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Some Exact Trace Formulæ

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§1. Introduction

In the study of the singularities of the trace of the wave operator on Riemannian manifolds there is a dichotomy between the constant curvature case and the general case. On an $(n+1)$-dimensional manifold of constant negative curvature, M one studies

$$\operatorname{tr}(\exp t(\Delta + \left(\frac{n}{2}\right)^2)^{\frac{1}{2}}). \tag{1.1}$$

In this case the Selberg trace formula provides an exact expression in terms of the lengths of the closed geodesics on M. In the general case one considers

$$\operatorname{tr}(\exp t(\Delta)^{\frac{1}{2}}). \tag{1.2}$$

As such, flat tori are the only examples where the trace in (1.2) is computed explicitly. In fact, these are the only examples where precise bounds for the error term in the Guillemin-Duistermaat formula are known. In this note we narrow the gap, a bit, by showing that there is a general procedure for obtaining an exact formula for

$$\operatorname{tr}(\exp t(L + \alpha^2))$$

whenever one has an exact formula for

$$\operatorname{tr}(\exp t(L)^{\frac{1}{2}}).$$

Here L can be taken as a shifted Laplace operator but the idea applies with much greater generality. As a special case we obtain the trace in (1.2) for the Laplace operator of a compact hyperbolic manifold. Our calculations serve, at least in this special case, to clarify the nature of the error term in the Guillemin-Duistermaat formula.

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§2. The solution operator for the shifted wave equation

Suppose that L is a self adjoint elliptic operator on a Hilbert space, H. Further assume that the Cauchy problem

$$\begin{aligned} u_{tt} &= (L+\alpha^2)u, \\ u(0) &= f, \\ u_t(0) &= g, \end{aligned} \tag{2.1}$$

has a solution $u \in C^2_{\text{loc}}(\mathbb{R}; H)$ for data f, g in a dense subspace of H. In this section we obtain a formula for the solution of the Cauchy problem

$$\begin{aligned} v_{tt} &= (L+\beta^2)v, \\ v(0) &= f, \\ v_t(0) &= g, \end{aligned} \tag{2.2}$$

in terms of the solution to (2.1). To simplify the calculations we assume that $g = 0$, in (2.1) and $f = 0$ in (2.2). The general case follows easily from this.

We make the following ansatz:

(2.3) The function $v(t)$ is given by a Volterra operator applied to $u(t)$, that is

$$v(t) = \int_0^t k(t,s)u(s)ds.$$

For L a Laplace operator, the finite propagation property of the wave equation forces such an ansatz. By applying the operators $L + \beta^2$ and ∂_t^2 to (2.3) and integrating by parts, we obtain that $k(t,s)$ must satisfy:

$$\begin{aligned} k_{tt} &= k_{ss} - (\alpha^2 - \beta^2)k \text{ for } |s| < |t|, \\ k_t(t,t) + k_s(t,t) &= 0, \\ k_s(t,0) &= 0. \end{aligned} \tag{2.4}$$

A solution to this differential equation is easily found by the method of descent from the wave equation in $\mathbb{R}^2$, see [John]. The kernel, k is given by

$$k(t,s;\alpha,\beta) = J_0(\sqrt{(\alpha^2-\beta^2)(t^2-s^2)}). \tag{2.5}$$

The explicit dependence on α, β will sometimes be suppressed. The J_0-Bessel function is defined as an infinite series by:

$$J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k}(k!)^2}. \tag{2.6}$$

From (2.6) it is clear that $k(t,s)$ is a smooth function of $s, t \in \mathbb{R}^2$ and that the choice of square root is immaterial. Differentiating shows that k satisfies the boundary conditions in (2.4).

Proposition 2.7. *Let $u(t) \in C^2_{\mathrm{loc}}(\mathbb{R}; H)$ satisfy (2.1) with $g = 0$ and define $k(t,s)$ by (2.5). Then $v(t)$ defined by (2.3) belongs to $C^3_{\mathrm{loc}}(\mathbb{R}; H)$ and satisfies*

$$v_{tt} = (L + \beta^2)v; \quad v(0) = 0,\, v_t(0) = f. \tag{2.8}$$

Proof. The only point requiring comment is the regularity of $v(t)$. This follows easily as the kernel is $\mathcal{C}^\infty$ and the domain of integration is compact. Thus the standard theory of vector valued integrals allows us to differentiate under the integral sign. The conclusion then follows by an application of the Cauchy-Schwarz inequality.

