} \epsilon_{jk\ell} X_\ell Y_j, \quad j, k = 1, 2, 3,$$

$$[X_j, Y_m] = [Y_m, Y_n] = 0, \quad j = 1, 2, 3, \ m, n = 1, 2, 3, 4,$$

where $\epsilon_{jk\ell}$ is the Levi-Civita symbol with $\epsilon_{123} = 1$.

Proof (*Outline*). This follows from Theorem 3.3, Lemma 3.2, Lemma 2.1 (iv), and the fact that $P_A^2 = P_A$. Q.E.D.

The vector space of all bounded linear operators on $\mathcal{H}$ is a Lie algebra with the Lie bracket $[\,\cdot\,,\,\cdot\,]$. We denote it by $\mathfrak{L}(\mathcal{H})$. Theorem 3.4 implies the following result.

Theorem 3.5 [10]. *Let $\mathfrak{M} \subset \mathfrak{L}(\mathcal{H})$ be the subspace spanned by $X_k Y_m, k = 1, 2, 3, m = 1, 2, 3, 4$. Then $\mathfrak{M}$ is a Lie subalgebra of $\mathfrak{L}(\mathcal{H})$.*

As is well-known, the Lie algebra $\mathfrak{su}\ (2, \mathbb{C})$ of the special unitary group $SU(2)$ is the set of 2×2 complex skew-Hermitian matrices of trace zero and has a basis $\{e_j\}_{j=1}^3$ which satisfy the commutation relations

$$[e_j, e_k] = \sum_{\ell=1}^{3} \epsilon_{jk\ell} e_\ell, \quad j, k = 1, 2, 3.$$

We define a linear map $\varrho : \mathfrak{su}\ (2, \mathbb{C}) \to \mathfrak{L}(\mathcal{H})$ by

$$\varrho(\sum_{j=1}^{3} \alpha_j e_j) = \sum_{j=1}^{3} \alpha_j X_j, \quad \alpha_j \in \mathbb{C}, j = 1, 2, 3.$$

Theorem 3.6 [10]. *Suppose that A and B are injective. Then ϱ is an isomorphism between $\mathfrak{su}\ (2, \mathbb{C})$ and $\mathfrak{M}$.*

Proof. We need only to note that, in the present case, $P_A = P_B = 1$. Q.E.D.

In the case where A and B are not necessarily injective, we can proceed as follows. Let

$$\mathcal{H}_0 = (\text{Ker } A + \text{Ker } B)^\perp$$

and define the operators A_0 and B_0 acting in $\mathcal{H}_0$ by

$$A_0 f = Af, \quad f \in D(A_0),$$
$$B_0 f = Bf, \quad f \in D(B_0)$$

with

$$D(A_0) = D(A) \cap \mathcal{H}_0, \quad D(B_0) = D(B) \cap \mathcal{H}_0.$$

It has been proven in [20] that A_0 and B_0 are injective, self-adjoint, and anticommute. We define the operators

$$X_1^{(0)} = i\frac{U_{A_0}}{2}, \quad X_2^{(0)} = i\frac{U_{B_0}}{2}, \quad X_3^{(0)} = i\frac{U_{C(A_0,B_0)}}{2}.$$

and the map $\varrho_0 : \mathfrak{su}\ (2, \mathbb{C}) \to \mathfrak{L}(\mathcal{H}_0)$ by

$$\varrho_0(\sum_{j=1}^{3} \alpha_j e_j) = \sum_{j=1}^{3} \alpha_j X_j^{(0)}, \quad \alpha_j \in \mathbb{C}, j = 1, 2, 3.$$

Applying Theorem 3.6 with A and B replaced by A_0 and B_0, respectively, we have the following result.

