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On Symmetry Groups of the MIC-Kepler Problem and Their Unitary Irreducible Representations

Toshihiro Iwai and Yoshio Uwano

It is well known that the quantized Kepler problem (i.e., the hydrogen atom) admits the symmetry groups, $SO(4)$, $E(3)$ (the Euclidean motion group), or $SO^+(1,3)$ (the proper Lorentz group), according as the energy is negative, zero, or positive (cf. [B-I]). The symmetry groups here stand for Lie groups which act unitarily irreducibly on the Hilbert spaces associated with the energy-spectrum for the Kepler problem. However, only a part of the unitary irreducible representations are realized as the symmetry group for the Kepler problem. A question now arises: Are the other unitary irreducible representations realizable as symmetry groups for a "modified" Kepler problem ?

This question is worked out in this article. Both in classical and quantum mechanics, the Kepler problem is generalized to the MIC-Kepler problem, the Kepler problem along with a centrifugal potential and Dirac's monopole field, which is named after McIntosh and Cisneros [MI-C]. It will be shown that the quantized MIC-Kepler problem exhausts almost all the unitary irreducible representations of $SU(2) \times SU(2)$, $\mathbf{R}^3 \ltimes SU(2)$, or $SL(2,\mathbf{C})$ as the symmetry group, according as the energy is negative, zero, or positive, which groups are the double covers of $SO(4)$, $E(3)$, and $SO^+(1,3)$, respectively. For $SL(2,\mathbf{C})$, the principal series representations are all realizable, but not the others.

§1. Setting up the quantized MIC-Kepler problem

The MIC-Kepler problem is to be defined as a reduced system of the conformal Kepler problem. Consider the principal $U(1)$ bundle $\pi : \mathbf{R}^4 - \{0\} \to \mathbf{R}^3 - \{0\}$ whose projection π and $U(1)$ action Φ_t are given, respectively, by

$$\pi(q) = (2(q_1q_3 + q_2q_4),\ 2(-q_1q_4 + q_2q_3),\ q_1^2 + q_2^2 - q_3^2 - q_4^2), \tag{1.1}$$

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and

$$\Phi_t : q \longmapsto T(t)q \tag{1.2}$$

with

$$T(t) = \begin{pmatrix} N & \\ & N \end{pmatrix} \quad \text{with} \quad N = \begin{pmatrix} \cos\frac{t}{2} & -\sin\frac{t}{2} \\ \sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix} \quad t \in [0, 4\pi], \tag{1.3}$$

where $(q_j)_{j=1,2,3,4}$ are the Cartesian coordinates in $\mathbf{R}^4$. The missing matrix entries are all zero, here and henceforth.

For any fixed integer m, let ρ_m be the unitary irreducible representation of $U(1)$ on $\mathbf{C}$,

$$\rho_m : T(t) \longmapsto e^{imt/2}, \quad t \in [0, 4\pi]. \tag{1.4}$$

Then the associated complex line bundle $L_m = (\mathbf{R}^4 - \{0\} \times_m \mathbf{C},\ \pi_m,\ \mathbf{R}^3 - \{0\})$ is formed through the representation ρ_m. Note that, contrary to the literature [K-N], the left action is under consideration.

The standard connection on $\mathbf{R}^4 - \{0\}$ gives rise to the linear connection ∇ for L_m, the curvature of which, Ω_m, takes the form

$$\Omega_m = \frac{im}{2r^3}(x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2), \tag{1.5}$$

where $\pi(q) = (x_j)_{j=1,2,3}$ are the Cartesian coordinates in $\mathbf{R}^3 - \{0\}$ and $r^2 = \sum_{j=1}^{j=3} x_j^2$. The Ω_m describes Dirac's monopole field of strength $-m/2$. The *MIC-Kepler problem* is then defined on the complex line bundle L_m.

Definition. The MIC-Kepler problem is a quantum system defined on L_m together with the Hamiltonian operator

$$\widehat{H}_m = -\frac{1}{2}\sum_{j=1}^{3} {\nabla_j}^2 + \frac{(m/2)^2}{2r^2} - \frac{k}{r} \tag{1.6}$$

acting on cross sections in L_m, where ∇_j stands for the covariant derivation, $\nabla_{\partial/\partial x_j}$, and k is a positive constant.

The reduction process giving this definition proceeds as follows: The *conformal Kepler problem* is defined as a quantum system with the Hamiltonian operator

$$\widehat{H} = -\frac{1}{2}\left(\frac{1}{4r}\sum_{\ell=1}^{4} \frac{\partial^2}{\partial q_\ell^2}\right) - \frac{k}{r} \tag{1.7}$$

acting on the functions on $\mathbf{R}^4 - \{0\}$, where $r = \sum_{\ell=1}^{\ell=4} q_\ell^2$.

A function $f(q)$ on $\mathbf{R}^4 - \{0\}$ is referred to as *ρ_m-equivariant*, if it satisfies

$$f(T(t)q) = e^{imt/2} f(q), \quad t \in [0, 4\pi]. \tag{1.8}$$

The ρ_m-equivariant functions are in one-to-one correspondence with the cross sections in L_m. Then, on denoting by q_m the correspondence of the ρ_m-equivariant functions to the cross sections in L_m, one has

$$\widehat{H}_m = q_m \circ \widehat{H} \circ q_m^{-1}, \tag{1.9}$$

which turns out to be expressed as (1.6).

Since our interest centers on quantum systems only, the adjective "quantized" is to be omitted. Further, for convenience' sake, we will often abbreviate the MIC-Kepler problem and the conformal Kepler problem to MICK and CK, respectively.

Equation (1.9) is the relation on the base of which we study symmetry groups for the MIC-Kepler problem in each case of energy, negative, zero, or positive. The procedure is as follows:

(1) Find symmetry groups of the conformal Kepler problem. As the results, the harmonic oscillator, a free particle, or the repulsive oscillator are associated with CK, according respectively as the energy of CK is negative, zero, or positive. These symmetry groups are represented in Hilbert spaces labeled with the energies of CK.
(2) Equation (1.9) shows that the subspace of ρ_m-equivariant functions in the representation space for the symmetry group of CK reduces to the Hilbert space of cross-sections in L_m associated with each of the spectra of the MIC-Kepler problem. Through this reduction, a symmetry group of MICK turns out to be given by a subgroup of the symmetry group of CK that leaves invariant each subspace of ρ_m-equivariant functions.
(3) Prove the irreducibility of the representations of the symmetry groups of MICK.

There is another way to study the quantized Kepler and MIC-Kepler problem. For negative energies, the geometric quantization method provides the negative energy eigenvalues [S, Ml-T, Ml]. However, the geometric quantization turns no attention to zero or positive energy, nor to the relation with representation of symmetry groups.

