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Deformation Spaces on Geometric Structures

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0. Introduction

In this note we shall study geometric structures on smooth manifolds and deformation spaces. In 1981 Thurston gave a lecture on projective structures on surfaces in which he has established the following structure theorems (unpublished):

I. There is a canonical decomposition by convex hulls on a hyperbolic surface S which admits a (one dimensional complex) projective structure.
II. There is an isomorphism between the deformation space $\mathbf{CP}^1(S_g)$ and the product $\mathcal{T}(S_g) \times \mathcal{ML}(S_g)$.

Here $\mathcal{T}(S_g)$ is the Teichmüller space of a closed orientable surface S_g of genus $g \geqq 2$ and $\mathcal{ML}(S_g)$ is the space of measured laminations.
Since there was considerable interest in the argument of proof and the key idea seems to be generalized in higher dimension, we have decided to write down an exposition of the above structure theorems (I), (II).

(Complex) projective structure on surfaces is equivalent to conformally flat structure on surfaces when we identify $\mathbf{C}P^1 = S^2$ and

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$\mathrm{PSL}_2(C) = PO(3,1)^0$. We shall generalize I to conformally flat structure on manifolds in arbitrary dimensions.
In Chapter 1, we show that there is a canonical decomposition of a conformally flat manifold. The above Thurston correspondence will be proved in Chapter 2. There is no originality concerning the argument of Chapter 2 except for a certain generalization. It is nothing but our interpretation of Thurston's lecture. In Chapter 3 we shall describe various projective structures by using Kleinian groups. In Chapter 4 we review (G, X)-structures and examine the properties of limit sets of (G, X)-manifolds. The deformation space for (G, X)-structures will be defined more generally. As an application, we study the deformation spaces of S^1 invariant geometric structures in Chapter 5. Particularly we treat spherical CR structures and conformally flat structures as such geometric structures.

The authors have been informed from Professor William Goldman, and Professor Sadayoshi Kojima that Kulkarni-Pinkall also showed the existence of canonical stratification of conformally flat manifolds (cf. [34]).

We would like to thank Professor William Goldman for showing us his note of the Thurston's lecture. And we also thank the referee to pointing out our mistakes in earlier draft.

1. Canonical decomposition of conformally flat manifolds

A conformally flat structure on a smooth n-manifold is a maximal collection of charts modelled on the standard n-sphere S^n whose coordinate changes lie in the group $\mathrm{Conf}(S^n)$ of conformal transformations of S^n. The group $\mathrm{Conf}(S^n)$ is isomorphic to the Lorentz group $PO(n+1,1)$. If a smooth n-manifold M admits a conformally flat structure, then by the monodromy argument there exists a developing pair (ρ, dev), where $\mathrm{dev} : \tilde{M} \to S^n$ is a conformal immersion and $\rho : \pi_1(M) \to \mathrm{Conf}(S^n)$ is a homomorphism such that $\mathrm{dev} \cdot \gamma = \rho(\gamma) \cdot \mathrm{dev}$ $(\gamma \in \pi_1(M))$. Here $\tilde{M}$ is the universal covering space of M and $\pi_1(M)$ is the fundamental group. The map dev is called a developing map and ρ is called a holonomy homomorphism, both unique up to an element of $\mathrm{Conf}(S^n)$. Remark that the term "conformal" means "in the category $(\mathrm{Conf}(S^n), S^n)$", which is different from the usual terminology when dim $= 2$.

1.1. Maximal balls

Definition 1.1.1. Let $\mathbf{H}^{n+1}$ be a (real) hyperbolic space with boundary $\partial\mathbf{H}^{n+1} = S^n$. The n dimensional sphere S^n is a conformally flat manifold by stereographic projection. A geometric k-sphere S^k is the boundary of a $(k+1)$ dimensional totally geodesic subspace of $\mathbf{H}^{n+1}$. A geometric ball is a domain of S^n bounded by a codimension one geometric sphere.

Definition 1.1.2. Let N be a conformally flat manifold. Given a conformal immersion $f : N \to S^n$, a geometric ball of N is an open subset U such that $f : U \to f(U)$ is a diffeomorphism onto a geometric ball of S^n. Then the set of geometric balls of N is partially ordered by inclusions. We call a maximal geometric ball a *maximal ball.*

The following is a generalization of the proposition due to Thurston.

Proposition 1.1.3. *Let $f : N \to S^n$ be a conformal immersion. Then either one of the following is true.*

(i) *N is conformally equivalent to the standard sphere S^n, a euclidean space $\mathbf{R}^n$, or a hyperbolic space $\mathbf{H}^n$.*
(ii) *Every point of N lies in a proper maximal ball.*

Proof. Suppose that (ii) is false. A point of N lies in some geometric ball but not in a maximal ball. And so there exists a sequence $U_1 \subset U_2 \subset \cdots \subset U_i \subset \cdots$ of geometric balls containing x. The union $W = \bigcup_{i=1}^{\infty} U_i$ is not a geometric ball. As f is injective on each U_i, f must map W isomorphically onto a euclidean space $\mathbf{R}^n (\approx S^n - \{\infty\})$. If $N \neq W$ then f maps the closure $\bar{W}$ isomorphically onto S^n. And thus it follows that $N = \bar{W}$. This proves (i). If some maximal ball U is not proper, then $N = U$. Since the image $f(U)$ is a geometric ball of S^n, it is conformally equivalent to a hyperbolic space $\mathbf{H}^n$. Q.E.D.

Let $f : N \to S^n$ be a conformal immersion. The spherical metric on S^n defines a Riemannian metric on N so that f is a local isometry. Let $\bar{N}$ be the metric completion of N. It is easy to see that f extends to a map $\bar{f} : \bar{N} \to S^n$. Recall that for a maximal ball U, $f : U \to B$ is a diffeomorphism onto an n dimensional ball B. Note that the closure of U in N is not compact by maximality. However we have

$$\bar{f} : \bar{U} \to \bar{B} \tag{1.1.4}$$

is a homeomorphism onto a closed ball of S^n. Denote by ∂U the boundary of $\bar{U}$ in $\bar{N}$. Let M^n be a closed conformally flat manifold and $(\phi, \mathrm{dev}) : (\pi_1(M), \tilde{M}) \to (\mathrm{Conf}(S^n), S^n)$ be the developing pair as in 1.1 of Chapter 1. Recall that dev is a conformal immersion. We have the following application.

Lemma 1.1.5. *Let M^n be a closed conformally flat manifold and $\mathcal{F}$ be the set of all maximal balls of the universal covering space $\tilde{M}$. If the boundaries of all the elements of $\mathcal{F}$ meet at a common point, then the developing map is a covering map.*

Proof. We put $\tilde{M} = N$, $\pi_1(M) = \Gamma$, $\mathrm{dev} = f$. Let $\bar{N}$ be the metric completion of N and $\bar{f} : \bar{N} \to S^n$ be the map extending f. Let x be a common point of ∂U for all $U \in \mathcal{F}$. Note that $x \notin N$, otherwise x would be an interior point of some maximal ball. Put $\bar{f}(x) = \infty$. We prove that f misses the point $\{\infty\}$. Suppose that there is $y \in N$ such that $f(y) = \infty$. The point y lies in some maximal ball U. Since $\bar{f} : \bar{U} \to \bar{B}$ is a homeomorphism and $x \in \partial U$, it is impossible. Therefore, as the developing map misses a point, it is a covering map onto its image (cf. [25],[34]). Q.E.D.

Proposition 1.1.6. *Let M^n be a closed conformally flat manifold. Suppose that $\mathcal{F}$ consists of finite elements (possibly empty) or the boundaries of all the elements of $\mathcal{F}$ meet at a finite number of common points. Then M is conformally equivalent to a spherical space form, a Hopf manifold, a euclidean space form, or a hyperbolic space form.*

Proof. Suppose the latter case. By the above lemma we have that $\mathrm{dev} : \tilde{M} \to \mathrm{dev}(\tilde{M}) \subset S^n - \{\infty\}$ is a covering map. Since the common points are finite, the fundamental group $\pi_1(M)$ has a subgroup π' of finite index those elements of which leave these points fixed. And so the holonomy subgroup $\phi(\pi')$ belongs to the similarity subgroup of $\mathrm{Conf}(S^n)$ which is the stabilizer at $\{\infty\}$ in S^n. Therefore dev is a homeomorphism of either $\mathbf{R}^n$ or $S^n - \{0, \infty\}$ (cf. [14]). In our case dev is a homeomorphism onto $S^n - \{0, \infty\}$ or M is a Hopf manifold.
For the remaining case, if $\mathcal{F}$ is empty then M is either a spherical space form or a euclidean space form. Suppose that $\mathcal{F}$ consists of finite elements. Then $\tilde{M}$ is covered by the union of those finite maximal balls U. It follows that the number of $\mathrm{dev}^{-1}(x)$ is finite for each $x \in S^n$. It is easy to see that $\mathrm{dev} : \tilde{M} \to \mathrm{dev}(\tilde{M})$ is a finite covering map. Passing to a subgroup of finite index in π if necessary we can assume that π leaves each element U of $\mathcal{F}$ invariant. Let Γ be the holonomy group to

π. Since the image of dev misses more than one point, we know that $\text{dev}(\tilde{M}) \subset S^n - L(\Gamma)$. On the other hand, note that Γ is discrete because $\text{dev} : U \to B$ is a homeomorphism. This implies that $L(\Gamma) \subset S^{n-1}$ where we view $\partial B = S^{n-1}$. Moreover Γ acts properly discontinuously on $S^n - L(\Gamma)$ by (1.1.5). In particular it follows that either $\text{dev}(\tilde{M}) = \mathbf{H}^n$ or $\text{dev}(\tilde{M}) = S^n - L(\Gamma)$. The former case implies that M is a hyperbolic space form. In the latter case the set of all maximal balls in $S^n - L(\Gamma)$ must be finite. However if we note that $S^{n-1} - L(\Gamma) \neq \emptyset$, it is easy to see that for any point $x \in S^{n-1} - L(\Gamma)$ there are infinitely many maximal balls containing x. This is impossible in this case. Q.E.D.

Note. *If M is a Hopf manifold, every maximal ball of $\tilde{M}$ meets at exactly two points.*

1.2. Decomposition of conformally flat manifolds

Let $f : N \to S^n$ be a conformal immersion and $\mathcal{F}$ the set of all maximal balls of N. Let U be an element of $\mathcal{F}$. Put

$$U_\infty = \bar{U} - N.$$

Then ∂U decomposes into a disjoint union of $\partial U \cap N$ and U_∞.

Definition 1.2.1. The set U_∞ is called an ideal set of U. The ideal set is a closed subset of $\bar{N}$. (For example, if $\bar{U}$ is a closed disk, then U_∞ may look like a Cantor set and $\partial U \cap N$ is a disjoint union of intervals. Since U has the natural Poincaré metric, U_∞ corresponds to a closed subset of points at infinity.)

Recall from (1.1.4) that $\bar{U}$ is conformally equivalent to a closed ball $\bar{B}$. We may form the convex hull $C(U_\infty)$ for U_∞ inside U. (Note that this can be defined when U_∞ contains more than one point.) Let $\mathbf{D}^{m+1} = \mathbf{H}^{m+1} \cup S^m$ be the compactification of a hyperbolic space $\mathbf{H}^{m+1}$. If K is a closed subset of S^m then we denote by $\mathcal{H}(K)$ the convex hull in $\mathbf{H}^{m+1}$. It is easy to check the following.

Lemma 1.2.2. *Let $P \subset \mathbf{H}^{m+1}$ be a totally geodesic hyperplane such that either one of the components of $S^m - \partial P$ does not meet K. Then $\mathcal{H}(K \cap \partial P) = \mathcal{H}(K) \cap P$.*

Using this lemma we define pleats on the boundary $\partial C(U_\infty)$ of $C(U_\infty)$. Given a closed convex set $\mathbf{C}$ of $\mathbf{D}^{n+1}$ there is a canonical retraction $\Phi_{\mathbf{C}} : \mathbf{D}^{n+1} \to \mathbf{C}$ called a closest point mapping. Recall that if $x \in S^n - \mathbf{C}$ there is a horoball centered at x disjoint from $\mathbf{C}$. Then

$\Phi_{\mathbf{C}}(x)$ is the point of the first contact when we increase the radius of this horoball continuously until it touches $\mathbf{C}$. See [8] for details. Denote by fU^c the complement of fU in S^n. If $C = \bar{\mathcal{H}}(fU^c)$ is the closure of the convex hull $\mathcal{H}(fU^c)$ in $\mathbf{H}^{n+1}$, then we have a map $\Phi_C : \mathbf{D}^{n+1} \to C$. Note that $\mathcal{H}(\partial(fU^c))$ is a totally geodesic subspace of $\mathbf{H}^{n+1}$, so we set $\mathcal{H}(\partial(fU^c)) = \mathbf{H}^n$. Since $\bar{\mathcal{H}}(\partial(fU^c)) \subset C$, the above map restricts to a map

$$\Phi_U : fU(= B) \to \mathbf{H}^n.$$

Note that it is a conformal diffeomorphism.

Definition 1.2.3. For a totally geodesic hyperplane $P \subset \mathbf{H}^n$ we call also $f^{-1}\Phi_U^{-1}(P)$ a totally geodesic hyperplane in U. Put $f^{-1}\Phi_U^{-1}(P) = Q$. If $P^1 < P^2 < \cdots < P^{n-2} < P^{n-1} = P$ is a chain of totally geodesic subspaces, then there exists a k dimensional totally geodesic subspace Q^k of U and similarly a chain $Q^1 < Q^2 < \cdots < Q^{n-2} < Q^{n-1} = Q$ and so on.

Since $\Phi_U(C(\bar{f}U_\infty \cap \partial P)) = \mathcal{H}(\bar{f}U_\infty \cap \partial P)$ and $\mathcal{H}(\bar{f}U_\infty \cap \partial P) = \mathcal{H}(\bar{f}U_\infty \cap P)$ by the above lemma, we have that $C(\bar{f}U_\infty \cap \partial P) = fC(U_\infty) \cap \Phi_U^{-1}(P)$. Noting that $fC(U_\infty \cap \partial Q) = C(\bar{f}U_\infty \cap \partial P)$, it is easy to check that $C(U_\infty \cap \partial Q) = C(U_\infty) \cap Q$. An iteration of this argument yields that

$$C(U_\infty \cap \partial Q^k) = C(U_\infty) \cap Q^k \ (k = 1, \cdots, n-1). \tag{1.2.4}$$

Definition 1.2.5. Let $Q^1 < Q^2 < \cdots < Q^{n-2} < Q^{n-1} = Q$ be a chain of totally geodesic subspaces in a maximal ball U. Suppose that either one of $\partial U - \partial Q$ does not meet U_∞. If $\mathrm{Int}\, C(U_\infty) \neq \emptyset$, (equivalently $\mathrm{Int}\, C(U_\infty) \cap U$ is open in U) and $C(U_\infty) \cap Q^k$ contains an open subset in Q^k then by (1.2.4) that $C(U_\infty \cap \partial Q^k)$ is said to be a k dimensional pleat of the boundary $\partial C(U_\infty)$ $(k = 1, 2, \cdots, n-1)$.
Put $\Lambda_U = \partial C(U_\infty)$. Choosing all possible geodesic hyperplanes Q in U and passing to all chains of geodesic subspaces $\{Q^k\}$, we obtain all pleats in Λ_U. The set Λ_U is composed of all possible pleats in dimension less than or equal to $n-1$. In the case that $\mathrm{Int}\, C(U_\infty) = \emptyset$, there exists a totally geodesic subspace Q' such that $C(U_\infty) = C(U_\infty) \cap Q'$ is open in Q'. We say that $C(U_\infty)$ is an m dimensional pleat if $\dim Q' = m$. Inductively we can define pleats of $\partial C(U_\infty)$ unless $m = 1$. Note that $C(U_\infty)$ is a one dimensional pleat if and only if U_∞ consists of a pair of points.