Theorem 3.7 [10]. *The map ϱ_0 is an isomorphism between $\mathfrak{su}\ (2, \mathbb{C})$ and the Lie algebra $\mathfrak{M}_0$ generated by $X_j^{(0)}, j = 1, 2, 3$.*

Theorem 3.7 implies that ϱ_0 is a faithful representation of $\mathfrak{su}\ (2, \mathbb{C})$ on the Hilbert space $\mathcal{H}_0$. If $\mathcal{H}_0$ is infinite dimensional, then ϱ_0 gives an infinite dimensional representation of $\mathfrak{su}\ (2, \mathbb{C})$. The structure of the representation ϱ_0 may be interesting. We have the following theorem.

Theorem 3.8 [10]. *Let $\mathcal{H}$ be separable and $\mathcal{H}_0$ be infinite dimensional. Then there exists a sequence $\{\mathcal{M}_n\}_{n=1}^{\infty}$ of subspaces in $\mathcal{H}_0$ with the following properties:*

(i) *For each m and n with $m \neq n$, $\mathcal{M}_m$ and $\mathcal{M}_n$ are orthogonal.*
(ii) $\mathcal{H}_0 = \oplus_{n=1}^{\infty} \mathcal{M}_n$.
(iii) *For all $n \in \mathbb{N}$,* $\dim \mathcal{M}_n = 2$ *and $\mathcal{M}_n$ is left invariant by* $\{X_j^{(0)}\}_{j=1}^{3}$.

In particular, the representation ϱ_0 is completely reducible with the highest weight of each irreducible component being $1/2$.

In concluding this section, we give a remark on a relevance of anticommuting self-adjoint operators to Clifford algebra theory. The Clifford algebra $\mathfrak{A}_n$ associated with the n-dimensional Euclidean space $\mathbb{R}^n$ is the algebra generated by elements $\gamma_j, j = 1, \cdots, n$, and identity 1 satisfying

$$\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{jk}, \quad j, k = 1, \cdots, n. \tag{3.1}$$

Let A and B be anticommuting self-adjoint operators in the Hilbert space $\mathcal{H}$ and define

$$\Gamma_1 = U_{A_0}, \quad \Gamma_2 = U_{B_0}, \quad \Gamma_3 = U_{C(A_0,B_0)}.$$

Then the operators $\Gamma_j, j = 1, 2, 3$, are self-adjoint on $\mathcal{H}_0$. Moreover we have

$$\Gamma_j\Gamma_k + \Gamma_k\Gamma_j = 2\delta_{jk}, \quad j, k = 1, 2, 3,$$

and Γ_j leaves $\mathcal{M}_n$ invariant. Let $\Gamma_j^{(n)}$ be the restriction of Γ_j to $\mathcal{M}_n$, so that we have

$$\Gamma_j = \bigoplus_{n=1}^{\infty} \Gamma_j^{(n)}.$$

Let $\mathfrak{C}_n$ be the algebra generated by $\Gamma_j^{(n)}, j = 1, 2, 3$. Then we have the following result.

Theorem 3.9 [10]. *For each $n = 1, 2, \cdots$, the algebra $\mathfrak{C}_n$ is the spin representation of $\mathfrak{A}_3$.*

IV. The sum of two anticommuting self-adjoint operators

Let A and B be anticommuting self-adjoint operators in the Hilbert space $\mathcal{H}$. As we have seen in Lemma 2.1 (vii), $A+B$ is self-adjoint. This section concerns more detailed properties of the operator $A + B$.

4.1. The case where B is injective

In this case, the partial isometry U_B is unitary with the spectrum $\sigma(U_B) = \{\pm 1\}$, so that we have the orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_+ \bigoplus \mathcal{H}_- = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} | f \in \mathcal{H}_+, g \in \mathcal{H}_- \right\} \tag{4.1}$$

with

$$\mathcal{H}_\pm = \text{Ker } (U_B \mp 1).$$

Theorem 4.1 [10]. *Let A and B be anticommuting self-adjoint operators in $\mathcal{H}$ and B be injective. Then A, B, and P_A have the following matrix representations with respect to (w.r.t.) the decomposition* (4.1)*:*

