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Foundations of Flat Conformal Structure

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Dedicated to Professor Masahisa Adachi

Introduction

A flat conformal structure on an n-dimensional manifold N is a maximal system of local charts taking values on S^n, with transition functions Moebius transformations. In short it is a geometric structure modelled on $(\mathcal{M}(S^n), S^n)$, where $\mathcal{M}(S^n)$ denotes the group of Moebius transformations on S^n. Equivalently, it is a conformal equivalence class of conformally flat Riemannian metrics on N if $n \geq 3$. See §1 for Liouville's theorem. By certain abuse we denote a flat conformal structure by the same letter as the underlying manifold.

In dimension 2, flat conformal structures are usually called projective structures and have been extensively studied by various authors in the field of function theory. Analytic methods such as the theory of quasiconformal maps often play crucial roles there. In dimension ≥ 3, however, the situation is quite different. Topology, instead of analysis, provides major tools of study.

The concept of flat conformal structures was first introduced by Kuiper ([35],[36],[37]) around 1950. Thereafter it had been forgotten for some time, until it was revived by Kulkarni ([40],[41],[42],[43]), related with his study of discrete group actions in general. Then came an important turning point when Fried ([13]) established a remarkable theorem concerning closed similarity manifolds. It solved a fundamental and annoying problem which one encounters in the primary stage of the theory, thereby making it possible to have a good grip on elementary flat conformal structures, with Goldman ([15]) and Kamishima ([25]) contributing significantly to this direction.

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At the same time various interesting examples have been piled up by many authors including Thurston [56], Bestvina-Cooper [4], Freedman-Skora [10], Gromov-Lawson-Thurston [19], Kuiper [38] and, quite recently,
Kapovich-Potyagailo [32], making the field even more active.

This article has two objectives. One is to provide the basic knowledge of flat conformal structures and to serve as an introductory guide of the field. The other is to show some new pieces of knowledge. §1 $\sim$ §3 are devoted to the former purpose, where the reader can find exposition of fundamental properties of Moebius transformations and flat conformal structures. No original results are included in these early sections. However for the full understanding of later sections, they are helpful, or even indispensable.

§4 and §5 are also mainly expository, though they include some slightly improved (new) results. Hereafter let N be a connected closed flat conformal manifold of dimension ≥ 3. In §4, we prove the following version of Fried's theorem.

Theorem (4.4). *If the holonomy group of N has a fixed point in S^n, then N is either S^n, an Euclidean space form or a Hopf manifold.*

Unlike the original theorem ([13]), we no longer postulate that the developing map misses the fixed point. This yields clearer understanding of the limit set (§5) and a wider range of applications. Using Theorem (4.4), various results (mostly known) can be proved by elementary and straightforward arguments. Although the proof of Theorem (4.4) is nothing but a small modification of the argument in [13], it might be worth while to record it. The same result was obtained independently by R. Miner [58], who mainly worked in the context of spherical CR structures.

In §5, we define the limit set $L(N)$ of a flat conformal manifold N. Five different ways are possible and in Theorem (5.18), they are shown to coincide eventually. Especially we get that the limit set defined by means of the holonomy group is identical to the one obtaind by looking at the behaviour of the developing map. (Most of these facts are already known to Kulkarni-Pinkall [43].) As immediate corollaries we have the followings.

Corollary (5.23). *If the developing map of N is not onto S^n, then it is a covering map onto its image.*

Corollary (5.24). *Suppose the following* (1) *and* (2).

(1) $S^n \setminus L(N)$ *is connected and the fundamental group* $\pi_1(S^n \setminus L(N))$ *is finitely generated.*

(2) *For any point* $x \in L(N)$, *there exists an arbitrarily small neighbourhood* U *of* x *such that* $U \setminus L(N)$ *is connected.*

Then the developing map is a covering map onto $S^n \setminus L(N)$.

In dimension 2, Corollary (5.23) is well known and easy to show using hyperbolic metric. For higher dimension, it was first proved by Kamishima. Again our method is short and straightforward. Corollary (5.24) can be found in Kulkarni-Pinkall [45], where condition (2) is mistakingly dropped. In §5, we also characterize those flat conformal manifolds whose developing maps are covering maps (onto the images) and whose holonomy groups are indiscrete. (Theorem (5.26).) In dimension 3, this was first obtained by Kamishima ([24]) and independently by Gusevskiǐ-Kapovich ([20]) in dimension 3.

N is called elementary if the limit set is finite. N is called a C-structure if it is a connected sum of elementary structures and is not itself elementary. In dimension 3, we have the following result.

Theorem (6.12). *Suppose* $\dim(N) = 3$. *Then* N *is a C-structure if and only if the limit set* $L(N)$ *is a tame Cantor set.*

Recall that a Cantor set Υ in S^n is called tame if there exists a self homeomorphism of S^n which carries Υ into S^1. Otherwise it is called wild.

The above theorem is proved along the argument of Kulkarni ([43]), in which Stalling's theorem ([54],[55]) concerning ends of groups plays a central part. The theory of ends are summarized in the appendix for the convenience of the reader.

After preparing Poincaré's polyhedral theorem in §7 (in the framework of flat conformal manifolds), we shall show the following theorem in §8.

Theorem (8.1). *There exists a flat conformal manifold* N *of dimension* 3 *whose limit set* $L(N)$ *is a wild Cantor set.*

This theorem is an improvement of the work of Bestvina-Cooper ([4]) who constructed such examples for open 3-manifolds. Our example in Theorem (8.1) is compact.

Literature concerning flat conformal structures is extensively collected in the reference, though not complete, of course.

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CONTENTS

§1. Conformal map and Liouville's theorem

In this section, we give definitions of a conformal map and a Moebius transformation of the n-sphere. After providing fundamental properties, we show that a locally defined conformal map is the restriction of a Moebius transformation if $n \geq 3$. (Liouville's theorem.)

Definition (1.1). A real $n \times n$ matrix A is called a *conformal matrix* if $A = \lambda P$ for $\lambda > 0$ and an orthogonal matrix P.

Thus A is conformal precisely when A preserves the angle of given two vectors. Notice that the products and the inverses of conformal matrices are again conformal.

Let $\widehat{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$ be the one point compactification of $\mathbf{R}^n$. Points in $\widehat{\mathbf{R}}^n$ is indicated by letters a, x and so forth. For $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$,

$$|x| = (\sum_{i=1}^{n} x_i^2)^{1/2}$$

denotes the Euclidean norm of x. To endow $\widehat{\mathbf{R}}^n$ the structure of an oriented manifold, the following local charts (U_i, q_i) are commonly used $(i = 1, 2)$.

$$\begin{cases} U_1 = \mathbf{R}^n, & q_1 = id : U_1 \longrightarrow \mathbf{R}^n, \\ U_2 = \widehat{\mathbf{R}}^n \setminus \{0\}, & q_2 : U_2 \longrightarrow \mathbf{R}^n, \end{cases}$$

where q_2 is defined by

$$q_2(x_1, \ldots, x_n) = \frac{1}{|x|^2}(x_1, \ldots, x_{n-1}, -x_n).$$

In the above definition and in all that follows, if the image of ∞ by a map is clear by the continuity, we do not explicitly state it. An important property of q_2 is that the differential matrix $D_a q_2$ at any point $a \in \mathbf{R}^n \setminus \{0\}$ is a conformal matrix. Verification is left to the reader.

Let U be a domain (i.e. a connected open subet) of $\widehat{\mathbf{R}}^n$.

Definition (1.2). A C^1 map $f : U \to \widehat{\mathbf{R}}^n$ is called a *conformal map* if the following condition is satisfied. For any $a \in U$, if $a \in U_i$ and $f(a) \in U_j$, then the differential $D_{q_i(a)}(q_j \circ f \circ q_i^{-1})$ is a conformal matrix.

Since for any $b \in \mathbf{R}^n \setminus \{0\}$, $D_b(q_2 \circ q_1^{-1})$ is a conformal matrix, Definition (1.2) is invariant under possible changes of local charts around a and $f(a)$. A conformal map is a submersion and thus has a local inverse, which is again a conformal map. Also the composite of two conformal maps is conformal.

Lemma (1.3). *Suppose $f : U \to \widehat{\mathbf{R}}^n$ is a C^1 submersion, where U is a domain of $\widehat{\mathbf{R}}^n$. If $D_a f$ is a conformal matrix for any $a \in U \cap \mathbf{R}^n \cap f^{-1}(\mathbf{R}^n)$, then f is a conformal map.*

Proof. This follows at once from the fact that the conformal matrices form a closed subset in the general linear group. Q.E.D.

Let us give examples of conformal maps. Let $0 < p < n$. By a *dimension p sphere* in $\widehat{\mathbf{R}}^n$, we mean either a dimension p metric sphere in $\mathbf{R}^n$ or a dimension p plane in $\mathbf{R}^n$ plus $\{\infty\}$. A dimension p sphere is sometimes called a codimension $n - p$ sphere.

Definition (1.4). Let σ be a codimension one sphere in $\widehat{\mathbf{R}}^n$. The *inversion at σ*

$$J_\sigma : \widehat{\mathbf{R}}^n \longrightarrow \widehat{\mathbf{R}}^n$$

is defined as follows.

(1) If σ is the sphere of radius r centered at a, then for any $x \in \mathbf{R}^n \setminus \{a\}$,

$$J_\sigma(x) = \frac{r^2}{|x-a|}(x-a) + a.$$

(2) If σ contains a codimension one plane, J_σ is the reflexion at that plane.

See Figure (1.1). The inversion is an orientation reversing involution with the fixed point set σ.

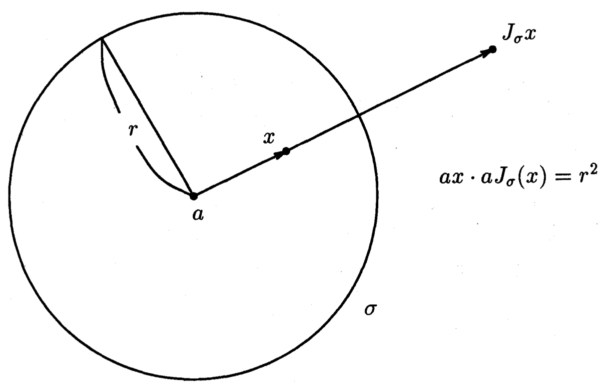

Figure (1.1)

Definition (1.5). Composite of inversions is called a *Moebius transformation.* The group of all the Moebius transformations of $\widehat{\mathbf{R}}^n$ is denoted by $\mathcal{M}(\widehat{\mathbf{R}}^n)$.

Proposition (1.6). *Moebius transformation is a conformal map and carries a sphere in* $\widehat{\mathbf{R}}^n$ *to a sphere of the same dimension.*

Proof. Computaion shows that an inversion is a conformal map. Also it is well known, very easy to show by Euclidean geometry, that an inversion maps a codimension one sphere to a codimension one sphere. Therefore a sphere of arbitrary dimension, the intersection of several codimension one spheres, is mapped to a sphere of the same dimension. The proposition follows from this. Q.E.D.

Proposition (1.7). *The following maps are Moebius transformations.*

(a) *Translation by* a, $x \mapsto x + a$.

(b) *Magnification by* $\lambda > 0$, $x \mapsto \lambda x$.

(c) *Orthogonal transformation by* $P \in O(n)$, $x \mapsto Px$.

Proof. Translation is the composite of two inversions at parallel planes. This shows (a). Likewise positive magnification is the composite of two inversions at concentric spheres and orthogonal transformation is the composite of several inversions at planes through 0, showing (b) and (c). Q.E.D.

Lemma (1.8). *Let* $f : \widehat{\mathbf{R}}^n \to \widehat{\mathbf{R}}^n$ *be a Moebius transformation. If* $f(0) = 0$, $f(\infty) = \infty$, $D_0 f = E$, *then* $f = \mathrm{id}$.

Proof. Moebius transformations carry circles to circles. Since f keeps 0 and ∞ fixed, f preserves the (singular) dimension one foliation $\mathcal{L}$ formed by the straight lines through 0. Since f is a conformal map, f also preserves the codimension one foliation $\mathcal{L}^{\perp}$ of spheres centered at 0. See Figure (1.2). Notice also that f keeps the leaf of $\mathcal{L}$ invariant, since $D_0 f = E$. Thus we obtain

$$f(x) = \frac{R}{r} x.$$

on the sphere $|x| = r$. The conformality of f implies

$$\frac{dR}{dr} = \frac{R}{r}.$$

Therefore we have $R = ar$. But $a = 1$ since $D_0 f = E$. This shows $f = \mathrm{id}$. Q.E.D.

Proposition (1.9).

(1) f *is a Moebius transformation such that* $f(\infty) = \infty$ *if and only if*

$$f(x) = Ax + b.$$

(2) f *is a Moebius transformtion such that* $f(\infty) \neq \infty$ *if and only if*

$$f(x) = AJ(x - b) + c.$$

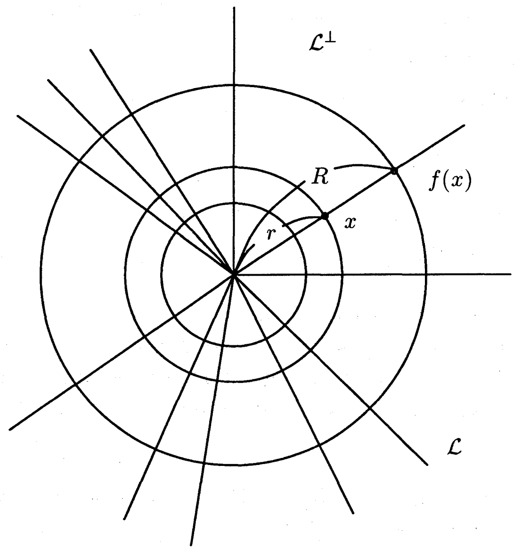

Figure (1.2)

Here A *is a conformal matrix,* b *and* c *are points of* $\mathbf{R}^n$ *and* J *is the inversion at the unit sphere* $\{|x| = 1\}$.

Proof. It is a direct consequence of Lemma (1.7) that the transformations of the above expressions are Moebius transformations. Conversely suppose that f is a Moebius transformation with $f(\infty) = \infty$. Let $f(0) = b$ and $D_0 f = A$. Define $g(x) = Ax + b$. Then $g^{-1} \circ f$ satisfies the hypothesis of Lemma (1.8). Thus $g = f$. This completes the proof of (1). On the other hand, suppose that f is a Moebius transformation with $f(\infty) \neq \infty$. Let $f(b) = \infty$. Define h by $h(x) = J(x - b)$. Then $f \circ h^{-1}$ is a Moebius transformation which keeps ∞ fixed. By (1), we have

$$f \circ h^{-1}(x) = Ax + c.$$

This completes the proof of (2). Q.E.D.

We shall finish this section with the following celebrated theorem of Liouville.

Theorem (1.10). *Let* $n \geq 3$. *Suppose* $f : U \to \widehat{\mathbf{R}}^n$ *is a conformal map, where* U *is a domain of* $\widehat{\mathbf{R}}^n$. *Then* f *is the restriction of a Moebius transformation.*

As a matter of fact, this theorem does not hold for $n = 2$. In fact the Riemann mapping theorem asserts the abundance of conformal maps which are not restrictions of Moebius transformations.

Theorem (1.10) was first proved by J. Liouville in his 1850 paper ([46]), under the additional assumption that f be of class C^3. Since then, it had been an open problem, astonishingly difficult, to weaken the differentiability assumption, until at last in 1969, P. Hartman gave a complete proof for C^1 maps ([21]).

Independently, F.W. Gehring, among others, developed the theory of quasiconformal maps in dimension ≥ 3. Specifically he defined 1-quasiconformal maps, which is a genaralization of conformal maps, where no differentiability assumption is made. In [14], Gehring showed that a locally defined 1-quasiconformal map is the restriction of a Moebius transformation.

However these results need involvement in deep general treatment and cannot be collected here. Instead, we give a simple elementary proof essentially due to R. Nevanlinna ([49]) assuming that the given conformal map f is C^3. (Nevanlinna postulated that f is C^4.)

Proof of Theorem (1.10). We use the following convention. x_i denotes the i-th coordinate of $\mathbf{R}^n$ and for $f : U \to \widehat{\mathbf{R}}^n$, f_{x_i}, $f_{x_i x_j}$ and so forth denote the first and the second partial derivatives and so forth. They are vectors of $\mathbf{R}^n$. In the first place, since f is conformal, we have

$$(f_{x_i}, f_{x_j}) = r^2 \delta_{ij},$$

where $r(x) = \|D_x f\|$ is the mapping norm of the Jacobi matrix. Differentiating by x_k, we get for $i = j$,

$$(f_{x_i x_k}, f_{x_i}) = r r_{x_k}$$

and for $i \neq j$,

$$(f_{x_i x_k}, f_{x_j}) + (f_{x_i}, f_{x_j x_k}) = 0.$$

For mutually distinct indices i, j and k, by permuting the indices, we have

$$(f_{x_i x_k}, f_{x_j}) = 0.$$

Since j can be any index except i and k and $f_{x_1}, \ldots, f_{x_n}$ are mutually orthogonal, we have

$$f_{x_i x_k} = \mu f_{x_i} + \nu f_{x_k},$$

where

$$\mu = (f_{x_i x_k}, f_{x_i})/r^2 = r_{x_k}/r$$

$$\nu = r_{x_i}/r.$$

Letting $\rho = 1/r$, we have

$$\rho f_{x_i x_k} + \rho_{x_i} f_{x_k} + \rho_{x_k} f_{x_i} = 0.$$

Differentiating by x_j, we obtain

$$\rho f_{x_i x_j x_k} + \rho_{x_j} f_{x_i x_k} + \rho_{x_i} f_{x_j x_k} + \rho_{x_k} f_{x_i x_j}$$

$$+\rho_{x_i x_j} f_{x_k} + \rho_{x_j x_k} f_{x_i} = 0.$$

By permutation of the indices, we obtain for $j \neq k$,

$$\rho_{x_j x_k} = 0.$$

By rotating the coordinates by 45 degrees in the (x_j, x_k)-plane, we have

$$\rho_{x_j x_j} = \rho_{x_k x_k}.$$

Now since $\rho_{x_j x_k} = 0$ for any $k \neq j$ ρ_{x_j} is constant on the hyperplane $\{x_j = c\}$. Thus it follows that $\rho_{x_j x_j}$ is constant on $\{x_j = c\}$. That is, $\rho_{x_1 x_1} = \cdots\cdots = \rho_{x_n x_n}$ is constant in U.

By composing f with a suitable Moebius transformation if necessary, we may assume that $0 \in U$ and $f(0) = \infty$. Then the image by f of an arbitrarily small ball $|x| < \varepsilon$ contains $|x| > K$ for some large $K > 0$. By the volume formula, this implies that $\rho(a_m) \to 0$ for some sequence $a_m \to 0$. On the other hand, since $\rho_{x_i x_j} = 2\alpha\delta_{ij}$ for some $\alpha > 0$, ρ is a quadratic function on $U \setminus \{0\}$, with the leading term $\alpha|x|^2$. Since ρ is positive valued on $U \setminus \{0\}$ and $\rho(a_m) \to 0$, we have

$$\rho(x) = \alpha|x|^2.$$

Notice that the same value of ρ is also attained by the inversion g which is defined by

$$g(x) = \frac{x}{\alpha|x|^2}.$$

Thus by the chain rule, the composite $h = g \circ f^{-1} : f(U) \to \widehat{\mathbf{R}}^n$ satisfies $\|D_p h\| = 1$ for any $p \in f(U) \setminus \{\infty\}$. That is, h is an isometry with respect to the Euclidean metric on $\mathbf{R}^n$. This implies that $h(x) = Px+b$ for some orthogonal matrix P and $b \in \mathbf{R}^n$. In fact, all that needs proof is that h is an affine transformation. But since

$$(h_{x_i}, h_{x_k}) = \delta_{ij},$$

by differentiating we get

$$(h_{x_i x_j}, h_{x_k}) = 0,$$

showing that $h_{x_i x_j} = 0$. This implies that h is an affine transformation. Thus h and hence f are the restrictions of Moebius transformations, as is required. Q.E.D.

§2. More on Moebius transformation

Denote by $\mathcal{M}(\widehat{\mathbf{R}}^n)$ the group of Moebius transformations of $\widehat{\mathbf{R}}^n$.

Lemma (2.1). *Let $f \in \mathcal{M}(\widehat{\mathbf{R}}^n)$ and let $\sigma \subset \widehat{\mathbf{R}}^n$ be a codimension one sphere. Then,*

$$f \circ J_\sigma \circ f^{-1} = J_{f(\sigma)}.$$

Proof. Clearly $g = f \circ J_\sigma \circ f^{-1} \circ J_{f(\sigma)}$ is an orientation preserving Moebius transformation which keeps points in $f(\sigma)$ fixed. Thus for an arbitrary Moebius transformation h such that $h(f(\sigma)) = \{x_n = 0\}$, we have that $k = h \circ g \circ h^{-1}$ keeps $\{x_n = 0\}$ pointwise fixed. Especially we obtain that $k(0) = 0$, $k(\infty) = \infty$ and $D_0 k = E$ since k is orientation preserving. Therefore by (1.8), we obtain $k = \mathrm{id}$. This shows (2.1). Q.E.D.