§2. The negative energy case and $SU(2) \times SU(2)$

2.1. A symmetry group of the conformal Kepler problem with negative energy

Following Procedure (1)–(3) presented in Section 1, we start with a symmetry group of CK with negative energy. It is of great help to associate CK with the four-dimensional harmonic oscillator, which is the quantum system with the Hamiltonian operator

$$\widehat{J}_\lambda = -\frac{1}{2}\sum_{j=1}^{4}\frac{\partial^2}{\partial {q_j}^2} + \frac{\lambda^2}{2}\sum_{j=1}^{4} q_j^2, \tag{2.1}$$

where λ is a positive parameter. The harmonic oscillator will be often abbreviated to HO, henceforth. $\widehat{H}$ and $\widehat{J}_\lambda$ satisfy

$$4r\left(\widehat{H} + \frac{\lambda^2}{8}\right) = \widehat{J}_\lambda - 4k. \tag{2.2}$$

This means that the eigenfunctions of CK with negative eigenvalue $-\lambda^2/8$ are obtained from eigenfunctions of HO with positive eigenvalue $4k$. Thus to find the symmetry group of CK for the eigenvalue $-\lambda^2/8$ is to find that of HO for the eigenvalue $4k$. Let us define the creation operator $(a_j^\dagger)_{j=1,2,3,4}$ for the harmonic oscillator by

$$\begin{aligned} a_1^\dagger &= \frac{1}{2\sqrt{\lambda}}\left(\lambda q_1 - i\lambda q_2 - \frac{\partial}{\partial q_1} + i\frac{\partial}{\partial q_2}\right), \\ a_2^\dagger &= \frac{1}{2\sqrt{\lambda}}\left(\lambda q_3 - i\lambda q_4 - \frac{\partial}{\partial q_3} + i\frac{\partial}{\partial q_4}\right), \\ a_3^\dagger &= \frac{1}{2\sqrt{\lambda}}\left(\lambda q_1 + i\lambda q_2 - \frac{\partial}{\partial q_1} - i\frac{\partial}{\partial q_2}\right), \\ a_4^\dagger &= \frac{1}{2\sqrt{\lambda}}\left(\lambda q_3 + i\lambda q_4 - \frac{\partial}{\partial q_3} - i\frac{\partial}{\partial q_4}\right). \end{aligned} \tag{2.3}$$

Then, by using multi-indices $\mathbf{k}$ denoting $k_1k_2k_3k_4$ ($k_j \geq 0$: integers, $j = 1, \cdots, 4$), the normalized eigenfunctions for HO, a basis in $L^2(\mathbf{R}^4)$, are expressed in the form

$$\psi_{\mathbf{k}}(q) = (\mathbf{k}!)^{-1/2}(a_1^\dagger)^{k_1}(a_2^\dagger)^{k_2}(a_3^\dagger)^{k_3}(a_4^\dagger)^{k_4}\psi_{\mathbf{0}}(q) \tag{2.4a}$$

with

$$\psi_0(q) = \sqrt{\lambda/\pi}\exp(-\lambda r/2), \tag{2.4b}$$

where $\mathbf{k}! = k_1!k_2!k_3!k_4!$. Equation (2.2) and the fact that $\psi_{\mathbf{k}}$ is associated with the eigenvalue $\lambda(n+2)$ with $k_1+\cdots+k_4=n$ are put together to provide the following.

Proposition 2.1. *The negative eigenvalues of the conformal Kepler problem are given by* $E_n = -2k^2/(n+2)^2$ *($n \geq 0$: integer), and their associated eigenspaces denoted by* S_n *are spanned by the functions* $\psi_{\mathbf{k}}$ *given by (2.4) subject to the conditions*

$$k_1+k_2+k_3+k_4=n \quad \text{and} \quad \lambda = 4k/(n+2). \tag{2.5}$$

We mention here that the Hilbert space structure of each S_n is determined by restricting the inner product

$$\langle f, g\rangle = \int_{\mathbf{R}^4} \overline{f(q)} g(q) 4r dq_1 dq_2 dq_3 dq_4. \tag{2.6}$$

to S_n. The restricted inner product is denoted by $\langle\ ,\ \rangle_n$. Note that with respect to $\langle\ ,\ \rangle$ the $\widehat{H}$ becomes a symmetric operator in $C_0^\infty(\mathbf{R}^4)$ (see [I]).

We note here that the relation similar to (2.2) holds also in classical theory, so that the well-known symmetry group $SU(4)$ of the harmonic oscillator turns out to be the symmetry group of CK (see [I, I-U1]). Hence, on "quantizing" the action of the classical symmetry group $SU(4)$ of CK, a symmetry group of CK with negative energy is to be derived so as to act on S_n. We thus obtain the following.

Proposition 2.2. *The conformal Kepler problem with negative energy admits* $SU(4)$ *as a symmetry group which acts unitarily irreducibly on* $(S_n, \langle\ ,\ \rangle_n)$ *in the manner*

$$(U_C^{(n)}\psi_{\mathbf{k}})(q) = (\mathbf{k}!)^{-1/2}(C^Ta^\dagger)_1^{k_1}(C^Ta^\dagger)_2^{k_2}(C^Ta^\dagger)_3^{k_3}(C^Ta^\dagger)_4^{k_4}\psi_0(q) \tag{2.7}$$

for $C \in SU(4)$ *and* $\psi_{\mathbf{k}} \in S_n$, *where* $a^\dagger$ *stands for the column vector of operators and* $(C^Ta^\dagger)_j$ $(j=1,2,3,4)$ *is the* j*-th component.*

2.2. A symmetry group of the MIC-Kepler problem with negative energy

We proceed to a symmetry group of MIC-Kepler problem with negative energy. As was stated in Procedure (2) in Section 1, the subgroup of the symmetry group $SU(4)$ leaving invariant the subspace $S_{n,m}$ of ρ_m-equivariant functions in S_n will become a symmetry group of MICK. We shall first look into $S_{n,m}$. From (2.3) and (2.7), the $U(1)$ action given by (1.2) proves to be expressed as

$$\psi_{\mathbf{k}}(T(t)q) = (U^{(n)}_{\widetilde{T}(t)}\psi_{\mathbf{k}})(q) = e^{i(-k_1-k_2+k_3+k_4)t/2}\psi_{\mathbf{k}}(q) \tag{2.8a}$$

with

$$\widetilde{T}(t) = \begin{pmatrix} e^{-it/2}I_2 & \\ & e^{it/2}I_2 \end{pmatrix} \quad (I_2 : \ 2\times 2 \text{ identity matrix}), \tag{2.8b}$$

This equation yields the following.