$$A = \begin{pmatrix} 0 & a^*M_- \\ aM_+ & 0 \end{pmatrix}, \quad B = \begin{pmatrix} B_+ & 0 \\ 0 & -B_- \end{pmatrix}, \tag{4.2}$$
$$P_A = \begin{pmatrix} a^*a & 0 \\ 0 & aa^* \end{pmatrix},$$

where a is a partial isometry from $\mathcal{H}_+$ to $\mathcal{H}_-$, B_+ (resp. B_-) and M_+ (resp. M_-) are commuting nonnegative self-adjoint operators in $\mathcal{H}_+$ (resp. $\mathcal{H}_-$), and $aB_+ \subset B_-a$.

This theorem is a generalization of [20, Corollary 3.3] which gives matrix representations of A and B similar to (4.2) in the case where both of A and B are injective.

We consider the diagonalization of $A + B$ w.r.t. the decomposition (4.1). By the commutativity of $|A|$ and $|B|$ [Lemma 2.1(iii)], we can define, via the functional calculus,

$$\Lambda = \mathrm{Arctan}(|A||B|^{-1}),$$

which is bounded and self-adjoint. Since $-iX_3$ and Λ are commuting bounded self-adjoint operators, $-iX_3\Lambda$ is bounded and self-adjoint. Hence the operator

$$V = e^{X_3\Lambda}$$

is unitary. It turns out that V implements the diagonalization of $A+B$ w.r.t. the decomposition (4.1):

Theorem 4.2 [10]. *Let A and B be anticommuting self-adjoint operators and B be injective. Then*

$$\begin{aligned} V(A+B)V^{-1} &= U_B(A^2+B^2)^{1/2} \\ &= \begin{pmatrix} (L_A^*L_A + B_+^2)^{1/2} & 0 \\ 0 & -(L_AL_A^* + B_-^2)^{1/2} \end{pmatrix}, \end{aligned} \tag{4.3}$$

where

$$L_A = aM_+.$$

Remark. Formula (4.3) can be regarded as an abstract and non-perturbative version of the so-called *Tani-Foldy-Wouthuysen transformation* of the usual Dirac operator in three space dimensions (e.g., [14]).

Theorem 4.2 can be proven by using the following lemma.

Lemma 4.3 [10].

$$VX_1V^{-1} = (1 - P_A)X_1 + P_A(X_1 \cos\Lambda + X_2 \sin\Lambda),$$
$$VX_2V^{-1} = (1 - P_A)X_2 + P_A(-X_1 \sin\Lambda + X_2 \cos\Lambda).$$

4.2. The case where B is not injective

In this case, we note the following fact.

Lemma 4.4 [10]. *The operator P_B commutes with A, V, U_B, and $(A^2 + B^2)^{1/2}$.*

Lemma 4.4 implies that A, B, V, U_B, and $(A^2+B^2)^{1/2}$ can be reduced to $(\text{Ker } B)^{\perp}$ in which B is injective. Thus we can apply the preceeding result in Section 4.1 to obtain the following theorem.

Theorem 4.5 [10]. *Let A and B be anticommuting self-adjoint operators. Then* (4.3) *holds on* $(\text{Ker } B)^{\perp}$.

Remark. In the case of abstract Dirac operators, results similar to Theorems 4.2 and 4.5 have been obtained in [22].

V. Characterization of anticommutativity of self-adjoint operators in connection with Clifford algebra

In Section III we have seen that two anticommuting self-adjoint operators are related to the Clifford algebra $\mathfrak{A}_3$. This fact suggests that it may be more natural to characterize anticommutativity of self-adjoint operators in connection with Clifford algebra. In fact, such a characterization is possible as we shall present below.

Let $\mathcal{H}$ be a Hilbert space. We say that $\{\gamma_j\}_{j=1}^n$ *is a self-adjoint representation of the Clifford algebra $\mathfrak{A}_n$ on $\mathcal{H}$ if each γ_j is a bounded self-adjoint operator on $\mathcal{H}$ satisfying* (3.1).

The first of the main results in this section is the following.