Let $\iota : \widehat{\mathbf{R}}^n \to \widehat{\mathbf{R}}^{n+1}$ be the standard embedding, i.e.,

$$\iota(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0).$$

As usual $\widehat{\mathbf{R}}^n$ is considered to be a subset of $\widehat{\mathbf{R}}^{n+1}$ by ι. Let σ be an $(n-1)$-dimensional sphere in $\widehat{\mathbf{R}}^n$. Then the inversion $J_\sigma : \widehat{\mathbf{R}}^n \to \widehat{\mathbf{R}}^n$ can be extended to the inversion $J_\tau : \widehat{\mathbf{R}}^{n+1} \to \widehat{\mathbf{R}}^{n+1}$ at the n-dimensional sphere τ orthogonal to $\widehat{\mathbf{R}}^n$ such that $\widehat{\mathbf{R}}^n \cap \tau = \sigma$. This yields an injection.

$$i : \mathcal{M}(\widehat{\mathbf{R}}^n) \to \mathcal{M}(\widehat{\mathbf{R}}^{n+1}).$$

Again $\mathcal{M}(\widehat{\mathbf{R}}^n)$ is considered to be a subgroup of $\mathcal{M}(\widehat{\mathbf{R}}^{n+1})$ by i.

On the other hand let

$$S^n = \{x \in \mathbf{R}^{n+1} \mid |x| = 1\}.$$

Let τ be an n-dimensional sphere in $\widehat{\mathbf{R}}^{n+1}$ which is perpendicular to S^n. Since inversions are conformal maps which send spheres to spheres, J_τ is a transformation which keeps S^n invariant. Composites of such inversions constitute a Lie group $\mathcal{M}(S^n)$ of *Moebius transformations of* S^n. Denote the inclusion by

$$j : \mathcal{M}(S^n) \longrightarrow \mathcal{M}(\widehat{\mathbf{R}}^{n+1}).$$

Define $v \in \mathcal{M}(\widehat{\mathbf{R}}^{n+1})$ by $v = T \circ J_2 \circ J_1$. where J_1 is the reflexion at the plane $x_{n+1} = -1/2$, J_2 is the inversion at the sphere $|x| = 2$ and T is the translation by $(0, \ldots, 0, 1)$. See Figure (2.1).

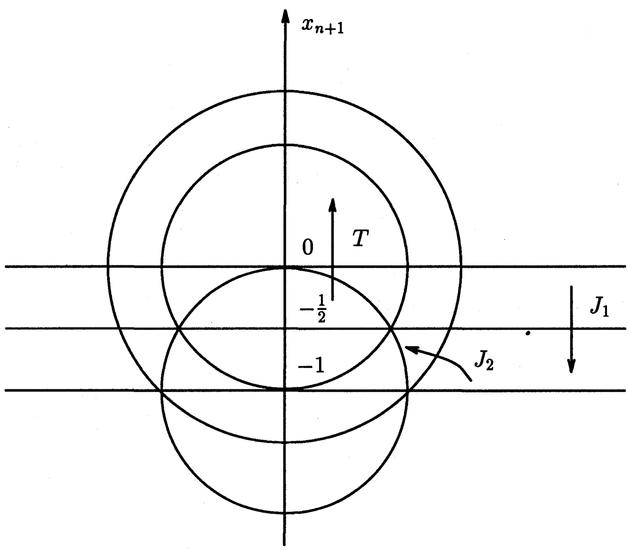

Figure (2.1)

Notice that $v(\widehat{\mathbf{R}}^n) = S^n$. Define

$$c_v : \mathcal{M}(\widehat{\mathbf{R}}^{n+1}) \to \mathcal{M}(\widehat{\mathbf{R}}^{n+1})$$

by

$$c_v(f) = v \circ f \circ v^{-1}.$$

Proposition (2.2). *c_v maps the subgroup $\mathcal{M}(\widehat{\mathbf{R}}^n)$ isomorphically onto the subgroup $\mathcal{M}(S^n)$.*

Proof. v maps an n-sphere τ perpendicular to $\widehat{\mathbf{R}}^n$ to the n-sphere $v(\tau)$ which is perpendicular to S^n. On the other hand it follows from (2.1) that $c_v(J_\tau) = J_{v(\tau)}$. This shows (2.2). Q.E.D.

Let

$$D^{n+1} = \{x \in \mathbf{R}^{n+1} \mid |x| < 1\},$$
$$H^{n+1} = \{x \in \mathbf{R}^{n+1} \mid x_{n+1} > 0\}.$$

Proposition (2.3). *We have*

$$\mathcal{M}(\widehat{\mathbf{R}}^n) = \{f \in \mathcal{M}(\widehat{\mathbf{R}}^{n+1}) \mid f(H^{n+1}) = H^{n+1}\},$$

$$\mathcal{M}(S^n) = \{f \in \mathcal{M}(\widehat{\mathbf{R}}^{n+1}) \mid f(D^{n+1}) = D^{n+1}\}.$$

Proof. By virtue of (2.2), it suffices to show the statement only for $\widehat{\mathbf{R}}^n$. (Notice that $v(H^{n+1}) = D^{n+1}$.) The inclusion $\subset$ is clear. Conversely, suppose that $f \in \mathcal{M}(\widehat{\mathbf{R}}^{n+1})$ satisfies that $f(H^{n+1}) = H^{n+1}$. First of all, consider the case where $f(\infty) = \infty$. Then by (1.9), $f(x) = \lambda P x + b$, where $\lambda > 0$, $P \in O(n+1)$ and $b \in \mathbf{R}^{n+1}$. Since $f(\mathbf{R}^n) = \mathbf{R}^n$, we have that $b \in \mathbf{R}^n$. Further since f preserves H^{n+1}, we also obtain that

$$P = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix},$$

where $Q \in O(n)$. Thus it follows from (1.7) that $f \in \mathcal{M}(\widehat{\mathbf{R}}^n)$. The remaining case can easily be reduced to this case. Details are left to the reader. Q.E.D.

We need some standard terminologies in geometry.

Definition (2.4). Two Riemannian metrics g_1 and g_2 on a manifold M are said to be *conformally equivalent* , if there exists a positive valued function μ on M such that $g_2 = \mu g_1$.

Definition (2.5). A C^1 map $f : (M_1, g_1) \to (M_2, g_2)$ of Riemannian manifolds is called a *Riemannian conformal map* if the induced metric $f^* g_2$ is conformally equivalent to g_1.

Riemannian conformal maps are usually called conformal maps in the literature. However in order to avoid confusion with Definition (1.2), we call them Riemannian conformal maps in this article. The following three Riemannian metrics are important in what follows.

Definition (2.6). Denote by g_E the *Euclidean metric* on $\mathbf{R}^n$, i.e. $g_E = \sum_{i=1}^n dx_i^2$, by g_S the *spherical metric* on S^n, that is, the restriction of the Euclidean metric on $\mathbf{R}^{n+1}$ to the submanifold S^n and by g_H the *hyperbolic metric* on D^n, i.e.,

$$g_H = \frac{4\sum_{i=1}^n dx_i^2}{(1-|x|^2)^2}.$$

It is well known that g_S has constant sectional curvature 1 and that (D^n, g_H) is a complete Riemannian manifold with constant sectional curvature -1.

Proposition (2.7). Let U be a domain in $\widehat{\mathbf{R}}^n$. A C^1 map $f : U \to \widehat{\mathbf{R}}^n$ is a conformal map in the sense of Definition (1.2) if and only if $v \circ f \circ v^{-1} : v(U) \to S^n$ is a Riemannian conformal map w.r.t. the spherical metric.

Proof. First notice that for a domain $U \subset \mathbf{R}^n$,

$$f : (U, g_E) \longrightarrow (\mathbf{R}^n, g_E)$$

is a Riemannian conformal map if and only if $D_a f$ is a conformal matrix for any $a \in U$. On the other hand, the following two maps

$$v : \mathbf{R}^n \to S^n,$$

$$v \circ q_2 : \mathbf{R}^n \to S^n$$

are Riemannian conformal maps from $(\mathbf{R}^n, g_E)$ to (S^n, g_S), where q_2 is the coordinate chart of $\widehat{\mathbf{R}}^n$ defined in §1. (2.7) follows from this. Q.E.D.

Thus Liouville's theorem can be rephrased as follows.

Let $U \subset S^n$ $(n \geq 3)$ be a domain. Then a Riemannian conformal map $f : U \to S^n$ w.r.t. the spherical metric is the restriction of a transformation in $\mathcal{M}(S^n)$.

Hereafter we focus our attention to the action of $\mathcal{M}(S^n)$ on S^n and D^{n+1}. Thus Moebius transformations are considered primarily as acting on S^n. However there are some occasions where the coordinates of $\widehat{\mathbf{R}}^n$ is more convenient. In what follows, frequent use will be made of the following lemma, which is a special case of (2.1). As before $J \in \mathcal{M}(\widehat{\mathbf{R}}^{n+1})$ denotes the inversion at S^n.

Lemma (2.8). *For* $f \in \mathcal{M}(S^n)$, *we have* $J \circ f = f \circ J$.

To study the action of $\mathcal{M}(S^n)$, the transformations are classified according to whether they preserve ∞ or not. In the first place, we have the following proposition.

Proposition (2.9). *For* $f \in \mathcal{M}(S^n)$, *the following statements are equivalent.*

(1) $f(\infty) = \infty$.
(2) $f(0) = 0$.
(3) f *induces an isometry of* (S^n, g_S).
(4) $f(x) = Px$ *for some* $P \in O(n+1)$.

Proof. By virtue of (2.8), We have (1) $\Leftrightarrow$ (2). (1) $\Rightarrow$ (4) follows from the expression of (1.9), (4) $\Rightarrow$ (1) and (4) $\Rightarrow$ (3) is clear and (3) $\Rightarrow$ (4) follows from the next lemma. Q.E.D.

Lemma (2.10). *Suppose that a Lie group* G *acts on a connected* n*-dimensional Riemannian manifold* N *transitively and isometrically. Suppose also that the first derivative gives an isomorphism* $G_x \cong O(n)$, *where* G_x *is the isotropy subgoup at some* $x \in N$. *Then* G *is precisely the group of all the isometries of* N.

Proof. For any isometry f, there exists a unique element $g \in G$ such that $g^{-1} \circ f(x) = x$ and $D_x(g^{-1} \circ f) = E$. Then $g^{-1} \circ f$ keeps any point on any geodesic ray at x fixed. That is, $g^{-1} \circ f = \mathrm{id}$.
Q.E.D.

Next for f with $f(\infty) \neq \infty$, we define the isometric sphere and use it to describe a geometric decomposition of f. For an $n \times n$ matrix A, $\|A\|$ denotes the mapping norm. In particular if A is a conformal matrix, then we have $\|A\| = (\det A)^{1/n}$.

Definition (2.11). For a transformation $f \in \mathcal{M}(\widehat{\mathbf{R}}^{n+1})$ with $f(\infty) \neq \infty$, the *isometric sphere* $I(f)$ of f is defined by

$$I(f) = \{x \in \mathbf{R}^{n+1} \mid \|D_x f\| = 1\}.$$

The isometric sphere cannot be defined for transformations which keep ∞ fixed. Recall that by (1.9), f can be expressed as

$$f(x) = \lambda P J(x - b) + c,$$

where $\lambda > 0$, $P \in O(n+1)$ and $b, c \in \mathbf{R}^{n+1}$. Note that $f(b) = \infty$ and $f(\infty) = c$. For $x \in \mathbf{R}^{n+1}$, we have

$$\|D_x f\| = \frac{\lambda}{|x-b|^2}.$$

Thus the isometric sphere $I(f)$ is the codimension one sphere of radius $\lambda^{1/2}$, centered at $f^{-1}(\infty)$. We summarize fundamental properties of isometric sphere in the following proposition. The proof is left to the reader.

Proposition (2.12). *For $f \in \mathcal{M}(\widehat{\mathbf{R}}^{n+1})$ such that $f(\infty) \neq \infty$, we have the following.*

(1) *The center of the isometric sphere $I(f)$ is the point $f^{-1}(\infty)$.*
(2) *f carries $I(f)$ to $I(f^{-1})$ and induces an isometry there. In particular, $I(f)$ and $I(f^{-1})$ have the same radius.*
(3) *f carries the interior of $I(f)$ to the exterior of $I(f^{-1})$.*
(4) *The interior of the isometric sphere $I(f)$ consists precisely of those points x for which $\|D_x f\| > 1$ holds.*

Proposition (2.13). *For $f \in \mathcal{M}(S^n)$ such that $f(\infty) \neq \infty$, the isometric sphere $I(f)$ is perpendicular to S^n.*

Proof. Since the action of f on S^n is not an isometry, there are points in S^n where the norms of the derivatives of f are less than or greater than 1. This implies that $I(f)$ intersects S^n in an $(n-1)$ sphere. f induces an isometry from $I(f)$ to $I(f^{-1})$ which sends the sphere $I(f) \cap S^n$ to the sphere $I(f^{-1}) \cap S^n$. Thus for $x \in I(f)$, the spherical distance in $I(f)$ between x and $I(f) \cap S^n$ coincides with the spherical distance in $I(f^{-1})$ between $f(x)$ and $I(f^{-1}) \cap S^n$. That is, for $x \in I(f)$, we have $|x| = |f(x)|$ and consequently $\|D_x J\| = \|D_{f(x)} J\|$. See Figure (2.2). Differentiating the equation $J \circ f = f \circ J$, we obtain that $\|D_x f\| = 1$ implies $\|D_{J(x)} f\| = 1$. That is, $J(I(f)) = I(f)$. This shows (2.13). Q.E.D.

Proposition (2.14). *A transformation $f \in \mathcal{M}(S^n)$ such that $f(\infty) \neq \infty$ can be decomposed as*

$$f = J_{\pi(f)} \circ J_{I(f)} \circ P(f),$$

where $P(f)$ is a transformation in $O(n+1)$ which preserves $I(f)$ and $\pi(f)$ is the bisector of the centers of $I(f)$ and $I(f^{-1})$ if $I(f) \neq I(f^{-1})$

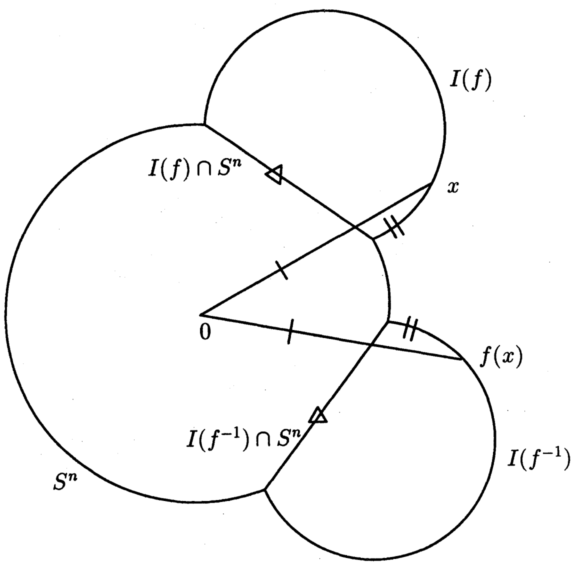

Figure (2.2)

and an arbitrary hyperplane which passes through the center of $I(f)$ *and* 0 *if* $I(f) = I(f^{-1})$. *See Figure* (2.3).

Proof. The transformation $g = J_{\pi(f)} \circ J_{I(f)}$ clearly carries $I(f)$ to $I(f^{-1})$ and there the norm of the differential is 1. That is, $I(g) = I(f)$ and $I(g^{-1}) = I(f^{-1})$. It follows that $g^{-1} \circ f$ preserves the sphere $I(f)$ and is an isometry there. Notice also that $g^{-1} \circ f$ preserves the interior of $I(f)$. Applying (2.9) to a transformation of $I(f)$, it follows that $g^{-1} \circ f = P(f)$ keeps ∞ fixed. Since $P(f)$ preserves S^n, $P(f)$ is a transformation in $O(n+1)$. Q.E.D.

It is a well known fact that $\mathcal{M}(S^n)$ is a Lie group of dimension $\frac{1}{2}(n+1)(n+2)$ with two connected components.

Definition (2.15). Let $\{f_k\}_{k=1,2,\ldots}$ be a sequence of elements of $\mathcal{M}(S^n)$. We say $f_k \to \infty$ if and only if for any compact subset C of $\mathcal{M}(S^n)$, there exists $k_0 > 0$ such that $f_k \notin C$ for $k \geq k_0$.

Thus $f_k \to \infty$ if and only if f_k has no subsequence which converges to an element of $\mathcal{M}(S^n)$.

For $f \in \mathcal{M}(S^n)$, we define

$$\|Df\|_{S^n} = \sup\{\|D_x f\| \mid x \in S^n\}.$$

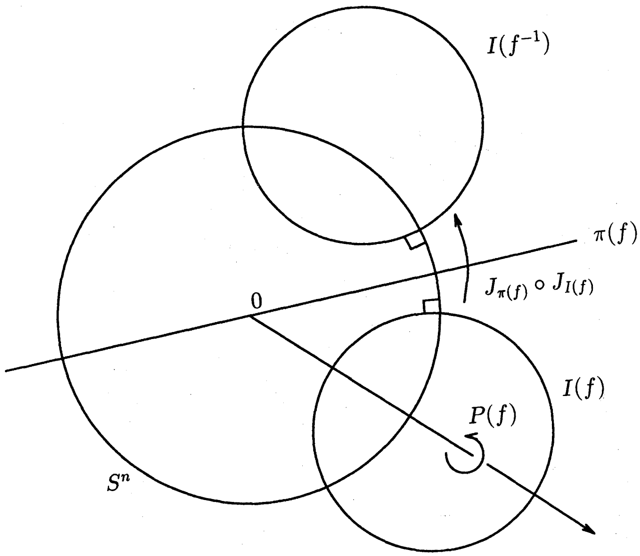

Figure (2.3)

Proposition (2.16). *For a sequence* $\{f_k\}$ *in* $\mathcal{M}(S^n)$, *the following conditions are equivalent.*

(1) $f_k \to \infty$.
(2) $\|Df_k\|_{S^n} \to \infty$.
(3) *Except for finite* k, $f_k(\infty) \neq \infty$ *and* radius $I(f_k) \to 0$.

Proof. First we shall show the equivalence of (2) and (3). Assume for simplicity that $f_k(\infty) \neq \infty$ for any k. Let

$$f_k(x) = r_k^2 P_k J(x - b_k) + c_k.$$

We have

$$\|D_x f_k\| = \frac{r_k^2}{|x - b_k|^2},$$

where $r_k =$ radius $I(f_k)$. Since $I(f_k)$ is perpendicular to S^n, we obtain

$$\|Df_k\|_{S^n} = \frac{r_k^2}{(\sqrt{1 + r_k^2} - 1)^2} = \frac{(\sqrt{1 + r_k^2} + 1)^2}{r_k^2}.$$

See Figure (2.4). From this follows the equivalence of (2) and (3).

Next, (2) $\Rightarrow$ (1) is obvious. To show the converse, we assume that (2) , hence (3), does not hold and will show that (1) fails, that is, f_k has

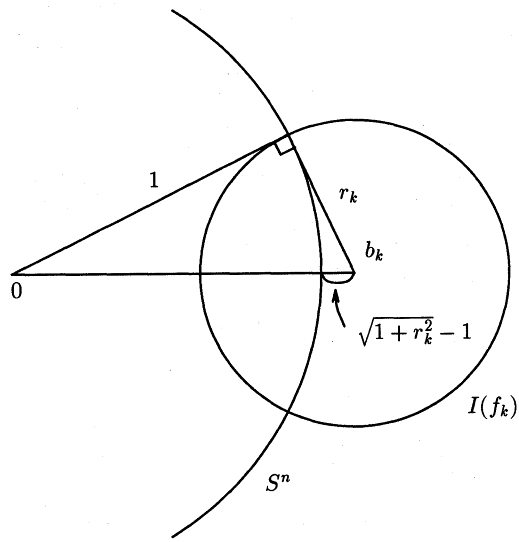

Figure (2.4)

a subsequence which converges in $\mathcal{M}(S^n)$. Thus in the course of the proof, we are free to pass to a subsequence, if necessary. If $f_k(\infty)=\infty$ for infinitely many k, then such f_k belongs to a compact subgroup $O(n+1)$ of $\mathcal{M}(S^n)$, showing that (1) does not hold. Therefore we may assume (passing to a subsequence) that $f_k(\infty)\neq\infty$ for any $k\geq 1$ and $r_k\to\rho$ for some $0<\rho\leq\infty$.

Assume for a while that $0<\rho<\infty$. Then in the decomposition of (2.14), the sphere $I(f_k)$ may be assumed to converge. That is, the inversion $J_{I(f_k)}$ converges in $\mathcal{M}(S^n)$. Likewise we may assume that $J_{\pi(f_k)}$ and $P(f_k)$ also converge in $\mathcal{M}(S^n)$. This shows that (1) does not hold.

Next consider the case where $\rho=\infty$. Notice that $\rho=\infty$ if and only if $f_k^{-1}(\infty)\to\infty$, since the sphere $I(f_k)$ centered at $f_k^{-1}(\infty)$ is always perpendicular to the fixed sphere S^n. Take an arbitrary transformation g of $\mathcal{M}(S^n)$ such that $g(b)=\infty$ for some $b\neq\infty$ and consider the sequence $f_k\circ g$. Then $g^{-1}\circ f_k^{-1}(\infty)\to b$. That is, radius $I(f_k\circ g)\to r$ $(0<r<\infty)$. Therefore this case can be reduced to the former case. Q.E.D.

Next we shall show that a Moebius transformation in $\mathcal{M}(S^n)$ induces an isometry of (D^{n+1},g_H). The key step is the following lemma.

Lemma (2.17). *Let* $f \in \mathcal{M}(S^n)$ *and let* $x \in \mathbf{R}^{n+1} \setminus S^n$. *Then*

$$\|D_x f\| = \frac{1 - |f(x)|^2}{1 - |x|^2}$$

Proof. Both hand sides decompose as products when f decomposes as a composite. Thus it is sufficient to show (2.17) only for the inversion J_τ at an n-dimensional sphere $\tau = \{|x-a| = r\}$ which is perpendicular to S^n. We have

$$J_\tau(x) = r^2 \frac{(x-a)}{|x-a|^2} + a$$

and

$$\|D_x J_\tau\| = \frac{r^2}{|x-a|^2}.$$

Since the sphere τ is perpendicular to S^n, we have

$$|a|^2 = 1 + r^2.$$

Then it is easy to show by calculation that

$$|J_\tau(x)|^2 - 1 = \frac{r^2}{|x-a|^2}(|x|^2 - 1).$$

This shows (2.17). Q.E.D.