Lemma 2.3. *The subspace $S_{n,m}$ of ρ_m-equivariant functions in S_n is spanned by the functions $\psi_{\mathbf{k}} \in S_n$ subject to*

$$k_1 + k_2 = \frac{n-m}{2}, \quad k_3 + k_4 = \frac{n+m}{2}, \tag{2.9}$$

where the integers n and m are simultaneously even or odd with $|m| \le n$. The $S_{n,m}$ is of dimension $(n-m+2)(n+m+2)/4$.

From the relation (1.9), we see that the $S_{n,m}$ reduces to the space of eigen-cross sections of $\widehat{H}_m$ with negative eigenvalue $E_n = -2k^2/(n+2)^2$. Indeed, for any $f \in S_{n,m}$, one has

$$\widehat{H}_m(q_m f) = q_m(\widehat{H} f) = E_n(q_m f). \tag{2.10}$$

We hence denote by $q_m(S_{n,m})$ the space of eigen-cross sections of $\widehat{H}_m$ derived from $S_{n,m}$. The Hilbert space structure $\langle\ ,\rangle_{n,m}$ is, of course, induced from the inner product $\langle\ ,\ \rangle_n$ as

$$\langle \gamma_1, \gamma_2\rangle_{n,m} = \langle q_m^{-1}\gamma_1, q_m^{-1}\gamma_2\rangle_n \tag{2.11}$$

for $\gamma_1, \gamma_2 \in q_m(S_{n,m})$. We now have the following.

Proposition 2.4. *The subspace $S_{n,m}$ of ρ_m-equivariant functions in S_n is mapped, through q_m, bijectively to the space of eigen-cross sections, $q_m(S_{n,m})$, for the MIC-Kepler problem with negative eigenvalue $E_n = -2k^2/(n+2)^2$, where n and m are simultaneously even or odd with $|m| \le n$.*

Now that we have the eigenspace $q_m(S_{n,m})$, we are ready to discuss a symmetry group of MICK with negative energy. In view of the course of obtaining $q_m(S_{n,m})$, we see that a subgroup of $SU(4)$ that leaves $S_{n,m}$ invariant turns into the symmetry group of MICK with negative energy. The largest subgroup of $SU(4)$ that leaves $S_{n,m}$ invariant is $S(U(2)\times U(2))$, which includes the $U(1)$ with the action (2.8). However, since the $U(1)$ action (2.8) is the identity in $q_m(S_{n,m})$ we had better get rid of this $U(1)$ action, so that we treat $SU(2)\times SU(2)$. For any $(C_1, C_2)\in SU(2)\times SU(2)$, we have, of course, the inclusion

$$(2.12)\qquad (C_1, C_2)\in SU(2)\times SU(2)\longmapsto \begin{pmatrix} C_1 & \\ & C_2 \end{pmatrix}\in SU(4).$$

In order to express the action of $SU(2)\times SU(2)$ in a concrete form, we rewrite the creation operators $a_1^\dagger$, $a_2^\dagger$, $a_3^\dagger$, and $a_4^\dagger$ as $A_1^\dagger$, $A_2^\dagger$, $B_1^\dagger$, and $B_2^\dagger$, respectively. Then Proposition 2.2 reduces to the following (cf. [I-U1]).

Proposition 2.5. *A subgroup $SU(2)\times SU(2)$ of $SU(4)$ acts unitarily on $S_{n,m}$, whose action is expressed, for $(C_1,\ C_2)\in SU(2)\times SU(2)$ and $\psi_{\mathbf{k}}\in S_{n,m}$, as*

$$(2.13)\quad (U^{(n,m)}_{(C_1,C_2)}\,\psi_{\mathbf{k}})(q) \\ = (\mathbf{k}!)^{-1/2}(C_1^TA^\dagger)_1^{k_1}(C_1^TA^\dagger)_2^{k_2}(C_2^TB^\dagger)_1^{k_3}(C_2^TB^\dagger)_2^{k_4}\psi_{\mathbf{0}}(q),$$

where $(C_1^TA^\dagger)_j$ and $(C_2^TB^\dagger)_j$ $(j=1,2)$ are the j-th components of the column vectors of operators $C_1^TA^\dagger$ and $C_2^TB^\dagger$, respectively.

Owing to Proposition 2.5, we can define well a unitary $SU(2)\times SU(2)$ action, $W^{(n,m)}$, on the eigenspace $q_m(S_{n,m})$ of the MIC-Kepler problem; for $\gamma\in q_m(S_{n,m})$ and $(C_1,C_2)\in SU(2)\times SU(2)$,

$$(2.14)\qquad W^{(n,m)}_{(C_1,C_2)}\gamma := (q_m\circ U^{(n,m)}_{(C_1,C_2)}\circ q_m^{-1})\gamma.$$

Proposition 2.6. *The group $SU(2)\times SU(2)$ has unitary representation on each of the eigenspace $q_m(S_{n,m})$ of the MIC-Kepler problem with the eigenvalue E_n, where n and m are simultaneously even or odd with $|m|\le n$.*

2.3. The irreducibility of the $SU(2)\times SU(2)$ representation

On recalling Lemma 2.3, the condition (2.9) enables us to identify $S_{n,m}$ with the tensor product of the space of homogeneous polynomials in $(A_j^\dagger)$ of degree $(n-m)/2$ and that in $(B_j^\dagger)$ of degree $(n+m)/2$. Then, it follows from (2.13) that each of the factor groups of $SU(2)\times SU(2)$ is represented irreducibly in homogeneous polynomial space of degree

$(n-m)/2$ in $A^\dagger$ and in that of degree $(n+m)/2$ in $B^\dagger$, so that the representation $U^{(n,m)}$ proves to be irreducible. Further, according to Wigner [W], the tensor product representations exhaust all the unitary irreducible representations of $SU(2) \times SU(2)$, up to isomorphisms. Owing to the unitary equivalence, the representations $W^{(n,m)}$ turn out to exhaust all the unitary irreducible representations of $SU(2) \times SU(2)$, up to isomorphisms. The results in this section is summarized as follows:

Theorem 2.7 [I-U1]. *The MIC-Kepler problem with negative energies admits $SU(2) \times SU(2)$ as a symmetry group. The representation of $SU(2) \times SU(2)$ on each of the eigenspaces $(q_m(S_{n,m}), \langle\ ,\ \rangle_{n,m})$ covers all the unitary irreducible representations of $SU(2) \times SU(2)$, up to isomorphisms, if n and m vary under the condition stated in Proposition 2.6.*

§3. The zero-energy case and $\mathbf{R}^3 \ltimes SU(2)$

3.1. A symmetry group of the conformal Kepler problem with zero-energy

In the case of zero-energy, we associate CK with a *four-dimensional free particle*; a quantum system with the Hamiltonian operator

$$\widehat{F} = -\frac{1}{2}\sum_{j=1}^{4} \frac{\partial^2}{\partial {q_j}^2}, \tag{3.1}$$

which can be extended to a self-adjoint operator in $L^2(\mathbf{R}^4)$. The free particle will be often abbreviated to FP, in what follows. $\widehat{H}$ and $\widehat{F}$ satisfy

$$4r\,\widehat{H} = \widehat{F} - 4k, \tag{3.2}$$

which implies that to study the $\widehat{H}$ with zero-spectrum is to study the $\widehat{F}$ with spectrum $4k$.