Theorem 5.1 [11]. *Let A and B be self-adjoint operators in a Hilbert space $\mathcal{H}$. Suppose that there exists a self-adjoint representation $\{\gamma_1, \gamma_2\}$ of $\mathfrak{A}_2$ on $\mathcal{H}$ such that each γ_j commutes with A and B. Then A and B anticommute if and only if*

$$e^{is\gamma_1 A}e^{it\gamma_2 B} = e^{it\gamma_2 B}e^{is\gamma_1 A}$$

for all $s, t \in \mathbb{R}$.

Remark. If γ_1 commutes with A, then $\gamma_1 A$ is self-adjoint with $\gamma_1 A = A\gamma_1$. The same holds for the pair $\{\gamma_2, B\}$. Hence $\exp(is\gamma_1 A)$ and $\exp(it\gamma_2 B)$ can be defined via the functional calculus.

Theorem 5.1 has some interesting consequences. We fix a self-adjoint representation $\{\gamma_1, \gamma_2\}$ of $\mathfrak{A}_2$ on a Hilbert space $\mathcal{K}$. We denote by $\mathcal{K}\otimes\mathcal{H}$ the tensor product of $\mathcal{K}$ and $\mathcal{H}$.

Theorem 5.2 [11]. *Let A and B be self-adjoint operators in a Hilbert space $\mathcal{H}$. Then A and B anticommute if and only if $\gamma_1 \otimes A$ and $\gamma_2 \otimes B$ commute in the Hilbert space $\mathcal{K} \otimes \mathcal{H}$.*

Remark. A simple example of $\mathcal{K}$ and $\{\gamma_1, \gamma_2\}$ is given by

$$\mathcal{K} = \mathbb{C}^2,$$
$$\gamma_1 = \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The matrices σ_1 and σ_2 are the first two of the so-called *Pauli matrices.*

We have a "dual" version of Theorem 5.2:

Theorem 5.3 [11]. *Let A and B be self-adjoint operators in a Hilbert space $\mathcal{H}$. Then A and B commute if and only if $\gamma_1 \otimes A$ and $\gamma_2 \otimes B$ anticommute in the Hilbert space $\mathcal{K} \otimes \mathcal{H}$.*

Remark. In the case where $\mathcal{K} = \mathbb{C}^2$ and $\gamma_j = \sigma_j, j = 1, 2$, the necessary condition in Theorem 5.3 has been proven in [13] by a method different from that in [11].

Theorem 5.3 can be applied to the self-adjointness problem of operators of Dirac's type. We first recall a basic result due to Vasilescu [23]:

Lemma 5.4 [23]. *Let $\{A_j\}_{j=1}^n$ be a family of mutually anticommuting self-adjoint operators in a Hilbert space ($n < \infty$). Then $\sum_{j=1}^n A_j$ is self-adjoint and*

$$\left(\sum_{j=1}^n A_j\right)^2 = \sum_{j=1}^n A_j^2.$$

Using this lemma and Theorem 5.3, we can prove the following fact:

Theorem 5.5 [11]. *Let $\{A_j\}_{j=1}^n$ be a family of mutually commuting self-adjoint operators in a Hilbert space $\mathcal{H}$ $(n < \infty)$. Let $\{\gamma_j\}_{j=1}^n$ be a self-adjoint representation of $\mathfrak{A}_n$ on a Hilbert space $\mathcal{K}$. Then the operator*

$$\not{D} := \sum_{j=1}^n \gamma_j \otimes A_j$$

is self-adjoint in $\mathcal{K} \otimes \mathcal{H}$ and

$$\not{D}^2 = \sum_{j=1}^n I \otimes A_j^2.$$

We next consider a countable family $\{A_n\}_{n=1}^\infty$ of self-adjoint operators. We can define the operator

$$A := \sum_{n=1}^\infty A_n$$

by the relation

$$D(A) = \left\{ f \in \bigcap_{n=1}^\infty D(A_n) \mid \mathrm{w} - \lim_{N\to\infty} \sum_{n=1}^N A_n f \text{ exists} \right\},$$

$$Af = \mathrm{w} - \lim_{N\to\infty} \sum_{n=1}^N A_n f, \quad f \in D(A).$$

The following lemma is an extension of Lemma 5.4.