Corollary (2.18). *An element* $f \in \mathcal{M}(S^n)$ *induces an isometry of* (D^{n+1}, g_H).

The converse can also be shown using (2.10), once we establish the following lemma.

Lemma (2.19). *For any point* $a \in D^{n+1}$, *there exists a transformation* $f \in \mathcal{M}(S^n)$ *such that* $f(0) = a$.

Proof. Let l be the radius through a. For any $x \in l$, let σ_x be the codimension one sphere perpendicular to l at x and orthogonal to S^n. Then $J_{\sigma_x} \in \mathcal{M}(S^n)$ sends 0 to some point in l. Clearly we have

$$\lim_{x \to 0} J_{\sigma_x}(0) = 0, \quad \lim_{x \to b} J_{\sigma_x}(0) = b,$$

where b is the end point of l. By the continuity of $J_{\sigma_x}(0)$, we obtain a point x in l such that $J_{\sigma_x}(0) = a$. Q.E.D.

Theorem (2.20). *$\mathcal{M}(S^n)$ is precisely the group of isometries of (D^{n+1}, g_H).*

Theorem (2.21). *In (D^{n+1}, g_H), the geodesics are the circles that are orthogonal to S^n. Denoting the distance in (D^{n+1}, g_H) by d_H, we also have for $a \in D^{n+1}$*

$$d_H(0,a) = \log \frac{1+|a|}{1-|a|}.$$

Proof. First let us find the shortest path combining 0 and a $(a \neq 0)$. Let $\gamma(t)$ be an arbitrary smooth arc such that $\gamma(0) = 0$ and $\gamma(1) = a$. Schwartz's inequality yields

$$||\gamma(t)|'| \leq |\gamma'(t)|.$$

Thus we have

$$\begin{aligned} \text{length}(\gamma) &= \int_0^1 \frac{2|\gamma'(t)|dt}{1-|\gamma(t)|^2} \geq \int_0^1 \frac{2||\gamma(t)|'|dt}{1-|\gamma(t)|^2} \\ &\geq \int_0^{|a|} \frac{2ds}{1-s^2} = \log \frac{1+|a|}{1-|a|}. \end{aligned}$$

This shows the last part of (2.21) and that the geodesic through 0 and a are the radius.

Now consider the general case. Let $a, b \in D^{n+1}$. By (2.19), there exists $f \in \mathcal{M}(S^n)$ such that $f(0) = a$. Since f^{-1} is an isometry, f^{-1} maps the geodesics to the geodesics. Further since f^{-1} is a Moebius transformation, f^{-1} maps the diameter through $f(b)$ to the circle through a and b which is orthogonal to S^n. Q.E.D.

Finally we shall classify transformations in $\mathcal{M}(S^n)$ according to its dynamics on $\mathrm{Cl}(D^{n+1})$. By (2.3), they keep $\mathrm{Cl}(D^{n+1})$ invariant, where Cl denotes the closure.

Proposition (2.22). *Let $f \in \mathcal{M}(S^n)$. For the induced transformation*

$$f : \mathrm{Cl}(D^{n+1}) \to \mathrm{Cl}(D^{n+1}),$$

we have the followings.

(1) *f has at least one fixed point in $\mathrm{Cl}(D^{n+1})$.*
(2) *If f has three or more fixed points in S^n, then f has a fixed point in D^{n+1}.*

Proof. (1) follows from Brouwer's fixed point theorem. To show (2), coordinates of $\widehat{\mathbf{R}}^n$ and H^{n+1} are more convenient. By conjugating

$$g = c_v^{-1}(f) \in \mathcal{M}(\widehat{\mathbf{R}}^n)$$

by a suitable element of $\mathcal{M}(\widehat{\mathbf{R}}^n)$, we may assume that g keeps fixed 0, ∞ and another point a. By (1.9), we have for $x \in \mathbf{R}^n$, $g(x) = \lambda Px$, where $\lambda > 0$ and

$$P = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix},$$

where $Q \in O(n)$. Since g also keeps a fixed, it follows that $\lambda = 1$. Thus for example, $(0, y) \in H^{n+1}$ $(y > 0)$ is fixed by g. This completes the proof of (2). Q.E.D.

Definition (2.23). $f \in \mathcal{M}(S^n)$ (resp. $\mathcal{M}(\widehat{\mathbf{R}}^n)$) is called *elliptic* if f has fixed points in D^{n+1} (resp. H^{n+1}), *loxodromic* if f is not elliptic and has exactly two fixed points in S^n (resp. $\widehat{\mathbf{R}}^n$) and *parabolic* otherwise.

Notice that by (2.22), a parabolic transformation has precisely one fixed point in S^n (resp. $\widehat{\mathbf{R}}^n$).

Next we shall describe the standard forms of conjugacy classes of these three types of transformations. For elliptic transformations, it is convenient to work with the coordinates of S^n and to conjugate so that 0 is the fixed point. However for the other types, the coordinates of $\widehat{\mathbf{R}}^n$ is preferable. Notice that parabolic (resp. loxodromic) transformations can be conjugated so that they keep ∞ (resp. ∞ and 0) fixed.

Proposition (2.24).

(1) *Let $f \in \mathcal{M}(S^n)$ be an elliptic transformation such that $f(0) = 0$. Then we have $f(x) = Px$ for some $P \in O(n+1)$.*

(2) *Let $f \in \mathcal{M}(\widehat{\mathbf{R}}^n)$ be a loxodromic transformation such that $f(\infty) = \infty$ and $f(0) = 0$. Then we have $f(x) = \lambda Px$ for some $\lambda \neq 1, > 0$ and $P \in O(n)$.*

(3) *Let $f \in \mathcal{M}(\widehat{\mathbf{R}}^n)$ be a parabolic transformation such that $f(\infty) = \infty$. Then by conjugating with a translation of $\mathbf{R}^n$, we have $f(x) = Px + b$ for some $P \in O(n)$ and $b \in \mathbf{R}^n \setminus \{0\}$ such that $Pb = b$.*

Proof. To show (2), notice that $\lambda \neq 1$ since otherwise f would fix points of the straight line perpendicular to $\mathbf{R}^n$ which passes through 0, contrary to the hypothesis that f is loxodromic.

To prove (3), let $f(x) = \lambda Px + b'$. Since f cannot have a fixed point in $\mathbf{R}^n$, we have $\lambda = 1$ and $b' \notin \mathrm{Image}(P - I)$. But $b' = (P-I)a + b$, for some $a \in \mathbf{R}^n$ and $b \perp \mathrm{Image}(P - I)$. It is a standard exercise in linear algebra to show $Pb = b$. Conjugating f by the translation by a, we obtain the transformation $x \longmapsto Px + b$, as is required. Q.E.D.

Definition (2.25). For a loxodromic transformation $f \in \mathcal{M}(S^n)$, the geodesic which combines the two fixed points of f is called an *axis* of f.

Definition (2.26). A codimension one sphere in $\mathrm{Cl}(D^{n+1})$ which is tangent to S^n at $a \in S^n$ is called a *horosphere* at a.

Proposition (2.27). *A loxodromic transformation of $\mathcal{M}(S^n)$ preserves its axis. A parabolic transformation preserves the horospheres at the fixed point.*

Proof. To prove the first part, notice that the standard form (2) of (2.24) preserves the x_{n+1}-axis in H^{n+1}. The transformation $v \in \mathcal{M}(\widehat{\mathbf{R}}^{n+1})$ (defined just before (2.2)) maps x_{n+1}-axis to a diameter in D^{n+1}. Any transformation of $\mathcal{M}(S^n)$ maps a diameter to a geodesic of D^{n+1}. Therefore by conjugating the standard form, we get the desired result. The latter part can be shown likewise. Notice that the standard form (3) of (2.24) preserves the plane $\{x_{n+1} = c\}$ $(c > 0)$, which is mapped by v to a horosphere. Q.E.D.

§3. Flat conformal structure

In this section we define a flat conformal structure, its developing map and holonomy homomorphism. We study their fundamental properties.

In the first place, we define a (G, X)-structure in general circumstances. Let X be a real analytic manifold and let G be a Lie group acting real analytically, transitively and effectively on X. In this study, all the group actions are to be on the left, unless otherwise specified. Let N be a connected topological manifold of the same dimension as X.

Definition (3.1). A collection $\mathcal{U} = \{(U_\alpha, q_\alpha)\}_{\alpha \in \Lambda}$ is called a (G, X)-*atlas* if

(1) $\{U_\alpha\}$ is an open covering of N.
(2) $q_\alpha : U_\alpha \to X$ is an embedding.
(3) For each component V of $U_\alpha \cap U_\beta$, there exists $g \in G$ such that $q_\beta(x) = g q_\alpha(x)$, $x \in V$.

An element (U_α, q_α) is called a $\mathcal{U}$*-chart.*

Definition (3.2). A maximal (G,X)-atlas is called a (G,X)-*structure* on N or a *geometric structure* vaguely. A manifold equipped with a (G,X)-structure is called a (G,X)-*manifold.*

Let $p : M \to N$ be a covering map.

Definition (3.3). Let $\{(U_\alpha, q_\alpha)\}_{\alpha\in\Lambda}$ be a (G,X)-atlas on N for a (G,X)-structure $\mathcal{U}$ such that U_α is homeomorphic to an n-ball. Let V_α^i be a connected component of $p^{-1}(U_\alpha)$. Then $\{(V_\alpha^i, q_\alpha \circ p)\}$ is a (G,X)-atlas on M. The (G,X)-structure which contains $\{(V_\alpha^i, q_\alpha \circ p)\}$ is called the *lift* of $\mathcal{U}$ by p and is denoted by $p^*\mathcal{U}$. Especially when p is a homeomorphism, $p^*\mathcal{U}$ and $\mathcal{U}$ are called *isomorphic.*

Given a (G,X)-structure $\mathcal{U}$ on N, the associated developing map and holonomy homomorphism are defined as follows.

Let $p : \widetilde{N} \to N$ be the universal covering space with the base point $x_0 \in \widetilde{N}$. Let $\pi_1(N)$ be the fundamental group at the base point $p(x_0)$. As usual, $\pi_1(N)$ is identified via x_0, with the group of deck transformations of $\widetilde{N}$. Denote by $\widetilde{\mathcal{U}}$ the lift of $\mathcal{U}$ by p. Fix once and for all a $\widetilde{\mathcal{U}}$-chart (U_0, q_0) around x_0.

Definition (3.4). A sequence $((U_i, q_i), g_i)$, $(1 \le i \le r)$ is called a *chart chain from* (U_0, q_0) if for $1 \le i \le r$, we have

(a) $(U_i, q_i) \in \widetilde{\mathcal{U}}$, $g_i \in G$,
(b) $U_{i-1} \cap U_i$ is nonempty and connected,
(c) $q_{i-1}(x) = g_i q_i(x)$, $x \in U_{i-1} \cap U_i$.

Given a chart chain as above, it is possible to extend the base map q_0 to a continuous map $D : U_0 \cup U_1 \to X$ by

$$D(x) = g_1 q_1(x), \quad x \in U_1.$$

Successively D can be extended to $U_0 \cup U_1 \cup U_2$ by

$$D(x) = g_1 g_2 q_2(x), \quad x \in U_2.$$

See Figure (3.1). This motivates the following definition.

Definition (3.5).

(1) The *developing map* $D : \widetilde{N} \to X$ *w.r.t. the base chart* (U_0, q_0) is defined by

$$D(x) = g_1 g_2 \cdots g_r \cdot q_r(x), \quad x \in \widetilde{N},$$

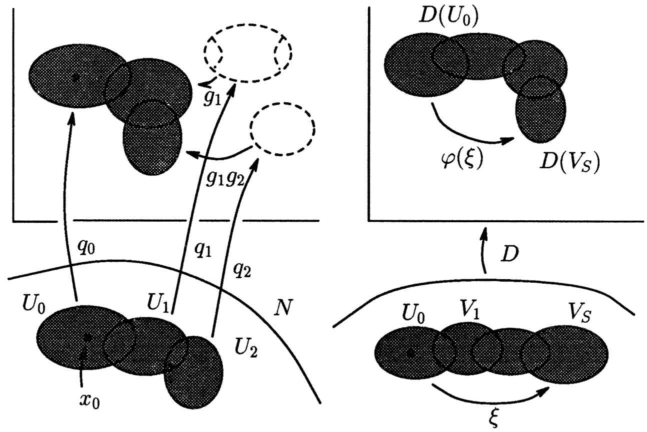

Figure (3.1)

where $((U_i, q_i), g_i)$, $(1 \le i \le r)$ is a chart chain from (U_0, q_0) such that $x \in U_r$.

(2) The *holonomy homomorphism* $\varphi : \pi_1(N) \to G$ *w.r.t. the base chart* (U_0, q_0) is defined by

$$\varphi(\xi) = h_1 h_2 \cdots h_s, \quad \xi \in \pi_1(N)$$

where $((V_j, p_j), h_j)$, $1 \le j \le s$ is a chart chain from (U_0, q_0) such that

$$(V_s, p_s) = (\xi U_0, q_0 \circ \xi^{-1}).$$

D and ϕ are well defined since $\widetilde{N}$ is simply connected. The proof is routine and is omitted. Also it is clear that D is a submersion (or immersion).

Definition (3.6). A pair (D, φ) is called a *DH pair* if the following is satisfied.

(1) $D : \widetilde{N} \to X$ is a submersion.
(2) $\varphi : \pi_1(N) \to G$ is a homomorphism.
(3) $D(\xi x) = \varphi(\xi) D(x)$, $\quad \xi \in \pi_1(N)$, $x \in \widetilde{N}$.

Proposition (3.7). *Let D and φ be the developing map and the holonomy homomorphism for a base chart (U_0, q_0). Then (D, φ) is a DH pair.*

Proof. To show

$$D(\xi x) = \varphi(\xi) D(x), \quad \xi \in \pi_1(N),\ x \in \widetilde{N},$$

let

$$\sigma = \{((U_i, q_i), g_i)\}_{1 \le i \le r}$$

be a chart chain from (U_0, q_0) such that $x \in U_r$ and let

$$\tau = \{((V_j, p_j), h_j)\}_{1 \le j \le s}$$

be a chart chain from (U_0, q_0) such that

$$(V_s, p_s) = (\xi U_0, q_0 \circ \xi^{-1}).$$

Let

$$\xi_\sharp \sigma = \{((\xi U_i, q_i \circ \xi^{-1}), g_i)\}.$$

$\xi_\sharp \sigma$ is a chart chain from $(\xi U_0, q_0 \circ \xi^{-1}) = (V_s, p_s)$. Thus τ followed by $\xi_\sharp \sigma$ is a chart chain from (U_0, q_0) to the point ξx. That is, we have

$$D(\xi x) = h_1 h_2 \cdots h_s \cdot g_1 g_2 \cdots g_r \cdot q_r \circ \xi^{-1}(\xi x) = \varphi(\xi) D(x).$$

Finally let us show that φ is a homomorphism. We have

$$\varphi(\xi_1 \xi_2) D(x) = D(\xi_1 \xi_2 x) = \varphi(\xi_1) D(\xi_2 x) = \varphi(\xi_1) \varphi(\xi_2) D(x).$$

It follows that $\varphi(\xi_1 \xi_2) = \varphi(\xi_1)\varphi(\xi_2)$, since the action of G on X is effective and real analytic. (Note that Image(D) is a domain since D is a submersion.) Likewise we have $\varphi(1) = 1$. Q.E.D.

Definition (3.8). Two DH pair (D, φ) and (D', φ') are said to be *equivalent* if there exists $g \in G$ such that $D'(x) = gD(x)$ and $\varphi'(\xi) = g\varphi(\xi)g^{-1}$ for $x \in \widetilde{N}$ and $\xi \in \pi_1(N)$.

Proposition (3.9). *The correspondence of (3.7) gives a bijection between the set of (G, X)-structures on N and the set of the equivalence classes of DH pairs.*

Proof. Let (D, φ) (resp. (D', φ')) be the DH pair associated to the base chart (U_0, q_0) (resp. (U_0', q_0'))of a given (G, X)-structure.

Consider a chart chain

$$((U_i, q_i), g_i), \quad 1 \leq i \leq r$$

from (U_0', q_0') such that $(U_r, q_r) = (U_0, q_o)$. Let $g = g_1 g_2 \cdots g_r$. Then it is easy to show that

$$D'(x) = gD(x), \quad \varphi'(\xi) = g\varphi(\xi)g^{-1}.$$

Conversely given an equivalence class of DH pairs, one can get a (G, X)-structure on N by restricting the developing map to small domains of $\widetilde{N}$ and projecting down by $p : \widetilde{N} \to N$. Q.E.D.

By certain abuse, (G, X)-structures are sometimes denoted by their DH pairs as $[D, \varphi]$.

Definition (3.10). For a (G, X)-structure $[D, \varphi]$ on N,

$$H = \text{ Image}(\varphi) \subset G$$

is called the *holonomy group* of $[D, \varphi]$.

By (3.9), the holonomy group of a (G, X)-structure is unique up to conjugations in G.

Let Γ be a discrete group which acts on N.

Definition (3.11). Γ is said to act *discontinuously* on N, if for any $x \in N$, there exists a neighbourhood U of x such that

$$\text{Card}\{\gamma \in \Gamma \mid \gamma U \cap U \neq \phi\} < \infty.$$

The proof of the following proposition is left to the reader.

Proposition (3.12). *Γ acts freely and discontinuously on N if and only if for any $x \in N$, there exists a neighbourhood U such that if $\gamma \neq 1$, then $\gamma U \cap U = \phi$.*

Suppose $N \to P$ be a regular covering with the group of deck trasformations Γ. Then the action of Γ on N is free and discontinuous. Conversely, if Γ acts freely and discontinuously on a manifold N, then the canonical projection $\pi : N \to N/\Gamma$ is a regular covering with the group of deck transformations Γ.

Proposition (3.13). *Suppose that* Γ *acts on* N *freely and discontinuously. Then*

$$\widetilde{\Gamma} = \{\widetilde{\gamma} : \widetilde{N} \to \widetilde{N} \mid \widetilde{\gamma} \textit{ is a lift of } \gamma, \quad \gamma \in \Gamma\}$$

acts on $\widetilde{N}$ *freely and discontinuously.* $\widetilde{\Gamma}$ *is the group of deck transformations of the following universal covering.*

$$\pi \circ p : \widetilde{N} \to N \to N/\Gamma.$$

We have the following exact sequence;

$$1 \to \pi_1(N) \to \widetilde{\Gamma} \to \Gamma \to 1.$$

Proof. We only show that the action of $\widetilde{\Gamma}$ is free and discontinuous. The rest is left to the reader. Let $x \in \widetilde{N}$. Take a small neighbourhood $\widetilde{U}$ of x such that

(1) $U = p(\widetilde{U})$ is evenly covered by p and
(2) $\gamma U \cap U = \phi$ if $\gamma \neq 1$, $\gamma \in \Gamma$.

Suppose $\widetilde{\gamma}(\widetilde{U}) \cap \widetilde{U} \neq \phi$ for $\widetilde{\gamma} \in \widetilde{\Gamma}$. Then we have $\gamma(U) \cap U \neq \phi$, where $\widetilde{\gamma}$ is a lift of γ. This shows that $\gamma = 1$ by (2). Thus $\widetilde{\gamma}$ is a deck transformation of p. But by (1), we have $\widetilde{\gamma} = 1$. Q.E.D.

Let $\mathcal{U}$ be a (G, X)-structure on N.

Definition (3.14). An action of Γ on N is called a *$\mathcal{U}$-action* if and only if for any $\gamma \in \Gamma$, we have $\gamma^*\mathcal{U} = \mathcal{U}$.

Suppose that an action of Γ on N is a free and discontinuous $\mathcal{U}$-action. As before, $\pi : N \to N/\Gamma$ is the canonical projection.

Definition (3.15). A (G, X)-strucure $\pi_*\mathcal{U}$, called the *projection* of $\mathcal{U}$, is defined as follows. Let (D, φ) be the DH pair associated to a base chart (U_0, q_0). Since the action of the lift $\widetilde{\Gamma}$ is a $\widetilde{\mathcal{U}}$-action, we have that $(\widetilde{\gamma}U_0, q_0 \circ \widetilde{\gamma}^{-1})$ is a $\widetilde{\mathcal{U}}$-chart for any $\widetilde{\gamma} \in \widetilde{\Gamma}$. Thus as in Definition (3.5) (2), we can define a homomorphism

$$\psi : \widetilde{\Gamma} \to G$$

by using a chart chain to $(\widetilde{\gamma}U_0, q_0 \circ \widetilde{\gamma}^{-1})$. Then (D, ψ) is a DH pair for N/Γ. $\pi_*\mathcal{U}$ is defined to be the (G, X)-structure corresponding to this DH pair.

Clearly $\psi : \widetilde{\Gamma} \to G$ is an extension of the holonomy homomorphism $\varphi : \pi_1(N) \to G$.

As is shown later, there are many examples of pair (G, X) such that the isotropy subgroup

$$G_x = \{g \in G \mid gx = x\}$$

is compact for any $x \in X$. Then the corresponding (G, X)-structures have the following striking feature.