We start with a review of $\widehat{F}$ with positive spectra. Let us denote by $\mathcal{S}(\mathbf{R}^4_q)$ and $\mathcal{S}(\mathbf{R}^4_u)$ the spaces of smooth rapidly decreasing functions on $\mathbf{R}^4_q$ and $\mathbf{R}^4_u$, respectively, where the subscripts q and u indicate the variables used in $\mathbf{R}^4$'s. For $\phi \in \mathcal{S}(\mathbf{R}^4_q)$, we denote its Fourier transform by $\widetilde{\phi} \in \mathcal{S}(\mathbf{R}^4_u)$. On using the polar coordinates, $u = \sqrt{2s}\omega$, with $\omega \in S^3$ and $s \geq 0$, the Fourier integral formula is put in the form

$$\phi(q) = \frac{1}{(2\pi)^2}\int_0^\infty 2s\,ds \int_{S^3} e^{iq\cdot\sqrt{2s}\omega}\,\widetilde{\phi}(\sqrt{2s}\omega)\,dS^3, \tag{3.3}$$

where dS^3 denotes the standard volume element of the three-sphere S^3. Further, Plancherel's theorem takes the form

$$(3.4) \qquad \|\phi\|^2_{L^2(\mathbf{R}^4_q)} = \|\widetilde{\phi}\|^2_{L^2(\mathbf{R}^4_u)} = \int_0^\infty 2s\,ds \int_{S^3} |\widetilde{\phi}(\sqrt{2s}\omega)|^2\,dS^3.$$

Thus, if we define the function space

$$(3.5) \qquad \mathcal{H}_s := \left\{ f(q) = \int_{S^3} e^{iq\cdot\sqrt{2s}\omega}\, F(\omega)dS^3;\ F \in L^2(S^3) \right\},$$

$L^2(\mathbf{R}^4_q)$ turns out to be decomposed into a direct integral

$$(3.6) \qquad L^2(\mathbf{R}^4_q) = \int_0^\infty \oplus \mathcal{H}_s\, 2s\,ds.$$

It is worth pointing out that any $f \in \mathcal{H}_s$ satisfies the Schrödinger equation; $\widehat{F}f = sf$ (see Helgason [H]). Moreover, the map $\kappa_s : L^2(S^3) \to \mathcal{H}_s$ given by $F(\omega) \mapsto f(q)$ through (3.5) makes $\mathcal{H}_s$ into a Hilbert space, so that one has the isomorphism, for all positive s,

$$(3.7) \qquad \mathcal{H}_s \cong L^2(S^3).$$

Summarizing the above, we have the following.

Proposition 3.1. *$L^2(\mathbf{R}^4_q)$ is decomposed into the direct integral of Hilbert spaces $\{\mathcal{H}_s\}$ each of which is isomorphic to $L^2(S^3)$ and associated with the spectrum s of $\widehat{F}$. Hence, $\mathcal{H}_{4k}$ is a Hilbert space associated with the zero-energy of the conformal Kepler problem.*

We proceed to study a symmetry group of FP to get a symmetry group of CK with zero-energy. In classical theory, the symmetry group of FP is known to be $\mathbf{R}^9 \ltimes SO(4)$, where $\mathbf{R}^9$ and $\ltimes$ denote the additive group of 4×4 traceless real symmetric matrices and the semi-direct product, respectively [I-U2]. In quantum theory, the action of $\mathbf{R}^9 \ltimes SO(4)$ is "quantized" to give a unitary representation in $L^2(\mathbf{R}^4_q)$ as

$$(3.8) \quad (X_{(M,g)}\phi)(q) = \frac{1}{(2\pi)^2} \int_{\mathbf{R}^4_u} e^{iq\cdot u} \exp\left(-\frac{i}{2}u \cdot Mu\right) \widetilde{\phi}(g^{-1}u)\,du,$$

where $\widetilde{\phi}$ is the Fourier transform of $\phi \in L^2(\mathbf{R}^4_q)$, and $(M, g) \in \mathbf{R}^9 \ltimes SO(4)$ (see [I-U2]). On applying (3.6) to (3.8), the representation $X_{(M,g)}$ gives rise to unitary representations $U^s_{(M,g)}$ of $\mathbf{R}^9 \ltimes SO(4)$ in each $\mathcal{H}_s$, which takes the form

$$(3.9)\qquad (U^s_{(M,g)}f)(q) = \int_{S^3} e^{iq\cdot\sqrt{2s}\omega} \exp\left(-\frac{i}{2}\omega\cdot M\omega\right) F(g^{-1}\omega)\,dS^3,$$

where $f \in \mathcal{H}_s$ is of the form (3.5).

Proposition 3.2. *The free particle admits* $\mathbf{R}^9 \ltimes SO(4)$ *as a symmetry group, which is represented unitarily as* (3.9) *in each Hilbert spaces* $\mathcal{H}_s$ *given by* (3.5).

Propositions 3.1 and 3.2 are put together to give the following.

Theorem 3.3. *The conformal Kepler problem with zero-energy admits* $\mathbf{R}^9 \ltimes SO(4)$ *as a symmetry group whose action is represented unitarily on the Hilbert space* $\mathcal{H}_{4k}$.

Remark. Because of (3.7), a unitary representation, denoted by V^s, of $\mathbf{R}^9 \ltimes SO(4)$ in $L^2(S^3)$ is also defined by

$$(3.10)\qquad U^s_{(M,g)} \circ \kappa_s = \kappa_s \circ V^s_{(M,g)} \quad ((M,g) \in \mathbf{R}^9 \ltimes SO(4)).$$

3.2. A symmetry group of the MIC-Kepler problem with zero-energy

We derive a symmetry group of MICK with zero-energy from the symmetry group of CK obtained in Theorem 3.3. A subgroup of the symmetry group, $\mathbf{R}^9 \ltimes SO(4)$, of CK that leaves the subspace of ρ_m-equivariant functions in $\mathcal{H}_{4k}$ is shown to be a symmetry group of MICK with zero-energy.

We study first the subspace of ρ_m-equivariant functions in $\mathcal{H}_s$, which subspace is denoted by $\mathcal{H}_{s,m}$. On carrying out a similar argument to that of negative energy case, we have the following.