Lemma 5.6 [23]. *Let $\{A_n\}_{n=1}^\infty$ be a family of mutually anticommuting self-adjoint operators in a Hilbert space $\mathcal{H}$ such that $D(\sum_{n=1}^\infty A_n)$ is dense in $\mathcal{H}$. Then $\sum_{n=1}^\infty A_n$ is self-adjoint and*

$$\left(\sum_{n=1}^\infty A_n \right)^2 = \sum_{n=1}^\infty A_n^2.$$

Using Lemma 5.6, we can obtain an extension of Theorem 5.5:

Theorem 5.7 [11]. *Let $\{A_n\}_{n=1}^\infty$ be a family of mutually commuting self-adjoint operators in a Hilbert space $\mathcal{H}$. Let $\{\gamma_n\}_{n=1}^\infty$ be a* *self-adjoint representation of* $\mathfrak{A}_\infty$ *on a Hilbert space* $\mathcal{K}$. *Suppose that* $D(\sum_{n=1}^\infty \gamma_n \otimes A_n)$ *is dense in* $\mathcal{K} \otimes \mathcal{H}$. *Then the operator*

$$\not{D}_\infty := \sum_{n=1}^\infty \gamma_n \otimes A_n$$

is self-adjoint in $\mathcal{K} \otimes \mathcal{H}$ *and*

$$\not{D}_\infty^2 = \sum_{n=1}^\infty I \otimes A_n^2.$$

The operator $\not{D}$ (resp. $\not{D}_\infty$) in Theorem 5.5 (resp. Theorem 5.7) gives a class of operators of Dirac's type in an abstract form. Hence Theorems 5.5 and 5.7 solve the self-adjointness problem for such Dirac operators. Examples to which Theorems 5.5 and 5.7 are applicable include: (i) the Dirac-Weyl operator with a strongly singular gauge potential [13] (cf. also [12]); (ii) classes of operators of Dirac's type in an abstract Boson-Fermion Fock space (infinite dimensional Dirac operators) [3, 4, 7, 9, 11].

VI. Anticommuting self-adjoint operators and supersymmetric quantum theory

As a final topic in this paper, we discuss a connection of the theory of anticommuting self-adjoint operators with supersymmetric quantum theory (SSQT).

We first give an abstract definition of SSQT (e.g., [1, 2, 4, 17, 25]). Let $N \geq 1$ be an integer. A SSQT with N-supersymmetry is defined to be a quadruple $\{\mathcal{H}, \{Q_n\}_{n=1}^N, H, N_F\}$ consisting of a Hilbert space $\mathcal{H}$, a set of self-adjoint operators $\{Q_n\}_{n=1}^N$ ("supercharges"), self-adjoint operators H ("supersymmetric Hamiltonian") and N_F ("Fermion number operator") acting in $\mathcal{H}$, which satisfies the following conditions:

(S.1) $N_F^2 = I$ (identity on $\mathcal{H}$) and $N_F \neq \pm I$.

(S.2) $H = Q_n^2, \quad n = 1, \cdots, N$.

(S.3) For each $n = 1, \cdots, N$, N_F leaves $D(Q_n)$ invariant and

$$N_F Q_n + Q_n N_F = 0 \quad \text{on } D(Q_n),\ n = 1, \cdots, N.$$

(S.4) For all $n, m = 1, \cdots, N$, with $n \neq m$,

$$(Q_n\psi, Q_m\phi) + (Q_m\psi, Q_n\phi) = 0, \quad \psi, \phi \in D(Q_n) \cap D(Q_m),$$

where $(\cdot, \cdot)$ is the inner product of $\mathcal{H}$.