Proposition (3.16). *Let N be a closed (G, X)-manifold. Suppose the isotropy subgroup G_x is compact for $x \in X$. Then the developing map $D : \widetilde{N} \to X$ is a covering map onto X. In particular, if X is simply connected, then D is a homeomorphism.*

Proof. Since G_x is compact, there exists a G_x-invariant, positive definite, symmetric, bilinear form on the tangent space $T_x X$. Distributing it by the action of G, we obtain a G-invariant Riemannian metric g of X. Since $\bar{g} = D^* g$ is $\pi_1(N)$-invariant, it projects down to a Riemannian metric on N. Therefore $\bar{g}$ is complete.

For small $\varepsilon > 0$, we have that D maps any 2ε-ball in $\widetilde{N}$ isometrically onto a 2ε-ball in X. Then clearly any ε-ball in X is evenly covered by D. Q.E.D.

We shall raise some examples of (G, X)-structures.

Example (3.17). Denote by $\mathrm{Isom}(S^n)$, $\mathrm{Isom}(\mathbf{R}^n)$ or $\mathrm{Isom}(D^n)$ the group of isometries of the Riemannian manifold (S^n, g_S), $(\mathbf{R}^n, g_E)$ or (D^n, g_H). The corresponding (G, X)- structure (resp. manifold) is called *spherical, Euclidean* or *hyperbolic* structure (resp. manifold). Specifically, closed spherical or Euclidean manifold is called *spherical* or *Euclidean space form.*

Notice that $\mathrm{Isom}(S^n) = O(n+1)$ and $\mathrm{Isom}(D^n) = \mathcal{M}(S^{n-1})$. $\mathrm{Isom}(\mathbf{R}^n)$ consists of transformations, called *Euclidean motions,*

$$x \mapsto Px + b, \quad (P \in O(n),\ b \in \mathbf{R}^n).$$

All the three satisfy the hypothesis of (3.16). Therefore if the manifolds are compact, their universal covering spaces can be identified with S^n (if $n > 1$), $\mathbf{R}^n$ or D^{n+1}. A spherical space form is isomorphic to S^n/Γ if $n > 1$, where Γ is a finite group of $SO(n+1)$. The following theorem is due to Bieberbach ([5]). A neat proof, quite short, is found in P. Buser ([6]).

Theorem (3.18). *An Euclidean space form has n-torus as a finite covering.*

The main object of our study is the following (G, X)-structure.

Definition (3.19). A $(\mathcal{M}(S^n), S^n)$-structure (resp. manifold) is called a *flat comformal structure* (resp. *manifold*). A group action preserving a flat conformal structure is called a *conformal action.*

There is another way to get to the same concept.

Definition (3.20). A Riemannian manifold (N, g) of dimension n is called *conformally flat* if for any point $x \in N$, there exist a neighbourhood U and an embedding $f : U \to \mathbf{R}^n$ such that $f^* g_E$ is conformally equivalent to $g|_U$.

Notice that the above definition does not change if we use as a model space (S^n, g_S) instead of $(\mathbf{R}^n, g_E)$. In fact, they are conformally equivalent as we saw in §2.

Now let $\mathcal{U}$ be a flat conformal structure on N. For each $\mathcal{U}$-chart (U_α, q_α), there is the induced Riemannian metric $q_\alpha^* g_S$ on U_α. In a component V of $U_\alpha \cap U_\beta$, we have $q_\beta = g \circ q_\alpha$ for some $g \in \mathcal{M}(S^n)$. Since g is a conformal map w.r.t. g_S, $q_\alpha^* g_S$ and $q_\beta^* g_S$ are conformally equivalent on V. Take a locally finite partition of unity $\{t_\alpha\}$ associated with the covering $\{U_\alpha\}$ of $\mathcal{U}$-charts. The Riemannian metric

$$g = \sum_\alpha t_\alpha q_\alpha^* g_S$$

is a conformally flat metric.

Conversely suppose $n \geq 3$. Let g be a conformally flat metric on an n-dimensional manifold N. Then we have a family $\{(U_\alpha, f_\alpha)\}$ such that $\{U_\alpha\}$ is an open covering of N, that f_α is an embedding of U_α into S^n and that $f_\alpha^* g_S$ is conformally equivalent to g. Thus for any component V of $U_\alpha \cap U_\beta$,

$$f_\beta \circ f_\alpha^{-1}|_{f_\alpha(V)} : f_\alpha(V) \to f_\beta(V)$$

is a Riemannian conformal map in (S^n, g_S). Thus by Liouville's theorem, we have that

$$f_\beta \circ f_\alpha^{-1}|_{f_\alpha(V)} \in \mathcal{M}(S^n).$$

We obtain a flat conformal structure. In summary, we have;

Proposition (3.21). *Flat conformal structure on a manifold N yields a conformally equivalence class of conformally flat metrics. Further if $n \geq 3$, this correspondence is bijective.*

For $n = 2$, the above two concepts are in fact different. In this dimension, flat conformal structure is often called (*complex*) *projective* *structure* since

$$(\mathcal{M}(S^2), S^2) = (PGL(2 : \mathbf{C}), \mathbf{C}P^1),$$

while conformally flat Riemannian metric corresponds to complex structures.

If $(G', X') \subset (G, X)$, that is, $G' \subset G$, $X' \subset X$ and the G'-action on X' is the restriction of the G-action on X, then, as a matter of fact, a (G', X')-structure is naturally considered as a (G, X)-structure. Thus spherical manifolds, Euclidean manifolds and hyperbolic manifolds are considered to be flat conformal manifolds. In fact we have the following inclusions of (G, X)-pairs.

$$(\mathrm{Isom}(S^n), S^n) \subset (\mathcal{M}(S^n), S^n).$$

$$(\mathrm{Isom}(\mathbf{R}^n), \mathbf{R}^n) \subset (\mathcal{M}(\widehat{\mathbf{R}}^n), \widehat{\mathbf{R}}^n) \underset{\approx}{\xrightarrow{c_v}} (\mathcal{M}(S^n), S^n).$$

$$(\mathrm{Isom}(D^n), D^n) \subset (\mathcal{M}(S^{n-1}), D^n) \subset (\mathcal{M}(\widehat{\mathbf{R}}^n), \widehat{\mathbf{R}}^n) \underset{\approx}{\xrightarrow{c_v}} (\mathcal{M}(S^n), S^n).$$

A significant feature of these examples is that the developing maps are homeomorphisms onto their images (except the case of $(\mathrm{Isom}(S^1), S^1)$). However for a point $a \in S^n$, the isotropy group $\mathcal{M}(S^n)_a$ is not compact. (Compare that $\mathcal{M}(S^n)_a$ is compact for $a \in D^{n+1}$.) Therefore flat conformal manifolds in general do not enjoy this kind of good properties. In fact there are many such examples as we shall show in what follows. We make the following definition.

Definition (3.22). Let $\mathcal{U} = [D, \varphi]$ be a flat conformal structure and let $H = \mathrm{Image}(\varphi)$ be the holonomy group. $\mathcal{U}$ is said to be of *type* 1 if D is a covering map onto its image and H is discrete, of *type* 2 if D is a covering map but H is indiscrete, of *type* 3 if H is discrete but D is not a covering map and *type* 4 otherwise.

Before starting the study of type 1 flat conformal structures, we need some preparations. Let Γ be a subgroup of $\mathcal{M}(S^n)$.

Definition (3.23). A subset $A \subset S^n$ is called Γ-invariant if $\gamma(A) = A$ for any $\gamma \in \Gamma$.

Definition (3.24). Let Ω_Γ be the set of points $x \in S^n$ such that there exists a neighbourhood U of x such that $\gamma U \cap U = \phi$ but for finitely many $\gamma \in \Gamma$. Ω_Γ is called the *domain of discontinuity* of Γ.

Ω_Γ is the maximal Γ-invariant open subset of S^n on which Γ acts discontinuously.

Definition (3.25). Γ is called a *Kleinian group* if $\Omega_\Gamma \neq \phi$.

Clearly we have:

Proposition (3.26). *A Kleinian group is discrete in* $\mathcal{M}(S^n)$.

It is known that the converse does not hold. However we have:

Proposition (3.27). *If* Γ *is discrete, then* Γ *acts on* D^{n+1} *discontinuously.*

Proof. Assume Γ is infinite and let $\Gamma = \{\gamma_n\}$. Since Γ is discrete, $\gamma_n \to \infty$, that is, $\gamma_n \notin O(n+1)$ for but finitely many n and radius $I(\gamma_n) \to 0$. It follows that any compact subset of D^{n+1} is outside $I(\gamma_n)$ except finite n. (Recall $I(\gamma_n) \cap S^n \neq \phi$.) (3.27) follows from (2.12)(3). Q.E.D.

The following fact is helpful in our study of flat conformal structures of type 1. The proof is more or less the same as (3.27). The reader will find it in §5, after the definition of limit set is made.

Corollary (5.16). *Suppose a discrete group* Γ *admits an invariant open set* Ω *such that* $S^n \setminus \Omega$ *is neither empty nor a singleton. Then* Γ *acts on* Ω *discontinuously.*

Flat conformal structure of type 1 is constructed as follows. Let Γ be a Kleinian group in $\mathcal{M}(S^n)$ which acts freely and discontinuously on a Γ-invariant domain Ω. The action is of course a conformal action on a flat conformal manifold Ω. Hence the quotient manifold Ω/Γ admits a flat conformal structure $\mathcal{U}$. The developing map D is the universal covering followed by the inclusion;

$$D : \widetilde{\Omega} \xrightarrow{\pi} \Omega \subset S^n$$

and the holonomy group is Γ. Concrete examples of this construction will be given in later sections.

Definition (3.28). The flat conformal structure (manifold) constructed as above is called a *Kleinian structure* (*manifold*).

Definition (3.29). Two flat conformal manifolds are called *commensurable* if they have isomorphic finite coverings.

Proposition (3.30). *Any type 1 flat conformal compact manifold* N *is commensurable to a Kleinian manifold.*

The proof needs the following theorem due to Selberg. See e.g. ([53]).

Theorem (3.31). *Any finitely generated subgroup of* $GL(n,\mathbf{R})$ *has a torsion free subgroup of finite index.*

As is well known, $\mathcal{M}(S^n)$ is isomorphic to the projectivised Lorentz group $PO(n+1,1)$. Thus (3.31) is applicable to a subgroup of $\mathcal{M}(S^n)$.

Proof of (3.30). If the developing map D is onto S^n, then D is a homeomorphism and N is isomorphic to a spherical space form. Likewise if D misses only one point, then N is isomorphic to an Euclidean space form. Otherwise, by (5.16), the holonomy group H acts on $\Omega =$ Image(D) discontinuously. Let Γ be a torsion free finite index subgroup of H. Γ acts on Ω freely. We have the following two covering maps.

$$p : \widetilde{N}/\varphi^{-1}(\Gamma) \to N,$$

$$\bar{D} : \widetilde{N}/\varphi^{-1}(\Gamma) \to \Omega/\Gamma.$$

p is a finite covering since $\varphi^{-1}(\Gamma)$ is a finite index subgroup of $\pi_1(N)$. Therefore $\widetilde{N}/\varphi^{-1}(\Gamma)$ is compact and $\bar{D}$ is also a finite covering.

Q.E.D.

One can show by examples that Proposition (3.30) cannot be sharpened in general.

Next an example of type 2 flat conformal structure is in order.

Example (3.32). Let $P(x) = \lambda R_\theta x$ be a conformal linear transformation on $\mathbf{R}^2$ ($\lambda > 0$, R_θ; the rotation by θ). For $t \in \mathbf{R}$, let

$$P^t(x) = \lambda^t R_{t\theta}(x).$$

Let Q be another conformal transformation which keeps 0 fixed such that $Q \neq P^t$ for any $t \in \mathbf{R}$.

Let $\mathbf{R}^2/\mathbf{Z}^2 = T^2$. Define $\varphi : \mathbf{Z}^2 \to \mathcal{M}(\widehat{\mathbf{R}}^2)$ by $\varphi(l,m) = P^l Q^m$ and $D : \mathbf{R}^2 \to \widehat{\mathbf{R}}^2$ by $D(x,y) = P^x Q^y a$ for some $a \in \mathbf{R} \setminus \{0\}$. Since $PQ = QP$, we have (D,φ) is a DH pair. D is clearly a covering map onto $\mathbf{R}^2 \setminus \{0\}$. But often $H =$ Image(φ) is not discrete, for example when $\lambda = 1$ and $\theta \notin \mathbf{Q}$.

See Figure (3.2). This example cannot be generalized to higher dimensions, since $\mathbf{R}^n \setminus \{0\}$ is simply connected if $n \geq 3$. However, in §5, we give examples of type 2 flat conformal compact manifolds of dimension ≥ 3 and give a characterization of such manifolds.

The following is an example of type 3 flat conformal structure.

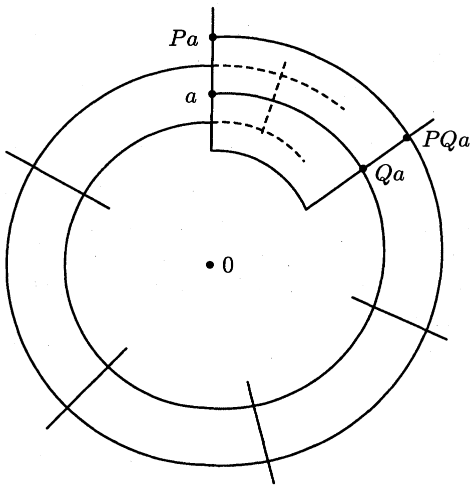

Figure (3.2)

Example (3.33). Let Σ be a closed Riemann surface of genus ≥ 2, that is, a hyperbolic manifold of dimension 2. The developing map D is a homeomorphism onto a disk in S^2. We shall alter D without changing the holonomy homomorphism φ. Let α be a simple closed geodesic in Σ and let V be the ε-neighbouhood of α for small $\varepsilon > 0$. Then a lift $\widetilde{V}$ of V in the universal covering $\widetilde{\Sigma} \cong D^2$ is the mutually disjoint ε-neighbourhood of a lift of α. See Figure (3.3). D is altered inside $\widetilde{V}$ to a new map D' in such a way that it coincides with D near the boundary of $\widetilde{V}$ and it goes extra once around S^2. Clearly D' can be constructed so that (D', φ) is a DH pair. See Figure (3.4). It is easy to show that D' is onto S^n. Thus it is not a covering map. For more detail, see Goldman ([16]). The same construction is possible for higher dimension if we start with a compact hyperbolic manifold which admits a totally geodesic closed submanifold of codimension 1. See Kourouniotis ([33]).

Finally an example of type 4.

Example (3.34). Prepare two copies of type 2 flat conformal manifolds N_1 and N_2 constructed in Example (3.34). Inside an atlas (U_i, q_i) of N_i, take a small disk V_i which is mapped by q_i to a metric disk in S^2. There exists an element $g \in \mathcal{M}(S^2)$ such that g maps V_1 to the exterior of V_2 . Consider the connected sum

$$N_1 \sharp N_2 = (N_1 \setminus \mathrm{Int} V_1) \cup (N_2 \setminus \mathrm{Int} V_2) / \sim .$$

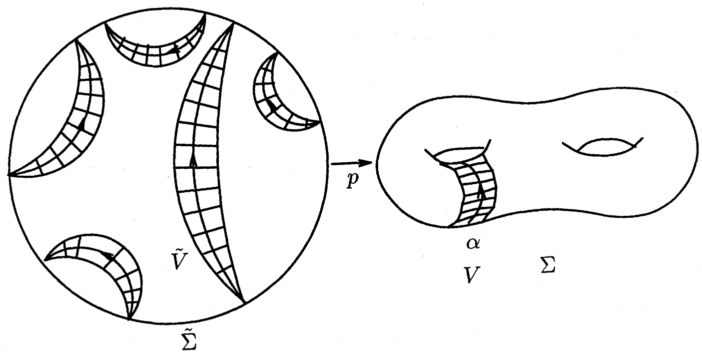

Figure (3.3)

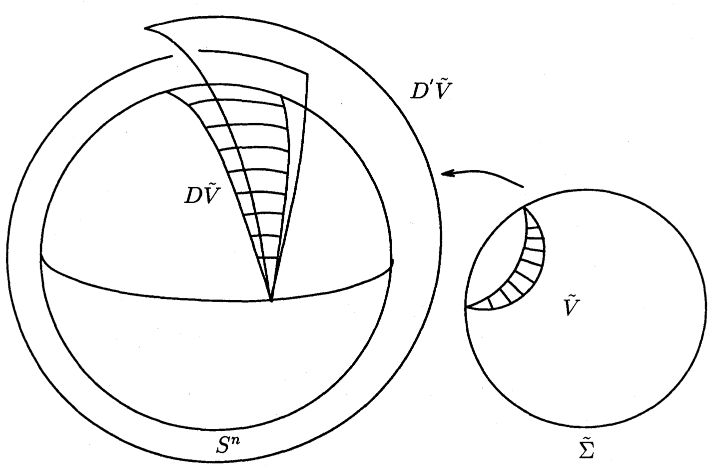

Figure (3.4)

If we chose the above identification appropriately, we obtain a continuous map

$$(g \circ q_1) \cup q_2 : (U_1 - \mathrm{Int}V_1) \cup (U_2 - \mathrm{Int}V_2)/\sim \longrightarrow S^2.$$

Using this we get in an obvious way a flat conformal structure on $N_1 \sharp N_2$. It is not difficult to show that the developing map of this structure is onto S^2 and therefore is not a covering map. The holonomy group is indiscrete since we started with type 2 examples.

The above operation, called *connected sum* of the structure, will be described in more detail in §6.

§4. Closed similarity manifolds

In this section we assume $n \geq 3$ and mainly work with $\widehat{\mathbf{R}}^n$, instead of S^n. As before $\mathcal{M}(\widehat{\mathbf{R}}^n)$ denotes the group of Moebius transformations of $\widehat{\mathbf{R}}^n$. As shown in §1, the isotropy subgroup $\mathcal{M}(\widehat{\mathbf{R}}^n)_\infty$ at ∞ consists of transformations

$$f(x) = \lambda P x + b, \quad \lambda > 0, \ P \in O(n), \ b \in \mathbf{R}^n.$$

λ (resp. P) is called the *norm* (resp. *orthogonal part*) of f and is denoted by $\|f\|$ (resp. $P(f)$). Clearly a transformation $f \in \mathcal{M}(\widehat{\mathbf{R}}^n)_\infty$ induces a transformation of $\mathbf{R}^n$. When viewed as a transformation of $\mathbf{R}^n$, f is called an *Euclidean similarity.* The group of Euclidean similarities is denoted by $ES(\mathbf{R}^n)$. We have an isomorphism

$$ES(\mathbf{R}^n) \approx \mathcal{M}(\widehat{\mathbf{R}}^n).$$

Definition (4.1). An $(ES(\mathbf{R}^n), \mathbf{R}^n)$-structure (manifold) is called a *similarity structure (manifold).*

Euclidean space forms are examples of similarity manifolds. Other examples are Hopf manifolds to be defined below.

Definition (4.2). A closed similarity manifold N is called a *Hopf manifold* if the developing map D is a homeomorphism onto $\mathbf{R}^n \setminus \{0\}$.

Then the holonomy group H is discrete and is contained in the isotropy subgroup $ES(\mathbf{R}^n)_0$. By taking norm and orthogonal part, we obtain the isomorphism

$$ES(\mathbf{R}^n)_0 \cong \mathbf{R}_+ \times O(n).$$

$\|H\| = \{\|f\| \mid f \in H\}$ is nontrivial since N is closed, and is discrete since $O(n)$ is compact. Therefore it is infinite cyclic. Let $\|h\|$ $(h \in H)$ be a generator. Since the kernel $\{\|h\| = 1\}$ is finite, $\langle h^2 \rangle$ is a finite index subgroup of H. Clearly $(\mathbf{R}^n \setminus \{0\})/\langle h^2 \rangle$ is homeomorphic to $S^{n-1} \times S^1$. Thus we have;

Proposition (4.3). *Hopf manifold has a finite covering which is homeomorphic to* $S^{n-1} \times S^1$.

In [13], Fried has shown that these two examples of similarity manifolds are the only examples. That is, an arbitrary similarity manifold is isomorphic to either an Euclidean space form or a Hopf manifold. See also Kuiper ([36]). The purpose of this section is to give an improved version of Fried's theorem. Instead of confining ourselves to similarity manifolds, we consider flat conformal manifolds in general.

Theorem (4.4). *Let* N *be a closed flat conformal manifold of dimension* ≥ 3 *such that the holonomy group* H *is contained in the isotropy subgroup* $\mathcal{M}(\widehat{\mathbf{R}}^n)_\infty$. *Then* N *is isomorphic to either* $\widehat{\mathbf{R}}^n$, *a Hopf manifold or an Euclidean space form.*

One can state the original Fried's theorem as a corollary.

Corollary (4.5). *Closed similarity manifold of dimension* ≥ 3 *is isomorphic to a Hopf manifold or an Euclidean space form.*

What is new in Theorem (4.4) is that the developing map is allowed to cover the point ∞, while in the original Fried's theorem (Corollary (4.5)) it is postulated to miss ∞. Although the difference is apparently not significant and the proof is in fact almost the same, Theorem (4.4) brings forth a far wider range of applications in practice (as far as flat conformal structures are concerned). To the best knowledge of the author, (4.4) cannot be found in the literature. Therefore it is obviously worth while to give a complete proof of (4.4).

The rest of this section is devoted to the proof of (4.4). In way of contradiction, we assume that N is isomorphic to neither of the three structures in (4.4). Denote by D the developing map , by φ the holonomy homomorphism and by H the holonomy group. The proof consists of three steps.

Step 1. Clearly $D^{-1}(\infty)$ is discrete and invariant by the deck transformation. Thus $N(\infty) = \pi(D^{-1}(\infty))$ is a finite set. Then $N^* = N \setminus N(\infty)$ is a similarity manifold.