Proposition 3.4. *The subspace,* $\mathcal{H}_{4k,m}$, *of* ρ_m*-equivariant functions in* $\mathcal{H}_{4k}$ *is mapped, through* q_m, *to the space of cross sections associated with the MIC-Kepler problem with zero-energy.*

We then understand that $q_m(\mathcal{H}_{4k,m})$ is the space associated with $\widehat{H}_m = 0$. The Hilbert space structure is defined on $q_m(\mathcal{H}_{s,m})$ by the inner product $\langle\ ,\ \rangle_{s,m}$ which is naturally induced from the inner product, say, $\langle\ ,\ \rangle_s$, in $\mathcal{H}_s$ as

$$(3.11)\qquad \langle\gamma_1, \gamma_2\rangle_{s,m} = \langle q_m^{-1}\gamma_1, q_m^{-1}\gamma_2\rangle_s,$$

where $\gamma_1, \gamma_2 \in q_m(\mathcal{H}_{s,m})$. We study further the structure of $\mathcal{H}_{s,m}$ and of $q_m(\mathcal{H}_{s,m})$. Specializing (3.9) in a subgroup $\{(0, T(t));\ t \in [0, 4\pi]\}$ of $\mathbf{R}^9 \ltimes SO(4)$, one has, for $f \in \mathcal{H}_s$,

$$f(T(t)q) = (U^s_{(0,T(-t))}f)(q) = \int_{S^3} e^{iq\cdot\sqrt{2s}\omega}\, F(T(t)\omega)dS^3, \tag{3.12}$$

where $f = \kappa_s(F)$ for $F \in L^2(S^3)$. This implies that f is ρ_m-equivariant in $\mathcal{H}_s$ if and only if F is ρ_m-equivariant in $L^2(S^3)$. We denote by $L^2(S^3)_m$ the space of ρ_m-equivariant functions in $L^2(S^3)$, which has a Hilbert space structure as a closed subspace of $L^2(S^3)$. Thus we have the isomorphisms,

$$q_m(\mathcal{H}_{s,m}) \cong \mathcal{H}_{s,m} \cong L^2(S^3)_m. \tag{3.13}$$

We are now in a position to find a symmetry group of MICK with zero-energy. Like in the negative energy case, a subgroup of $\mathbf{R}^9 \ltimes SO(4)$ that leaves $\mathcal{H}_{s,m}$ invariant proves to be a subgroup which is commutative with the $U(1)$ action (3.12). We can show that the largest one of such subgroups is isomorphic to $\mathbf{R}^3 \ltimes U(2)$. However, since this subgroup includes the $U(1) \cong \{(0, T(t));\ t \in [0, 4\pi]\}$, we choose to take $\mathbf{R}^3 \ltimes SU(2)$ after eliminating the $U(1)$ from $\mathbf{R}^3 \ltimes U(2)$. We have to notice here how the $\mathbf{R}^3 \ltimes SU(2)$ is represented as pairs of 4×4 matrices in $\mathbf{R}^9 \ltimes SO(4)$: Let a map β be defined as

$$\beta : \begin{pmatrix} a_1 + ib_1 & a_2 + ib_2 \\ a_3 + ib_3 & a_4 + ib_4 \end{pmatrix} \longmapsto Z = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix} \tag{3.14}$$

$$\text{with} \quad Z_j = \begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix} \quad (j = 1, 2, 3, 4).$$

Then $\mathbf{R}^3 \ltimes SU(2)$ is represented as a subgroup of $\mathbf{R}^9 \ltimes SO(4)$, $\beta(\mathbf{R}^3) \ltimes \beta(SU(2))$. We thus obtain the following.

Proposition 3.5. *The semi-direct product group $\mathbf{R}^3 \ltimes SU(2)$ acts unitarily on $\mathcal{H}_{s,m}$, where $\mathbf{R}^3$ indicates the additive group of 2×2 traceless Hermitian matrices.*

By $U^{(s,m)}$ we denote the induced action of $\mathbf{R}^3 \ltimes SU(2)$ on $\mathcal{H}_{s,m}$. Then, the action of $\mathbf{R}^3 \ltimes SU(2)$ on the Hilbert space $q_m(\mathcal{H}_{s,m})$ is defined, for $\gamma \in q_m(\mathcal{H}_{s,m})$, by

$$W^{(s,m)}_{(M,g)}\gamma = (q_m \circ U^{(s,m)}_{(M,g)} \circ q_m^{-1})\gamma, \tag{3.15}$$

where $(M, g) \in \mathbf{R}^3 \ltimes SU(2)$ is represented as a pair of 4×4 real matrices (cf. (3.14)). This action is unitary, of course.

Proposition 3.6. *The MIC-Kepler problem with zero-energy admits the semi-direct product group* $\mathbf{R}^3 \ltimes SU(2)$ *as a symmetry group, which is unitarily represented in* $q_m(\mathcal{H}_{s,m})$ *in the manner of* (3.15) *together with* (3.9).

In closing Section 3.2, we make a mention of the unitary representation V^s of $\mathbf{R}^9 \ltimes SO(4)$ in $L^2(S^3)$ (see (3.10)). From (3.10) and (3.13), we can define a unitary representation $V^{(s,m)}$ of $\mathbf{R}^3 \ltimes SU(2)$ on $L^2(S^3)_m$ by

$$U^{(s,m)}_{(M,g)} \circ \kappa_s = \kappa_s \circ V^{(s,m)}_{(M,g)}, \tag{3.16}$$

where $(M, g) \in \mathbf{R}^3 \ltimes SU(2)$ is represented as a pair of 4×4 real matrices. The representations $W^{(s,m)}$ and $V^{(s,m)}$ are unitarily equivalent on account of (3.15) and (3.16).

3.3. Relation to the Mackey's induced representation

In this section, the representation $W^{(s,m)}$ of $\mathbf{R}^3 \ltimes SU(2)$ is shown to be equivalent to the *Mackey*'s induced representation. Since $W^{(s,m)}$ is equivalent to $V^{(m,s)}$, we choose to show the equivalence between $V^{(s,m)}$ and Mackey's representation. According to Mackey [Mk], the induced representation of $\mathbf{R}^3 \ltimes SU(2)$ is realized on the Hilbert space of functions on the group manifold $\mathbf{R}^3 \ltimes SU(2)$.