Note that (S.3) means that N_F and Q_n anticommute in a "naive" sense, while (S.4) shows that Q_n and Q_m $(n \neq m)$ anticommute in the sense of quadratic form on $D(Q_n) \cap D(Q_m)$. It is natural to ask if they anticommute in the proper sense given in the Introduction.

The following fact is known.

Lemma 6.1 [23]. *Let T be a bounded self-adjoint operator and Q be a self-adjoint operator in a Hilbert space. Suppose that T leaves $D(Q)$ invariant and*

$$TQ + QT = 0 \quad \text{on } D(Q).$$

Then T and Q anticommute.

Applying Lemma 6.1 to $T = N_F$ and $Q = Q_n$, we have the following result.

Proposition 6.2. *In any SSQT $\{\mathcal{H}, \{Q_n\}_{n=1}^N, H, N_F\}$, each Q_n and N_F anticommute.*

As for (S.4), we can apply the following theorem.

Theorem 6.3. *Let Q_1 and Q_2 be self-adjoint operators in a Hilbert space $\mathcal{H}$ such that*

$$Q_1^2 = Q_2^2$$

and

$$(Q_1\psi, Q_2\phi) + (Q_2\psi, Q_1\phi) = 0, \quad \psi, \phi \in D(Q_1) \cap D(Q_2). \tag{6.1}$$

Then Q_1 and Q_2 anticommute.

Proof. We have $L \equiv |Q_1| = |Q_2|$. Hence $D(Q_1) = D(Q_2) = D(L)$ and the polar decompositions of Q_1 and Q_2 are given by

$$Q_1 = U_1 L, \quad Q_2 = U_2 L,$$

where $U_j = U_{Q_j}$. Putting these formulas into (6.1), we have

$$(U_1\widetilde{\psi}, U_2\widetilde{\phi}) + (U_2\widetilde{\psi}, U_1\widetilde{\phi}) = 0 \tag{6.2}$$

with $\widetilde{\psi} = L\psi, \widetilde{\phi} = L\phi, \quad \psi, \phi \in D(L)$.

We first consider the case where L is injective and hence so is Q_j $(j = 1, 2)$. Then U_1 and U_2 are unitary, self-adjoint and Ran L is dense in $\mathcal{H}$. Hence (6.2) implies that

$$U_jU_k + U_kU_j = 2\delta_{jk}, \quad j, k = 1, 2.$$

Let $\mathcal{D} = \cup_{n=1}^{\infty} \text{Ran } E_L([0,n])$. Then $\mathcal{D}$ is dense in $\mathcal{H}$. Since U_j commutes with L, U_1 and U_2 leave $\mathcal{D}$ invariant and hence so do Q_1 and Q_2. It is easy to see that $\mathcal{D}$ is a set of entire analytic vectors for each Q_j and $Q_1Q_2 + Q_2Q_1 = 0$ on $\mathcal{D}$. Hence we can apply [20, Proposition 5.2] to conclude that Q_1 and Q_2 anticommute.

In the case where L is not injective, Q_1 and Q_2 are reduced to $\mathcal{H}_0 \equiv (\text{Ker } L)^{\perp} = (\text{Ker } Q_1)^{\perp} = (\text{Ker } Q_2)^{\perp}$. We can apply the preceeding result to $\widetilde{Q}_j \equiv Q_j \restriction \mathcal{H}_0$ to conclude that $\widetilde{Q}_1$ and $\widetilde{Q}_2$ anticommute. This implies the anticommutativity of Q_1 and Q_2 in $\mathcal{H}$. Q.E.D.

Theorem 6.3 gives the following result.

Proposition 6.4. *In any SSQT* $\{\mathcal{H}, \{Q_n\}_{n=1}^N, H, N_F\}$, Q_n *and* Q_m $(n, m = 1, \cdots, N, n \neq m)$ *anticommute.*

Remark. The SSQT considered above is a non-relativistic one. In relativistic cases, condition (S.2) have to be replaced by a more complicated one (e.g., [9, 24]).

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