Definition (4.6). A domain $U^* \subset \widetilde{N}^* = \pi^{-1}(N^*)$ is called a *copy* of $U \subset \mathbf{R}^n$ if $D|_{U^*} : U^* \to U$ is a homeomorphism.

Points in $\widetilde{N}^*$ are denoted by a^*, x^* and so forth and their images by D by a, x and so forth. Thus , $B^*(a^*, r)$ denotes a copy containing $a^* \in \widetilde{N}^*$ of $B(a, r)$, the open ball of radius r centered at a. We call a^* and r the *center* and *radius* of $B^*(a^*, r)$.

Definition (4.7). A closed subset $l^* \subset \widetilde{N}^*$ is called a *complete half line* if for any copy of ball $B^* \subset \widetilde{N}^*$, $B^* \cap l^*$ is mapped by D to $B \cap k$, where k is a complete half line in $\mathbf{R}^n$.

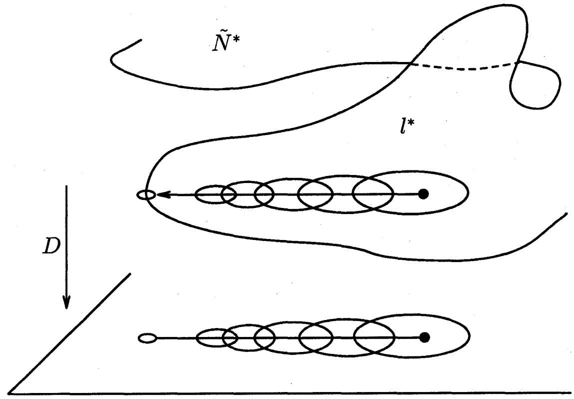

Figure (4.1)

See Figure (4.1).

By certain abuse, the parametrization of a complete half line l^* is denoted by the same letter, as

$$l^* : [0, \infty) \to l^*.$$

Notice that given any point $x^* \in \widetilde{N}^*$ and a tangent vector v at x^*, there exists a unique complete half line l^* such that $l^*(0) = x^*$ and tangent to v. Clearly deck transformation carries a complete half line to a complete half line.

Definition (4.8). A complete half line l^* is called *short* if $D(l^*)$ is not a complete half line in $\mathbf{R}^n$.

Claim (4.9). *Given a short complete half line l^*, there exists a neighbourhood U of $N(\infty)$ such that $\pi(l^*) \cap U = \phi$.*

Proof. For any point $c_i \in N(\infty)$, choose a compact neighbourhood U_i such that

(a) $\pi(l^*(0)) \notin U_i$,
(b) U_i is evenly covered by π,
(c) For any component E^* of $\pi^{-1}(U_i)$, there exist $a \in \mathbf{R}^n$ and $R > 0$ such that the following map is a homeomorphism.

$$D|_{E^*} : E^* \to E = E(a, R) = \{|x - a| \geq R\} \cup \{\infty\}$$

Notice that in (c), if one component of $\pi^{-1}U_i$ is mapped to $E(a,R)$, then all the other is also mapped to some $E(a',R')$. Thus (c) is attained if one chooses U_i small and appropriate.

Let us show that $l^* \cap E^*$ is empty. If not, the image $l \cap E$ is a half line starting at ∂E. ($l = D(l^*)$.) Since l^* is short, l is not a complete half line of $\mathbf{R}^n$. Then one can choose a ball $B \subset E \setminus \{\infty\}$ centered at the point $\lim_{t\to\infty} l(t)$. Then B has a copy B^* in E^*. But $B \cap l$ is not the intersection of B with a complete half line in $\mathbf{R}^n$. See Figure (4.2). This contradicts the hypothesis that l^* is complete. Let $U = \cup_i U_i$. We have $\pi(l^*) \cap U = \phi$. Q.E.D.

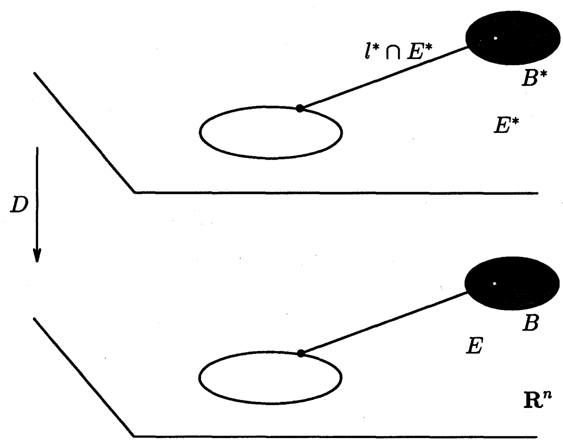

Figure (4.2)

For any $x^* \in \widetilde{N}^*$, let $r(x^*)$ be the maximal radius of a copy of ball centered at x^*. See Figure (4.3).

Claim (4.10). $r(x^*) < \infty$.

Proof. If not, x^* is contained in a copy of $\mathbf{R}^n$, say P. If $P = \widetilde{N}$, then N would be an Euclidean space form, contradicting the hypothesis. Suppose $P \neq \widetilde{N}$. Take a point $y^* \in \mathrm{Fr}(P)$ and a sequence $\{y_n^*\} \subset P$ such that $y_n^* \to y^*$. Clearly we have $D(y_n^*) \to \infty$. It follows from the continuity of D that $D(y^*) = \infty$. Therefore there is a neighbourhood Q of y^* which is mapped by D homeomorphically onto $E(0,R)$ for some large $R > 0$. Then $D : P \cup Q \to \widehat{\mathbf{R}}^n$ is a

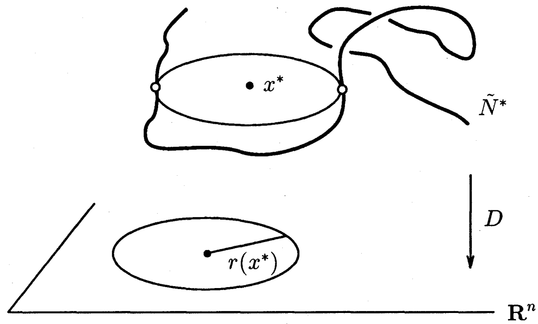

Figure (4.3)

homeomorphism. We have

$$N \cong \widetilde{N} = P \cup Q \cong \widehat{\mathbf{R}}^n.$$

Again a contradiction. Q.E.D.

Definition (4.11). The *Fried metric* is a continuous Riemannian metric on $\widetilde{N}^*$ defined by

$$g_F = \frac{D^* g_E}{r(x^*)^2} \quad \text{on} \quad T_{x^*}\widetilde{N}^*.$$

Let ξ be a deck transformation of $\widetilde{N}$ and $x^* \in \widetilde{N}^*$. We have

$$\xi(B^*(x^*, r)) = B^*(\xi x^*, \|\varphi(\xi)\| r).$$

This shows $r(\xi x^*) = \|\varphi(\xi)\| r(x^*)$. That is, the deck transformation ξ is an isometry for the Fried metric g_F. Thus g_F induces a Riemannian metric of N^*, which is also called the Fried metric. The distance functions of Fried metrics both on $\widetilde{N}^*$ and on N^* are denoted by d_F.

The following is the aim of Step 1.

Claim (4.12). *Let $B^* = B^*(a^*, r(a^*))$ be the maximal copy of ball centered at $a^* \in \widetilde{N}^*$. Then there exists a copy of half space H^* such that $B^*(a^*, r(a^*)) \subset H^*$.*

Proof. For simplicity let us assume $r(a^*) = 1$ and $D(a^*) = e_n = (0, \cdots, 0, 1)$. By D, we identify B^* with $B = \{|x - e_n| < 1\}$. By this identifidation, we consider the function r or the Fried metric g_F to be defined on B. By the maximality of B^*, there exists a radius l^* which is a complete half line. Assume

$$l = D(l^*) = \{x_1 = 0, \cdots, x_{n-1} = 0, 0 < x_n \leq 1\}.$$

See Figure (4.4).

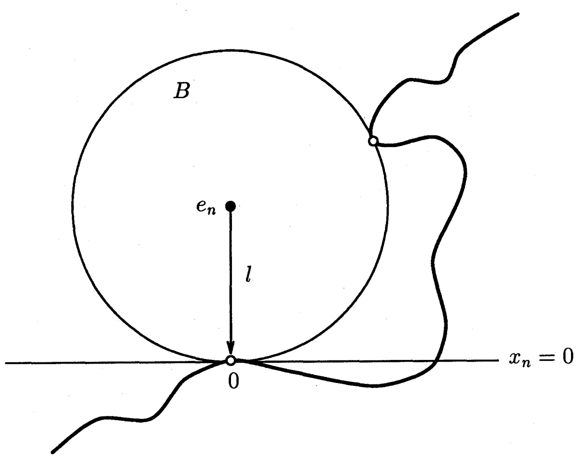

Figure (4.4)

Let us study for a while the Fried metric on B. First of all for any $x_0 \in B$, we have $r(x_0) \leq |x_0|$. In fact if not, the origin 0 is contained in

$$A = B \cup \{|x - x_0| < r(x_0)\}.$$

A has a copy containing a^*. This contradicts the completeness of l^*. Thus we have $g_F \geq g_G$ on B, where

$$g_G = \frac{g_E}{|x|^2}.$$

For any $x \in B$, let $\theta = \theta(x)$ be the angle of the vector $\overrightarrow{0x}$ and l. We have

Subclaim (4.12.1). $\qquad d_F(x, l) \geq d_G(x, l) = \theta.$

Proof. Let $\gamma(t)$ be a smooth path in B combining x and a point in l. Denote by $\text{length}_G(\gamma)$ the length of γ w.r.t. d_G. Let

$$\gamma(t) = |\gamma(t)| p(t).$$

We have

$$|\gamma'(t)| \geq |\gamma(t)||p'(t)|.$$

In fact, since $|p(t)| = 1$, we have $(p(t), p'(t)) = 0$ and

$$\gamma'(t) = |\gamma(t)|' p(t) + |\gamma(t)| p'(t).$$

Therefore we obtain the following equality.

$$\mathrm{length}_G(\gamma) = \int_0^1 \frac{|\gamma'(t)|}{|\gamma(t)|} dt \geq \int_0^1 |p'(t)| dt \geq \theta.$$

On the other hand it is easy to show that for a suitable choice of γ, one has $\mathrm{length}_G \gamma = \theta$. Q.E.D.

Now by Claim (4.9), There exist a compact submanifold $N_C = N - \mathrm{Int} U$ which contains $\pi(l^*)$ and a sequence $t_i \uparrow \infty$ such that for some $c \in N_C$,

$$d_F(\pi(l^*(t_i)), c) \downarrow 0.$$

Also assume that $d_F(\pi(l^*(t_1)), c)$ is sufficiently small. Then by (4.12.1), there exists a point $b^* \in B^*$ such that $c = \pi(b^*)$ and

$$d_F(l^*(t_1), b^*) = d_F(\pi(l^*(t_1)), c).$$

Now there exists a sequence $\{\xi_i\}$ of deck transformations such that

$$d_F(l^*(t_i), \xi_i b^*) \downarrow 0.$$

See Figure (4.5).

Thus passing to the model $B \subset \mathbf{R}^n$, we may assume the following. Let $f_i = \varphi(\xi_i) \in ES(\mathbf{R}^n)$ and $b = D(b^*) \in B$.

(1) $f_i(b) \in B$.
(2) $\theta(f_i(b)) \to 0$.
(3) $f_i(b) \to 0$.
(4) $P(f_i) \to P_0 \in O(n)$.
(5) $\|f_i\| \to 0$.

Notice that (5) follows from (3) since

$$\|f_i\| = \frac{r(f_i(b))}{r(b)} \leq \frac{|f_i(b)|}{r(b)} \to 0.$$

See Figure (4.6).

Now for $i \gg 1$, taking $j \gg i$, we may assume

(6) $P(f_i f_j^{-1})$ is very near E ,
(7) $\|f_i f_j^{-1}\|$ is very large.

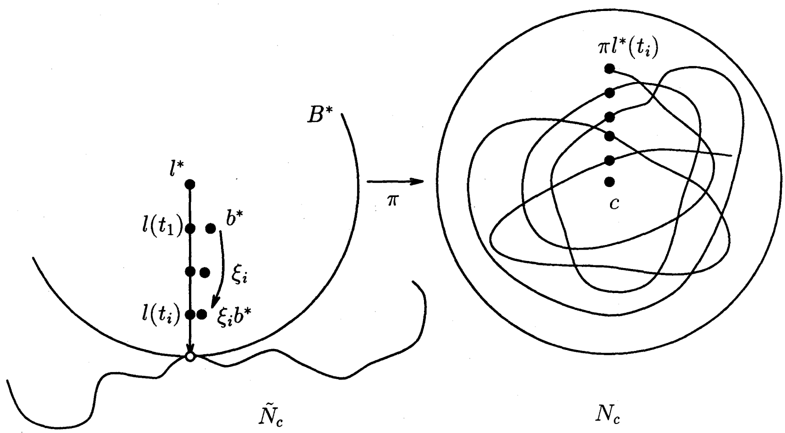

Figure (4.5)

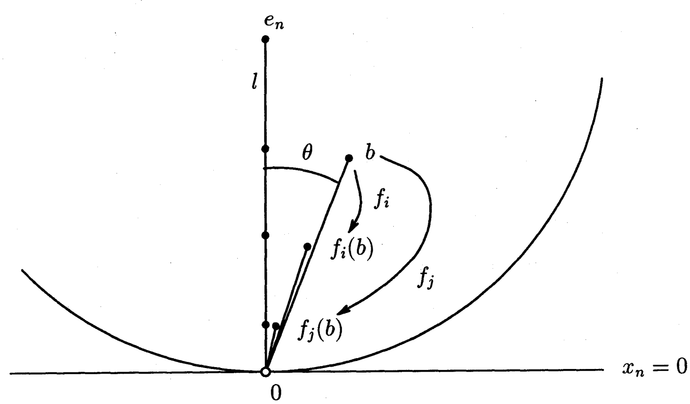

Figure (4.6)

Next by (2), $\overrightarrow{0f_j(b)}$ is almost parallel to $\overrightarrow{e_n}$ and is almost perpendicular to ∂B. Applying $f_i f_j^{-1}$, we still have that

(8) $\overrightarrow{f_i f_j^{-1}(0) f_i(b)}$ is almost parallel to $\overrightarrow{e_n}$,

(9) $\overrightarrow{f_i f_j^{-1}(0) f_i(b)}$ is almost perpendicular to $f_i f_j^{-1}(\partial B)$.

In fact (8) follows from (6) and (9) from the fact that $f_i f_j^{-1}$ is an

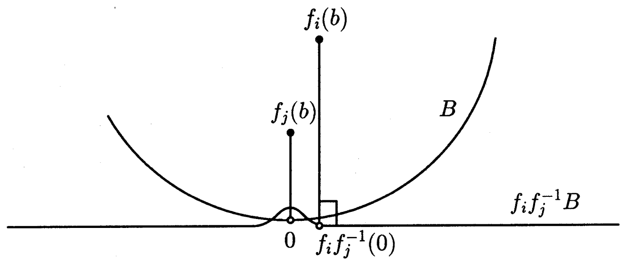

Figure (4.7)

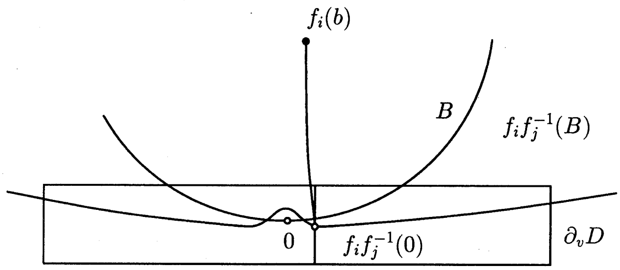

Figure (4.8)

Euclidean similarity. See Figure (4.7).

On the other hand, notice that $\xi_i\xi_j^{-1}B^* \cap B^*$ is nonempty and $\xi_i\xi_j^{-1}B^* \cup B^*$ is a copy of $f_i f_j^{-1}B \cup B$. Therefore by the completeness of l^*, we have that

(10) $f_i f_j^{-1}(0) \notin B$,

(11) $0 \notin f_i f_j^{-1}(B)$.

Let

$$f_i f_j^{-1}(0) = (\alpha_1, \cdots, \alpha_n)$$

and for $M \gg 1$ and $0 < \varepsilon \ll 1$, let

$$D = \{|x_i - \alpha_i| \leq M \ (1 \leq i \leq n-1), \ |x_n - \alpha_n| \leq \varepsilon\}.$$

Then by (5), (8) and (9), (taking $j \gg i \gg 1$ even greater) we have

$$\partial(f_i f_j^{-1}B) \cap \partial D = \partial(f_i f_j^{-1}B) \cap \partial_v D,$$

where ∂_v denotes the vertical boundary. See Figure (4.8).

We have by (11) that $\alpha_n > -\varepsilon$ and by (10) that $\alpha_n < \varepsilon$. It also follows that $f_i f_j^{-1}(0)$ is very near 0.

This shows that any x in the half space $\{x_n > 0\}$ is in $f_i f_j^{-1} B$ for some $j \gg i \gg 1$. Since $f_i f_j^{-1} B \cup B$ has a copy containing a^*, the proof of (4.12) is now complete. Q.E.D.

Step 2. In Step 1, for any point $a^* \in \widetilde{N}^*$, we have found a copy of half space containing $B^*(a^*, r(a^*))$. We have;

Claim (4.13). *A copy of half space H^* containing $B^*(a^*, r(a^*))$ is unique.*

Proof. Clearly H is tangent to $B(a, r(a^*))$ and the radius to the point of tangency is the developing image of a short complete line. In other words, there exists a unique short complete line in $\widetilde{N}^*$ which is contained in $B^*(a^*, r(a^*))$. This shows the uniqueness of H^*. Q.E.D.

Definition (4.14). H^* of (4.13) is denoted by $H^*(a^*)$ and its image by D by $H(a^*)$. The point of tangency of $H(a^*)$ and $B(a, r(a^*))$ is denoted by $p(a^*)$.

Notice that maximal copy of half space containing a^* may not be unique. Since D is a submersion,

$$D|_{\,\mathrm{Cl}H^*(a^*)} : \mathrm{Cl}H^*(a^*) \to \mathbf{R}^n$$

is injective and $D(\mathrm{Fr}H^*(a^*))$ is an open subset of $\partial H(a^*)$.

Definition (4.15). For $a^* \in \widetilde{N}^*$, denote

$$L(a^*) = \partial H(a^*) \setminus D(\mathrm{Fr}H^*(a^*)) \subset \mathbf{R}^n.$$

In other words, $x \in L(a^*)$ if and only if $x = \lim_{t\to\infty} l(t)$ for some short complete line l^* such that $l^*(0) = a^*$. See Figure (4.9).

For $b \in \mathrm{Cl}H(a^*) \setminus L(a^*) \subset \mathbf{R}^n$, we denote by b^* the unique point in $\mathrm{Cl}H^*(a^*) \subset \widetilde{N}^*$ such that $D(b^*) = b$.

Claim (4.16). *For $b \in \mathrm{Cl}H(a^*) \setminus L(a^*)$, $\partial H(b^*)$ passes through $p(a^*)$.*

Proof. Suppose not. We have $a \notin H(b^*)$ since $H(a^*) \cup H(b^*)$ has a copy in $\widetilde{N}^*$. Consider the transformation

$$f_j f_i^{-1} = (f_i f_j^{-1})^{-1} \in ES(\mathbf{R}^n)$$

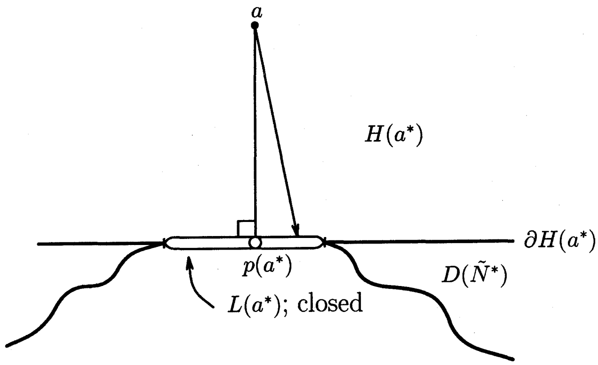

Figure (4.9)

of Step 1. Recall that $\|f_j f_i^{-1}\|$ is very small, $P(f_j f_i^{-1})$ is very near E and $f_j f_i^{-1}$ has a fixed point near $p(a^*)$. Thus $f_j f_i^{-1}(\partial H(b^*))$ is almost parallel to $\partial H(b^*)$ and much near $p(a^*)$. Clearly

$$H(a^*) \cup H(b^*) \cup f_j f_i^{-1}(H(b^*))$$

has a copy in $\widetilde{N}^*$. This contradicts that $p(b^*) \in L(b^*)$. See Figure (4.10). Q.E.D.

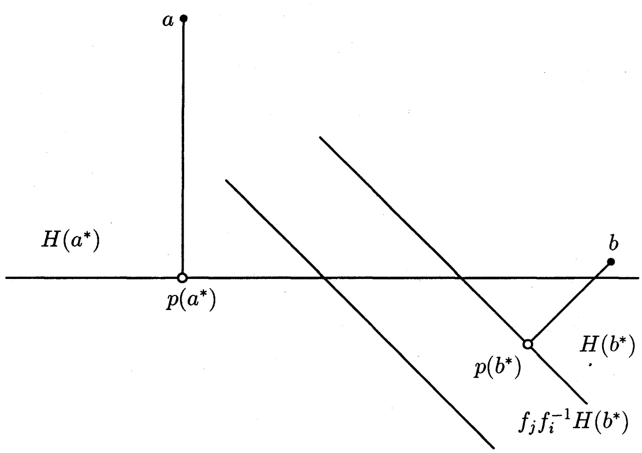

Figure (4.10)

Claim (4.17). *$L(a^*)$ is an affine subspace of $\mathbf{R}^n$.*

Proof. Let $x, y \in L(a^*)$. Clearly $x = p(c^*)$ for some $c^* \in H(a^*)$. Likewise for y. If a point b on the line passing x and y does not belong to $L(a^*)$, apply (4.16) to b. Then $\partial H(b^*)$ passes through x and y, that is, through b. A contradiction. Q.E.D.