Let α be a bijection of S^3 to $SU(2)$ given by

$$\alpha : \omega \in S^3 \longmapsto \begin{pmatrix} \omega_1 + i\omega_2 & -\omega_3 + i\omega_4 \\ \omega_3 + i\omega_4 & \omega_1 - i\omega_2 \end{pmatrix}. \tag{3.17}$$

Using α, we can define an injection $A^{(s,m)}$ of $L^2(S^3)_m$ to a space of functions on $\mathbf{R}^3 \ltimes SU(2)$; for $F \in L^2(S^3)_m$, $A^{(s,m)}$ is given by

$$(A^{(s,m)} F)(M, g) = \exp(isv \cdot g^{-1} M g\, v)\, F(\alpha^{-1}(g)) \tag{3.18}$$

with $v = (1, 0)^T$, where $(M, g) \in \mathbf{R}^3 \ltimes SU(2)$ indicates a pair of 2×2 complex matrices. It is easy to prove that for a subgroup $\mathbf{R}^3 \ltimes U(1)$ acting on $\mathbf{R}^3 \ltimes SU(2)$ to the right, functions $A^{(s,m)}F$ are subject to the transformation

$$\begin{aligned} A^{(s,m)}F((M,g)(N,u(t))) &= \chi_{s,m}(N,\, u(t))^{-1}(A^{(s,m)}F)(M,g) \\ \text{with} \quad \chi_{s,m}(N,\, u(t)) &= \exp(-isv \cdot N\, v)\, e^{-imt/2}, \end{aligned} \tag{3.19}$$

where $u(t)$ is the 2×2 matrix given by $u(t) = \mathrm{diag}(e^{it/2}, e^{-it/2})$, and the $\chi_{s,m}$ is known as an irreducible representation of $\mathbf{R}^3 \ltimes U(1)$ on

C. Equations (3.18) and (3.19) imply that $L^2(S^3)_m$ is mapped to the space of $\chi_{s,m}$-equivariant functions on $\mathbf{R}^3 \ltimes SU(2)$, which space can be made into a Hilbert space, and is isomorphic to the Hilbert space of L^2-cross sections in a complex line bundle over the quotient space $(\mathbf{R}^3 \ltimes SU(2))/(\mathbf{R}^3 \ltimes U(1)) \cong S^2$. We denote by $L^2(\mathbf{R}^3 \ltimes SU(2))_{s,m}$ the Hilbert space of $\chi_{s,m}$-equivariant functions on $\mathbf{R}^3 \ltimes SU(2)$.

Let $T^{(s,m)}$ be the Mackey's induced representation of $\mathbf{R}^3 \ltimes SU(2)$ in $L^2(\mathbf{R}^3 \ltimes SU(2))_{s,m}$. Then a straightforward calculation shows that $A^{(s,m)}$ gives an intertwining operator between $T^{(s,m)}$ and $V^{(s,m)}$;

$$T^{(s,m)}_{(M,g)} \circ A^{(s,m)} = A^{(s,m)} \circ V^{(s,m)}_{\mathrm{Re}(M,g)} \quad ((M,g) \in \mathbf{R}^3 \ltimes SU(2)), \tag{3.20}$$

where $\mathrm{Re}(M,g)$ denotes the real matrix representation (see (3.14)). Since $T^{(s,m)}$ is known to be irreducible and to exhaust all the unitary irreducible representations, up to isomorphisms, we have the following.

Theorem 3.7. *The unitary representation $(W^{(4k,m)}, q_m(\mathcal{H}_{4k,m}))$ of the symmetry group $\mathbf{R}^3 \ltimes SU(2)$ of the MICK is irreducible. The $W^{(4k,m)}$ exhausts all the unitary irreducible representations of $\mathbf{R}^3 \ltimes SU(2)$, up to isomorphisms, if the parameter k and m range over all the positive real numbers and the integers, respectively.*

§4. The positive energy case and $SL(2,\mathbf{C})$

4.1. A symmetry group of the conformal Kepler problem with positive energies

In the positive energy case, the four-dimensional repulsive oscillator is associated with the conformal Kepler problem. The repulsive oscillator is a quantum system with the Hamiltonian operator

$$\widehat{R}_\lambda = -\frac{1}{2}\sum_{j=1}^{4}\frac{\partial^2}{\partial {q_j}^2} - \frac{\lambda^2}{2}\sum_{j=1}^{4} q_j^2, \tag{4.1}$$

where λ is a positive parameter. From now on, the repulsive oscillator will be often abbreviated to RO. The $\widehat{H}$ and $\widehat{R}_\lambda$ satisfy the relation

$$4r\left(\widehat{H} - \frac{\lambda^2}{8}\right) = \widehat{R}_\lambda - 4k. \tag{4.2}$$

Like in the zero-energy case, studying $\widehat{H}$ with positive spectrum $\lambda^2/8$ amounts to studying $\widehat{R}_\lambda$ with positive spectrum $4k$.

We start by studying the repulsive oscillator. On physical grounds, it is better for us to introduce a unitary operator ξ of $L^2(\mathbf{R}^4_q)$ to $L^2(\mathbf{R}^4_u)$, which is expressed as the integral transform (see [I-U3])

$$(4.3)\quad (\xi\phi)(u) = \frac{\lambda}{2\pi^2}\int_{\mathbf{R}^4_q} \exp\left\{\frac{i}{2}(u\cdot u - 2\sqrt{2\lambda}\,u\cdot q + \lambda q\cdot q)\right\}\phi(q)\,dq.$$

Then the ξ maps $\widehat{R}_\lambda$ into

$$(4.4)\qquad \widehat{L}_\lambda := \xi\circ\widehat{R}_\lambda\circ\xi^{-1} = \frac{\lambda}{i}\left(\sum_{j=1}^{4} u_j\frac{\partial}{\partial u_j} + 2\right),$$

where the domains of $\widehat{R}_\lambda$ and $\widehat{L}_\lambda$ are considered, say, as the spaces of smooth functions of rapid descent on $\mathbf{R}^4_q$ and $\mathbf{R}^4_u$, respectively. The $\widehat{L}_\lambda$ is easy to treat. In fact, $\widehat{L}_\lambda/\lambda$ is immediately integrated to give a one-parameter group of unitary transformations D_t on $L^2(\mathbf{R}^4_u)$;

$$(4.5)\qquad (D_t\phi)(u) = e^{2t}\phi(e^t u).$$

The generator of D_t should be a self-adjoint extension of $\widehat{L}_\lambda/\lambda$, which we denote by the same letter. The unitary operator F_t defined by $F_t = \xi^{-1}\circ D_t\circ\xi$ then have the generator that is a self-adjoint extension of $\widehat{R}_\lambda/\lambda$, which we denote by the same letter. Hence we have the following.

Lemma 4.1. *The repulsive oscillator* $(\widehat{R}_\lambda, L^2(\mathbf{R}^4_q))$ *is unitarily equivalent to the quantum system* $(\widehat{L}_\lambda, L^2(\mathbf{R}^4_u))$.