Let $f : N \to S^n$ be a conformal immersion as before. Using the spherical metric of S^n, N admits a Riemannian metric such that f is a local isometry. Recall that $\bar{N}$ is a metric completion and $\bar{f} : \bar{N} \to S^n$ is a map extending f. Choose a point x in N. Let $W(x)$ be the union of all maximal balls containing x.

Lemma 1.2.6. *If $\overline{W(x)}$ is the closure of $W(x)$ in $\bar{N}$ then $\overline{W(x)}$ is compact.*

Proof. Let ρ (resp. ρ_0) be the distance function of $\bar{N}$ (resp. S^n). Since f maps $W(x)$ injectively, f is a homeomorphism of $W(x)$ onto its image Ω. Let $\{p_i\}$ be an arbitrary sequence of $\overline{W(x)}$. Choose a sequence $\{q_i\}$ in $W(x)$ such that $\rho(p_i, q_i) < 1/i$. Since $\bar{\Omega}$ is compact, the sequence $\{f(q_i)\}$ has an accumulation point and so $\{f(q_i)\}$ is Cauchy. Given $\varepsilon > 0$, choose δ such that $0 < \delta < \varepsilon$. Suppose that $\rho_0(f(q_i), f(q_j)) < \delta$. If δ is sufficiently small, then there exists a maximal ball in Ω containing the points $f(q_i)$,$f(q_j)$. And so Ω contains a minimizing geodesic between $f(q_i)$ and $f(q_j)$. It implies that $\rho_0(f(q_i), f(q_j)) = \rho(q_i, q_j)$. In particular the sequence $\{q_i\}$ is Cauchy. Since $\bar{N}$ is complete, the sequence $\{q_i\}$ has a limit point q. And thus we have $\lim p_i = q$. Hence $\overline{W(x)}$ is compact.
Q.E.D.

Theorem 1.2.7. *Let $f : N \to S^n$ be a conformal immersion and $\mathcal{F}$ the nonempty set of all maximal balls. Then every point of N lies in the convex hull $C(U_\infty)$ for a unique element $U \in \mathcal{F}$.*

Proof. Choose a point x in N and let $W(x)$ be as above. Put $W(x)_\infty = \overline{W(x)} - N$. Note that it contains more than one point for otherwise there are no maximal balls containing x. Since $W(x)_\infty$ is a closed subset of $\overline{W(x)}$, $W(x)_\infty$ is compact by the above lemma. And so $\bar{f}(W(x)_\infty)$ is a closed subset of S^n. If $\mathcal{H} = \mathcal{H}(\bar{f}(W(x)_\infty))$ is the convex hull in $\mathbf{H}^{n+1}$, then we have a closest point mapping $\Phi_{\mathcal{H}} : \mathbf{D}^{n+1} \to \mathcal{H}$. Now there is a unique totally geodesic hyperplane P through $\Phi_{\mathcal{H}}(f(x))$ perpendicular to the geodesic from $f(x)$ to $\Phi_{\mathcal{H}}(f(x))$. Then we have from (1.2.2) that

$$\Phi_{\mathcal{H}}(f(x)) \in \mathcal{H}(\bar{f}(W(x)_\infty)) \cap P = \mathcal{H}(\bar{f}(W(x)_\infty) \cap \partial P).$$

Let B be a geometric ball containing $f(x)$ such that $\partial B = \partial P$.The set $U = f^{-1}(B)$ is a maximal ball containing x because $B \subset f(W(x))$. As $\bar{f}\partial U = \partial f U = \partial P$, it follows that

$$\begin{aligned}\mathcal{H}(\bar{f}(W(x)_\infty) \cap \partial P) &= \mathcal{H}(\bar{f}(W(x)_\infty \cap \partial U)) \\ &= \mathcal{H}(\bar{f}(U_\infty)).\end{aligned}$$

Since $\Phi_U(fC(U_\infty)) = \Phi_U(C(\bar{f}U_\infty)) = \mathcal{H}(\bar{f}U_\infty)$ and $\Phi_{\mathcal{H}}|fU = \Phi_U$, it follows that $\Phi_U(f(x)) = \Phi_{\mathcal{H}}(f(x)) \in \Phi_U(f(C(U_\infty)))$ and hence $x \in C(U_\infty)$. The proof of the uniqueness is the converse of the above argument. Q.E.D.

Corollary 1.2.8.

(1) *The family* $\{C(U_\infty);\ U \in \mathcal{F}\}$ *consists of disjoint subsets.*
(2) *The set* $\cup\Lambda_U$ *is closed in* N.

Proof. There exists a unique element of $\mathcal{F}$ such that each point of N lies in its convex hull by the above theorem. This implies that $\{C(U_\infty);\ U \in \mathcal{F}\}$ are disjoint. For (2), it suffices to show that if a sequence $\{x_i\} \in \Lambda_{U_i}$ $(U_i \in \mathcal{F})$ converges to $x \in N$, then there exists an element $U \in \mathcal{F}$ such that $x \in \Lambda_U$. Recall that $\Lambda_U = C(U_\infty) - \operatorname{int} C(U_\infty)$. There exists $U \in \mathcal{F}$ such that $x \in C(U_\infty)$. If x is not contained in Λ_U, then $x \in \operatorname{int} C(U_\infty)$. It follows that for sufficiently large i, Λ_{U_i} meets with $C(U_\infty)$. This contradicts that $\{C(U_\infty);\ U \in \mathcal{F}\}$ are disjoint.
Q.E.D.

Let M^n be a closed conformally flat manifold and $\tilde{\mathcal{F}}$ the set of all maximal balls of the universal covering space $\tilde{M}$. It is obvious that $\tilde{\mathcal{F}}$, the family $\{C(U_\infty)\}$ and the set $\cup\Lambda_U$ are invariant under the fundamental group π.

Corollary 1.2.9. *Let* M^n *be a closed conformally flat manifold. Suppose that* $\tilde{\mathcal{F}}$ *is not empty. Then the universal covering space* $\tilde{M}$ *supports a* π *invariant canonical decomposition* $\{C(U_\infty); U \in \mathcal{F}\}$.

2. Thurston parametrization of projective structure on surfaces

In this chapter we shall prove the Thurston isomorphism II stated in Introduction. Recall that (complex) projective structure on surfaces is equivalent to conformally flat structure on surfaces when we identify $(S^2, \mathrm{Conf}(S^2)^0)$ with $(\mathbf{CP}^1, \mathrm{PSL}_2(C))$. As before, given a projective structure on a surface S we have a developing pair $(\phi, \mathrm{dev}) : (\pi_1(S), \tilde{S}) \to (\mathrm{PSL}_2(C), \mathbf{C}P^1)$ up to conjugation by elements of $\mathrm{PSL}_2(C)$.

2.1. Deformation spaces on surfaces

Suppose that S_g is a closed orientable surface of genus $g \geqq 1$. For the brevity we set $S = S_g$ and $\Gamma = \pi_1(S_g)$. A surface Σ is a hyperbolic surface if the universal covering space is conformally equivalent to a hyperbolic plane $\mathbf{H}^2$. Consider the subspace $\Omega^+(S)$ (cf. 4.3.4);

$$\Omega^+(S) = \{(\phi, \mathrm{dev}) : (\Gamma, \tilde{S}) \to (\mathrm{PSL}_2(\mathbf{C}), \mathbf{CP}^1)\}/\,\mathrm{Diff}^0(S),$$

where dev are orientation-preserving immersions.
The topology on $\Omega^+(S)$ is given by the following subbasis:

(1) $\mathcal{N}(U) = U/\sim$ where U is an open subset in $\mathrm{Map}(\tilde{S}, \mathbf{CP}^1)$ with the compact-open topology.
(2) $\mathcal{N}(K) = \{\mathrm{dev} \in \mathrm{Map}(\tilde{S}, \mathbf{CP}^1)|\ \mathrm{dev}\,|K$ is an embedding for a compact subset $K \subset \tilde{S}\}/\sim$.

Put

$$\mathbf{CP}^1(S)^+ = \mathrm{PSL}_2(\mathbf{C}) \setminus \Omega^+(S).$$

Definition 2.1.1. Let S_g be a hyperbolic surface. The space $\mathbf{CP}^1(S_g)^+$ is called the deformation space of projective structures or $\mathbf{CP}^1$-structures on S_g. $\mathcal{T}(S_g)$ is the usual Teichmüller space.

Thurston has introduced the notion of geodesic laminations on surfaces. (Cf. [9],[40],[8].) Namely, a geodesic lamination on a hyperbolic surface Σ is a closed subset consisting of a disjoint union of simple geodesics. Let Λ be a geodesic lamination on Σ. By a transversal we mean an embedding $\ell : [0,1] \to \Sigma$ such that at each t where $\ell(t) \in \Lambda$ the map ℓ is transverse to the leaf through $\ell(t)$.
A transverse measure on Λ is a function μ which assigns to each transversal ℓ a Radon measure $\mu(\ell)$ on $[0,1]$ supported by $\{t \in [0,1] \mid \ell(t) \in \Lambda\}$ which is compatible under the canonical homeomorphisms between nearby transversals. We call the pair (Λ, μ) a measured geodesic lamination of Σ.

Definition 2.1.2. $\mathcal{ML}(\Sigma)$ is the space of measured geodesic laminations on Σ, equipped with the weak $*$ topology.

If f is a homeomorphism of a closed hyperbolic surface Σ onto Σ' then f induces a homeomorphism $f : \mathcal{ML}(\Sigma) \to \mathcal{ML}(\Sigma')$. (See [8],[9].)

Let S_g be a hyperbolic surface. Note that $\mathcal{T}(S_g)$ is homeomorphic to $\mathbf{R}^{6g-6}$. In this case the space of measured laminations $\mathcal{ML}(S_g)$ is also homeomorphic to the real vector space of dimension $6g-6$. Moreover, $\mathbf{CP}^1(S_g)^+$ can be identified with the cotangent bundle of $\mathcal{T}(S_g)$. (Compare [5].) In contrast to this identification we have a new parametrization on $\mathbf{CP}^1(S_g)^+$.

Theorem (Thurston). *Let S_g be a hyperbolic surface. Then there exists a homeomorphism*

$$\Theta : \mathbf{CP}^1(S_g)^+ \longrightarrow \mathcal{T}(S_g) \times \mathcal{ML}(S_g).$$

See [16],[17],[15],[13] for the related topics. The rest of this section is devoted to the proof of this theorem.

2.2. Locally convex pleated maps

Let $f : N \to \mathbf{CP}^1$ be a conformal immersion. We have shown in Theorem 1.2.7 that every point x of N lies in the convex hull $C(U_\infty)$ of a unique maximal ball U. Let $\mathcal{F}$ be the set of maximal balls. For each U there is a closest point mapping $\Phi_U : f(U) \longrightarrow \mathbf{H}^2(\subset \mathbf{H}^3)$. (See Section 1.) Set $\Psi(x) = \Phi_U(f(x))$ if $x \in C(U_\infty)$. By uniqueness it defines a well defined map

$$\Psi : N \longrightarrow \mathbf{H}^3. \tag{2.2.1}$$

It is obvious that Ψ is a continuous map. Note that in our case each $C(U_\infty)$ is either a region or a (one dimensional) pleat, the image $\Psi(C(U_\infty))$ lies on a geodesic or in a totally geodesic hyperplane of $\mathbf{H}^3$.

Definition 2.2.2. Given an arbitrary conformal immersion $f : N \to \mathbf{CP}^1$, we have passed from it to a map $\Psi : N \longrightarrow \mathbf{H}^3$. The map Ψ is called a pleated map.

2.3. Assignment of $\mathbf{CP}^1(S_g)^+$ to $\mathcal{T}(S_g)$

In general a pleated map Ψ is not locally injective. By definition (2.2.2), Ψ is injective on each $C(U_\infty)$ for $U \in \mathcal{F}$. On the other hand, we consider a point $x \in N$ such that there is a sequence $\{x_i\}$ converging to x such that $\Psi(x_i) = \Psi(x)$. Since each x_i lies in a distinct $C(U^i_\infty)$ and Ψ is injective on $C(U^i_\infty)$ for sufficiently large i, all $C(U^i_\infty)$ have the same dimension equal to 1. The map Ψ fails to be injective on the union of those $C(U^i_\infty)$. Each ideal set U^i_∞ is a locally constant pair of points in $\bar{N}$.

Definition 2.3.1. Denote by $\mathcal{B}$ the set of those $C(U^i_\infty)$ on which Ψ fails to be injective.

Ψ is locally injective on $N - \mathcal{B}$. Let N' be the space obtained from $N - \mathcal{B}$ by identifying the boundaries of each component of $N - \mathcal{B}$ which have the same Ψ image. Let $\eta : N \to N'$ be the resulting collapse map which is clearly a homotopy equivalence. Since each component of $N - \mathcal{B}$ is isometric to a hyperbolic region with boundary composed of complete geodesics, the image N' supports a complete hyperbolic metric. Moreover if g is an conformal automorphism of N then it leaves $N - \mathcal{B}$ invariant. When g stabilizes a component, it acts as isometries with respect to a hyperbolic metric of that component. Otherwise g translates one component to another component preserving boundary geodesics. Therefore the map g induces a hyperbolic isometry $\theta(g) \in \mathrm{PSL}_2(\mathbf{R})$ on $\mathbf{H}^2$ where we put $N' = \mathbf{H}^2$. The map η satisfies that $\eta \circ g = \theta(g) \circ \eta$. Now given a projective structure (ϕ, dev) in $\mathbf{CP}^1(S_g)^+$, we apply the above argument to $(\phi, \mathrm{dev}) : (\Gamma, \tilde{S}) \to (\mathrm{PSL}_2(\mathbf{C}), \mathbf{CP}^1)$. Then it induces an equivariant homotopy equivalence $(\theta, \eta) : (\Gamma, \tilde{S}) \to (\mathrm{PSL}_2(\mathbf{R}), \mathbf{H}^2)$. The map η induces a homotopy equivalence of S onto $\mathbf{H}^2/\theta(\Gamma)$. Within the homotopy class of η, there is a diffeomorphism $h : S \to \mathbf{H}^2/\theta(\Gamma)$ up to an element of $\mathrm{Diff}^0(S)$. Hence a projective structure (ϕ, dev) defines a well defined element $[S, h]$ of $\mathcal{T}(S_g)$.