Claim (4.18). *The correspondence $a^* \mapsto L(a^*)$ is locally constant.*

Proof. Take $b^* \in H^*(a^*)$. Then $\partial H(b^*)$ passes through $p(a^*)$ by (4.16) and $p(a^*) \in L(b^*)$. Since $L(b^*) \cap H(a^*) = \phi$, we have $L(b^*) \subset \partial H(a^*) \cap \partial H(b^*)$. Likewise, $L(a^*) \subset \partial H(a^*) \cap \partial H(b^*)$. It follows easily that $L(a^*) = L(b^*)$. Q.E.D.

Since $\widetilde{N}^*$ is connected, $L(a^*)$ is independent of the choice of $a^* \in \widetilde{N}^*$. Denote $L = L(a^*)$.

Claim (4.19). *The developing map D is a covering map onto a component of $\mathbf{R}^n \setminus L$.*

Proof. Clearly no points of $\widetilde{N}^*$ are mapped by D into L. Also we have that points in $\mathbf{R}^n \setminus L$ are evenly covered by D. Let us consider the point ∞. For $\dim L \geq 1$, $\infty \in \mathrm{Cl}L$ cannot be in Image(D). For $\dim(L) = 0$ (say $L = \{0\}$), if $\infty \in$ Image(D), then one can show that

$$D : \widetilde{N} \to \widehat{\mathbf{R}}^n \setminus \{0\}$$

is a homeomorphism. But $H \subset ES(\mathbf{R}^n)$ has ∞ as a fixed point. A contradiction. Q.E.D.

Step 3.

Lemma (4.20). *Let $\Gamma = \langle f, g \rangle \subset ES(\mathbf{R}^n)$, where*

(1) $\|f\| \neq 1, \quad f(a) = a \quad (a \in \mathbf{R}^n)$,
(2) $g(a) \neq a$.

Then Γ is indiscrete.

Proof. Assume $\|f\| < 1$. Let $h = g \circ f \circ g^{-1}$. Then $\|h\| = \|f\|$ and $h(g(a)) = g(a)$. Let $h_n = f^n \circ h \circ f^{-n}$. We have $\|h_n\| = \|h\| = \|f\|$, the fixed point of h_n is $f^n(g(a))$ and $f^n(g(a)) \to a \ (n \to \infty)$. That is, $h_n \to f$. This shows (4.20). Q.E.D.

Now we shall complete the proof of Theorem (4.4). First of all if $L = \{0\}$, then by Step 2, $D : \widetilde{N} \to \mathbf{R}^n \setminus \{0\}$ is a homeomorphism. That is, N is a Hopf manifold. This contradicts our hypothesis.

Consider the case $\dim L \neq n-2$. Suppose for simplicity that $L = \mathbf{R}^q$. By (4.19), D is a homeomorphism onto a component V of $\mathbf{R}^n \setminus \mathbf{R}^q$. Thus the holonomy group H must be discrete. By Step 1, there exists $f \in H$ such that $\|f\| \neq 1$. Clearly $f(L) = L$. Assume $f(0) = 0$. By (4.20), we have $g(0) = 0$ for any $g \in H$. Therefore $g(\mathbf{R}^{n-q}) = \mathbf{R}^{n-q}$, where $\mathbf{R}^{n-q}$ is the orthogonal complement of $\mathbf{R}^q$. When identified by D , Fried metric is given by

$$g_F = \frac{g_E}{|x_2|^2},$$

where $x = (x_1, x_2)$ $(x_1 \in \mathbf{R}^q, x_2 \in \mathbf{R}^{n-q})$. Since $N = N^*$ is compact, d_F is totally bounded. That is, there exists $K > 0$ such that for any $x, y \in V$, $d_F(x, gy) < K$ for some $g \in H$. But this is impossible if we choose $y \in \mathbf{R}^{n-q} \cap V$ and x_1 as large as desired.

Finally suppose $L = \mathbf{R}^{n-2}$. This case needs extra care. Since $\mathbf{R}^n \setminus \mathbf{R}^{n-2}$ is not simply connected, D is not a homeomorphism and H may not be discrete. Denote by $R_\theta \in ES(\mathbf{R}^n)$ the rotation by angle θ around $\mathbf{R}^{n-2}$. Let

$$\mathrm{Stab}(\mathbf{R}^{n-2}) = \{f \in ES(\mathbf{R}^n) \mid f(\mathbf{R}^{n-2}) = \mathbf{R}^{n-2}\}.$$

Notice that R_θ commutes with an element of $\mathrm{Stab}(\mathbf{R}^{n-2})$. Let

$$H^{n-1} = \{x_{n-1} > 0, x_n = 0\}.$$

Define a homeomorphism

$$h : H^{n-1} \times S^1 \to \mathbf{R}^n \setminus \mathbf{R}^{n-2}$$

by $h(x,t) = R_{2\pi t}x$. The universal covering of $\mathbf{R}^n \setminus \mathbf{R}^{n-2}$ is identified with $H^{n-1} \times \mathbf{R}$. Then as is easily shown, the lift of $\mathrm{Stab}(\mathbf{R}^{n-2})$ is identified with $ES(\mathbf{R}^{n-2}) \times \mathbf{R}$. That is, we have the following equivariant mapping of (G, X)-pairs

$$(ES(\mathbf{R}^{n-2}) \times \mathbf{R},\ H^{n-1} \times \mathbf{R}) \longrightarrow (\mathrm{Stab}(\mathbf{R}^{n-2}),\ \mathbf{R}^n \setminus \mathbf{R}^{n-2}).$$

The DH-pair

$$(D, \varphi) : (\widetilde{N}, \pi_1(N)) \longrightarrow (\mathbf{R}^n \setminus \mathbf{R}^{n-2},\ \mathrm{Stab}(\mathbf{R}^{n-2}))$$

clearly lifts to a DH-pair

$$(D', \varphi') : (\widetilde{N}, \pi_1(N)) \longrightarrow (H^{n-1} \times \mathbf{R},\ ES(\mathbf{R}^{n-2}) \times \mathbf{R}).$$

Since D' is a homeomorphism, the image $\widetilde{H} = \varphi'(\pi_1(N))$ is discrete. As before, $\widetilde{H}$ contains

$$(f, t) \in ES(\mathbf{R}^{n-2}) \times \mathbf{R}$$

such that $\|f\| \neq 1$. Let $f(0) = 0$. Since $ES(\mathbf{R}^{n-2})$ and $\mathbf{R}$ commute, the argument of (4.20) is also valid and we have $g(0) = 0$ for any $(g, s) \in \widetilde{H}$. The rest of the proof is similar.

§5. Limit set

The purpose of this section is to define limit set for flat conformal manifolds of an arbitrary type. In this section flat conformal manifolds are to be connected and compact, unless otherwise specified.

First of all consider an arbitrary subgroup Γ of $\mathcal{M}(S^n)$. (Γ may not be discrete. It may not be even finitely generated.) Let us begin by defining the limit set for the group Γ by looking at its action on S^n. There are four different ways and all of them are natural and useful.

Definition (5.1). Let $L_F = L_F(\Gamma)$ be the closure of the set of the fixed points of loxodromic or parabolic elements of Γ.

Definition (5.2). Let $L_J = L_J(\Gamma)$ be the set of points $x \in S^n$ such that for any neighbourhood U of x, the family $\{f|_U\}_{f \in \Gamma}$ is not equicontinuous.

Definition (5.3). Let $L_P = L_P(\Gamma)$ be the set of points $x \in S^n$ such that for any neighbourhood U of x, the set $\{f \in \Gamma \mid fU \cap U \neq \phi\}$ is not precompact in $\mathcal{M}(S^n)$.

By definition L_F, L_J and L_P are closed Γ-invariant subsets of S^n. Of course L_J is an analogy of Julia set in one dimensional complex dynamical system. Notice that if Γ is discrete, then $S^n \setminus L_P$ coincides with the domain of discontinuity Ω_Γ defined in (3.24).

Definition (5.4). Let $L_\omega = L_\omega(\Gamma)$ be the set of accumulation points in S^n of the orbit Γa of a certain point $a \in D^{n+1}$.

This definition is independent of the choice of $a \in D^{n+1}$. In fact, for another point $b \in D^{n+1}$ and for $\gamma_k \in \Gamma$, we have $d_H(\gamma_k(a), \gamma_k(b)) =$

$d_H(a, b)$, where d_H denotes the hyperbolic distance. By the difference between the hyperbolic distance and the Euclidean distance, we have

$$\lim_{k\to\infty} \gamma_k(a) = x \iff \lim_{k\to\infty} \gamma_k(b) = x.$$

Also note that L_ω is closed and Γ-invariant. In fact if $\lim_{k\to\infty} \gamma_k(a) = x$, then we have $\lim_{k\to\infty} \gamma\gamma_k(a) = \gamma(x)$ for $\gamma \in \Gamma$. Below we shall prove the *minimality* of L_ω.

Definition (5.5). Let A be a Γ-invariant closed subset of S^n such that $\mathrm{Card}(A) \geq 2$. The *convex hull* of A, denoted by $C(A)$, is defined to be the convex hull in (D^{n+1}, g_H) of all the geodesics combining two points of A.

Clearly $C(A)$ is a closed Γ-invariant subset of D^{n+1}. See Figure (5.1).

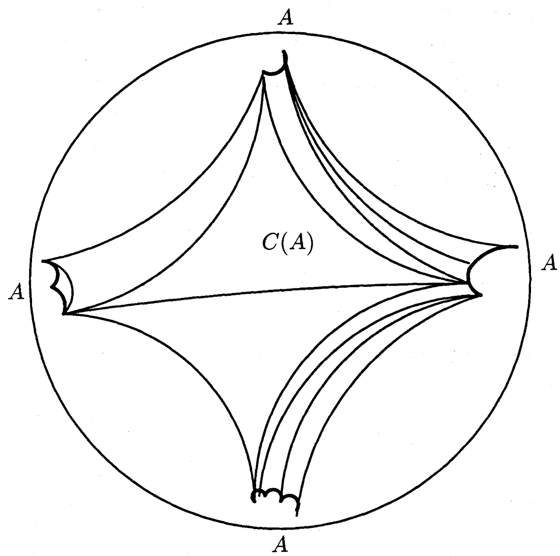

Figure (5.1)

Lemma (5.6). *Let A be an arbitrary Γ-invariant closed set such that* $\mathrm{Card}(A) \geq 2$. *Then we have* $L_\omega(\Gamma) \subset A$.

Proof. Take the point $a \in D^{n+1}$ of (5.4) inside $C(A)$. Then the orbit of a cannot evade $C(A)$. This shows (5.6). Q.E.D.

Corollary (5.7). *If* Γ *has no fixed point in* S^n, *then* $L_\omega(\Gamma)$ *is the unique minimal set, i.e., it is contained in any nonempty closed* Γ*-invariant subset of* S^n.

For $\gamma \in \mathcal{M}(S^n)$, denote by $CI(\gamma)$ the convex hull in $\mathbf{R}^{n+1}$ of the isometric sphere $I(\gamma)$.

Lemma (5.8). *For* $\{\gamma_k\} \subset \Gamma \subset \mathcal{M}(S^n)$ *such that* $\gamma_k \to \infty$, *we have* $d(CI(\gamma_k), L_\omega(\Gamma)) \to 0$.

Proof. For the properties of isometric spheres, see (2.11)$\sim$(2.16). We shall prove (5.8) by establishing $d(CI(\gamma_k^{-1}), L_\omega) \to 0$. Recall that $\gamma_k \to \infty$ if and only if $\operatorname{radius} I(\gamma_k^{-1}) = \operatorname{radius} I(\gamma_k) \to 0$ and that $I(\gamma_k)$ is always orthogonal to S^n. Therefore given a point $a \in D^{n+1}$, we have $a \notin CI(\gamma_k)$ for large k. That is, $\gamma_k a \in CI(\gamma_k^{-1})$. See Figure (5.2). Since $d(\gamma_k a, L_\omega) \to 0$, it follows that $d(CI(\gamma_k^{-1}), L_\omega) \to 0$.

Q.E.D.

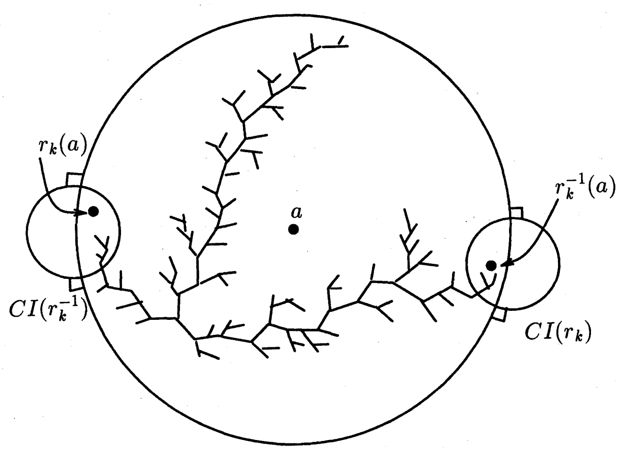

Figure (5.2)

Definition (5.9). Two points $x, y \in L_\omega$ are called *dual* in case there exists $\gamma_k \in \Gamma$ such that $\gamma_k(a) \to x$ and $\gamma_k^{-1}(a) \to y$ $(a \in D^{n+1})$.

This is also independent of the choice of a. For $x \in L_\omega$, let D_x be the set of points in L_ω which are dual to x. Diagonal argument shows that D_x is a closed subset. Also if $\gamma_k a \to x$ and $\gamma_k^{-1} a \to y$, then for $\gamma \in \Gamma$, $\gamma_k\gamma^{-1}a \to x$ and $\gamma\gamma_k^{-1}a \to \gamma y$. That is, D_x is Γ-invariant.

Lemma (5.10). *If* Γ *has no fixed point in* S^n, *then any two points of* L_ω *are dual.*

Proof. For $L_\omega \neq \phi$, the condition assures that $\mathrm{Card}(D_x) \geq 2$. Since D_x is closed and Γ-invariant, we have by (5.6) that $D_x \supset L_\omega$. Q.E.D.

Proposition (5.11). *If* Γ *has no fixed point in* S^n, *then for any pair of distinct points* $x, y \in L_\omega(\Gamma)$, *there exists a loxodromic transformation whose two fixed points are arbitrarily near* x *and* y.

Proof. By (5.10), we have that x and y are dual. Let $\gamma_k(a) \to x$ and $\gamma_k^{-1}(a) \to y$ $(\gamma_k \in \Gamma,\ a \in D^{n+1})$. We have clearly $\gamma_k \to \infty$. By applying the argument of (5.8), we obtain that $CI(\gamma_k^{-1})$ is sufficiently near x and $CI(\gamma_k)$ is sufficiently near y. Since $x \neq y$, we may assume that $CI(\gamma_k^{-1}) \cap CI(\gamma_k) = \phi$. It is easy to show that γ_k is a loxodromic transformation with one fixed point in $CI(\gamma_k^{-1})$ and the other in $CI(\gamma_k)$. This shows (5.11). Q.E.D.

Lemma (5.12). $L_\omega(\Gamma) = \phi$ *if and only if* Γ *is precompact.*

Proof. This follows at once from the fact that for any $a \in D^{n+1}$, the isotropy subgroup of $\mathcal{M}(S^n)$ at a is isomorphic to a compact group $O(n+1)$. Q.E.D.

Proposition (5.13). *A subgroup* Γ *of* $\mathcal{M}(S^n)$ *is precompact if and only if it has a common fixed point in* D^{n+1}. *In particular, maximal compact subgroups of* $\mathcal{M}(S^n)$ *are conjugate to* $O(n+1)$.

Proof. The if part is trivial. Let us show that a compact subgroup Γ has a fixed point in D^{n+1}. (Pass to $\mathrm{Cl}\Gamma$ if Γ is noncompact.) Choose an arbitrary point $a \in D^{n+1}$. Let $d = \mathrm{diam}_H(\Gamma a)$ and let $d_H(a, ga) = d$ $(g \in \Gamma)$. Let a_1 be the middle point of a and ga. For any $h \in \Gamma$, consider the hyperbolic tetrahedron with vertices a, ga, ha and hga. All the edges have length $\leq d$. Easy hyperbolic trigonometry shows $d(a_1, ha_1) \leq cd$ for some (computable) $c \in (0,1)$. That is, $\mathrm{diam}_H(\Gamma a_1) \leq cd$. Likewise construct a_2, a_3 etc. Let $a_\infty = \lim_{k\to\infty} a_k$. We have $\mathrm{diam}_H(\Gamma a_\infty) = 0$. That is, a_∞ is a fixed point of Γ. Q.E.D.

Corollary (5.14). *Unless* Γ *has a fixed point in* $D^{n+1} \cup S^n$, Γ *contains a loxodromic transformation.*

Proof. The condition implies that $\mathrm{Card} L_\omega \geq 2$. Therefore (5.14) follows from (5.11). Q.E.D.

Theorem (5.15). *For an arbitrary subgroup* $\Gamma \subset \mathcal{M}(S^n)$, *we have*

$$L_F(\Gamma) \subset L_J(\Gamma) = L_P(\Gamma) = L_\omega(\Gamma).$$

Moreover unless $L_F(\Gamma) = \phi$ *and* $L_\omega(\Gamma)$ *is a singleton, we have* $L_F(\Gamma) = L_\omega(\Gamma)$.

Proof. $L_F \subset L_J \cap L_P$: This follows at once from the local models of loxodromic and parabolic transformations.

$L_J \cup L_P \subset L_\omega$: Suppose $x \notin L_\omega$. Then by (5.8), for small $\varepsilon > 0$ and for a small neighbourhood U of x, we have that $CI(\gamma) \cap U = \phi$ if radius $I(\gamma) < \varepsilon$ and $\gamma \in \Gamma$. But the set of γ such that radius $I(\gamma) \geq \varepsilon$ is precompact by (2.16). It follows from (2.12) that $x \notin L_J \cup L_P$.

We shall divide the proof of the remaining part into four cases.

Case 1. Γ has no fixed point in $D^{n+1} \cup S^n$.

By (5.14) we have $L_F \neq \phi$. Therefore it follows from (5.7) that $L_\omega \subset L_F$. Together with the inclusion we have already established, we obtain that $L_F = L_J = L_P = L_\omega$.

Case 2. Γ has a fixed point in D^{n+1}.

By (5.12) and (5.13), this is equivalent to $L_\omega = \phi$. We have $L_F = L_J = L_P = L_\omega = \phi$.

Case 3. Γ has a fixed point $y \in S^n$ and that $L_\omega \setminus \{y\} \neq \phi$.

Let $x \in L_\omega \setminus \{y\}$. Notice that parabolic and elliptic transformations of the isotropy group Γ_y keep horospheres at y invariant. Therefore there must exist loxodromic transformations $\gamma_n \in \Gamma$ such that $\gamma_n a \to x$ $(a \in D^{n+1})$. Then $\gamma_n^{-1} a \to y$. That is, we have $y \in L_\omega$ and $L_\omega \subset L_F$, showing that $L_F = L_J = L_P = L_\omega$.

Case 4. $L_\omega = \{y\}$.

This is the only case where we cannot prove $L_\omega \subset L_F$. In order to complete the proof of (5.15), it suffices to show that $y \in L_F \cap L_P$. Since $L_\omega \neq \phi$, there exists a sequence $\{\gamma_k\} \subset \Gamma$ such that $\gamma_k \to \infty$. Since $\gamma_k y = y$ and γ_k are not loxodromic, we have $y \in CI(\gamma_k) \cup CI(\gamma_k^{-1})$

and $CI(\gamma_k) \cap CI(\gamma_k^{-1}) \neq \phi$. Hence for any neighbourhood U of y, $CI(\gamma_k) \subset U$ for sufficiently large $k > 0$. But by (2.12) and (2.16), we have that $\{\gamma_k|_U\}$ is not equicontinuous. That is, $y \in L_J$. Clearly we have $\gamma_k(U) \cap U \neq \phi$. Therefore $y \in L_P$. Q.E.D.

The following corollary was already used in §3.

Corollary (5.16). *Suppose a discrete group* Γ *admits an invariant open set* Ω *such that* $S^n \setminus \Omega$ *is neither empty nor a singleton. Then* Γ *acts on* Ω *discontinuously.*

Proof. Since Γ is discrete, $S^n \setminus L_P$ coincides with the domain of discontinuity. By (5.6), we have $S^n \setminus \Omega \supset L_\omega = L_P$. Therefore Ω is contained in the domain of discontinuity. Q.E.D.

We will give an example of Γ for which $L_F(\Gamma) = \phi$ and $L_\omega(\Gamma)$ is a singleton. The same example can be found in Kulkarni ([44]).

Example (5.17). Let us work with $\mathcal{M}(\widehat{\mathbf{R}}^4)$. We shall construct a subgroup Γ such that $L_F(\Gamma) = \phi$ and that $L_\omega(\Gamma) = \{\infty\}$. Equivalently, the group Γ consists purely of elliptic elements, keeps ∞ fixed and does not have a fixed point in H^5. By (1.9) and (2.24), any element $f \in \Gamma$ has the form

$$(*) \qquad f(x) = Px + b \quad (P \in O(4), \quad b \in \mathbf{R}^4).$$

Notice that f is elliptic if and only if f has a fixed point $a \in \mathbf{R}^4$. In fact, then, the point $(a, x) \in H^5$ $(x > 0)$ is kept fixed by the extended action of f. Likewise the group Γ has a fixed point in H^5 if and only if it has a fixed point in $\mathbf{R}^4$. Therefore our purpose is to construct a group Γ consisting of transformations f of $(*)$ such that

$f \in \Gamma$ has a fixed point in $\mathbf{R}^4$.