If we define

$$v'(t) = u(t) + \int_0^t k_t(t,s)u(s)ds \tag{2.9}$$

then $v'(t)$ solves the same Cauchy problem as u with respect to the shifted operator:

$$v'_{tt} = (L + \beta^2)v'; \quad v'(0) = f,\, v'_t(0) = 0. \tag{2.10}$$

As a corollary of the Proposition we can express the solution kernel for (2.10) in terms of the solution kernel for (2.1). Denote them by $R^\beta(p,q;t)$ and $R^\alpha(p,q;t)$ respectively, then

$$R^\beta(p,q;t) = R^\alpha(p,q;t) + \int_0^t k_t(t,s;\alpha,\beta)R^\alpha(p,q;s)\,ds. \tag{2.11}$$

This formula or (2.9) can of course be applied with α and β interchanged. The uniqueness of the solution to the Cauchy problem, (2.1), implies that

$$u(t) = \partial_t \int_0^t k(t,s;\beta,\alpha) v'(s)\, ds. \tag{2.12}$$

The proposition applies if L is taken to be a second order self adjoint, elliptic pseudodifferential operator. If L is not a differential operator then it does have finite propagation speed; however analysis of the singular support of the solution operators would indicate that a relation like (2.11) should still hold.

As a second corollary of the proposition we have the following calculus identity

Corollary 2.13. *For α, β, λ complex numbers the following identity holds:*

$$\frac{\sin t\sqrt{\lambda^2-\beta^2}}{\sqrt{\lambda^2-\beta^2}} = \int_0^t J_0\left(\sqrt{(\alpha^2-\beta^2)(t^2-s^2)}\right) \cos s\sqrt{\lambda^2-\alpha^2}\, ds. \tag{2.14}$$

Remark. This formula is classical and can be found in [MOS, p. 411].

Proof. For $\alpha, \beta, \lambda \in \mathbb{R}$, (2.14) follows from the spectral theorem and Proposition 2.7 applied with $L = \partial_x^2$. Both sides of the equation are entire functions of these parameters and thus the equation holds in general.

§3. The Selberg formula for the shifted wave equation

In this section we use the method of Lax and Phillips and the results of §2 to derive an exact formula for

$$\operatorname{tr} \exp t(L+\beta^2)^{\frac{1}{2}},$$

for L the Laplacian of a hyperbolic manifold. For simplicity we only consider three dimensional, compact manifolds, though the method applies equally well to other dimensions and the finite volume case.

To begin we recall that the solution to the Cauchy problem:

$$\begin{aligned} u_{tt} &= (\Delta_{\mathbb{H}^3} + 1)u, \\ u(0) &= f, \\ u_t(0) &= 0, \end{aligned} \tag{3.1}$$

is given by

$$u(p,t) = \partial_t\Big[\frac{1}{4\pi \sinh t} \int\limits_{d(p,q)=t} f(q)\, d\sigma(q)\Big], \tag{3.2}$$

see [LP1]. Here $d\sigma(q)$ is the volume element induced on the sphere of radius t. It follows from (2.11) that the solution to

$$\begin{aligned} v_{tt} &= (\Delta_{\mathbb{H}^3} + \beta^2)u, \\ v(0) &= f, \\ v_t(0) &= 0, \end{aligned} \tag{3.3}$$

is given by

$$v(p,t) = \partial_t\Big[\int_0^t k(t,s;1,\beta)\partial_s\Big[\frac{1}{4\pi \sinh s} \int\limits_{d(p,q)=s} f(q)\, d\sigma(q)\Big]\, ds\Big]. \tag{3.4}$$

Let Γ be a co-compact subgroup of $\mathrm{Aut}(\mathbb{H}^3)$. If f is automorphic with respect to Γ then the solutions u and v to (3.1) and (3.3) respectively are as well. They can be reexpressed as integrals over a fundamental domain, $\mathcal{F}_\Gamma$, for the action of Γ. We concentrate on the expression for v:

(3.5)

$$v(p,t) = \sum_{\gamma\in\Gamma} \partial_t\Big[\int_0^t k(t,s;1,\beta)\partial_s\Big[\frac{1}{4\pi \sinh s} \int\limits_{\mathcal{F}_\Gamma \cap \{d(\gamma p,q)=s\}} f(q)\, d\sigma(q)\Big]\, ds\Big].$$

We let $R^\beta_\Gamma(p,q;t)$ denote the kernel of the operator appearing on the right hand side of (3.5). Let $h \in \mathcal{C}^\infty_c(\mathbb{R})$ be an even function. A simple calculation using the functional calculus shows that the kernel of the operator

$$K_h = \int_{-\infty}^{\infty} h(t) \exp t(\Delta_\Gamma + \beta^2)^{\frac{1}{2}}\, dt \tag{3.6}$$

is given by

$$\kappa_h(p,q) = \int_{-\infty}^{\infty} h(t) R^\beta_\Gamma(p,q;t)\, dt. \tag{3.7}$$

The trace formula is obtained by using (3.6) and (3.7) to compute the $\operatorname{tr} K_h$ in two different ways.

First we do the calculation using the spectral representation of Δ_Γ. Since Γ is co-compact, it follows from the classical theory of self adjoint, elliptic operators that the L^2-spectrum of Δ_Γ is a decreasing sequence without finite points of accumulation:

$$\sigma(\Delta_\Gamma) = \{0 = -\lambda_0 > -\lambda_1 \geq -\lambda_2 \geq \cdots\}. \tag{3.8}$$

Let $\{\phi_n; n = 0, 1, \dots\}$ be the corresponding eigenfunctions.

Since h is even and smooth of compact support it follows that the spectral representation of K_h is given by

$$K_h = \sum_{n=0}^{\infty} \widehat{h}(\sqrt{\lambda_n - \beta^2})\phi_n(p)\phi_n(q). \tag{3.9}$$

As h is even so is $\widehat{h}$, its Fourier transform, and thus the choice of square root in (3.9) is immaterial. The representation in (3.9) affords our first computation of $\operatorname{tr} K_h$:

$$\operatorname{tr} K_h = \sum_{n=0}^{\infty} \widehat{h}(\sqrt{\lambda_n - \beta^2}). \tag{3.10}$$

The second computation follows from (3.7) and the classical fact that

$$\operatorname{tr} K_h = \int_{\Gamma\backslash\mathbb{H}^3} \kappa_h(p,p)\, \mathrm{dVol}. \tag{3.11}$$

As a preliminary step we define a Bessel transform of h by

$$\widetilde{h}(s) = -\int_s^{\infty} h'(t)k(t,s;1,\beta)\, dt. \tag{3.12}$$

As is apparent from (3.12), the map $h \longrightarrow \widetilde{h}$ carries even functions to even functions and smooth functions to smooth functions. If $\overline{\operatorname{supp}}(h)$ is the convex hull of the support of h then

$$\overline{\operatorname{supp}}(h) = \overline{\operatorname{supp}}(\widetilde{h}). \tag{3.13}$$

We considered more detailed properties of this transform in the next section.

After an integration by parts and an interchange of the t and s integrals in (3.5) and (3.7) we obtain

$$K_h f(p) = 2\sum_{\gamma\in\Gamma} \int_0^\infty \widetilde{h}(s)\partial_s \left[\frac{1}{4\pi \sinh s} \int_{\mathcal{F}_\Gamma \cap \{d(\gamma p, q)=t\}} f(q)\, d\sigma(q) \right] ds.$$

Integrating by parts in s and observing that it is simply a radial variable, we obtain:

$$K_h f(p) = -\frac{1}{2\pi} \sum_{\gamma\in\Gamma} \int_{\mathcal{F}_\Gamma} \frac{\widetilde{h}_s(d(\gamma p, q))}{\sinh(d(\gamma p, q))} f(q) \ \mathrm{dVol}\,.$$