2.4. Canonical measure on circular lamination

Let $f : N \to \mathbf{CP}^1$ be as before. Recall from (1.3.2) that the subset $\tilde{\Lambda}_1 = \cup\{\Lambda_U \mid U \in \mathcal{F}\}$ is closed and consists of a disjoint union of (one dimensional) pleats. In order to define the canonical measure $\tilde{\mu}_1(\ell)$ on a transversal ℓ for $\tilde{\Lambda}_1$, it suffices to specify the nondecreasing function $\varphi(t) = \int_{[0,t]} \tilde{\mu}_1(\ell)dt$ whose derivative is equal to $\tilde{\mu}_1(\ell)$. For each $t \in [0,1]$, let $U^t \in \mathcal{F}$ be a unique maximal ball such that $\ell(t) \in C(U^t_\infty)$. If $s, t \in [0,1]$ are sufficiently close, the balls U^s and U^t must intersect. Let $\Theta(s,t)$ denote the *dihedral* angle of intersection of the circles ∂U^s, ∂U^t, measured inside one ball and outside the other. The function $\varphi(t)$ is then defined as the infimum of all Θ-sum

$$\Theta(0, t_1) + \Theta(t_1, t_2) + \cdots + \Theta(t_n, t)$$

over all subdivisions $0 < t_1 < t_2 < \cdots < t_n < t$ of $[0, t]$. The following is the elementary calculation of the trigonometry.

Lemma 2.4.1. *If $r < s < t$ are sufficiently close, then $\Theta(r, s)$, $\Theta(s,t)$ and $\Theta(r,t)$ are defined and $\Theta(r,s) + \Theta(s,t) \leqq \Theta(r,t)$.*

With this lemma a nondecreasing function can be defined as

$$\varphi(t) = \lim \sum \Theta(0, t_1) + \Theta(t_1, t_2) + \cdots + \Theta(t_n, t)$$

where $\sum$ runs over all subdivisions of $[0, t]$ and lim is taken as the mesh of subdivisions goes to zero. Therefore the derivative $\varphi\prime = \tilde{\mu}_1(\ell)$ is a measure on a transversal ℓ for $\tilde{\Lambda}_1$. Finally the compatiblity of the measure on the various transversals is deduced from the following remark. The leaf through $\ell(t)$ determines the ball U^t and the measure on ℓ is determined by the angles made by the ∂U^t. Thus corresponding transversals determine the same measure.

Let $\Psi : N \to \mathbf{H}^3$ be a pleated map for an immersion $f : N \to \mathbf{CP}^1$. Let $\mathcal{B}$ be the set as in (2.3.1). It follows that $\eta(\mathcal{B}) = \eta(\tilde{\Lambda}_1)$. If we put $\eta(\tilde{\Lambda}_1) = \tilde{\Lambda}_2$, then $\tilde{\Lambda}_2$ is a geodesic lamination on N' (=$\mathbf{H}^2$). Moreover let ℓ be a transversal to $\tilde{\Lambda}_2$, then $\eta^{-1}(\ell)$ is also a transversal to $\tilde{\Lambda}_1$. Set $\tilde{\mu}_2(\ell) = \tilde{\mu}_1(\eta^{-1}(\ell))$. We have a measure $\tilde{\mu}_2$ on $\tilde{\Lambda}_2$. And thus $(\tilde{\Lambda}_2, \tilde{\mu}_2)$ is a measured geodesic lamination on N'. As before suppose that $(\phi, \mathrm{dev}) : (\Gamma, \tilde{S}) \to (\mathrm{PSL}_2(\mathbf{C}), \mathbf{CP}^1)$ is a developing pair. The above argument implies that there is a measured geodesic lamination $(\tilde{\Lambda}_2, \tilde{\mu}_2)$ over $(\theta(\Gamma), \mathbf{H}^2)$. It is easy to see that $(\tilde{\Lambda}_2, \tilde{\mu}_2)$ is invariant under the group $\theta(\Gamma)$. That is, $\theta(\gamma)(\tilde{\Lambda}_2) = (\tilde{\Lambda}_2)$ and $\tilde{\mu}_2(\theta(\gamma)(\ell)) = \tilde{\mu}_2(\ell)$. It induces a measured geodesic lamination (Λ_2, μ_2) on $\mathbf{H}^2/\theta(\Gamma)$. There is a diffeomorphism $h : S \to \mathbf{H}^2/\theta(\Gamma)$ as above. Then we have a geodesic measured lamination (Λ, μ) on S such that $h(\Lambda) = \Lambda_2$ and $\mu(\ell) = \mu_2(h(\ell))$. Hence it defines an element $(\Lambda, \mu) \in \mathcal{ML}(S)$.

2.5. Thurston correspondence

We have a well defined map $\Theta : \mathbf{CP}^1(S_g) \to \mathcal{T}(S_g) \times \mathcal{ML}(S_g)$,

$$\Theta((\phi, \mathrm{dev})) = ([S, h], (\Lambda, \mu)).$$

It is easy to see that Θ is injective, because given two projective structures which have the same image in $\mathcal{T}(S_g) \times \mathcal{ML}(S_g)$. The coincidence on the first summand implies that each developing map coincides outside each $\mathcal{B}$. But the second summand measures the difference on $\mathcal{B}$ and so two developing maps coincide on the whole $\tilde{S}$.
Let $\mathcal{ML}(S_g, \mathcal{S})$ be the subspace of $\mathcal{ML}(S_g)$ such that every lamination consists of compact leaves. If $\overline{\mathcal{ML}_h(M, \mathcal{S}^{n-1})}$ is the closure in $\mathcal{ML}(S_g)$, then it is known that $\overline{\mathcal{ML}_h(S_g, \mathcal{S})} = \mathcal{ML}(S_g)$. We show that there is a map

$$\mathfrak{S} : \mathcal{T}(S_g) \times \mathcal{ML}(S_g) \longrightarrow \mathbf{CP}^1(S_g) \tag{2.5.1}$$

such that $\Theta \cdot \mathfrak{S} = \mathrm{id}$.
To prove this we need some preliminaries.

Definition 2.5.2 (cf. [16]). Let $\alpha > 0$ be any mumber and $W_\alpha = \{z \in \mathbf{C} | 0 \leqq \mathrm{Im}\, z \leqq \alpha\pi\}$.

Let s be the stereographic projection which maps $\mathbf{C}^*$ onto $S^2-\{\infty\}$. Then we define a map $\xi : \mathbf{C} \to S^2$ to be the exponential map $\exp : \mathbf{C} \to \mathbf{C}^*$ followed by s.
Put $\xi(W_\alpha) = C_\alpha$. Both W_α and C_α are conformally flat manifolds with boundary. We call W_α an α-pile and C_α an α-crescent.

Proof of (2.5.1). Let $([S,h],(\Lambda,\mu))$ be a representative element of $\mathcal{T}(S_g) \times \mathcal{ML}(S_g)$. We suppose first that $(\Lambda,\mu) \in \mathcal{ML}(S_g,\mathcal{S})$. The map $(\theta,\tilde{h})$ maps a Γ invariant measured geodesic lamination $(\tilde{\Lambda},\tilde{\mu})$ onto a $\theta(\Gamma)$ invariant measured geodesic lamination $(\tilde{\Lambda}',\tilde{\mu}')$. The map $\tilde{h}$ is a homeomorphism of $\tilde{S}$ onto $\mathbf{H}^2$ where $\mathbf{H}^2$ is viewed as the upper hemisphere of S^2. Cut $\mathbf{H}^2$ along $\tilde{\Lambda}'$ and then insert the crescents C_α and glue them along the boundary components. Here these angles α come from those of the measure $\tilde{\mu}'$. Similarly cut $\tilde{S}$ along $\tilde{\Lambda}$ and insert the piles W_α and then glue along the corresponding boundary components by the map $\tilde{h}^{-1} \circ \xi$. The resulting manifold $\tilde{S}'$ is invariant under an action of Γ and thus the orbit space is still homeomorphic to S. Since both $\tilde{h}$ and ξ are projective immersions, combined with these maps, we have a well defined projective immersion $\mathrm{dev} : \tilde{S} \to S^2 = \mathbf{CP}^1$ and since the group Γ acts as projective transformations with respect to this structure on S', there is a holonomy homomorphism ϕ. If we set $\mathfrak{S}([S,h],(\Lambda,\mu)) = (\phi,\mathrm{dev})$, then the map is well defined on $\mathcal{T}(S_g) \times \mathcal{ML}(S_g,\mathcal{S})$ such that $\Theta \cdot \mathfrak{S} = \mathrm{id}$ on $\mathcal{T}(S_g) \times \mathcal{ML}(S_g,\mathcal{S})$. For an element $(\Lambda,\mu) \in \mathcal{ML}(S_g)$ there is a sequence $\{(\Lambda_i,\mu_i)\} \in \mathcal{ML}(S_g,\mathcal{S})$ that converges to (Λ,μ). Let $[S,h]$ be an arbitrary element of $\mathcal{T}(S_g)$ and fix it once. The map $\mathfrak{S}$ maps $([S,h],(\Lambda_i,\mu_i))$ to a sequence of projective structures $\{(\phi_i,\mathrm{dev}_i)\}$. Recalling the topology of $\mathbf{CP}^1(S_g)$ from (2.1) and by the fact that each dev_i coincides with the map $\tilde{h}$ outside $\tilde{\Lambda}_i$, the sequence of developing maps $\{\mathrm{dev}_i\}$ converges to a map on each compact set of $\tilde{S}$. And so it is easy to see that it converges to a map $\mathrm{dev} : \tilde{S} \to S^2 = \mathbf{CP}^1$ which is obviously a projective immersion. The projective immersion dev determines a holonomy homomorphism ϕ up to conjugation. Setting $\mathfrak{S}([S,h],(\Lambda,\mu)) = (\phi,\mathrm{dev})$, we obtain a continuous map $\mathfrak{S} : \mathcal{T}(S_g) \times \mathcal{ML}(S_g) \longrightarrow \mathbf{CP}^1(S_g)$ such that $\Theta \cdot \mathfrak{S} = \mathrm{id}$. Q.E.D.

2.6. Modular space of projective structures

Recall that $\mathbf{CP}^1(S)^+ = \mathrm{PSL}_2\,\mathbf{C} \setminus \Omega(S)^+ / \mathrm{Diff}^0(S)$. Then the space $\mathcal{M}\mathbf{CP}^1(S)^+ = \mathrm{PSL}_2\,\mathbf{C} \setminus \Omega(S)^+ / \mathrm{Diff}^+(S)$ is called the modular space of projective structures. On the other hand, the Teichmüller space $\mathcal{T}(S)$ is defined alternately to be $\mathbf{R}(\Gamma,\mathrm{PSL}_2\,\mathbf{R}) / \mathrm{PGL}_2\,\mathbf{R}$. And so $\mathcal{T}(S)$ is identified with the quotient space of sense-preserving discrete faithful representations, $\mathcal{T}(\Gamma) = \mathbf{R}^+(\Gamma, \mathrm{PSL}_2\,\mathbf{R})/\,\mathrm{PSL}_2\,\mathbf{R}$. There is also a similar identification $\mathcal{ML}(S) = \mathcal{ML}(\Gamma)$. Since each element of $\mathrm{Diff}^+(S)/\,\mathrm{Diff}^0(S)$ maps $\mathcal{ML}(S)$ onto itself, there is an action of $\mathrm{Out}^+(\Gamma)$ on $\mathcal{ML}(\Gamma)$.

Corollary 2.6.1. *There is a commutative diagram on which* $\mathrm{Out}^+(\Gamma)$ *acts diagonally.*

$$\begin{array}{ccc}
\mathrm{Diff}^+(S)/\,\mathrm{Diff}^0(S) & \longrightarrow & \mathrm{Out}^+(\Gamma) \\
\downarrow & & \downarrow \\
\mathbf{CP}^1(S)^+ & \xrightarrow{\Theta} & \mathcal{T}(\Gamma) \times \mathcal{ML}(\Gamma) \\
\downarrow & & \downarrow \\
\mathcal{M}\mathbf{CP}^1(S)^+ & \xrightarrow{\hat{\Theta}} & \mathcal{T}(\Gamma) \times \mathcal{ML}(\Gamma)/\,\mathrm{Out}^+(\Gamma).
\end{array}$$

If we recall that $\mathrm{Out}^+(\Gamma)$ acts properly discontinuously on $\mathcal{T}(\Gamma) \times \mathcal{ML}(\Gamma)$, it follows that

Corollary 2.6.2. $\mathrm{Diff}^+(S)/\,\mathrm{Diff}^0(S)$ *acts properly discontinuously on* $\mathbf{CP}^1(S)^+$.

3. Projective structures on surfaces and holonomy function groups

3.1 Subspaces of $\mathbf{CP}^1(S_g)^+$

As before S is a closed orientable surface S_g of genus $g \geqq 2$ and $\Gamma = \pi_1(S_g)$. Recall that

$$\mathbf{CP}^1(S)^+ = \mathrm{PSL}_2(\mathbf{C}) \setminus \Omega^+(S),$$

where $\Omega^+(S)$ is the deformation space of orientation-preserving developing maps. (See Chapter 3.)

Definition 3.1.1. Let $P : \Omega^+(S) \to \mathbf{CP}^1(S)^+$ be the canonical projection. Let $\mathbf{CP}^1(S)^+_0$ be the subspace of $\mathbf{CP}^1(S)^+$ consisting of injective developing maps. And $\mathbf{CP}^1(S)^+_1$ is the subspace of $\mathbf{CP}^1(S)^+$ consisting of nonsurjective developing maps. Let $\Omega^+(S)_i = P^{-1}(\mathbf{CP}^1(S)^+_i)$ $(i = 0, 1)$.

Proposition 3.1.2.

(i) $\mathbf{CP}^1(S)^+_0$ *is a closed subspace of* $\mathbf{CP}^1(S)^+$.

(ii) $\mathbf{CP}^1(S)^+_1$ *is a closed subspace of* $\mathbf{CP}^1(S)^+$

Proof. Suppose that a sequence $\{\mathrm{dev}_i\}$ in $\mathbf{CP}^1(S)^+_0$ converges to a developing map $\{\mathrm{dev}\}$.
Suppose that dev is not injective and $\mathrm{dev}(x) = \mathrm{dev}(y)$ for $x \neq y$ in $\tilde{S}$. There exists a compact neighborhood K of x which does not contain y and is mapped homeomorphically onto a closed ball $\mathrm{dev}(K)$. By the above topology on $\mathbf{CP}^1(S)^+$, dev_i is also an embedding on K for sufficiently large i. Let ρ be the spherical metric of $\mathbf{CP}^1$ and let ℓ_i be the shortest circular arc from $\mathrm{dev}_i(x)$ to $\mathrm{dev}_i(y)$. Since $\mathrm{dev}\,|K$ is an embedding for sufficiently large i, there is a sequence of points $\{p_i\} \in \mathbf{CP}^1$ each of which is the first contact of ℓ_i to $\partial\,\mathrm{dev}_i(K) = \mathrm{dev}_i(\partial K)$. There exists a sequence of points $\{z_i\} \in \partial K$ such that $\mathrm{dev}_i(z_i) = p_i$. Since ℓ_i gives the distance $\rho(\mathrm{dev}_i(x), \mathrm{dev}_i(y))$, it follows that $\rho(\mathrm{dev}_i(x), \mathrm{dev}_i(z_i)) \leqq \rho(\mathrm{dev}_i(x), \mathrm{dev}_i(y))$. The sequence $\{z_i\} \in \partial K$ converges to some point $z \in \partial K$. Then the above inequality yields that

$$\rho(\mathrm{dev}(x), \mathrm{dev}(z)) \leqq \rho(\mathrm{dev}(x), \mathrm{dev}(y)) = 0,$$

while x is an interior point of K, which is a contradiction. This proves (i).