Γ does not have a fixed point in $\mathbf{R}^4$.

First of all let us show that there exist $P, Q \in SO(4)$ such that for any nontrivial reduced word $w(P, Q)$, we have $|w(P, Q) - E| \neq 0$. Notice that for a (possibly real) algebraic group G, if G contains a free group of two generators, then for any nontrivial reduced word $w(x, y)$, the equation $w(x, y) = \mathrm{id}$ defines a proper subvariety (that is, a subvariety of positive codimension) of $G \times G$. The converse also holds since the complements of subvarieties of positive codimension are open dense subsets and their countable intersection is nonempty. Therefore a real algebraic group contains a free subgroup of two generators if and only if its complexification does. Now it is well known that $SO(2, 1)$

has a free subgroup of two generators. Clearly $SO(2,1)_{\mathbf{C}} = SO(3)_{\mathbf{C}}$. Therefore by the above consideration, $SO(3)$, hence its universal covering $SU(2)$, has a free group of two generators also. Considering the inclusion of $SU(2)$ into $SO(4)$, we obtain the desired P and Q.

Let

$$f : x \mapsto Px \quad \text{and} \quad g : x \mapsto Qx + b \ \ (b \neq 0).$$

Now $\Gamma = \langle f, g \rangle$ consists purely of elliptic transformations, since any element of Γ has the linear part without eigenvalue 1 and hence has a fixed point in $\mathbf{R}^4$. However f and g have no common fixed points in $\mathbf{R}^4$.

As a matter of fact, (5.17) implies that $L_F = L_\omega$ does not hold in higher dimension. However in low dimension, we have;

Theorem (5.18). *For* $\Gamma \subset \mathcal{M}(\widehat{\mathbf{R}}^n)$ $(n \leq 3)$*, we have*

$$L_F(\Gamma) = L_J(\Gamma) = L_P(\Gamma) = L_\omega(\Gamma).$$

Proof. All that need proof is that if $L_\omega = \{\infty\}$, then $L_F = \{\infty\}$. Equivalently, if Γ keeps ∞ fixed and if Γ does not have a fixed point in $\mathbf{R}^n$, then Γ contains nonelliptic transformations.

First of all for $n = 1$, there exist no elliptic transformations that keep ∞ fixed and there is nothing to prove.

For $n = 2$, assume that $f, g \in \mathcal{M}(\widehat{\mathbf{R}}^2)_\infty$ have no common fixed points in $\mathbf{R}^3$. Computation shows that $[f, g] = fgf^{-1}g^{-1}$ is parabolic, since the linear parts commute.

Finally let $n = 3$. It clearly suffices to verify for a group Γ consisting of orientation preserving transformations. Orientation preserving elliptic transformations in $\mathcal{M}(\widehat{\mathbf{R}}^3)_\infty$ are rotations around their axes. Let us show first that if two rotations f, g have disjoint axes, then the group $\langle f, g \rangle$ they generate contains a parabolic transformations. In fact, if the axes are parallel, then $[f, g]$ is parabolic. Suppose they are not parallel and assume for contradiction that fg^{-1} has a fixed point $x \in \mathbf{R}^n$. Then we have $f(x) = g(x) = y$. By Euclidean geometry, we have that the bisector of x and y contains the axes of f and g. (See Figure (5.3).) A contradiction.

Therefore if $\Gamma \subset \mathcal{M}(\widehat{\mathbf{R}}^3)_\infty$ is purely elliptic and have no common fixed points, then all the axes of transformarions of Γ must lie in a plane and all their rotation angles must be π. Therefore there exists an index two subgroup of Γ consisting of parabolic transformations. This contradiction shows (5.18). Q.E.D.

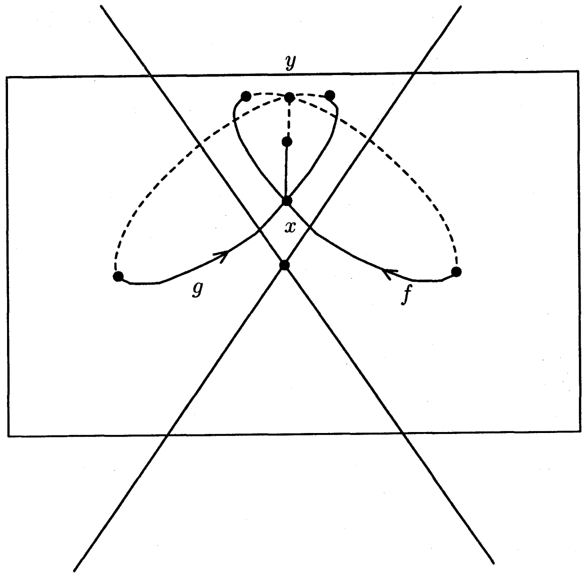

Figure (5.3)

Now let N be a connected closed flat conformal manifold modeled on $(\mathcal{M}(S^n), S^n)$. As before denote by D the developing map, by φ the holonomy homomorphism and by H the holonomy group. Hereafter by certain abuse, we consider a flat conformal manifold N to be equipped with a particular choice of developing map, holonomy homomorphism and holonomy groups. Our purpose is to define the limit set of N. So far, we already had four kinds of limit set in terms of the holonomy group H. For a flat conformal manifold, they are denoted by $L_F(N) = L_F(H)$ and so forth. We need one more definition, which is obtained by looking at the developing map.

Definition (5.19). Let $L_O = L_O(N)$ be the set of points x such that for any compact neighbourhood $\overline{U}$ of x, the inverse image $D^{-1}(\overline{U})$ has a nonempty and noncompact component.

As is shown easily, L_O is precisely the set of points which are not evenly covered by D.

For general closed (G, X)-manifolds, Kulkarni-Pinkall([45]) defined L_J and L_P and showed $L_J \supset L_P$ and $L_J \supset L_O$. They also showed that $L_J = L_P$ for closed flat conformal manifolds. The following is an elaboration of their rerult.

Theorem (5.20). *For a connected compact flat conformal manifold N, we have*

$$L_F(N) = L_\omega(N) = L_J(N) = L_P(N) = L_O(N).$$

Proof. $L_F = L_\omega$: If not, we have $L_F = \phi$, $L_\omega = \{a\}$ and H has a fixed point $a \in S^n$. But then by (4.4), N is isomorphic to either S^n, a Hopf manifold or a Euclidean space form. In any case we have $L_F = L_\omega$.

$L_\omega \subset L_O$: If $\mathrm{Card} L_O \geq 2$, then this follows at once from (5.6). If $L_O = \{a\}$, then a is a fixed point of H and again by (4.4), we have $L_\omega = L_O$. If $L_O = \phi$, then D is a covering map onto S^n . That is, N is a spherical space form and we have $L_\omega = \phi$.

$L_O \subset L_J$: Denote by $\overline{B}(x,r)$ the closed disk centered at $x \in S^n$ of radius $r > 0$ w.r.t. the spherical metric. The proof is by contradiction. Suppose $b \in L_O \setminus L_J$. That is, we assume

(1) For some $r_k \downarrow 0$, $D^{-1}\overline{B}(b, r_k)$ has a noncompact component E_k.

(2) $\{f|_{\overline{B}(b,r_1)}\}_{f\in H}$ is equicontinuous.

Choose $a_k \in E_k$. (Note that $D(a_k) \to b$.) Then since N is compact, there exists $\xi_k \in \pi_1(N)$ such that $\xi_k a_k$ is in some compact region of $\widetilde{N}$. Assume $\xi_k a_k \to c$. Choose a compact neighbourhood $\overline{V}$ of c such that

$$D|_{\overline{V}} : \overline{V} \to \overline{B}(D(c), 2\varepsilon)$$

is a homeomorphism for some $\varepsilon > 0$. Assume also $D(\xi_k a_k) \in \overline{B}(D(c), \varepsilon)$ for any $k > 0$. Choose $\delta > 0$ so

$$x, y \in \overline{B}(b, r_1), \quad d(x,y) < 2\delta \implies d(f(x), f(y)) < \varepsilon \quad \text{for any } f \in H.$$

For $r_k < \delta$, we have

$$\begin{aligned}
&\overline{B}(b, r_k) \subset \overline{B}(D(a_k), 2\delta), \\
&\varphi(\xi_k)(\overline{B}(D(a_k), 2\delta)) \subset \overline{B}(D(\xi_k a_k), \varepsilon), \\
&\overline{B}(D(\xi_k a_k), \varepsilon) \subset \overline{B}(D(c), 2\varepsilon).
\end{aligned}$$

Therefore

$$\varphi(\xi_k)(\overline{B}(b, r_k)) \subset \overline{B}(D(c), 2\varepsilon),$$

Now $\xi_k E_k$ is the component containing $\xi_k a_k$ of $D^{-1}(\varphi(\xi_k)(\overline{B}(b, r_k))$ and is contained in $\overline{V}$. Therefore $\xi_k E_k$, hence E_k, is compact. A contradiction. Q.E.D.

Definition (5.21). The set in (5.20) is called the *limit set* of N and is denoted by $L = L(N)$.

We summarize fundamental properties of L in the following proposition.

Proposition (5.22). *$L(N)$ is a closed H-invariant subset of S^n. Further if N is not isomorphic to a Hopf manifold, then $L(N)$ is the unique minimal set. In particular, if $L(N) \neq S^n$, then we have* $\mathrm{Int}L(N) = \phi$.

Below we shall give applications of (5.20). The first one (5.23) is originally due to Kamishima([25]). See also Gusevskiĭ-Kapovich([20]).

Corollary (5.23). *If the developing map D of a connected compact flat conformal manifold is not onto S^n, then D is a covering map onto its image.*

Proof. We need only consider the case where N is not a Hopf manifold. Then by (5.22), we have $L = L_O$ is contained in the complement of Image(D). That is, Image(D) is evenly covered by D.

Q.E.D.

The next application is found in Kulkarni-Pinkall ([45]), in which condition(2) below is mistakingly dropped.

Corollary (5.24). *Let N be a connected compact flat conformal manifold and let $\Omega = S^n \setminus L(N)$. Suppose*

(1) *Ω is connected and its fundamental group $\pi_1(\Omega)$ is finitely generated.*
(2) *For any point $x \in S^n$, there exists an arbitrarily small neighbourhood U such that $U \setminus L$ is connected.*

Then the developing map D is a covering map onto its image.

Proof. First of all let us prove that $D^{-1}(\Omega)$ is connected. In fact given any two points $a, b \in D^{-1}(\Omega)$, choose a path p in $\widetilde{N}$ joining a and b. The path p is covered by a finite union of small open set V_i. We may assume by (2) that $V_i \setminus D^{-1}(L) \approx D(V_i) \setminus L$ is connected. Then we can make a small change of p within $\bigcup_i V_i$ fixing the boundary points so that p is contained in $D^{-1}(\Omega)$. Therefore $D^{-1}(\Omega)$ is connected.

Now by (4.4), we need only consider the case where H has no fixed points in S^n. We need only show that $D(\widetilde{N}) \cap L = \phi$. Suppose the contrary. Choose a small compact ball $\overline{V}$ such that D is a homeomorphism on $\overline{V}$, that $\mathrm{Int}D(\overline{V}) \cap L \neq \phi$ and that $\mathrm{Ext}D(\overline{V}) \cap L \neq \phi$. Since $\pi_1(\Omega)$ is finitely generated, it is supported on some compact subset K of Ω. By (5.11), there exists a loxodromic transformation $f \in H$ with an attracting fixed point in $\mathrm{Int}D(\overline{V})$ and with a repelling fixed point outside $D(\overline{V})$. We have $f^n(K) \subset D(\overline{V})$ for some $n > 0$. Therefore $\pi_1(\Omega)$ is supported on $D(\overline{V}) \cap \Omega$. Now D gives a homeomorphism from $\overline{V} \cap D^{-1}(\Omega)$ onto $D(\overline{V}) \cap \Omega$. This shows $D_* : \pi_1(D^{-1}(\Omega)) \to \pi_1(\Omega)$ is an epimorphism. Since points in Ω are evenly covered by D, D gives a homeomorphism from $D^{-1}(\Omega)$ onto Ω. However $D^{-1}(D(\overline{V}))$ has a noncompact component, which is of course disjoint from $\overline{V}$. A contradiction. Q.E.D.

The condition (2) of (5.24) is in fact necessary. For, let Σ be a closed flat conformal 2-manifold corresponding to a B-group Γ ([3]). That is, $\Omega = S^n \setminus L$ is connected and simply connected and Σ is isomorphic to Ω/Γ. Apply the construction of (3.37) to Σ. We obtain a flat conformal structure with the same holonomy group and surjective developing map. All this is of course well known. For more general treatment, see e.g. Goldman ([16]).

We shall finish this section by studying type 2 flat conformal structures, i.e., with the developing maps covering maps and with indiscrete holonomy groups. First we give examples in dimension ≥ 3. (2-dimensional examples were already given in (3.32).) For our purpose the coordinates of $\widehat{\mathbf{R}}^n$ is convenient.

Consider $\widehat{\mathbf{R}}^{n-2} \subset \widehat{\mathbf{R}}^n$. As before, denote by $R_\theta \in \mathcal{M}(\widehat{\mathbf{R}}^n)$ the rotation by angle θ around $\widehat{\mathbf{R}}^{n-2}$. Let

$$H^{n-1} = \{x_{n-1} > 0, x_n = 0\}.$$

Define

$$h : H^{n-1} \times \mathbf{R} \to \widehat{\mathbf{R}}^n \setminus \widehat{\mathbf{R}}^{n-2}$$

by $h(x,t) = R_{2\pi t}x$. h is a universal covering. By (2.3), we have

$$\mathcal{M}(\widehat{\mathbf{R}}^{n-2}) = \{g \in \mathcal{M}(\widehat{\mathbf{R}}^n) \mid g(H^{n-1}) = H^{n-1}\}.$$

Let us define

$$\mathcal{S}(\widehat{\mathbf{R}}^{n-2}) = \{f \in \mathcal{M}(\widehat{\mathbf{R}}^n) \mid f(\widehat{\mathbf{R}}^{n-2}) = \widehat{\mathbf{R}}^{n-2}\}.$$

An element $f \in \mathcal{S}(\widehat{\mathbf{R}}^{n-2})$ carries H^{n-1} to a half plane bounded by $\widehat{\mathbf{R}}^{n-2}$. Clearly f is determined by $f(H^{n-1})$ and $f|_{\widehat{\mathbf{R}}^{n-2}}$. This shows that f commutes with R_θ. Therefore we have the epimorphism

$$\psi : \mathcal{M}(\widehat{\mathbf{R}}^{n-2}) \times \mathbf{R} \longrightarrow \mathcal{S}(\widehat{\mathbf{R}}^{n-2})$$

defined by $\psi(g,t) = R_{2\pi t}g$.

Consider the diagonal action of $\mathcal{M}(\widehat{\mathbf{R}}^{n-2}) \times \mathbf{R}$ on $H^{n-1} \times \mathbf{R}$. Then

$$(\psi, h) : (\mathcal{M}(\widehat{\mathbf{R}}^{n-2}) \times \mathbf{R},\ H^{n-1} \times \mathbf{R}) \longrightarrow (\mathcal{S}(\widehat{\mathbf{R}}^{n-2}),\ \widehat{\mathbf{R}}^{n} \setminus \widehat{\mathbf{R}}^{n-2})$$

is an equivariant mapping of (G, X)-pairs.

Example (5.25). Let $\Gamma \subset \mathcal{M}(\widehat{\mathbf{R}}^{n-2})$ be a discrete subgroup which acts freely on H^{n-1}. Suppose $M = H^{n-1}/\Gamma$ is compact. For any $\theta \in \mathbf{R} \setminus \{0\}$ and any homomorphism $\mu : \Gamma \to \mathbf{R}$, define

$$\overline{\varphi} : \Gamma \times \mathbf{Z} \to \mathcal{M}(\widehat{\mathbf{R}}^{n-2}) \times \mathbf{R}$$

by $\overline{\varphi}(\gamma, m) = (\gamma, \mu(\gamma) + m\theta)$. Using a triangulation of H^{n-1}/Γ, one can construct a continuous map $u : H^{n-1} \to \mathbf{R}$ such that $u(\gamma x) = \mu(\gamma) + u(x)$ for $\gamma \in \Gamma$ and $x \in H^{n-1}$. Define a homeomorphism

$$\overline{D} : \widetilde{M} \times \mathbf{R} \to H^{n-1} \times \mathbf{R}$$

by $\overline{D}(x,t) = (x, u(x) + \theta t)$, where we identify the universal covering $\widetilde{M}$ with H^{n-1}. Let $\varphi = \psi \circ \overline{\varphi}$ and $D = h \circ \overline{D}$. Then (D, φ) is clearly a DH pair for $M \times S^1$. Therefore it defines a flat conformal structure on $M \times S^1$. Since $\overline{D}$ is a homeomorphism, D is a covering map onto $\widehat{\mathbf{R}}^{n} \setminus \widehat{\mathbf{R}}^{n-2}$ and the holonomy group H is indiscrete e.g., if we choose $\theta \in \mathbf{R} \setminus \mathbf{Q}$. (Moreover for a suitable choice of μ, the "rotation part" of H is not even infinite cyclic.) Thus this is a type 2 flat conformal structure.

Conversely we have the following theorem which was first obtained by Gusevskiĭ-Kapovich ([20]) in dimension 3.

Theorem (5.26). *Suppose N is a type 2 connected closed flat conformal manifold modeled on $(\mathcal{M}(\widehat{\mathbf{R}}^{n}), \widehat{\mathbf{R}}^{n})$, where $n \geq 3$. Then by changing the DH pair within the equivalence class, we have $L(N) =$*

$\widehat{\mathbf{R}}^{n-2}$. *Moreover* N *is a hyperbolic manifold bundle over the circle whose holonomy map is an isometry.*

Proof. *Step* 1. $L(N) = \widehat{\mathbf{R}}^{n-2}$.

Let $\mathrm{Cl}H_0$ be the identity component of the closure $\mathrm{Cl}H$ of H. Since H is indiscrete, we have $\mathrm{Cl}H_0 \neq \{1\}$.

Case 1. $\mathrm{Cl}H_0$ *is noncompact.* In this case $L \equiv L_\omega(\mathrm{Cl}H_0)$ is nonempty. Notice that $\mathrm{Cl}H_0$ is a normal subgroup of $\mathrm{Cl}H$. This implies that L is invariant by the action of $\mathrm{Cl}H$ and hence by H. Therefore by (5.22), we have $L(N) \subset L$. On the other hand, it is easy to show that

$$L = L_\omega(\mathrm{Cl}H_0) \subset L_\omega(\mathrm{Cl}H) = L_\omega(H) = L(N).$$

Therefore we have $L = L(N)$.

Also by (4.4), we obtain that $\mathrm{Card}L \geq 3$. In fact, if for example $\mathrm{Card}L = 2$, then (4.4) implies that N or its double covering is a Hopf manifold, contrary to our hypothesis.

Let us show next that there is no fixed point of $\mathrm{Cl}H_0$ in L. Suppose on the contrary that there exists one, say x. Then for any $h \in H$, hx is also a fixed point of $\mathrm{Cl}H_0$, since $\mathrm{Cl}H_0$ is a normal subgroup of $\mathrm{Cl}H$. On the other hand, the orbit Hx is dense in L and therefore has cardinality ≥ 3. That is, there exist at least three fixed points of $\mathrm{Cl}H_0$ in L. This implies by the argument of (2.22) that $\mathrm{Cl}H_0$ has a fixed point in D^{n+1}, contradicting the assumption that $\mathrm{Cl}H_0$ is noncompact.

By (5.6) this implies that any $\mathrm{Cl}H_0$-orbit K in L is dense in L. Notice that K is an injectively immersed submanifold in $\widehat{\mathbf{R}}^n$. By (5.11), there exists a loxodromic transformation $f \in H$. We may assume for simplicity that $f(x) = \lambda P x$, $(\lambda > 1,\ P \in O(n))$ and that $0 \in K$. Clearly K is kept invariant by f. Now the smoothness of K at 0 implies that $K = \widehat{\mathbf{R}}^k$ for some $1 \leq k \leq n$. ($K = \mathbf{R}^k$ implies that ∞ is a fixed point of $\mathrm{Cl}H_0$, contradicting the above observation.) This shows $L = \widehat{\mathbf{R}}^k$. Since the developing map D is a covering map onto its image, we have $D(\widetilde{N}) \cap L(N) = \phi$. In particular we obtain $k \neq n$. Finally we have $k = n-2$, since otherwise D is a homeomorphism onto a connected component of $\widehat{\mathbf{R}}^n \setminus \widehat{\mathbf{R}}^k$ and H must be discrete.

Case 2. $\mathrm{Cl}H_0$ *is compact.* Here the coordinates of S^n is convenient. First of all by (5.13), we may assume $\mathrm{Cl}H_0 \subset O(n+1)$. If 0 is the unique fixed point of $\mathrm{Cl}H_0$, then 0 is also a fixed point of H.

That is, $H \subset O(n+1)$. A contradiction. Therefore the fixed point set of $\mathrm{Cl}H_0$ in S^n is S^k $(0 \leq k \leq n)$. Since $\mathrm{Cl}H_0$ is nontrivial, we have $k \neq n$. Likewise we obtain $k \neq n-1$. In fact since $\mathrm{Cl}H_0$ is connected, we have $\mathrm{Cl}H_0 \subset SO(n+1)$. Therefore if $k = n-1$, then $\mathrm{Cl}H_0$ is trivial.