We may choose to study the system $(D_t, L^2(\mathbf{R}^4_u))$, instead of $(\widehat{L}_\lambda, L^2(\mathbf{R}^4_u))$. For $\phi\in\mathcal{S}(\mathbf{R}^4_u)$, the space of smooth functions of rapid descent on $\mathbf{R}^4_u$, set

$$(4.6)\qquad (P_s\phi)(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ist}e^{2t}\phi(e^t u)\,dt.$$

Then, from the Fourier integral formula, one obtains a decomposition

$$(4.7)\qquad \phi(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(P_s\phi)(u)\,ds.$$

It is worth noting that $P_s\phi$ satisfies a homogeneity condition

$$(4.8)\qquad (P_s\phi)(\epsilon u) = \epsilon^{is-2}(P_s\phi)(u)\quad (u\in\mathbf{R}^4_u,\ \epsilon>0).$$

Equation (4.7) together with Plancherel's theorem results in a direct integral decomposition

$$L^2(\mathbf{R}^4_u) = \int_{-\infty}^{\infty} \oplus \mathcal{H}_s(\mathbf{R}^4_u)\, ds, \tag{4.9}$$

where $\mathcal{H}_s(\mathbf{R}^4_u)$ are a one-parameter family of Hilbert spaces defined by

$$\mathcal{H}_s(\mathbf{R}^4_u) = \{f \in L^2_{loc}(\mathbf{R}^4_u - \{0\});\quad f(\epsilon u) = \epsilon^{is-2} f(u),\quad \epsilon > 0,\quad \text{and} \quad \int_{S^3} |f|^2\, dS^3 < +\infty\}. \tag{4.10}$$

The Hilbert space structure of $\mathcal{H}_s(\mathbf{R}^4_u)$ is, of course, defined, for $f \in \mathcal{H}_s(\mathbf{R}^4_u)$, by

$$\|f\|^2_{\mathcal{H}_s(\mathbf{R}^4_u)} = \int_{S^3} |f(\omega)|^2\, dS^3. \tag{4.11}$$

Further, the homogeneity condition in (4.10) makes any $f \in \mathcal{H}_s(\mathbf{R}^4_u)$ be determined by its restriction to S^3, so that one has the isomorphism

$$\mathcal{H}_s(\mathbf{R}^4_u) \cong L^2(S^3). \tag{4.12}$$

Further, that homogeneity condition along with (4.8) and $\epsilon = e^t$ gives rise to the equation, $\widehat{L}_\lambda f = \lambda s f$, for a smooth function f in $\mathcal{H}_s(\mathbf{R}^4_u)$. We thus have the following.

Proposition 4.2. *The Hilbert space $\mathcal{H}_s(\mathbf{R}^4_u)$ which is isomorphic to $L^2(S^3)$ is associated with the spectrum λs of $\widehat{L}_\lambda$.*

We turn to a symmetry group of $(D_t,\, L^2(\mathbf{R}^4_u))$. On "quantizing" an $SL(4,\mathbf{R})$ action on the phase space in classical theory (see [I-U3]), a unitary action of $SL(4,\mathbf{R})$ is derived on $L^2(\mathbf{R}^4_u)$, which is given, for $\phi \in L^2(\mathbf{R}^4_u)$, by

$$Y_g : \phi(u) \longmapsto \phi(g^{-1}u) \quad (g \in SL(4,\mathbf{R})). \tag{4.13}$$

Since $P_s \circ Y_g = Y_g \circ P_s$, one can restrict Y to $\mathcal{H}_s(\mathbf{R}^4_u)$ to define a unitary representation U^s in $\mathcal{H}_s(\mathbf{R}^4_u)$ by

$$(U^s_g f)(u) = f(g^{-1}u) = |g^{-1}u|^{is-2} f\left(\frac{g^{-1}u}{|g^{-1}u|}\right) \quad (g \in SL(4,\mathbf{R})). \tag{4.14}$$

We thus have the following.

Proposition 4.3. *$SL(4,\mathbf{R})$ is a symmetry group of the system $(D_t,\ L^2(\mathbf{R}^4_u))$, which group is unitarily represented in $\mathcal{H}_s(\mathbf{R}^4_u)$ as* (4.14).

The $SL(4,\mathbf{R})$ turns into a symmetry group of RO on the unitary equivalence between RO and $(\widehat{L}_\lambda,\ L^2(\mathbf{R}^4_u))$ (see Lemma 4.1). The representation space of $SL(4,\mathbf{R})$ for RO is, however, not easy to identify, since we cannot apply the unitary operator ξ^{-1} (cf. (4.3)) directly to $\mathcal{H}_s(\mathbf{R}^4_u)$ because of $\mathcal{H}_s(\mathbf{R}^4_u) \not\subseteq L^2(\mathbf{R}^4_u)$. An alternative way to get that space is to view $f \in \mathcal{H}_s(\mathbf{R}^4_u)$ as a tempered distribution. Then, it can be shown by calculation that, for $f \in \mathcal{H}_s(\mathbf{R}^4_u)$, there exists a unique function $h(q)$ on $\mathbf{R}^4_q$ which satisfies

$$T_f(\xi\phi) = T_h(\phi) \qquad (\phi \in \mathcal{S}(\mathbf{R}^4_q)) \tag{4.15}$$

(see [I-U3]), where T_f and T_h stand for the tempered distributions associated with f and h, respectively. Moreover, $h(q)$ proves to satisfy $\widehat{R}_\lambda h = \lambda s\, h$. Therefore, on denoting by η_s the map, determined by (4.15), of f to $h(q)$, the space $\eta_s(\mathcal{H}_s(\mathbf{R}^4_u))$ of functions on $\mathbf{R}^4_q$ is what we have looked for as a representation space of $SL(4,\mathbf{R})$, which space will be denoted by $\mathcal{K}_s(\mathbf{R}^4_q)$ henceforth. The Hilbert space structure for $\mathcal{K}_s(\mathbf{R}^4_q)$ is defined from that for $\mathcal{H}_s(\mathbf{R}^4_u)$ through η_s. Then it follows that

$$\mathcal{K}_s(\mathbf{R}^4_q) \cong \mathcal{H}_s(\mathbf{R}^4_u) \cong L^2(S^3). \tag{4.16}$$

A unitary representation V^s of $SL(4,\mathbf{R})$ in $\mathcal{K}_s(\mathbf{R}^4_q)$ is now induced from the representation U^s in $\mathcal{H}_s(\mathbf{R}^4_u)$;

$$V^s = \eta_s \circ U^s \circ \eta_s^{-1}. \tag{4.17}$$

Proposition 4.4. *The repulsive oscillator admits $SL(4,\mathbf{R})$ as a symmetry group, which has a unitary representation in the Hilbert space $\mathcal{K}_s(\mathbf{R}^4_q)$ associated the spectrum λs of $\widehat{R}_\lambda$.*

Proposition 4.4, in turn, provides the symmetry group of CK. If we set $\lambda s = 4k$ in accordance with (4.2), the $\mathcal{K}_{4k/\lambda}(\mathbf{R}^4_q)$ turns into the carrier space of the unitary representation of the symmetry group $SL(4,\mathbf{R})$ of CK.