As a kernel defined on $\mathcal{F}_\Gamma \times \mathcal{F}_\Gamma$, K_h is given by

$$\kappa_h(p,q) = -\frac{1}{2\pi} \sum_{\gamma\in\Gamma} \frac{\widetilde{h}_s(d(\gamma p, q))}{\sinh(d(\gamma p, q))} \ \mathrm{dVol}\,. \tag{3.14}$$

From this point the derivation of the trace proceeds exactly as in the treatment given in [LP2]. The result of these calculations is summarized in the following theorem:

Theorem 3.15. *Let $h \in C_c^\infty(\mathbb{R})$ be an even function with compact support and let $\Gamma \subset \mathrm{Aut}(\mathbb{H}^3)$ be torsion free and co-compact. Let the spectrum of the Laplace operator induced by the constant curvature metric on $\Gamma\backslash\mathbb{H}^3$ be denoted by $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$ and $\{\gamma\}_p$ denote a list of primitive, nontrivial conjugacy classes in Γ. For each complex number, β, the following identity holds:*

$$\begin{aligned} &\sum_{n=0}^\infty \widehat{h}(\sqrt{(\lambda_n - \beta^2)}) \\ &= \frac{1}{2\pi} \mathrm{Vol}(\Gamma\backslash\mathbb{H}^3)\widetilde{h}_{ss}(0) + \frac{1}{4}\sum_{k=1}^\infty \sum_{\{\gamma\}_p} \frac{l_\gamma \widetilde{h}(k l_\gamma)}{|\sinh \frac{k}{2}(l_\gamma + i\phi_\gamma)|^2}. \end{aligned} \tag{3.16}$$

Here $l_\gamma + i\phi_\gamma$ is the complex translation length of γ. The function $\widetilde{h}(s)$ is defined by (3.12).

Proof. The proof of this theorem follows from (3.14) and (3.10) by well known computations, see [LP2]. We only note that for each β there is a different 'optimal' family of test functions. The choice is primarily dictated by the necessity of evaluating the Fourier transform of h at points off of the real line. If follows from the Paley-Wiener theorem and (3.13) that both sides of (3.16) are absolutely convergent for any value of β provided $h \in \mathcal{C}_c^\infty(\mathbb{R})$.

§4. The Bessel transform and the Guillemin-Duistermaat formula

In this section we consider the continuity properties of the trace functional as the parameter β is varied. The functional

$$f \longrightarrow \int_{-\infty}^{\infty} f(t)e^t \cos(e^t)\, dt, \tag{4.1}$$

is continuous on Schwarz class functions, $\mathcal{S}$. This is clear because the weight factor is the derivative of $\sin(e^t)$. This example exhibits the phenomenon of exponential cancellation in $\mathcal{S}'$. As we shall see the 'right hand sides' of the trace formulæ (3.16), exhibit such cancellation for values of $\beta \in [0,1)$. Thus the classical Selberg formula is the only case for which both sides of the trace formula are absolutely convergent for an 'optimal' family of test functions. We place optimal in quotes as the properties of the Fourier transform on spaces of functions with finite rates of decay make it difficult to fix an 'optimal' space precisely. Typically one gets the correct exponential order of decay but must give up powers in a polynomial correction factor.

For $\mu \in \mathbb{N}$ and $\nu \in \mathbb{R}_+ \cup \{0\}$, the two families of function spaces, $\mathcal{S}_{\mu,\nu}, \mathcal{T}^{\mu,\nu}$, are defined by

$$\begin{aligned}\mathcal{S}_{\mu,\nu} = \{f : f \text{ is holomorphic in } |\Im z| < \nu, \text{ continuous in } |\Im z| \leq \nu \\ \text{and satisfies } \sup_{|\Im z| \leq \nu} (1+|z|)^{\mu+1}|f(z)| < \infty\}\end{aligned}$$

and

$$\mathcal{T}^{\mu,\nu} = \{h : h \text{ has } \mu+1 \text{ derivatives in } L^1(\mathbb{R}; e^{\nu|x|}\, dx)\}.$$

The Banach norms are defined by

$$\begin{aligned}\|f\|_{\mu,\nu} &= \sup_{|\Im z| \leq \mu} (1+|z|)^{\mu+1}|f(z)| \\ \|h\|^{\mu,\nu} &= \sum_{j=0}^{\mu+1} \int_{-\infty}^{\infty} |\partial_x^j h(x)|\, e^{\nu|x|}\, dx.\end{aligned}$$

Both families are easily seen to be complete in their respective norms. The $\mathcal{T}^{\mu,\nu}$-spaces are the closure of $\mathcal{C}_c^\infty(\mathbb{R})$ in their respective norms.