Consider (ii). Suppose that a sequence $\{(\phi_i, d_i)\}$ converges to an element $\{(\phi, d)\}$ in $\mathbf{CP}^1(S)^+$. We can assume that the closure $\overline{\phi(\Gamma)}$ is neither a finite group nor a subgroup of the group of similarity transformations $\mathrm{Sim}(\mathbf{R}^2)$, for otherwise S would be covered by a sphere or a torus respectively. In particular $\phi(\Gamma)$ contains a loxodromic element. If $\phi(\gamma)$ is a loxodromic element for some $\gamma \in \Gamma$, there is a point x in $\mathbf{CP}^1$ such that $\phi(\gamma)x = x$. (Note that there exist exactly two points.) Let $L(\phi(\Gamma))$ be the limit set for $\phi(\Gamma)$. (See 4.1 or [11] for the definition.) It follows that $x \in L(\phi(\Gamma))$. The trace formula (cf. [4]) implies that g is either elliptic or parabolic if and only if $|\,\mathrm{tr}^2 g| \in [0, 4]$ for $g \in \mathrm{PSL}_2(\mathbf{C})$. Since $\phi_i(\gamma) \to \phi(\gamma)$ and $\phi(\gamma)$ is loxodromic, it follows that $\phi_i(\gamma)$ is also loxodromic for sufficiently large i. And so there exists a point x_i such that $\phi_i(\gamma)x_i = x_i$ for each i. Note that $\{x_i\} \in L(\phi_i(\Gamma))$ and $d_i(\tilde{S}) \cap L(\phi_i(\Gamma)) = \emptyset$. (See for example [25].)
The sequence $\{x_i\}$ has an accumulation point. Since $\phi_i(\gamma) \to \phi(\gamma)$ and $\phi_i(\gamma)x_i = x_i$, $\phi(\gamma)$ fixes that accumulation point. We can assume that $\lim x_i = x$.

We claim that d misses the point x. Suppose not. Let $d(p) = x$ for some $p \in \tilde{S}$. Choose a compact neighborhood C of p in $\tilde{S}$ and a closed ball $\bar{B}$ centered at x in $\mathbf{CP}^1$ so that $d : C \to \bar{B}$ is a diffeomorphism. We note that for sufficiently large i, $x_i \in \bar{B}$ and $d_i|C$ is an embedding.

Case 1. x lies inside $d_i(C)$. Since $x_i \notin d_i(C)$, it implies that $\rho(x_i, x) \geqq \operatorname{dist}(x_i, d_i(C))$ and let $\{a_i\} \in \partial C$ be the sequence of points which attains the distance $\operatorname{dist}(x_i, d_i(C))$, i.e., $\rho(x_i, d_i(a_i)) = \operatorname{dist}(x_i, d_i(C))$. The sequence $\{a_i\}$ has a limit point $a \in \partial C$. Since it follows that $d(a) \in \partial B$, we obtain that $\rho(x, d(a)) > 0$. On the other hand, the above inequality yields that $0 \geqq \rho(x, d(a))$, which is a contradiction.
Case 2. x lies outside $d_i(C)$. Since $p \in C$, it follows $\operatorname{dist}(x, d_i(C)) \leqq \rho(x, d_i(p))$. Note that $\lim \operatorname{dist}(x, d_i(C)) = 0$ because $\lim d_i(p) = d(p) = x$. Similarly as above we have a sequence of the points $\{b_i\} \in \partial C$ such that $\rho(x, d_i(b_i)) = \operatorname{dist}(x, d_i(C))$. As $\lim b_i = b$ for some point $b \in \partial C$, it follows that $\lim d_i(b_i) = d(b) \in \partial B$. And so $0 < \rho(x, d(b)) = \lim \operatorname{dist}(x, d_i(C))$, being a contradiction. Therefore d misses the point x.

By virtue of the theorem of [25] we have that d is a covering map. This shows (ii). Q.E.D.

3.2. Description of Kleinian groups by projective structures

Let G be a Kleinian group, i.e., a finitely generated discrete subgroup of $\mathrm{PSL}_2\,\mathbf{C}$. Put $\Omega = \Omega(G) = S^2 - L(G)$. Recall that
G is a function group if there is a component Ω_0 of Ω invariant under G.
G is a quasi-Fuchsian group if $\Omega = \Omega_0 \cup \Omega_1$ (i.e., consists of two components). As the special case if $\partial\Omega_0 (= \partial\Omega_1)$ is a round circle then G is a Fuchsian group.
G is a b-group if Ω has only one invariant simply connected component.
Let $S = \mathbf{H}^2/\Gamma$ be a closed orientable surface.

Definition 3.2.1.
$\mathcal{F}(\Gamma) = \{\theta \in \operatorname{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbf{C}) \mid \theta(\Gamma)$ is a function group$\}$.
$\mathcal{B}(\Gamma) = \{\theta \in \operatorname{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbf{C}) \mid \theta : \Gamma \to \theta(\Gamma)$ is an isomorphism, $\theta(\Gamma)$ is a Kleinian group and an invariant component is simply connected.$\}$
$\mathcal{R}_2(\Gamma) = \{\theta \in \operatorname{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbf{C}) \mid \theta(\Gamma)$ is quasi-conformally equivalent to $\Gamma.\}$ (i.e., the set of quasi-Fuchsian groups).
Let

$$P : \operatorname{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbf{C}) \to \operatorname{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbf{C}) / \mathrm{PSL}_2\,\mathbf{C}$$

be the canonical projection and put

$$F(\Gamma) = P(\mathcal{F}(\Gamma)), \quad B(\Gamma) = P(\mathcal{B}(\Gamma)), \text{ and } T_2\Gamma = P(\mathcal{R}_2(\Gamma)).$$

Note that $\operatorname{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbf{C}) / \mathrm{PSL}_2\,\mathbf{C}$ is connected but not a Hausdorff space.

Recall that $H : \Omega^+(S) \to \mathrm{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbf{C})$ is the map which assigns to an oriented projective structure its holonomy representation. The map H induces a holonomy map hol : $\mathbf{CP}^1(S)^+ \to \mathrm{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbf{C})/\,\mathrm{PSL}_2\,\mathbf{C}$.

Corollary 3.2.2. *The holonomy map* hol *maps* $\mathbf{CP}^1(S)^+_1 \to F(\Gamma)$. *In particular the holonomy groups are discrete.*

Proposition 3.2.3. *The holonomy map* hol *defines a homeomorphism of* $\mathbf{CP}^1(S)^+_0$ *onto* $B(\Gamma)$.

Proof. Let $\theta \in \mathcal{B}(\Gamma)$ be a representative element of $B(\Gamma)$ with an invariant simply connected component Ω_0. Since $\theta \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbf{C})$, there is an orientation-preserving conformal homeomorphism $f : \Omega_0 \to \mathbf{H}^2$ such that $f\theta(\Gamma)f^{-1}$ is Fuchsian. Let $\psi : \Gamma \to f\theta(\Gamma)f^{-1}$ be an isomorhism defined by $\psi(\gamma) = f\theta(\gamma)f^{-1}$. Then it is well known that there is a quasi-conformal homeomorphism $h : \mathbf{H}^2 \to \mathbf{H}^2$ which induces ψ. Put $\mathrm{dev} = f^{-1} \circ h$. It is easy to see that $[\theta, \mathrm{dev}]$ is an element of $\mathbf{CP}^1(S)^+_0$.

Let $P : \Omega^+_0(S) \to \mathbf{CP}^1(S)^+_0$ be the canonical projection of the deformation spaces (cf. (3.1.1)). We will show that H maps $\Omega^+_0(S)$ onto $\mathcal{B}(\Gamma)$. If (ϕ, dev) is an element of $\Omega^+_0(S)$, then it follows that $\phi \in \mathcal{F}$. If suffices to check that $\phi(\Gamma)$ has an invariant simply connected component of $\Omega = S^2 - L(\phi(\Gamma))$. Since dev is injective, $\phi(\Gamma)$ has a simply connected domain $\mathrm{dev}(\tilde{S})$ which sits in Ω. Let Ω_0 be an invariant maximal component in Ω containing $\mathrm{dev}(\tilde{S})$. Since $\phi(\Gamma)$ acts properly discontinuously and freely on Ω_0, we can choose a $\phi(\Gamma)$ invariant Riemannian metric on Ω_0. The map dev is a covering map because S is compact. Since $\phi : \Gamma \to \phi(\Gamma)$ is an isomorphism, dev must be an isometry. And thus $\mathrm{dev}(\tilde{S}) = \Omega_0$. We prove that H is injective. For (ϕ_i, dev_i) $(i = 1, 2)$, suppose that $H(\phi_1, \mathrm{dev}_1) = H(\phi_2, \mathrm{dev}_2)$, i.e., $\phi_1 = \phi_2$. Then, $\mathrm{dev}_1(\tilde{S}) = \mathrm{dev}_2(\tilde{S})$. For this, if not then $\phi(\Gamma) = \phi_1(\Gamma) = \phi_2(\Gamma)$ has at least two invariant components, i.e., $\mathrm{dev}_1(\tilde{S})$, $\mathrm{dev}_2(\tilde{S})$ (cf. [5]). Hence $\phi(\Gamma)$ is quasi-Fuchsian. However since both dev_1 and dev_2 are orientation-preserving, it is impossible. Put $\tilde{f} = \mathrm{dev}_2^{-1} \circ \mathrm{dev}_1$. Then it follows that $\tilde{f} \circ \gamma = \gamma \circ \tilde{f}$ for $\gamma \in \Gamma$. Therefore $f \in \mathrm{Diff}^0(S)$ and $[\phi_2, \mathrm{dev}_2] \circ f = [\phi_1, \mathrm{dev}_1]$.
And hence H is a one-to-one continuous map. Since H is a local homeomorphism by the Holonomy theorem 4.3.9 (cf. Chapter 4), it follows that H is a homeomorphism of $\Omega_0(S)$ onto $\mathcal{B}(\Gamma)$. Since the action of $\mathrm{PSL}_2\,\mathbf{C}$ on both $\Omega_0(S)$ and $\mathcal{B}(\Gamma)$ is free, it implies that hol is a homeomorphism. Q.E.D.

Let $\mathcal{R}_2(\Gamma)$ be the space of quasi-Fuchsian groups in $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbf{C})$ as in (3.2.1). If $\overline{\mathcal{R}_2(\Gamma)}$ is the closure of $\mathcal{R}_2(\Gamma)$ in $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbf{C})$, then we put $\partial\mathcal{R}_2(\Gamma) = \overline{\mathcal{R}_2(\Gamma)} - \mathcal{R}_2(\Gamma)$.

Definition 3.2.4. Define the following subspaces

$$\begin{aligned}\Omega^+(S, qf) &= \{(\phi, \mathrm{dev}) \in \Omega^+(S) \mid \phi \in \mathcal{R}_2(\Gamma)\},\\ \mathbf{CP}^1(S, qf)^+ &= P(\Omega^+(S, qf)),\end{aligned}$$

and

$$\begin{aligned}\Omega^+(S, \partial) &= \{(\phi,\ \mathrm{dev}) \in \Omega^+(S) \mid \phi \in \partial\mathcal{R}_2(\Gamma)\},\\ \mathbf{CP}^1(S, \partial)^+ &= P(\Omega^+(S, \partial)).\end{aligned}$$

$\mathbf{CP}^1(S, qf)^+$ (resp. $\mathbf{CP}^1(S, \partial)^+$) is called the deformation space of (oriented) projective structures with quasi-Fuchsian (resp. boundary) holonomy.

We have the following subspaces of $\mathbf{CP}^1(S, qf)^+$ (resp. $\mathbf{CP}^1(S, \partial)^+$) whose developing maps are injective;

$$\begin{aligned}\mathbf{CP}^1(S, qf)_0^+ &= \mathbf{CP}^1(S, qf)^+ \cap \mathbf{CP}^1(S)_0^+,\\ \mathbf{CP}^1(S, \partial)_0^+ &= \mathbf{CP}^1(S, \partial)^+ \cap \mathbf{CP}^1(S)_0^+.\end{aligned} \tag{3.2.5}$$

The simultaneous uniformization of Bers ([5]) is stated as follows.

Corollary 3.2.6 (Bers). $\mathbf{CP}^1(S, qf)_0^+ \approx \mathcal{T}(\Gamma) \times \Delta$.
Here Δ is an open cell contained in $\mathcal{ML}(S)$.

3.3. Insertion of annuli and operation on projective structures with boundary holonomy

An insertion of annuli (more generally, a grafting) produces a new structure from a given projective structure. (See Goldman [16].) Especially, let $\Omega_0^+(S, qf)$ be the space of projective structures with quasi-Fuchsian holonomy groups and with injective developing maps. Let $\mathcal{C}$ the set of all isotopy classes of a disjoint collection of homotopically nontrivial simple closed curves on S. Let $\mathcal{ML}(2\mathbf{Z})$ denote the set of measured geodesic laminations μ supported on a disjoint union of closed geodesics lying in $\mathcal{C}$ and together with 2π times positive integer weights. Then, each $\sigma \in \mathcal{ML}(2\mathbf{Z})$ defines an operation $\sharp$ which assigns to a structure of $\Omega_0(S, qf)$ a structure with surjective developing map.
Goldman ([16]) has shown that

Theorem 3.3.1. $\mathbf{CP}^1(S, qf)^+ \approx \mathbf{CP}^1(S, qf)_0^+ \times \mathcal{ML}(2\mathbf{Z})$.

It follows also that

$$\Omega^+(S, qf) \approx \Omega_0^+(S, qf) \times \mathcal{ML}(2\mathbf{Z}). \tag{3.3.2}$$

If $x \in \Omega_0^+(S, qf)$ and $\sigma \in \mathcal{ML}(2\mathbf{Z})$, then $x\sharp\sigma$ is a new structure with surjective developing map and with the same holonomy as that of x. It lies in one component of $\Omega^+(S, qf)$ different from $\Omega_0(S, qf)$. And so it follows that $\Omega^+(S, qf) = \bigcup_{\sigma \in \mathcal{ML}(2\mathbf{Z})} (\Omega_0^+(S, qf)\sharp\sigma)$ for which $\Omega_0^+(S, qf)\sharp\sigma$ is one component homomorphic to $\Omega_0^+(S, qf)$.

Let $\Omega_0^+(S, \partial)$ be the space of oriented projective structures with boundary holonomy and with injective developing maps (cf. (3.2.5)). The operation $\sharp$ can be defined on $\Omega_0^+(S, \partial)$. We shall prove the similar result for $\Omega^+(S, \partial)$.