Theorem 4.5. *The conformal Kepler problem with positive energy $\lambda^2/8$ admits $SL(4,\mathbf{R})$ as a symmetry group, which is unitarily represented in the Hilbert space $\mathcal{K}_{4k/\lambda}(\mathbf{R}^4_q)$.*

4.2. A symmetry group of the MIC-Kepler problem with positive energy

Like in the negative and the zero-energy cases, we study first a Hilbert space of cross sections in L_m associated with MICK with positive energy. Let $\mathcal{H}_{s,m}(\mathbf{R}_u^4)$ and $\mathcal{K}_{s,m}(\mathbf{R}_q^4)$ be the spaces of ρ_m-equivariant functions in $\mathcal{H}_s(\mathbf{R}_u^4)$ and in $\mathcal{K}_s(\mathbf{R}_q^4)$, respectively. In the positive energy case, we obtain the following proposition similar to Propositions 2.4 and 3.4.

Proposition 4.6. *The space $\mathcal{K}_{4k/\lambda,\, m}(\mathbf{R}_q^4)$ is mapped, through q_m, injectively to a space of cross sections in L_m in association with the positive spectrum $\lambda^2/8$ of the MIC-Kepler problem.*

In view of Proposition 4.6, we will denote by $q_m(\mathcal{K}_{s,m})$ the space of cross sections in L_m mapped from $\mathcal{K}_{s,m}(\mathbf{R}_q^4)$. The Hilbert space structure of $q_m(\mathcal{K}_{s,m})$, of course, comes from that of $\mathcal{K}_{s,m}(\mathbf{R}_q^4)$. We study the structure of $\mathcal{K}_{s,m}(\mathbf{R}_q^4)$ and $q_m(\mathcal{K}_{s,m})$ further. From (4.15) it follows that f in $\mathcal{H}_s(\mathbf{R}_u^4)$ is ρ_m-equivariant if and only if h in $\mathcal{K}_s(\mathbf{R}_q^4)$ is ρ_m-equivariant, hence $\mathcal{H}_{s,m}(\mathbf{R}_u^4) \cong \mathcal{K}_{s,m}(\mathbf{R}_q^4)$. Combined with (4.12), this fact yields

$$q_m(\mathcal{K}_{s,m}) \cong \mathcal{K}_{s,m}(\mathbf{R}_q^4) \cong \mathcal{H}_{s,m}(\mathbf{R}_u^4) \cong L^2(S^3)_m. \tag{4.18}$$

We proceed to a symmetry group of MICK with positive energy. A subgroup of the symmetry group $SL(4,\mathbf{R})$ of CK that leaves $\mathcal{K}_{4k/\lambda,m}$ $(\mathbf{R}_q^4)$ invariant turns into a symmetry group of MICK with energy $\lambda^2/8$. Like in the cases of negative and zero-energies, the subgroup to be looked for is a subgroup commutative with the $U(1) \cong \{T(t),\ t \in [0, 4\pi]\}$ $\subset SL(4,\mathbf{R})$. As a result, we have a real representation of $SL(2,\mathbf{C})$ in $SL(4,\mathbf{R})$, which is given by $\beta(g)$ for $g \in SL(2,\mathbf{C})$ and β in (3.14).

Proposition 4.7. *A real representation of $SL(2,\mathbf{C})$ in $SL(4,\mathbf{R})$ acts unitarily on $\mathcal{K}_{s,m}(\mathbf{R}_q^4)$.*

By $V^{(s,m)}$ we denote the restriction of V^s to $\mathcal{K}_{s,m}(\mathbf{R}_q^4)$. We can then define the unitary representation of $SL(2,\mathbf{C})$, denoted by $W^{(s,m)}$, in the Hilbert space $q_m(\mathcal{K}_{s,m})$ by

$$W_g^{(s,m)} = q_m \circ V_g^{(s,m)} \circ q_m^{-1}, \tag{4.19}$$

where $g \in SL(2,\mathbf{C})$ is represented in a 4×4 real matrix form. Thus we have the following.

Theorem 4.8. *The MIC-Kepler problem with positive energy admits $SL(2,\mathbf{C})$ as a symmetry group, which is unitarily represented in the Hilbert space $q_m(\mathcal{K}_{4k/\lambda,m})$ associated with $\widehat{H}_m = \lambda^2/8$.*

In conclusion, we give another unitary representation of $SL(2,\mathbf{C})$ in $L^2(S^3)_m$, which representation is unitarily equivalent to the representation $W^{(s,m)}$ in $q_m(\mathcal{K}_{s,m})$. The isomorphism (4.18) enables us to define $D^{(s,m)}$ through (4.14) together with the restriction map $f \mapsto F := f\,|_{S^3}$;

$$(D_g^{(s,m)}F)(\omega) = |g^{-1}\omega|^{is-2}F\left(\frac{g^{-1}\omega}{|g^{-1}\omega|}\right), \tag{4.20}$$

where $F \in L^2(S^3)_m$, and $g \in SL(2,\mathbf{C})$ is represented in the 4×4 real matrix form.

4.3. Relation to principal series representations of $SL(2,\mathbf{C})$

We show that the $W^{(s,m)}$ is unitarily equivalent to the so-called principal series representation of $SL(2,\mathbf{C})$. On account of the equivalence between $D^{(s,m)}$ and $W^{(s,m)}$, we choose to deal with $D^{(s,m)}$. If we introduce the complex vector space structure $\mathbf{C}^2$ into $\mathbf{R}_u^4$ by $u_1 + iu_2 = z_1$, and $u_3 + iu_4 = z_2$, the defining condition for $f \in \mathcal{H}_{s,m}(\mathbf{R}_u^4)$ is put in the form

$$f(\alpha z) = \alpha^{n_1}\overline{\alpha}^{n_2} f(z) \quad (\alpha \in \mathbf{C}-\{0\}) \tag{4.21a}$$

with

$$n_1 = \frac{1}{2}(is-2+m), \quad n_2 = \frac{1}{2}(is-2-m). \tag{4.21b}$$

This is identical with the condition required for the principal series representation due to Gel'fand et. al., which is denoted by $T_{(n_1+1,n_2+1)}$ ([G-G-V]). Indeed, one easily gets the equivalence

$$D_g^{(s,m)} = T_{(n_1+1,n_2+1)}((g^{-1})^T) \qquad \text{for} \qquad g \in SL(2,\mathbf{C}).$$

Since this principal series representation is irreducible, we have the following.

Theorem 4.9. *The unitary representation $W^{(4k/\lambda,m)}$ of $SL(2,\mathbf{C})$ in $q_m(\mathcal{K}_{4k/\lambda,m})$ is irreducible and exhausts all the principal series of unitary irreducible representations of $SL(2,\mathbf{C})$, up to isomorphisms, as λ and m range over all the positive real numbers and the integers, respectively.*

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Department of Applied Mathematics and Physics
Kyoto University
Kyoto 606-01
Japan