Proposition 4.2. *The Fourier transform defines a continuous map:*

$$\mathcal{F} : \mathcal{T}^{\mu,\nu} \longrightarrow \mathcal{S}_{\mu,\nu}. \tag{4.3}$$

Proof. Let $h \in \mathcal{C}_c^\infty(\mathbb{R})$; integrating by parts we obtain that

$$\mathcal{F}(h)(z) = \int_{-\infty}^{\infty} \frac{\partial_x^{\mu+1} h(x)}{(iz)^{\mu+1}} e^{-izx}\, dx, \tag{4.4}$$

for $z \in \mathbb{C} \setminus 0$. This formula implies that there exists a constant C such that

$$\|\mathcal{F}(h)\|_{\mu,\nu} \leq C\|h\|^{\mu,\nu}. \tag{4.5}$$

Since $\mathcal{C}_c^\infty(\mathbb{R})$ is dense in $\mathcal{T}^{\mu,\nu}$, the proposition follows from (4.5).

The map (4.3) is not surjective.

As an immediate consequence of (4.2) we conclude that the adjoint of the Fourier transform defines a continuous map

$$\begin{gathered} \mathcal{F}' : (\mathcal{S}_{\mu,\nu})' \longrightarrow (\mathcal{T}^{\mu,\nu})', \\ \langle \mathcal{F}'(l), g\rangle = \langle l, \mathcal{F}(g)\rangle. \end{gathered} \tag{4.6}$$

Returning to the study of trace formulæ we restrict our considerations to the closed subspaces of even functions, $\mathcal{S}^e_{\mu,\nu}, \mathcal{T}_e^{\mu,\nu}$. Recall that the counting function,

$$N_\Gamma(\lambda) = \#\{n : \lambda_n \leq \lambda\},$$

satisfies

$$N_\Gamma(\lambda) \sim \lambda^{\frac{3}{2}}, \tag{4.7}$$

see, [H]. Thus, for $\beta \in [0,1]$, the linear functionals

$$l_\beta(f) = \sum_{n=0}^{\infty} f(\sqrt{\lambda_n - \beta^2}) \in (\mathcal{S}^e_{\mu,\beta})', \text{ provided } \mu > 3. \tag{4.8}$$

As a consequence of (4.7) and (4.8) we obtain that $\mathcal{F}'(l_\beta) \in (\mathcal{T}_e^{\mu,\beta})'$. For even functions $h \in \mathcal{C}_c^\infty(\mathbb{R})$, (3.16) implies that

$$\begin{aligned}\mathcal{F}'(l_\beta)(h) &= \frac{1}{2\pi}\operatorname{Vol}(\Gamma\backslash\mathbb{H}^3)\widetilde{h}_{ss}(0)\\ &\quad+\frac{1}{4}\sum_{k=1}^{\infty}\sum_{\{\gamma\}_p}\frac{l_\gamma\widetilde{h}(kl_\gamma)}{|\sinh\frac{k}{2}(l_\gamma+i\phi_\gamma)|^2},\end{aligned}\tag{4.9}$$

where $\widetilde{h}$ is given by (3.12).

As $\mathcal{C}_c^\infty(\mathbb{R})$ is dense in $\mathcal{T}^{\mu,\beta}$, (4.9) is valid for all $h \in \mathcal{T}_e^{\mu,\nu}$, at least in the sense that if $h_n \in \mathcal{C}_c^\infty(\mathbb{R})$ converges to h in the $T^{\mu,\nu}$-topology then

$$\mathcal{F}'(l_\beta)(h) = \lim_{n\to\infty}\mathcal{F}'(l_\beta)(h_n).\tag{4.10}$$