Proposition 3.3.3.

$$\Omega_0^+(S, \partial) \times \mathcal{ML}(2\mathbf{Z}) \approx \Omega^+(S, \partial).$$

In order to prove this proposition, we need the following lemmata.

Lemma 3.3.4. *The holonomy map*

$$H : \overline{\Omega^+(S, qf)} \to \overline{\mathcal{R}_2(\Gamma)} \textit{ is locally injective.}$$

Proof. If we note that $\mathcal{ML}(2\mathbf{Z})$ is discrete in $\mathcal{ML}(S)$, then $\overline{\Omega^+(S, qf)} \approx \overline{\Omega_0^+(S, qf)} \times \mathcal{ML}(2\mathbf{Z})$. We prove that $H : \overline{\Omega_0^+(S, qf)} \to \overline{\mathcal{R}_2(\Gamma)}$ is injective. Let $x,\ y \in \partial\Omega_0^+(S, qf)(= \overline{\Omega_0^+(S, qf)}(S) - \Omega_0^+(S, qf))$ and suppose $H(x) = H(y)$. First note that $H(x) \in \partial\mathcal{R}_2(\Gamma) = \overline{\mathcal{R}_2(\Gamma)} - \mathcal{R}_2(\Gamma)$ since H is a local homeomorphism and by the definition (3.2.4). There are neighborhoods U,V of x,y respectively such that $H : U \to W$, $H : V \to W$ are homeomorphisms where W is a neighborhood of $H(x)$. Since $W \cap \mathcal{R}_2(\Gamma) \neq \emptyset$ is open, there are $a \in U$, $b \in V$ so that $H(a) = H(b)$ in $W \cap \mathcal{R}_2(\Gamma)$. Since $H|\Omega_0^+(S, qf)$ is a homeomorphim, it follows that $a = b$, i.e., $U \cap V \neq \emptyset$. It implies that $x = y$.

We note that nearby structures outside $\overline{\Omega_0(S, qf)}$ do not have quasi-Fuchsian holonomy groups. Namely, for $x \in \partial\Omega_0^+(S, qf)$, there is a neighborfood U of x such that for any $z \in U - \overline{\Omega_0^+(S, qf)}$, $H(z) \notin \mathcal{R}_2(\Gamma)$.
Q.E.D.

Lemma 3.3.5. $\partial\Omega^+(S, qf) = \Omega^+(S, \partial)$.

Proof. Using the above lemma, we have for $x \in \partial(\Omega^+(S, qf))$ that $H(x) \in \partial\mathcal{R}_2(\Gamma)$. By the definition 3.2.4 it follows that $\partial(\Omega^+(S, qf)) \subset \Omega^+(S, \partial)$. Let $x \in \Omega^+(S, \partial)$ so that $H(x) \in \partial\mathcal{R}_2(\Gamma)$. Let U be any neighborhood of x in $\Omega^+(S)$. Since H is a local homeomorphism, there exists a neighborhood W of x contained in U such that $H : W \to H(W)$ is a homeomorphism. As $H(x) \in \partial\mathcal{R}_2(\Gamma)$, we have $H(U) \cap \mathcal{R}_2(\Gamma) \neq \emptyset$. Choose $y \in U$ with $H(y) \in \mathcal{R}_2(\Gamma)$. Again by the definition 3.2.4, it follows that $y \in \Omega^+(S, qf)$, or $U \cap \Omega^+(S, qf) \neq \emptyset$. And hence $x \in \overline{\Omega^+(S, qf)}$. Since $H(x)$ is not a quasi-Fuchsian group, $x \in \partial(\Omega^+(S, qf))$. Q.E.D.

Proof of (3.3.3). Since $\overline{\Omega^+(S, qf)} \approx \overline{\Omega_0^+(S, qf)} \times \mathcal{ML}(2\mathbf{Z})$, it implies that

$$\partial\Omega^+(S, qf) \approx \partial\Omega_0^+(S, qf) \times \mathcal{ML}(2\mathbf{Z}). \tag{3.3.6}$$

On the other hand, $\Omega_0^+(S)$ is a closed subspace of $\Omega^+(S)$ by Proposition 3.1.2. It is noted that $\overline{\Omega_0^+(S, qf)} \subset \Omega_0^+(S)$. And so we have that $\partial\Omega_0^+(S, qf)) \subset \Omega_0^+(S)$. In view of (3.3.6), the subspace of $\partial\Omega^+(S, qf)$ consisting of injective developing maps, $(\partial\Omega^+(S, qf))_0 = \partial\Omega_0^+(S, qf)$. By Lemma 3.3.5 it follows that $\Omega_0^+(S, \partial) = (\partial\Omega^+(S, qf))_0$ and by (3.3.6) that $\Omega^+(S, \partial) \approx \Omega_0^+(S, \partial) \times \mathcal{ML}(2\mathbf{Z})$. Q.E.D.

4. (G, X)-structures

4.1. Limit sets in (G, X)

Recall that a geometric structure on a smooth n-manifold is a maximal collection of charts modelled on a simply connected n dimensional homogeneous space X of a Lie group G whose coordinate changes are restrictions of transformations from G. We call such a structure a (G, X)-structure. A manifold with this structure is called a (G, X)-manifold. Suppose that a smooth connected n-manifold M admits a (G, X)-structure. Then there exists a developing pair (ρ, dev), where $\mathrm{dev} : \tilde{M} \to X$ is a "structure-preserving" immersion and $\rho : \pi_1(M) \to G$ is a homomorphism (both unique up to elements of G). The group $\Gamma = \rho(\pi_1(M))$ is called the holonomy group for M.

In particular the developing pair (ρ, dev) is an invariant of the (G, X)-structure. In fact this developing map and holonomy give us a powerful tool in understanding the topology of (G, X)-manifolds. The first question arises when the developing map is a covering map onto its

image. In order to study this problem we introduce the notion of limit sets in (G, X) due to Kulkarni [33].
We consider the following sets. Let Γ be a subgroup of G.

(4.1.1)

Λ_0 = the closure of the set $\{x \in X |$ the stabilizer Γ_x is an infinite subgroup$\}$.
Λ_1 = the set of cluster points of $\{\gamma y | \gamma \in \Gamma\}$ where $y \in X - \Lambda_0$.
Λ_2 = the set of cluster points of $\{\gamma K | \gamma \in \Gamma\}$ where K is a compact subset of $X - \{\Lambda_0, \Lambda_1\}$.

Then the set $\Lambda = \Lambda(\Gamma) = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2$ is said to be the limit set of Γ. And the set $\Omega = X - \Lambda$ is called the domain of discontinuity for Γ. (Compare also [39] for further results of limit sets.) It is the fundamental result that

Proposition 4.1.2. *If $\Omega \neq \emptyset$, then Γ acts properly discontinuously on Ω. In particular Γ is discrete in G.*

(4.1.3) We examine another limit sets to our use. Let Y be a complete simply connected Riemannian manifold of nonpositive sectional curvature. Then there is a compactification $\bar{Y} = \partial Y \cup Y$ of Y. The space $\bar{Y}$, equipped with the cone topology, is homeomorphic to the closed ball and the boundary ∂Y is the set of points at infinity consisting of the equivalence classes of asymptotic geodesics. The group of isometries $\mathrm{Iso}(Y)$ extends to a topological action on its boundary. For example, recall that the n-sphere S^n is viewed as the ideal boundary of the real hyperbolic space $\mathbf{H}^{n+1}$. Similarly S^{2n+1} is the ideal boundary of the complex hyperbolic space $\mathbf{H}_{\mathbf{C}}^{n+1}$. Moreover, when Y is a hyperbolic space $\mathbf{H}^n$ or $\mathbf{H}_{\mathbf{C}}^{n+1}$, $\mathrm{Iso}(Y)$ acts as conformal (resp. CR) automorphisms of the sphere. We write $\mathrm{Iso}(Y) = \mathrm{Conf}(S^n)$ or $\mathrm{Aut}_{CR}(S^{2n+1})$ respectively.

Definition 4.1.4. For a subgroup Γ of $\mathrm{Iso}(Y)$ the limit set $L(\Gamma) \subset \partial Y$ is defined to be the set of cluster points of the orbit $\Gamma \cdot x$ for $x \in Y$.

As to the relation between the above limit set Λ, we have (cf. [25])

Proposition 4.1.5. *Let Γ be a discrete subgroup of either $\mathrm{Conf}(S^n)$ or $\mathrm{Aut}_{CR}(S^{2n+1})$. Then it follows that*

$$\Lambda(\Gamma) = L(\Gamma).$$

4.2. Application to developing maps

Suppose that M is a closed connected (G, X)-manifold. Let us be given a Γ invariant closed subset F in X. Suppose that there exist a component Y of the complement $X-F$ and a component N of $\text{dev}^{-1}(Y)$. We then have the restriction of the developing map $\text{dev}: N \longrightarrow Y$. We have proved the following result in [18] (cf. also in [16],[19]).

Lemma 4.2.1. *Under the above hypothesis, suppose that Y admits a Γ invariant complete Riemannian metric. Then the developing map* $\text{dev}: N \longrightarrow Y$ *is a covering map.*

As an application, we shall prove that;

Proposition 4.2.2. *Let Y be a Γ invariant closed subset of X with Hausdorff dimension $k < n-1$. Suppose that the complement $X-Y$ admits a Γ invariant complete Riemannian metric.*

(i) *if $k < n-2$, then* $\text{dev}: \tilde{M} \to X - \Lambda$ *is a homeomorphism.*

(ii) *for $n-2 \leqq k < n-1$, assume that either* $\text{dev}^{-1}(Y) = \emptyset$ *or* $\text{dev}_*: \pi_1(\tilde{M} - \text{dev}^{-1}(Y)) \to \pi_1(X-Y)$ *is surjective. Then* $\text{dev}: \tilde{M} \to X - Y$ *is a covering map, or* $\text{dev}: \tilde{M} \to X - \Lambda$ *is a homeomorphism.*

Proof. Note first that $\tilde{M} - \text{dev}^{-1}(Y)$ is connected since the Hausdorff dimension k is less than $n-1$ (cf. [18]). Moreover if $k < n-2$, $X-Y$ is 1-connected. We have from Lemma 4.2.1 that $\text{dev}: \tilde{M} - \text{dev}^{-1}(Y) \to X - Y$ is a covering map. As above if $k < n-2$, $\text{dev}: \tilde{M} - \text{dev}^{-1}(Y) \to X - Y$ is a homeomorphism. If $n-2 \leqq k < n-1$ then according to that $\text{dev}^{-1}(Y) = \emptyset$ or $\text{dev}^{-1}(Y) \neq \emptyset$ under the surjectivity assumption it follows that $\text{dev}: \tilde{M} \to X - Y$ is a covering map or $\text{dev}: \tilde{M} - \text{dev}^{-1}(Y) \to X - Y$ is a homeomorphism. Since dev is an immersion and $k < n-1$, for any point x in $\tilde{M}$ there exists a neighborhood U of x in $\tilde{M}$ such that $\text{dev}(U) \cap (X-Y) \neq \emptyset$. This implies that $\text{dev}: \tilde{M} \to \text{dev}(\tilde{M})$ is injective. Hence Γ acts properly discontinuously on $\text{dev}(\tilde{M})$ which shows that $\text{dev}(\tilde{M}) \cap \Lambda = \emptyset$. Since Γ acts properly discontinuously on $X - \Lambda$ by Proposition 4.1.2, it follows that $\text{dev}(\tilde{M}) = X - \Lambda$. Q.E.D.

4.3. Deformation space of (G, X)-structures and the Holonomy theorem

In this section we shall examine the structure of the deformation space of (G, X)-structures invariant under Lie groups. Let H be a connected Lie group acting on a smooth closed $(2n+1)$-manifold M.

4.3.1. *The deformation space $\mathcal{T}(H, M)$ is a space of H invariant marked (G, X)-structures on manifolds homeomorphic to M. $\mathcal{T}(H, M)$*

consists of equivalence classes of equivariant diffeomorphisms $f : M \to M'$ *from the action* (H, M) *to* H *invariant* (G, X)*-manifolds* M'. *Two such diffeomorphisms* $f_i : M \to M_i$ $(i = 1, 2)$ *are equivalent if there is an equivariant isomorphism (i.e., a* (G, X)*-structure preserving diffeomorphism)* $h : M_1 \to M_2$ *such that* $h \circ f_1$ *is isotopic to* f_2.

$$\begin{array}{ccc} M & \xrightarrow{f_1} & M_1 \\ {\scriptstyle f_2}\searrow & \simeq & \downarrow\ h \\ & & M_2 \end{array}$$

Note that it is not necessarily assumed to be *equivariantly isotopic.* On the other hand if M is a (G, X)-manifold then there is a developing pair $(\rho, \mathrm{dev}) : (\mathrm{Aut}(\tilde{M}), \tilde{M}) \to (G, X)$ such that $\pi \subset \mathrm{Aut}(\tilde{M})$, where $\pi = \pi_1(M)$ and $\mathrm{Aut}(\tilde{M})$ is a group of (G, X)-isomorphisms of $\tilde{M}$.

4.3.2. $\hat{\Omega}(H, M)$ *is the space consisting of all possible developing pairs* (ρ, dev) *which satisfy that* (ρ, dev) *represents an* H *invariant* (G, X)*-structure on* M *and such that if one forgets the structure then the action* (H, M) *is smoothly equivalent to the original action* (H, M). *That is, the action of each element of* $\hat{\Omega}(H, M)$ *is topologically unique but geometrically distinct.*

The topology on $\hat{\Omega}(H, M)$ *is given by the following subbasis* (cf. [8]).

(1) $\mathcal{N}(U) = \{U\}$ *where* U *is an open subset of* $\mathrm{Maps}(\tilde{M}, X)$ *in the compact open topology of* $\mathrm{Maps}(\tilde{M}, X)$.

(2) $\mathcal{N}(K) = \{\mathrm{dev} \in \hat{\Omega}(H, M) \mid \mathrm{dev}|K$ *is an embedding for a compact subset* $K \subset \tilde{M}\}$.

4.3.3. *We introduce a subgroup* $\widehat{\mathrm{Diff}}(H, M)$ *of* $\mathrm{Diff}(\tilde{M})$. *Let* $\mathrm{Diff}(H, M)$ *be the group of equivariant diffeomorphisms of* M *onto itself. Denote by* $\mathrm{Diff}^0(H, M)$ *the subgroup of* $\mathrm{Diff}(H, M)$ *whose elements are isotopic to the identity map. Consider the following exact sequences of the diffeomorphism groups, where* $N_{\mathrm{Diff}(\tilde{M})}(\pi)$ *(resp.* $C_{\mathrm{Diff}(\tilde{M})}(\pi)$*) is the normalizer (resp. centralizer) of* π *in* $\mathrm{Diff}(\tilde{M})$*;*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi & \longrightarrow & N_{\mathrm{Diff}(\tilde{M})}(\pi) & \xrightarrow{\eta} & \mathrm{Diff}(M) & \longrightarrow & 1 \\ & & & & \uparrow & & \uparrow & & \\ & & & & C_{\mathrm{Diff}(\tilde{M})}(\pi) & \longrightarrow & \mathrm{Diff}^0(M) & & \end{array}.$$

Put $\widehat{\mathrm{Diff}}(H,M)=\eta^{-1}(\mathrm{Diff}(H,M))$ *and let* $\widehat{\mathrm{Diff}}^0(H,M)$ *be the identity component. It follows easily that* $\eta(\widehat{\mathrm{Diff}}^0(H,M)) = \mathrm{Diff}^0(H,M)$ *and* $\widehat{\mathrm{Diff}}^0(H,M) \subset C_{\mathrm{Diff}(\tilde{M})}(\pi)$.

4.3.4. *The actions on* $\hat{\Omega}(H,M)$. *The natural right action of* $\widehat{\mathrm{Diff}}(H,M)$ *and the left action of* G *on* $\hat{\Omega}(H,M)$ *are defined by setting*

$$(\rho,\mathrm{dev})\circ\tilde{f} = (\rho\circ\mu(\tilde{f}),\mathrm{dev}\circ\tilde{f}),$$
$$g\circ(\rho,\mathrm{dev}) = (g\circ\rho\circ g^{-1}, g\circ\mathrm{dev}),$$

where $\mu(\tilde{f}):\pi\longrightarrow\pi$ is an isomorphism defined by $\mu(\tilde{f})(\gamma)=\tilde{f}\circ\gamma\circ\tilde{f}^{-1}$. Obviously both actions commute.

It is noted that two developing pairs (ρ_i,dev_i) $(i=1,2)$ represent the same structure on M if and only if there exists an element $g\in G$ such that $g\circ\mathrm{dev}_1=\mathrm{dev}_2$. Put

$$\Omega(H,M)=\hat{\Omega}(H,M)/\widehat{\mathrm{Diff}}^0(H,M).$$

The action of G induces an action of $\Omega(H,M)$. Then it is easy to show that

Lemma 4.3.5. *The elements of* $\mathcal{T}(H,M)$ *are in one-to-one correspondence with the orbits of* $G\setminus\Omega(H,M)$.

Definition 4.3.6. The space $G\backslash\Omega(H,M)$ equipped with the quotient topology is called the deformation space $\mathcal{T}(H,M)$ of H invariant (G,X)-structures on M.

Note that if one choose the trivial group as H then $\mathcal{T}(M)=\mathcal{T}(\{1\},M)$ is the usual deformation space. If $f:M\longrightarrow M'$ is a representative element of $\mathcal{T}(H,M)$ then there is a developing pair $(\rho,\mathrm{dev}):(\pi_1(M'),\tilde{M}')\longrightarrow(G,X)$. We have the holonomy representation $\rho\circ f_\sharp:\pi\longrightarrow G$ up to conjugate by an element of G. Let $\hat{H}\subset\widehat{\mathrm{Diff}}^0(H,M)$ be a closed connected subgroup such that $\eta(\hat{H})=H$. Note that the group $\hat{H}$ centralizes π and f is equivariant (cf. (4.3.1)). The group $\rho(\hat{H})$ centralizes $\rho\circ f_\sharp(\pi)$. Here we assume that

(4.3.7 $*$). *there exist a group* $K\subset G$ *and an isomorphism* ϕ : $\hat{H}\longrightarrow K$ *for which every representation* ρ *satisfies that* $g\circ\rho\circ g^{-1}=\phi$ *for some* $g\in G$.

It is noted that $\rho \circ f_\sharp(\pi)$ lies in the centralizer $C_G(K)$ up to conjugation. Let $\hat{\Omega}_0(H,M)$ be the subset of $\hat{\Omega}(H,M)$ whose holonomy representations ρ lie in $C_G(K)$ and $\rho|\hat{H} = \phi$. Put $\Omega_0(H,M) = \hat{\Omega}_0(H,M)/\widehat{\mathrm{Diff}}^0(H,M)$. The projection $\Omega_0(H,M) \to \mathcal{T}(H,M)$ is surjective by (4.3.7 $*$). Moreover we assume that

(4.3.7 $**$). *If two such representations of π are conjugate in G then they are conjugate by an element of $C_G(K)$.*

We then obtain a map hol : $\mathcal{T}(H,M) \to \mathrm{Hom}(\pi, C_G(K))/C_G(K)$ which assigns to a marked structure its holonomy representation. By the definition hol lifts to a map $\widehat{\mathrm{hol}} : \Omega_0(H,M) \to \mathrm{Hom}(\pi, C_G(K))$ which makes the following diagram commute.

$$
\begin{array}{ccc}
\Omega_0(H,M) & \xrightarrow{\widehat{\mathrm{hol}}} & \mathrm{Hom}(\pi, C_G(K)) \\
\downarrow & & \downarrow \\
\mathcal{T}(H,M) & \xrightarrow{\mathrm{hol}} & \mathrm{Hom}(\pi, C_G(K))/C_G(K).
\end{array}
\tag{4.3.8}
$$

If $H = \{1\}$ then it implies that $K = \{1\}$ and so $C_G(K) = G$. We have the usual holonomy map hol : $\mathcal{T}(M) \to \mathrm{Hom}(\pi, G)/G$. It has been proved by Lok ([37]) (see also [24],[48]) that $\widehat{\mathrm{hol}} : \Omega(M) \to \mathrm{Hom}(\pi, G)$ is a local homeomorphism. We prove also that

Holonomy Theorem 4.3.9. $\widehat{\mathrm{hol}} : \Omega_0(H,M) \to \mathrm{Hom}(\pi, C_G(K))$ *is a local homeomorphism.*

Proof. If we prove that the canonical map $\Omega_0(H,M) = \hat{\Omega}_0(H,M)/\widehat{\mathrm{Diff}}^0(H,M) \to \Omega(M) = \hat{\Omega}(M)/\widehat{\mathrm{Diff}}^0(M)$ is injective, then the holonomy map $\widehat{\mathrm{hol}} : \Omega(M) \to \mathrm{Hom}(\pi, G)$ restricts to a holonomy map $\widehat{\mathrm{hol}} : \Omega_0(H,M) \to \mathrm{Hom}(\pi, C_G(K))$. And so it is a local homeomorphism. Now suppose that two elements (ρ, dev) and (ρ', dev') represent the same element in $\hat{\Omega}(M)/\widehat{\mathrm{Diff}}^0(M)$. There exists an element $\tilde{f} \in \widehat{\mathrm{Diff}}^0(M)$ such that $\mathrm{dev}' = \mathrm{dev} \circ \tilde{f}$. Since $\rho|\hat{H} = \rho'|\hat{H}$ by (4.3.7 $*$), it follows that $\mathrm{dev} = \mathrm{dev} \circ (h\tilde{f}h^{-1}\tilde{f}^{-1})$ for each $h \in \hat{H}$. As $\hat{H}$ is connected and the map dev is a local homeomorphism, this equality implies that $\tilde{f} \circ h = h \circ \tilde{f}$ for every $h \in \hat{H}$. It follows that $\tilde{f} \in \widehat{\mathrm{Diff}}^0(H,M)$ by the definition 4.3.3. Hence the canonical map is injective. Q.E.D.

Remark **4.3.10**. Two assumptions of (4.3.7) will be satisfied when we consider semifree circle actions of H over H invariant spherical CR structures and H invariant conformally flat structures. We shall see this in the next chapter.

5. S^1 invariant geometric structures

5.1. Description of deformation spaces $\mathcal{T}(S^1, M)$

In this section we examine deformation spaces of S^1 invariant spherical CR structures and S^1 invariant conformally flat structures. Namely,

5.1.1. *Let* $H = S^1$.

(1) $(G, X) = (PU(n+1,1), S^{2n+1})$. *The corresponding space* $\mathcal{T}(H, M) = \mathcal{CR}(S^1, M)$ *is the deformation space of* S^1 *invariant spherical* CR *structures on* M *by the definition* 4.3.1.

(2) $(G, X) = (PO(n+1,1), S^n)$. *As before, the corresponding space* $\mathcal{T}(H, M) = \mathcal{CO}(S^1, M)$ *is the deformation space of* S^1 *invariant conformally flat structures on* M.

5.1.2. Let M be a closed $(2n+1)$-manifold. We suppose that the action (S^1, M) has the following properties for the CR case.

(i) M has a fixed point.

(ii) The orbit space M^* is a complex Kleinian orbifold $\mathbf{D}^{2n} - L(\pi^*)/\pi^*$.

Recall that the complex hyperbolic group $PU(n,1)$ acts on $\mathbf{D}^{2n}$ by biholomorphic transformations of $\mathbf{H}^n_{\mathbf{C}}$ and CR transformations of S^{2n-1}. The group $\pi^* \subset PU(n,1)$ and recall that $L(\pi^*)$ is the limit set of π^* in S^{2n-1}. By (i) the fixed point set F is homeomorphic to the ideal boundary $S^{2n-1} - L(\pi^*)/\pi^*$. For the conformal case, the action (S^1, M) on a closed n-manifold M has the same property as (i), but instead of (ii) we suppose

(ii)$'$ the orbit space M^* is a Kleinian orbifold $\mathbf{D}^{n-1} - L(\pi^*)/\pi^*$.

Recall from (4.3.2) that every element of $\Omega(S^1, M)$ represents an H invariant $(PU(n+1,1), S^{2n+1})$-structure on M and the CR action (H, M) is topologically equivalent to the action (S^1, M) of (5.1.2). Since M has a fixed point, it is noted that a lift $\hat{H}$ of H to $\tilde{M}$ is isomorphic to $\hat{H} = H$ (cf. (4.3.6)).
We have shown the topological rigidity of developing maps (cf. [18],[27]).

Proposition 5.1.3. *Let M be a closed spherical CR (resp. conformally flat) manifold with CR (resp. conformal) circle actions. Suppose that the action (S^1, M) has the property of* (5.1.2). *Put $\pi = \pi_1(M)$. If (ρ, dev) is the developing pair of an S^1 invariant (G, X)-structure on M where $(G, X) = (PU(n+1,1), S^{2n+1})$, $(PO(n+1,1), S^n)$ respectively, then the developing map* dev *maps homeomorphically onto the following subset of X up to an element of G;*

(1) $(\rho, \mathrm{dev}) : (S^1, \pi, \tilde{M}) \to (U(1), U(n,1), S^{2n+1} - L(\rho(\pi)))$.
(2) $(\rho, \mathrm{dev}) : (S^1, \pi, \tilde{M}) \to (SO(2), SO(n-1,1)^0 \times SO(2), S^n - L(\rho(\pi)))$.

Here $\rho : \pi \to \rho(\pi) \subset U(n,1)$ (resp. $PO(n-1,1)^0 \times SO(2)$) is an isomorphism and $L(\rho(\pi))$ is the limit set of $\rho(\pi)$ lying in S^{2n-1} (resp. S^{n-2}).

Proof. Since M has a fixed point by the condition (i) of (5.1.2), we have a lift of action $(S^1, \tilde{M})$ such that $\tilde{M}$ has a fixed point (cf. [6]). Then it follows from Proposition 3 and Note 2 of [27] that

$$\mathrm{dev} : \tilde{M} \to S^{2n+1} - L(\rho(\pi)) \text{ is homeomorphic}$$

and

$$\rho : (S^1, \pi) \to (\rho(S^1), \rho(\pi)) \subset (U(n-m), P(U(m+1,1) \times U(n-m))).$$

is an isomorphism for some $m \leqq n-1$. Moreover the limit set $L(\rho(\pi)) \subset S^{2m+1}$ and $S^{2m+1} - L(\rho(\pi))$ is the fixed point set of $\rho(S^1)$. In particular we have that $M \approx S^{2n+1} - L(\rho(\pi))/\rho(\pi)$ and $\mathrm{Fix}(S^1, M) = S^{2m+1} - L(\rho(\pi))/\rho(\pi)$. On the other hand the CR action (S^1, M) is topologically equivalent to the action of (5.1.2) which implies that $\mathrm{Fix}(S^1, M) \approx S^{2n-1} - L(\rho(\pi^*))/\pi^*$. Hence $m = n-1$. It follows that $\rho(S^1) = U(1)$ and $P(U(n,1) \times U(1)) = U(n,1)$. The fixed point set of $U(1)$ is $S^{2n-1} - L(\rho(\pi))$ in this case. The similar result holds for the conformal case when we note the results of [19],[26]. Q.E.D.

We shall check that the conditions of (4.3.7) are satisfied for $\mathcal{CR}(S^1, M)$ and $\mathcal{CO}(S^1, M)$.

Remark **5.1.4.**
(1) Let (ρ, dev) be a spherical CR structure on M. Each $(g \circ \rho \circ g^{-1}, g \circ \mathrm{dev})$ for $g \in G (= PU(n,1))$ represents the same structure as (ρ, dev) by the definition. The structure on M does not depend on the choice of geometric $(2n-1)$-sphere S^{2n-1} such that $L(\rho(\pi)) \subset S^{2n-1}$ by

Proposition 5.1.3. When we choose $K = U(1)$ as K of (4.3.7 $*$), for every representation ρ there exists $g \in G$ such that $g \circ \rho \circ g^{-1} : S^1 \to U(1)$ is an isomorphism. And so the condition (4.3.7 $*$) is satisfied for $\mathcal{CR}(S^1, M)$, similarly for $\mathcal{CO}(S^1, M)$ if we choose $K = SO(2)$. Then it is easy to see that the centralizer

$$C_G(K) = \begin{cases} U(n,1) \text{ for } G = PU(n+1,1) \\ SO(n-1,1)^0 \times SO(2) \text{ for } G = PO(n+1,1). \end{cases}$$

Recall that $\Omega_0^{CR}(S^1, M)$ is the subspace of $\Omega^{CR}(S^1, M)$ whose holonomy represntations belong to $U(n,1)$ (cf. (4.3.7)). It is easy to see that two such pairs (ρ, dev), (ρ', dev') represent the same structure if and only if there is an element $h \in U(n,1)$ such that $\mathrm{dev}' = h \circ \mathrm{dev}$ and $h \circ \rho \circ h^{-1} = \rho'$. The condition (4.3.7 $**$) is satisfied by this fact. As in (4.3.7), $\Omega_0(S^1, M) \to \mathcal{CR}(S^1, M)$ is surjective. We have the commutative diagram from (4.3.9)

$$(5.1.5) \qquad \begin{array}{ccc} \Omega_0^{CR}(S^1, M) & \xrightarrow{\widehat{\mathrm{hol}}} & \mathrm{Hom}(\pi, U(n,1)) \\ \downarrow & & \downarrow \\ \mathcal{CR}(S^1, M) & \xrightarrow{\mathrm{hol}} & R(\pi, U(n,1))/U(n,1), \end{array}$$

similarly for $\mathcal{CO}(S^1, M)$.