The counting function for the lengths of the prime geodesics, $\pi_\Gamma(x)$ has the asymptotic form

$$\pi_\Gamma(x) \sim \frac{Ce^{2x}}{x},\tag{4.11}$$

see [S]. From (4.11) and consideration of the coefficients appearing on the right hand side of (3.16), it is clear that for the sum on the right side of (4.10) to converge absolutely, it is necessary that

$$\widetilde{h}_s(s) \sim e^{-s}m(s), \text{ as } s \to \infty.\tag{4.12}$$

Here $m(s)$ is a function tending to zero sufficiently rapidly to make $m(s)s^{-1}$ integrable. This, in general, is not the case, as $\widetilde{h}(s)$ has essentially the same exponential rate of decay as $h(x)$. To show this we need to further analyze the Bessel transforms defined by (3.12).

To that end we compute the Fourier transform of

$$\widetilde{h}(s;\eta) = -\int_s^\infty h'(t)J_0(\eta\sqrt{t^2-s^2})\,dt,\ \eta \in \mathbb{R}.$$

Since $\widetilde{h}$ is an even function of s,

$$\begin{aligned}(\widetilde{h})\widehat{\ }(\xi;\eta) &= \int_{-\infty}^{\infty} e^{-is\xi}\widetilde{h}(s;\eta)\,ds\\ &= 2\int_0^\infty \cos(s\xi)\widetilde{h}(s)\,ds.\end{aligned}\tag{4.13}$$

Changing the order of the integrations in (4.13) we recognize the s-integral as an instance of (2.1):

$$(4.14)\qquad \begin{aligned}(\widetilde{h})\hat{}(\xi;\eta) &= -2\int_0^\infty h'(t)\int_0^t \cos(s\xi)J_0(\eta\sqrt{t^2-s^2})\,ds\,dt\\ &= -2\int_0^\infty h'(t)\frac{\sin t\sqrt{\xi^2+\eta^2}}{\sqrt{\xi^2+\eta^2}}\,dt.\end{aligned}$$

Integrating by parts, and once again taking advantage of the fact that $h(t)$ is even we obtain

$$(4.15)\qquad (\widetilde{h})\hat{}(\xi;\eta) = \widehat{h}(\sqrt{\xi^2+\eta^2}).$$

Using (4.15) we can investigate the analyticity properties of $(\widetilde{h})\hat{}(\xi;\eta)$ and thereby the exponential rate of decay of $\widetilde{h}(s;\eta)$. A straightforward calculation establishes

Lemma 4.16. *If $h \in L^1(\mathbb{R}; e^{-\nu|x|}\,dx)$ then $\widehat{h}(\xi)$ has an analytic extension to*

$$\mathcal{D}_\nu = \{\xi : |\Im\xi| < \nu\}$$

and is continuous in $\overline{\mathcal{D}_\nu}$.

On the other hand, using the inverse Fourier transform, one easily establishes the existence of functions in $L^1(\mathbb{R}; e^{-\nu|x|}\,dx)$ whose Fourier transforms have $\partial\mathcal{D}_\nu$ as a natural boundary. In fact such functions are dense in $\mathcal{T}^{\mu,\nu}, \forall_{\mu,\nu}$.

Suppose $\widehat{h}(\xi)$ is the Fourier transform of an even function in $L^1(\mathbb{R}; e^{-\nu|x|}\,dx)$ with natural boundary $\partial\mathcal{D}_\nu$. For real η, $\widehat{h}(\sqrt{\eta^2+\xi^2})$ is also analytic in $\mathcal{D}_\nu$. As the Fourier transform is also even, the square root does not introduce a singularity. Examination of the conformal map $\sqrt{\xi^2+\eta^2} \longrightarrow \xi$ shows that the domain of analyticity of $\widehat{h}(\sqrt{\eta^2+\xi^2})$ increases with η. However it never includes a strip $\mathcal{D}_{\nu'}$ for a $\nu' > \nu$. From Lemma 4.16 we conclude that, for such a function, the transforms, $\widetilde{h}(s;\eta), \eta \in \mathbb{R}$, cannot have larger exponential rates of decay than ν. Thus the distribution on the right hand side of (3.16) displays an exponential cancellation similar to that in (4.1).