(2) If $\tilde{M}^*$ is the orbit space of S^1 then the action $(\pi, \tilde{M})$ induces an action of π on $\tilde{M}^*$. Let $(\pi^*, \tilde{M}^*)$ be its action. The induced map $\pi \to \pi^*$ is an isomorphism. Let $U(1) \to U(n,1) \to PU(n,1)$ be the exact sequence for the CR case. The projection P maps $\rho(\pi)$ isomorphically onto its image $\rho(\pi)^*$. The homomorphism ρ induces an isomorphism $\rho^* : \pi^* \to \rho(\pi)^*$ such that the diagram is commutative:

$$\begin{array}{ccc} \pi & \xrightarrow{\rho} & \rho(\pi) \\ \downarrow & & \downarrow \\ \pi^* & \xrightarrow{\rho^*} & \rho(\pi)^* \end{array} .$$

Definition 5.1.6. $R_{CR}(\pi^*)$ is the subspace of $\mathrm{Hom}(\pi^*, PU(n,1))$ such that for each element ρ^* there exists a homeomorphism $f^* : \mathbf{D}^{2n} \to \mathbf{D}^{2n}$ such that $\rho^*(\alpha) = f^* \circ \alpha \circ f^{*-1}$ $(\alpha \in \pi^*)$ and in addition the restriction $f^* | \mathbf{H}_{\mathbf{C}}^n$ is a smooth map. Note that $\rho^* : \pi^* \to \rho^*(\pi^*)$ is an isomorphism and $\rho^*(\pi^*)$ is discrete in $PU(n,1)$. $R_{CO}(\pi^*)$ is defined similarly to be the subspace of $\mathrm{Hom}(\pi^*, PO(n-1,1)^0)$.

Remark **5.1.7**. Given an isomorphism $\rho^* : \pi^* \to \rho^*(\pi^*) \subset \mathrm{Hom}(\pi^*, PU(n,1))$, it does not always exist such a homeomorphism $f^* : \mathbf{D}^{2n} \to \mathbf{D}^{2n}$. However, for example $n = 1$ ($PU(1,1) \approx PO(2,1) \approx \mathrm{PSL}_2(\mathbf{R})$), and ρ^* is type-preserving (cf. [p.302, 23]), then it is well known that there exists a quasiconformal homeomorphism $f^* : \mathbf{D}^2 \to \mathbf{D}^2$ which induces ρ^*. In this case the space $R_{CR}(\pi^*)$ is alternatively defined to be the set of those elements consisting of type-preserving discrete faithful representations of π^* into $PU(1,1)$. Note that $R_{CR}(\pi^*) \approx R_{CO}(\pi^*)$ in this case (cf. [26].)

Definition 5.1.8. Let $R_{CR}(\pi)$ be the subspace of $\mathrm{Hom}(\pi, U(n,1))$ whose elements project down to $R_{CR}(\pi^*)$. If we note the exact sequence, $\mathrm{Hom}(\pi, U(1)) \to \mathrm{Hom}(\pi, U(n,1)) \to \mathrm{Hom}(\pi^*, PU(n,1))$, then it follows that

$$R_{CR}(\pi) = R_{CR}(\pi^*) \times \mathrm{Hom}(\pi, U(1)). \tag{5.1.9}$$

Similarly,

$$R_{CO}(\pi) = R_{C0}(\pi^*) \times \mathrm{Hom}(\pi, SO(2)). \tag{5.1.10}$$

Lemma 5.1.11. $\widehat{\mathrm{hol}}$ *maps* $\Omega_0^{CR}(S^1, M)$ *into* $R_{CR}(\pi)$*, similarly for* $\Omega_0^{CO}(S^1, M)$.

Proof. Let (ρ, dev) be a representative element of $\Omega_0^{CR}(S^1, M)$. We know that $(\rho, \mathrm{dev}) : (S^1, \pi, \tilde{M}) \to (U(1), U(n,1), S^{2n+1} - L(\rho(\pi)))$ is homeomorphic. Then (ρ, dev) induces a homeomorphism

$$(\rho^*, \mathrm{dev}^*) : (\pi^*, \tilde{M}^*) \to (PU(n,1), \mathbf{D}^{2n} - L(\rho^*(\pi^*))).$$

Note from (ii) of (5.1.2) that $\tilde{M}^* = \mathbf{D}^{2n} - L(\rho^*(\pi^*))$. In particular $\mathrm{dev}^* : \mathbf{H}^n_{\mathbf{C}}$ $(= \mathrm{Int}\,\tilde{M}^*) \to \mathbf{H}^n_{\mathbf{C}}$ is homeomorphic. Since dev^* is still an immersion, the complete metric of $\mathbf{H}^n_{\mathbf{C}}$ with $\mathrm{Iso}(\mathbf{H}^n_{\mathbf{C}}) = PU(n,1)$ induces a Riemannian metric such that dev^* is a local isometry. And hence $\mathrm{dev}^* : \mathbf{H}^n_{\mathbf{C}} \to \mathbf{H}^n_{\mathbf{C}}$ is an isometry. The space $\tilde{M}^*$ has a compactification $\mathbf{D}^{2n} = \tilde{M}^* \cup L(\pi^*)$. The isometry dev^* extends to a homeomorphism $f^* : \mathbf{D}^{2n} \to \mathbf{D}^{2n}$ for which $f^*(L(\pi^*)) = L(\rho^*(\pi^*))$ and $\rho^*(\alpha) = f^* \circ \alpha \circ f^{*-1}$ $(\alpha \in \pi^*)$. It follows by the definition 5.1.7 that $\rho^* \in R_{CR}(\pi^*)$ and thus $\rho \in R_{CR}(\pi)$. Q.E.D.

The diagram (5.1.5) reduces to the following commutative one.

$$
\begin{array}{ccc}
\Omega_0^{CR}(S^1, M) & \xrightarrow{\widehat{\mathrm{hol}}} & R_{CR}(\pi) \\
\downarrow & & \downarrow \\
\mathcal{CR}(S^1, M) & \xrightarrow{\mathrm{hol}} & R_{CR}(\pi)/U(n,1).
\end{array}
\tag{5.1.12}
$$

Since $\mathrm{Hom}(\pi, U(1))$ is a k dimensional torus for some k, it follows from (5.1.9) that

(5.1.13) $R_{CR}(\pi)/U(n,1) = R_{CR}(\pi^*)/PU(n,1) \times T^k$.
(5.1.14) $R_{CO}(\pi)/SO(n-1,1)^0 \times SO(2) = R_{CO}(\pi^*)/SO(n-1,1)^0 \times T^k$.

5.2. Structure of deformation spaces $\mathcal{T}(S^1, M)$

There is the natural homomorphism $\varphi : \mathrm{Diff}(S^1, M) \to \mathrm{Out}(\Gamma)$. Note that Ker φ contains the subgroup $\mathrm{Diff}^0(S^1, M)$. Recall that there exists a right action of $\mathrm{Diff}(S^1, M)/\mathrm{Diff}^0(S^1, M)$ on $\mathcal{T}(S^1, M)$. We examine the structure of $\mathcal{T}(S^1, M)$ in terms of representation spaces, where $\mathcal{T}(S^1, M) = \mathcal{CR}(S^1, M)$ or $\mathcal{CO}(S^1, M)$.

Proposition 5.2.1. *Let*

$$\mathrm{hol} : \mathcal{CR}(S^1, M) \to R_{CR}(\pi^*)/PU(n,1) \times T^k$$

and

$$\mathrm{hol} : \mathcal{CO}(S^1, M) \to R_{CO}(\pi^*)/SO(n-1,1)^0)/SO(n-1,1)^0 \times T^k$$

be the holonomy map respectively. Put $G = \mathrm{Ker}\,\varphi/\mathrm{Diff}^0(S^1, M)$. *If the fundamental group* π *is torsionfree, then*

(1) hol *is surjective.*
(2) *Each fiber of* hol *consists of the* G*-orbit.*
(3) *There exists a neighborhood* U *for each point of* $\mathcal{T}(S^1, M)$ *such that* $\mathrm{hol}(U)$ *is open.*

Proof. We prove for the CR case. (1) Given $\rho \in R_{CR}(\pi)$, $\rho(\pi)$ is discrete in $U(n,1)$ and $L(\rho(\pi)) \subset S^{2n-1}$. Then the group $\rho(\pi)$ acts properly discontinuously on $S^{2n-1} - L(\rho(\pi))$. Since $\rho(\pi)$ is torsionfree, it acts freely. We obtain a spherical CR manifold $M(\rho) = S^{2n-1} - L(\rho(\pi))/\rho(\pi)$. It is noted that $U(1)$ acts on $M(\rho)$ by CR automorphisms. We then show that M is diffeomorphic to $M(\rho)$. For this, let ρ^* be an element of $R_{CR}(\pi^*)$ induced from ρ. There is a homeomorphism

$f^* : \mathbf{D}^{2n} \to \mathbf{D}^{2n}$ such that $\rho^*(\pi^*) = f^*\pi f^{*-1}$. If we note that $M(\rho)^* = \mathbf{D}^{2n} - L(\rho^*(\pi^*))/\rho(\pi^*)$ then the map f^* induces a homeomorphism $h^* : M^* \to M(\rho)^*$. Consider the following diagram (cf. (5.1.1));

$$\begin{array}{ccccc} & & \pi & \xrightarrow{\approx} & \pi^* \\ & & \downarrow & & \downarrow \\ S^1 & \longrightarrow & \tilde{M} - \tilde{F} & \longrightarrow & \mathrm{Int}\, M^*(= \mathbf{H}^n_{\mathbf{C}}) \\ \| & & \downarrow & & \downarrow \\ S^1 & \longrightarrow & M - F & \longrightarrow & \mathbf{H}^n_{\mathbf{C}}/\pi^*, \end{array}$$

where $\tilde{F} \approx F$ is the fixed point set of S^1. It follows that $M - F = \mathbf{H}^n_{\mathbf{C}}/\pi^* \times S^1$. The same is true for $M(\rho)$. Then we can find an equivariant homeomorphism $h_1 : M - F \to M(\rho) - F(\rho)$ which induces $h^*|\,\mathrm{Int}\, M^*$. Here $F(\rho)$ is the fixed point set of $U(1)$ in $M(\rho)$. Since $F \approx \partial M^* \xrightarrow{h^*} \partial M(\rho)^* \approx F(\rho)$, we can choose a homeomorphism $h_2 : F \to F(\rho)$ which covers h^*. Combining h_1 and h_2, it is easy to construct an equivariant homeomorphism $h : M \to M(\rho)$. Therefore M admits an S^1 invariant spherical CR structure which is mapped by hol to ρ. This proves (1).

(2) Suppose that hol $([\rho, \mathrm{dev}]) =$ hol $([\rho', \mathrm{dev}'])$. Then it follows that $\rho' = g \circ \rho \circ g^{-1}$ for some $g \in U(n,1)$. Since dev : $\tilde{M} \to S^{2n+1} - L(\rho(\pi))$, and dev$'$: $\tilde{M} \to S^{2n+1} - L(\rho'(\pi))$ are homeomorphisms, we can put $\tilde{f} = (\mathrm{dev}')^{-1} \circ g \circ \mathrm{dev}$. It is easy to see that $\tilde{f}$ induces an element $f \in \mathrm{Diff}(S^1, M)$ such that $\varphi(f) = 1$. Hence $(f) \in G$. By definition we have that $[\rho', \mathrm{dev}'] \circ (f) = [\rho, \mathrm{dev}]$.

(3) It follows from the Holonomy theorem 4.3.9 that there exists a neighborhood $\tilde{U}$ in $\Omega^{CR}_0(S^1, M)$ for which $\widehat{\mathrm{hol}}(\tilde{U})$ is open in $R(\pi, U(n,1))$. Let U be the image of $\tilde{U}$ in $\mathcal{CR}(S^1, M)$. Since vertical arrows are open maps in the diagram (5.1.6), we obtain that hol(U) is open. It can be shown similarly for $\mathcal{CO}(S^1, M)$. Q.E.D.

Corollary 5.2.2. *Suppose that* $\hat{\varphi} : \mathrm{Diff}(S^1, M)/\mathrm{Diff}^0(S^1, M) \to \mathrm{Out}(\pi)$ *is injective. Then* $\mathcal{CR}(S^1, M)$ *is homeomorphic to* $R_{CR}(\pi^*)/PU(n,1) \times T^k$ *(Similarly,* $\mathcal{CO}(S^1, M)$ *is homeomorphic to* $R_{CO}(\pi^*)/SO(n-1,1)^0 \times T^k$*).*

See [26] for examples of this Corollary. (Indeed, Ker $\varphi = \mathrm{Diff}^0(S^1, M)$ if $\dim M = 3$.) Recall that there exists a right action of $\mathrm{Diff}(S^1, M)/\mathrm{Diff}^0(S^1, M)$ on $\Omega(S^1, M)$ (cf. (4.3.4)). Let $G =$

$\operatorname{Ker}\varphi/\operatorname{Diff}^0(S^1, M)$ be as before. In order to study the action of G on $\Omega(S^1, M)$, we need the following lemma.

Lemma 5.2.3. *Suppose that π is not virtually solvable.*

(1) *$U(n,1)$ acts properly on $R_{CR}(\pi)$.*
(2) *$SO(n-1,1)^0 \times SO(2)$ acts properly on $R_{CO}(\pi)$.*

Proof. We prove (1). Recall that $R_{CR}(\pi) = R(\pi, U(n,1)) \approx R(\pi^*, PU(n,1)) \times T^k$. Let $P : R(\pi^*, PU(n,1)) \times T^k \to R(\pi^*, PU(n,1))$ be the projection. Given a compact subset K of $R(\pi^*, PU(n,1)) \times T^k$, put $K^* = P(K)$. Let $\zeta_{U(n,1)}(K) = \{g \in U(n,1) |\ g \cdot K \cap K \neq \emptyset\}$ where those elements of $U(n,1)$ act by conjugation on $R(\pi, U(n,1))$. Recall that $P : U(n,1) \to PU(n,1)$ is the projection with kernel isomorphic to $U(1)$. Then it follows that $\zeta_{U(n,1)}(K) \subset P^{-1}(\zeta_{PU(n,1)}(K^*)) \approx \zeta_{PU(n,1)}(K^*) \times U(1)$. Since $\zeta_{U(n,1)}(K)$ is a closed subset in $U(n,1)$, it suffices to show that $\zeta_{PU(n,1)}(K^*)$ is compact. By the hypothesis, $\pi \approx \pi^*$ is not virtually solvable. Then the set $R(\pi^*, PU(n,1))$ consists of stable representations in the sense of Johnson-Millson ([p.53, 24]). And so it follows from Proposition 1.1 ([24]) that $PU(n,1)$ acts properly on the subset $R(\pi^*, PU(n,1))$. Hence $\zeta_{PU(n,1)}(K^*)$ is compact.
(2) follows similarly when we note from Proposition 1.1 ([24]) that the set $R(\pi^*, SO(n-1,1)^0)$ consists of stable representations. Q.E.D.