Of particular interest is $\beta = 0$, in this case, formula (3.16) is an exact version of the Guillemin-Duistermaat formula. We have, in effect, summed the error terms. Instead of getting a tractable right hand side, we have a tempered distribution similar to (4.1). For a manifold of negative curvature, there is an formula, due to Margulis, for the asymptotic distribution of the lengths of closed geodesics:

$$\pi(x) \sim \frac{ce^{hx}}{x}. \tag{4.17}$$

Here h is the topological entropy of the geodesic flow and c is a constant, see [M]. The discussion above and (4.17) suggest that one might have better luck trying to derive an 'exact' trace formula, in this context, for $\operatorname{tr} e^{t\sqrt{\Delta+h^2}}$. For this distribution there is a possibility that the error term in the Guillemin-Duistermaat formula might indeed be of 'lower order.'

As a final result we construct the inverse to (3.12) on functions of compact support.

Proposition 4.18. *If $h \in C_c^\infty(\mathbb{R})$ is even and $\widetilde{h}(s)$ is defined by (3.12) then*

$$h(t) = -\partial_t \int_t^\infty I_0(\sqrt{(\alpha^2-\beta^2)(t^2-s^2)})\widetilde{h}_s(s)\,ds. \tag{4.19}$$

Here α, β are any complex numbers and $I_0(z) = J_0(iz)$.

Proof. This is a simple consequence of (2.12). Set

$$u(x,t) = \frac{1}{2}(f(x+t) + f(x-t)).$$

Then $u(x,t)$ is the solution of the Cauchy problem:

$$\begin{aligned} u_{tt} &= u_{xx}, \\ u(x,0) &= f(x), \\ u_t(x,0) &= 0. \end{aligned} \tag{4.20}$$

Suppose further that $f(x)$ is compactly supported so that $u(x,\cdot)$ is as well for every value of x.

As u solves the Cauchy problem, (4.20), we conclude from Proposition 2.7 and (2.12) that for even $h \in C_c^\infty(\mathbb{R})$ and every $x \in \mathbb{R}$,

$$\begin{aligned} &\int_0^\infty h(t)u(x,t)\,dt \\ &= \int_0^\infty h(t)\partial_t\Big[\int_0^t k(t,s;\alpha,\beta)\partial_s\Big[\int_0^s k(s,r;\beta,\alpha)u(x,r)\,dr\Big]ds\Big]dt. \end{aligned} \tag{4.21}$$

As $h \in \mathcal{C}_c^\infty(\mathbb{R})$ we are free to integrate by parts and interchange the orders of the integrations. Integrating by parts in both t and s and reversing the order of the integrations in (4.21), leads to the identity

$$
\begin{aligned}
&\int_0^\infty h(t)u(x,t)\,dt \\
(4.22)\qquad &= \int_0^\infty u(x,r) \int_r^\infty k(s,r;\beta,\alpha)\partial_s\Big[\int_s^\infty k(t,s;\alpha,\beta)h'(t),dt\Big]ds\,dr \\
&= -\int_0^\infty u(x,r)\Big[\int_r^\infty k(s,r;\beta,\alpha)\widetilde{h}_s(s)\,ds\Big]\,dr.
\end{aligned}
$$

For $x = 0$ (4.22) states:

$$
\begin{aligned}
&\int_0^\infty h(t)\frac{1}{2}(f(t)+f(-t))\,dt \\
(4.23)\qquad &= -\int_0^\infty \frac{1}{2}(f(r)+f(-r))\Big[\int_r^\infty k(s,r;\beta,\alpha)\widetilde{h}_s(s)\,ds\Big]\,dr.
\end{aligned}
$$

Since $h(t)$ is an even function the proposition follows from (4.23) and the arbitrary nature of $f(t)$.

The inversion formula holds more generally than for $h \in \mathcal{C}_c^\infty(\mathbb{R})$. However if $\alpha^2 - \beta^2 \in \mathbb{R}_+$ then the kernel in (4.19) grows exponentially and thus some restrictions are required for this integral to make sense. The formula for $\widehat{\widetilde{h}}(s)$ makes it clear that in the absence of exponential decay it is unlikely that this transform has a reasonable inverse.

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