Proposition 5.2.4. *Suppose that π is not virtually solvable. Let $G = \operatorname{Ker}\varphi/\operatorname{Diff}^0(S^1, M)$ be as before. Then G acts properly discontinuously on $\mathcal{T}(S^1, M)$ where $\mathcal{T}(S^1, M) = \mathcal{CR}(S^1, M)$ or $\mathcal{CO}(S^1, M)$.*

Proof. When K is a compact subset of $\mathcal{T}(S^1, M)$, it has only to be shown that $\zeta_G(K) = \{(f) \in G | K \circ (f) \cap K \neq \emptyset\}$ is compact. Suppose we have sequences $\{f_i\} \in G$ and $[\rho_i, \mathrm{dev}_i], [\rho'_i, \mathrm{dev}'_i] \in K$ such that $[\rho_i, \mathrm{dev}_i] \circ (f_i) = [\rho'_i, \mathrm{dev}'_i]$ where $\{[\rho_i, \mathrm{dev}_i]\}$ and $\{[\rho'_i, \mathrm{dev}'_i]\}$ converge to some $[\rho, \mathrm{dev}]$ and $[\rho', \mathrm{dev}']$ in K respectively. Then by the remark (1) of (5.1.5) there exists a sequence $\{g_i\} \in U(n,1)$ (resp. $SO(n-1,1)^0 \times SO(2)$) such that (i) $g_i \circ \mathrm{dev}'_i = \mathrm{dev}_i \circ \tilde{f}_i$, (ii) $g_i \circ \rho'_i \circ g_i^{-1} = \rho_i \circ \mu(\tilde{f}_i)$. Since each f_i lies in Ker φ, it follows that (ii)$'$ $g_i \circ \rho'_i \circ g_i^{-1} = \rho_i$. We note that $\{\rho_i\}, \{\rho'_i\} \in R_{CR}(\pi)$ (resp. $R_{CO}(\pi)$), and $\{\rho_i\}$ (resp. $\{\rho'_i\}$) $\to \rho$ (resp. ρ'). By Lemma 5.2.3, (ii)$'$ implies that the sequence $\{g_i\}$ converges to some $g \in U(n,1)$ (resp. $SO(n-1,1)^0 \times SO(2)$).

On the other hand, the maps dev_i, dev'_i induce homeomorphisms $\hat{\mathrm{dev}}_i : M \to S^m - L(\rho_i(\Gamma))/\rho_i(\Gamma)$, $\hat{\mathrm{dev}}'_i : M \to S^m - L(\rho'_i(\Gamma))/\rho'_i(\Gamma)$, where $m = 2n+1$ or n. Each g_i defines a homeomorphism $\hat{g}_i : S^m - L(\rho_i'(\Gamma))/\rho_i'(\Gamma) \to S^m - L(\rho_i(\Gamma))/\rho_i(\Gamma)$. Therefore we obtain from (i) that $f_i = (\hat{\mathrm{dev}}_i)^{-1}(\hat{g}_i \circ \hat{\mathrm{dev}}_i')$. Since M is compact, $(\hat{\mathrm{dev}})^{-1}(\hat{g} \circ \hat{\mathrm{dev}}')$ is also defined so that $\{f_i\} \to (\hat{\mathrm{dev}})^{-1}(\hat{g} \circ \hat{\mathrm{dev}}')$. Put $f = (\hat{\mathrm{dev}})^{-1}(\hat{g} \circ \hat{\mathrm{dev}}') : M \to M$. Since each $f_i \in \mathrm{Ker}\,\varphi$, it follows that f represents an element of G. Hence $\zeta_G(K)$ is compact. Q.E.D.

For example, $G = \mathrm{Ker}\,\varphi / \mathrm{Diff}^0(S^1, M)$ is trivial if dim $M = 3$ (cf. [26]). However in general there are examples in higher dimentions for which G is nontrivial. For them we have the following.

Proposition 5.2.5. *Suppose that π is not virtually solvable. Then G acts freely on $\mathcal{T}(S^1, M)$, where $\mathcal{T}(S^1, M) = \mathcal{CR}(S^1, M)$ or $\mathcal{CO}(S^1, M)$.*

Proof. We prove the case that $\mathcal{T}(S^1, M) = \mathcal{CR}(S^1, M)$. Suppose that $[\rho, \mathrm{dev}] \circ (f) = [\rho, \mathrm{dev}]$. Then there exists an element $g \in U(n,1)$ such that (1) $g \circ \mathrm{dev} = \mathrm{dev} \circ \tilde{f}$, (2) $g \circ \rho \circ g^{-1} = \rho \circ \mu(\tilde{f}) = \rho$. If ρ^* is the corresponding element in $R(\Gamma^*, PU(n,1))$ then (2) implies that (3) $g^* \circ \rho^* \circ g^{*-1} = \rho^*$ for $g^* \in PU(n,1)$. The group $\rho^*(\Gamma^*)$ acts invariantly in $\mathbf{H}^n_{\mathbf{C}}$. Suppose that $\rho^*(\Gamma^*)$ leaves invariant a totally geodesic subspace $\mathbf{H}^k_{\mathbf{C}}$ of $\mathbf{H}^n_{\mathbf{C}}$ for $1 \leqq k \leqq n$. Then $\rho^*(\Gamma^*)$ leaves S^{2k-1} invariant so that it belongs to the subgroup $\mathrm{Aut}_{CR}(S^{2n-1}, S^{2k-1}) = P(U(k,1) \times U(n-k))$. Let $Q : P(U(k,1) \times U(n-k)) \to PU(k,1)$ be the projection whose kernel is isomorphic to $U(n-k)$. We can assume that k is the smallest dimension. And so $Q(\rho^*(\Gamma^*))$ is Zariski-dense in $PU(k,1)$. The condition (3) implies that g^* leaves also S^{2k-1}. It implies that $g^* \in P(U(k,1) \times U(n-k))$. Then the element $Q(g^*)$ centralizes the group $Q(\rho^*(\Gamma^*))$ and so does its algebraic closure. Since the algebraic closure is $PU(k,1)$ by the above remark, $Q(g^*)$ must be the identity map.
In particular we obtain that $g^* \in U(n-k)$. As $U(n,1) = P(U(n,1) \times U(1))$, it follows that $g \in U(n-k) \times U(1)$ $(= P(\mathcal{Z}(k,1) \times U(n-k) \times U(1))$ $)$ where $\mathcal{Z}(k,1)$ is the center of $U(k,1)$. On the other hand, $\mathrm{dev} : \tilde{M} \to S^{2n+1} - L(\rho(\pi))$ is homeomorphic and by (1) it follows that $\tilde{f} = (\mathrm{dev})^{-1} \circ g \circ \mathrm{dev}$. It is noted that $L(\rho(\pi)) = L(\rho^*(\pi^*)) \subset S^{2k-1}$ and S^{2k-1} is the fixed point set of $U(n-k)$. We can choose a path c in $U(n-k)$ between g^* and the identity map. By the above remark there is a lift $\tilde{c}$ of the path c starting at g with its endpoint $\tilde{c}(1) \in U(1)$. Since dev is equivariant with respect to S^1 and $U(1)$ actions, we conclude that $\tilde{f}$ is isotopic to $\tilde{c}(1)$. It is easy to check that $\tilde{f}$ is isotopic to the identity map of $\tilde{M}$. Hence f belongs to $\mathrm{Diff}^0(S^1, M)$. That is, $(f) \equiv 1$ in G.

We can prove similary for the case that $\mathcal{T}(S^1, M) = \mathcal{CO}(SO(2), M)$.

Q.E.D.

Corollary 5.2.6. *Let M be a closed S^1 invariant spherical CR manifold of dimension $2n+1$ (resp. a closed S^1 invariant conformally flat n-manifold). Suppose that the orbit space M^* is a complex Kleinian orbifold $\mathbf{D}^{2n} - L(\pi^*)/\pi^*$ with nonempty boundary (resp. a Kleinian orbifold $\mathbf{D}^{n-1} - L(\pi^*)/\pi^*$ with nonempty boundary) and π^* is torsionfree.*

If $\pi_1(M)$ is not virtually solvable, then

(1) hol : $\mathcal{CR}(S^1, M) \to R_{CR}(\pi^*)/PU(n,1) \times T^k$ *is a covering map whose fiber is isomorphic to G.*

(2) hol : $\mathcal{CO}(S^1, M) \to R_{CO}(\pi^*)/SO(n-1,1)^0 \times T^k$ *is a covering map whose fiber is isomorphic to G.*

Proof. The group G acts properly discontinuously and freely on $\mathcal{T}(U(1), M)$ by Lemma 5.2.3 and Proposition 5.2.4. Thus there exists a neighborhood U in $\mathcal{T}(U(1), M)$ such that $U \circ g \cap U = \emptyset$ if and only if $g \neq 1$ for $g \in G$. Then the result follows from Proposition 5.2.1.

Q.E.D.

References

[1] B. Apanasov, Two kinds of deformation of conformal structure which are distinct from bending, Proceedings of the Thirteenth Rolf Nevalinna Colloquium and Teichmüller theory Conference, Helsinki (1987).

[2] ———, Deformations of conformal structures on hyperbolic manifolds, (1990), to appear.

[3] W. Ballmann, M. Gromov and V. Schroeder, "Manifolds of nonpositive curvature", Progress in Math. 61, Birkhäuser, 1985.

[4] A. Beardon, "The geometry of discrete subgoups", Springer, New York, 1983.

[5] L. Bers, On boundaries of Teichmüller spaces and on Kleinian groups I, Ann. of Math., **91** (1970), 570–600.

[6] G. Bredon, "Introduction to compact transformation groups", Academic Press, New York, 1972.

[7] D.M. Burns and S. Schnider, Spherical hypersurfaces in complex manifolds, Invent. Math., **33** (1976), 223–246.

[8] R. Canary, D.B.A. Epstein and P. Green, Notes on notes of Thurston, London Mathematical Society Lecture Notes, **111** (1987), 3–92, Cambridge University Press.

[9] A. Casson and S. Bleiler, "Automorphisms of surfaces after Nielsen and Thurston", London Mathematical Society Student Texts 9, Cambridge University Press, 1988.

[10] P.J. Cheeger and D. Ebin, "Comparison Theorems in Riemannian Geometry", North-Holland Publ. Co., Amsterdam, 1975.

[11] S. Chen and L. Greenberg, Hyperbolic Spaces, Contribution to Analysis (1974), 49–87, Academic Press, New York.

[12] P.E. Conner and F. Raymond, Injective operations of the toral groups, Topology, **10** (1971), 283–296.

[13] D.B.A. Epstein and A. Marden, Convex hulls in hyperbolic spaces, a theorem of Sullivan,and measured pleated surfaces, London Mathematical Society Lecture Notes, **111** (1987), 114–253, Cambridge University Press.

[14] D. Fried, Closed similarity manifolds, Comment. Math. Helv., **55** (1980), 576–582.

[15] D. Gallo, W. Goldman and R. Porter, Projective structures with monodromy in $PSL(2, \mathbf{R})$, Preprint (1989).

[16] W. Goldman, Projective structures with Fuchsian holonomy, J. Diff. Geom., **25** (1987), 297–326.

[17] ——, Geometric Structures on manifolds and varieties of representations, Contemporary Math., **74** (1988), Proceedings of the Conference on Representation Theory.

[18] W. Goldman and Y. Kamishima, Topological rigidity of developing maps with applications to conformally flat structures, Contemporary Math. A.M.S., **74** (1988), 199–203.

[19] ——, Conformal automorphisms and conformally flat manifolds, Trans. A.M.S., **323** (1991), 797–810.

[20] W. Goldman and J. Millson, The deformation theory of representations of fundamental groups of compact Kähler Manifolds, Publ. I.H.E.S., to appear.

[21] W. Goldman and J. Millson, Local rigidity of discrete groups acting on complex hyperbolic space, Invent. Math., **88** (1987), 495–520.

[22] R. Gunning, Special coordinate coverings of Riemann surfaces, Math. Ann., **170** (1967), 67–85.

[23] W. Harvey, "Discrete groups and automorphic functions", Academic Press, 1972.

[24] D. Johnson and J. Millson, Deformation spaces associated to compact hyperbolic manifolds, Discrete groups in Geometry and Analysis, Progress in Math, **67** (1987), 48–106.

[25] Y. Kamishima, Conformally flat manifolds whose development maps are not surjective, Trans. A.M.S., **294** (1986), 607–623.

[26] ——, Conformally flat 3-manifolds with compact automorphism groups, J. London Math. Soc. 38(2) (1988), 367–378.

[27] Y. Kamishima and T. Tusboi, *CR*-structures on Seifert manifolds, Invent. Math., **104** (1991), 149–163.

[28] M. Kapovich, Flat conformal structures on 3-manifolds, Preprint (1989).
[29] ———, Deformation spaces of flat conformal structures, Preprint (1989).
[30] C. Korouniotis, Deformations of hyperbolic manifolds, Math. Proc. Cambridge Phil. Soc., **98** (1985), 247–261.
[31] I. Kra, A generalization of a theorem of Poincaré, Proc. of A.M.S., **27** (1971), 299–302.
[32] R. Kulkarni, On the principle of uniformization, J. Diff. Geom., **13** (1978), 109–138.
[33] ———, Groups with domains of discontinuity, Math. Ann., **237** (1978), 253–272.
[34] R. Kulkarni and U. Pinkall, Uniformization of geometric structures with applications to conformal geometry, Proceedings, Diff. Geom. Peñscola , Lecture notes in Math., **1209** (1986), 140–209, Springer-Verlag.
[35] ———, A canonical metric for Möbius structures and its applications, Preprint.
[36] R. Kulkarni and K.B. Lee and F. Raymond, Deformation spaces for Seifert manifolds, Proceedings, Special year in Geometry and Topology, Univ. of Maryland, Lecture notes in Math., **1167** (1985), 180–216, Springer-Verlag.
[37] L. Lok, Deformations of locally homogeneous spaces and Kleinian groups, Doctoral dissertation (1984), Columbia University.
[38] A. Marden, The geometry of finitely generated Kleinian groups, Ann. of Math., **99** (1974), 383–462.
[39] S. Matsumoto, Foudations of flat conformal structure, in these proceedings.
[40] J. Morgan and P. Shalen, Valuations, trees, and degenerations of hyperbolic structures, I, Ann. of Math, **120** (1984), 401–476.
[41] ———, "Degenerations of hyperbolic structures, II: Measured Laminations,Trees and 3-Manifolds", Preprint, 1986.
[42] P. Orlik, Seifert manifolds, Lecture Notes in Math., **291** (1972), Springer-Verlag.
[43] P. Orlik and F. Raymond, Actions of $SO(2)$ on 3-manifolds, Proceedings of the Conference on Transformation groups, Lecture Notes in Math., **299** (1968), 297–318, Springer-Verlag.
[44] F. Raymond, Classification of the actions of the circle on 3-manifolds, Trans. Amer. Math. Soc., **31** (1968), 51–78.
[45] G.P. Scott, The geometries of 3-manifolds, Bull. London Math., **15** (1983), 401–487.
[46] D. Sullivan and W. Thurston, Manifolds with canonical coordinates:Some exapmles, L'ens. Math., **29** (1983), 15–25.
[47] Ser P. Tan, Representations of surfaces into $\mathrm{PSL}(2, \mathbf{R})$ and Geometric structures, Doctoral dissertation (1988), University of Maryland.
[48] W. Thurston, "The geometry and topology of three manifolds", preprint.
[49] J. Wolf, "Spaces of constant curvature", McGraw-Hill, Inc., 1967.

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