Notice that S^k is H-invariant and therefore $L(N) \subset S^k$. Let us show $L(N) = S^k$. Since $k \leq n-2$, we obtain as in the proof of (5.24) that $\widetilde{N} \setminus D^{-1}(S^k)$ is connected. In way of contradiction take a point $x \in S^k \setminus L(N)$. Consider an ε-neighbourhood V of x in S^n such that $V \cap L(N) = \phi$.

Then we have as well that $D^{-1}(V \setminus S^k)$ is connected. That is, $D^{-1}(V)$ is connected. This shows that D is a homeomorphism, contrary to our hypothesis. Therefore we have $L(N) = S^k$. As before we obtain $k = n-2$.

Step 2. We shall show the last part of (5.26). Since $L(N) = \widehat{\mathbf{R}}^{n-2}$, the DH pair (D, φ) lifts to $(\overline{D}, \overline{\varphi})$, where

$$\overline{D} : \widetilde{N} \to H^{n-1} \times \mathbf{R},$$
$$\overline{\varphi} : \pi_1(N) \to \mathcal{M}(\widehat{\mathbf{R}}^{n-2}) \times \mathbf{R}.$$

Denote by p_i the canonical projection to the i-th factor. Consider a small perturbation $\overline{\varphi}'$ of $\overline{\varphi}$ such that $p_1 \circ \overline{\varphi}' = p_1 \circ \overline{\varphi}$ and $p_2 \circ \overline{\varphi}'(\pi_1(N)) \subset \mathbf{Q}$. Let $\varphi' = \psi \circ \overline{\varphi}'$. Then there exists a submersion $D' : \widetilde{N} \to \widehat{\mathbf{R}}^n$ such that (D', φ') is a DH pair. (See Thurston [56] Chapt. 5 or Canary-Epstein-Green [9] Chapt. 1.) The limit set of the new DH pair (D', φ') is also $\widehat{\mathbf{R}}^{n-2}$, since we have altered $\overline{\varphi}$ only in the $\mathbf{R}$-direction. Therefore by (5.24), D' is a covering map onto $\widehat{\mathbf{R}}^n \setminus \widehat{\mathbf{R}}^{n-2}$. That is, D' lifts to a homeomorphism

$$\overline{D}' : \widetilde{N} \to H^{n-1} \times \mathbf{R}.$$

Since $p_2 \circ \overline{\varphi}'(\pi_1(N)) \subset \mathbf{Q}$, we have that $p_2 \circ \overline{\varphi}'(\pi_1(N))$ is infinite cyclic with a generator θ. Let $\Gamma = \mathrm{Ker}(p_2 \circ \overline{\varphi}')$. We have an exact sequence

$$1 \to \Gamma \to \pi_1(N) \to \theta\mathbf{Z} \to 1.$$

Correspondingly we have a bundle structure of N with fiber $H^{n-1}/p_1\overline{\varphi}'(\Gamma)$ over $\mathbf{R}/\theta\mathbf{Z} \cong S^1$. Clearly the monodromy map is an isometry of a hyperbolic manifold. Q.E.D.

§6. Elementary structure and C-structure

Also in this section flat conformal manifolds are to be connected and compact unless otherwise specified. The dimension is always ≥ 3.

Definition (6.1). A flat conformal manifold N is called *elementary* if and only if $\mathrm{Card}L(N) \leq 2$.

As applications of (5.20), we have the following characterizations of elementary flat conformal manifolds.

Proposition (6.2). *The following conditions are equivalent.*

(1) $L(N) = \phi$.
(2) *The holonomy group* H *consists purely of elliptic transformations.*
(3) N *is a spherical space form.*

Proposition (6.3). *The following conditions are equivalent.*

(1) $L(N)$ *is a singleton.*
(2) H *contains parabolic transformations and no loxodromic transformations.*
(3) N *is an Euclidean space form.*

Proof. All that need proof is (2) $\Longrightarrow$ (1). (2) implies $L(N) \neq \phi$. By (5.14), we have $H \subset \mathcal{M}(S^n)_a$, $a \in S^n$. This shows (1). Q.E.D.

Proposition (6.4). *CardL*$(N) = 2$ *if and only if* N *or its double covering is a Hopf manifold.*

Proposition (6.5). *If* $\mathrm{Card}L(N) \geq 3$, *then* $L(N)$ *is a perfect set.*

Proof. The assumption implies by (5.14) the existence of a loxodromic elememt $f \in H$. Then at least one point of $L = L(N)$ is not fixed by f. Since L is invariant by f, we obtain that L is an infinite set. Therefore the derived set L' is nonempty. By the minimality of L ((5.22)), we have $L' = L$. That is, L is perfect. Q.E.D.

Theorem (6.6). *If the holonomy group* H *of a connected compact flat conformal manifold* N *does not contain a free group of two generators, then* N *is elementary.*

Proof. Suppose on the contrary that N is nonelementary.
Then $\mathrm{Card}L(N) = \infty$ and therefore by (5.11), there exist two loxodromic transformations $f, g \in H$ with disjoint fixed points. Now it

is a well known fact that f^n and g^n generate a free group for large n. Q.E.D.

(6.6) was first proved under the hypothesis that H is virtually nilpotent by Goldman([15]) and then by Kamishima ([25]) when H is virtually solvable. It is well known that for matrix group virtual solvability is equivalent to the condition of (6.6). See Tits ([57]).

Corollary (6.7). *If $\pi_1(M)$ does not contain a free group of two generators and if a compact connected manifold M does not have as a finite covering S^n, $S^{n-1} \times S^1$ or T^n, then M does not admit a flat conformal structure.*

(6.7) forbids many manifolds to admit flat conformal structures, e.g., 3-manifolds with Nil or Solv geometry.

Given two flat conformal manifolds N_1 and N_2, a new flat conformal manifold, called connected sum, is obtained in the following way. This operation was first introduced by Kulkarni ([42]).

Inside a conformal atlas (U_i, q_i) of N_i, choose a closed ball B_i. Assume that there exists $f \in \mathcal{M}(S^n)$ such that $f(\mathrm{Int} q_1(B_1)) = S^1 \setminus q_2(B_2)$. See Figure (6.1). (This is always possible e.g., if we choose B_i so that $q_i(B_i)$ is a metric ball.) Define a homeomorphism $h : \partial B_1 \to \partial B_2$ so that $q_2 \circ h = f \circ q_1$. Then

$$(f \circ q_1) \cup q_2 : (U_1 \setminus \mathrm{Int} B_1) \cup (U_2 \setminus \mathrm{Int} B_2)/h \to S^n$$

is a well defined embedding. Using $(f \circ q_1) \cup q_2$ and other small charts in N_i, we can define a flat conformal structure on the connected sum

$$(N_1 \setminus \mathrm{Int} B_1) \cup (N_2 \setminus \mathrm{Int} B_2)/h.$$

Definition (6.8). The flat conformal structure constructed in this way is called a *connected sum* of N_1 and N_2 and is denoted by $N_1 \sharp N_2$.

Notice that $N_1 \sharp N_2$ is not uniquely determined. For example if we fix $B_2 \subset N_2$ and make $B_1 \subset N_1$ much smaller, then the resultant connected sum would be different as a flat conformal structure.

One can also define the operation of connected sum of more than two structures.

Definition (6.9). A flat conformal structure (manifold) is called a C-*structure* (C-*manifold*) if it is a connected sum of finitely many elementary structures and is not itself elementary.

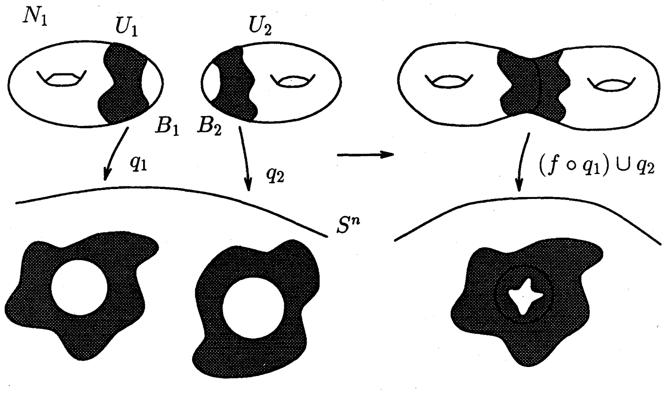

Figure (6.1)

It is easy to show that as a flat conformal manifold, $N \sharp S^n$ is isomorphic to N. This is called a trivial connected sum. There is only one case where a nontrivial connected sum of elementary flat conformal manifolds becomes again elementary, that is, when $N_1 = N_2 = \mathbf{R}P^n$, the real projective space. In this case $\pi_1(N_1 \sharp N_2)$ is isomorphic to the infinite dihedral group $\mathbf{Z}_2 * \mathbf{Z}_2$. One can show directly that $N_1 \sharp N_2$ has a Hopf manifold $S^{n-1} \times S^1$ as a double covering. In all the other cases the fundamental group of a connected sum contains a free group of two generators and therefore it cannot be elementary.

Definition (6.10). A Cantor set $\Upsilon \subset S^n$ is called *tame* if and only if there exists a homeomorphism $h : S^n \to S^n$ such that $h(\Upsilon) \subset S^1$. Otherwise Υ is called *wild.*

Proposition (6.11). *A C-structure N is of type 1. The limit set $L(N)$ is a tame Cantor set.*

Proof. For the first part of (6.11), it suffices to show the following; If the developing maps of the flat conformal structure N_1 and N_2 are injective, then the developing map of their connected sum is also injective. Let $S \subset N_1 \sharp N_2$ be the (n-1)-sphere on which the connected sum is made. S splits $N_1 \sharp N_2$ into two parts M_i such that $M_i \subset N_i$. Take a base point in $\mathrm{Int} M_1$. We have

$$\pi_1(N_1 \sharp N_2) \cong \pi_1(N_1) * \pi_1(N_2).$$

The element of $\pi_1(N_1 \sharp N_2)$ is represented uniquely as a reduced word of elements of $\pi_1(N_1)$ and $\pi_1(N_2)$. Consider the universal covering

$$\pi : \widetilde{N_1 \sharp N_2} \to N_1 \sharp N_2.$$

Choose one component of $\pi^{-1}(M_1)$ and denote it by $\widetilde{M_1}$. Choose one boundary component of $\widetilde{M_1}$ and denote it by $\widetilde{S}$. The component of $\pi^{-1}(M_2)$ which has $\widetilde{S}$ as a boundary component is denoted by $\widetilde{M_2}$. The boundary of $\widetilde{M_1}$ consists precisely of those components of $\pi^{-1}(S)$ which are of the form $\xi\widetilde{S}$ $(\xi \in \pi_1(M_1))$. $\xi\widetilde{M_2}$ is adjacent to $\widetilde{M_1}$ if and only if $\xi \in \pi_1(M_1)$. Now by the assumption the developing map D is injective on $\widetilde{M_1}$ and on $\xi\widetilde{M_2}$. $D(\widetilde{M_1})$ and $D(\xi\widetilde{M_2})$ are in the opposit sides of the sphere $D(\widetilde{S})$. This shows that D is injective on

$$\widetilde{M_1} \cup (\cup_{\xi \in \pi_1(M_1)} \xi\widetilde{M_2}).$$

Boundary component of $\xi\widetilde{M_2}$ except $\xi\widetilde{S}$ are of the form $\xi\eta\widetilde{S}$ $(\eta \in \pi_1(M_2) \setminus \{1\})$. Again $D(\xi\widetilde{M_2})$ and $D(\xi\eta\widetilde{M_1})$ are in the opposite sides of the sphere $D(\xi\eta\widetilde{S})$. Therefore D is injective on the union of $\widetilde{M_1}$, $\xi\widetilde{M_2}$ and $\xi\eta\widetilde{M}^1$ ($\xi \in \pi_1(M_1)$, $\eta \in \pi_1(M_2) \setminus \{1\}$). An induction on the length of the word of $\pi_1(N_1 \sharp N_2)$ yields that the developing map D of a C-structure N is injective. We also have that Image(D) is contained in the complement of the limit set $L = L(N)$ and the holonomy homomorphism is an isomorphism onto a discrete group H.

Next we shall show that L is totally disconnected. Once this is established, we have by (6.5) that L is a Cantor set. For simplicity, we prove this only for the connected sum N of *two* elementary structures N_1 and N_2. We use the same notations as before. Choose a base point $x_0 \in D(\mathrm{Int}\widetilde{M_1})$ and consider the family of disjoint topological spheres

$$\mathcal{S} = \{\varphi(\zeta)D(\widetilde{S}) \mid \zeta \in \pi_1(N)\}.$$

A point $x \in S^n \setminus \mathrm{Image}(D)$ is called *accessible* if there exists a path p in S^n combining x_0 and x such that p intersects finitely many spheres in $\mathcal{S}$. Accessible points consists precisely of the H-orbits of the points in $L(N_1) \cup L(N_2)$. See Figure (6.2). (We made the convension that $D|_{\widetilde{M_i}}$ coincides with the restriction of the developing map of N_i. This is always possible if we change the DH pairs of N_2 within the equivalence classes.)

Therefore accessible points are at most countable in number. Let $x \in S^n \setminus \mathrm{Image}(D)$ be a nonaccessible point. Then there are infinitely many nested spheres $\varphi(\zeta_i)D(\widetilde{S})$ $(i \geq 1)$ which separates x from x_0.

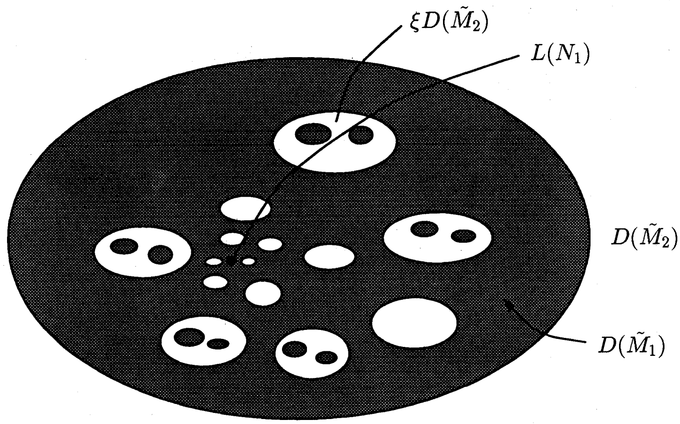

Figure (6.2)

Since ζ_i is distinct and H is discrete, we have $\varphi(\zeta_i) \to \infty$. Therefore by (5.8), we have

$$\mathrm{diam}\varphi(\zeta_i)D(\widetilde{S}) \to 0$$

since $D(\widetilde{S}) \cap L = \phi$. This shows that the component of $S^n \setminus \mathrm{Image}(D)$ at a nonaccessible point x is a singleton. Since accessible points are at most countable, this shows that $S^n \setminus \mathrm{Image}(D)$, hence L, is totally disconnected.

At this point we have obtained that $L = S^n \setminus \mathrm{Image}(D)$, since $S^n \setminus \mathrm{Image}(D)$, having no interior, is not evenly covered by D.

Finally to show the tameness of L, we have to define a homeomorphism $h : S^n \to S^n$ such that $h(L) \subset S^1$. First of all, define h on $\mathrm{Cl}D(\widetilde{M}_1)$ so that h carries all the boundary components to spheres intersecting S^1 and that h carries all the accessible points in $\mathrm{Cl}D(\widetilde{M}_1)$ into S^1. Next extend h to the adjacent components. Proceeding like this we can define the homomorphism h on the whole of S^n. Details are left to the reader. Q.E.D.

In dimension 3, we have the converse of (6.11).

Theorem (6.12). *Let N be a connected compact flat conformal manifold of dimension 3 such that the limit set $L(N)$ is a tame Cantor set. Then N is a C-manifold.*

Proof. We employ a method of Kulkarni ([43]) based upon the study of ends of a group. The necessary parts of the theory of ends are summarized in Appendix.

First of all notice that a tame Cantor set $L(N)$ satisfies the conditions of (5.24). Especially $\Omega = S^n \setminus L(N)$ is simply connected. Therefore the developing map $D : \widetilde{N} \to \Omega$ is a homeomorphism and the holonomy group H acts on Ω freely and discontinuously. That is, we have an isomorphism $N \cong \Omega/H$ of flat conformal structures. By Selberg's theorem (3.31), H has a torsion free subgroup H' of finite index. $N' = \Omega/H'$ is a finitely sheeted covering of N and therefore a compact manifold. Clearly we have $L(N') = L(N)$. By (A.4) of Appendix, we have

$$\mathrm{Card}\mathcal{E}(H') = \mathrm{Card}\mathcal{E}(\Omega) = \infty.$$

Therefore by Stalling's theorem (A.9), we obtain that H' is a nontrivial free product. Consequently N' decomposes as a nontrivial connected sum (as a manifold). See e.g. Hempel ([22]). Thus we have that $\pi_2(N) = \pi_2(N') \ncong 1$. By sphere theorem, this implies that N is reducible. It follows from (6.6) that N is not homeomorphic to $S^2 \times S^1$, since N is not an elementary structure. Therefore N is nonprime, that is, decomposes as a nontrivial connected sum $N = N_1 \sharp N_2$ as a manifold.

Let $S \subset N$ be the two sphere on which the connected sum is made. Let $N = M_1 \cup_S M_2$ and $N_i = M_i \cup_S B_i$, where B_i is homeomorphic to the closed 3-ball. Let $\pi : \widetilde{N} \to N$ be the universal covering and let $\widetilde{S}$ be a lift of S to $\widetilde{N}$. Denote by $\widetilde{M_i}$ the connected component of $\pi^{-1}(M_i)$ which has $\widetilde{S}$ as a boundary component. All the boundary components of $\widetilde{M_i}$ is of the form $\xi\widetilde{S}$ $(\xi \in \pi_1(N_i))$. Since $D|_{\widetilde{M_i}}$ is a homeomorphism, it extends in an equivariant way to $\widetilde{N_i} = \widetilde{M_i} \cup_{\xi\widetilde{S}} \xi\widetilde{B}_1$. From this we obtain a flat conformal structure on N_i, showing that the given structure on N is a connected sum of these two structures. Clearly we have $L(N_i) \subset L(N)$. Therefore either $\mathrm{Card}L(N_i) \leq 2$ or $L(N_i)$ is again a tame Cantor set. In the latter case, apply the whole argument once again to N_i. It is well known in 3-manifold theory that this process terminates. We obtain that N is a C-structure. Q.E.D.

It is unknown whether (6.12) holds in dimensin ≥ 4. In §8, we shall give an example of flat conformal 3-manifold whose limit set is a wild Cantor set. By (6.11), this is not a C-manifold.

§7. Poincaré polyhedron theorem

This section is devoted to the exposition of a fundamental theorem of Poincaré. It will be given in its simplest form, which is sufficient for our purpose of the next section. More general treatment is found e.g. in Maskit [47] in the framework of hyperbolic geometry.

Let T_i, T_i' $(1 \leq i \leq m)$ be metric $(n-1)$-spheres in S^n. Assume that any pair of them either intersect in an $(n-2)$-sphere or are disjoint and that any triple do not intersect at all. Let $\mathcal{E} = \{e_j\}$ be the family of $(n-2)$-spheres of the intersections. Let P be a component of the complement of the union of all T_i and T_i'. Assume that any element of $\mathcal{E}$ is contained in ∂P. See Figure (7.1).

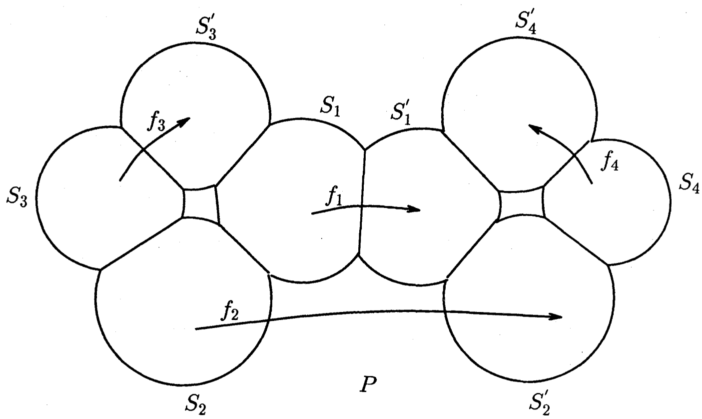

Figure (7.1)

Let $S_i = T_i \cap \partial P$ and $S_i' = T_i' \cap \partial P$. They are puctured $(n-1)$-spheres. Let $\mathcal{S} = \{S_i, S_i'\}$. An element of $\mathcal{S}$ or $\mathcal{E}$ is called respectively a *side* or an *edge* of P. Our first hypothesis is this.

(H.1). *For each i, there exists $f_i \in \mathcal{M}(S^n)$ such that $f_i(S_i) = S_i'$ and $f_i(P) \cap P = \phi$.*

Fix f_i once and for all and let $\mathcal{F} = \{f_i, f_i^{-1}\}$. An element of $\mathcal{F}$ is called a *side pairing transformation.* A side pairing transformation, say f_i, sends an edge e in ∂S_i to an edge e' in $\partial S_i'$. We call e and e' are related. This relation generates an equivalence relation in $\mathcal{E}$. $\mathcal{E}$ is partitioned into equivalence classes, called *cycles.* Each cycle $\mathcal{C}$ can be cyclically ordered as

$$\mathcal{C} = \{e_1 , \ldots, e_{p-1}, e_p = e_0\}$$

in such a way that for each $1 \le \nu \le p$, there exists $f_\nu \in \mathcal{F}$ such that $f_\nu(e_{\nu-1}) = e_\nu$. Let

$$f_C = f_p \circ \cdots\cdots \circ f_1.$$

Clearly $f_C(e_0) = e_0$. For each cycle C, f_C is well defined up to inverse and conjugation. See Figure (7.2).

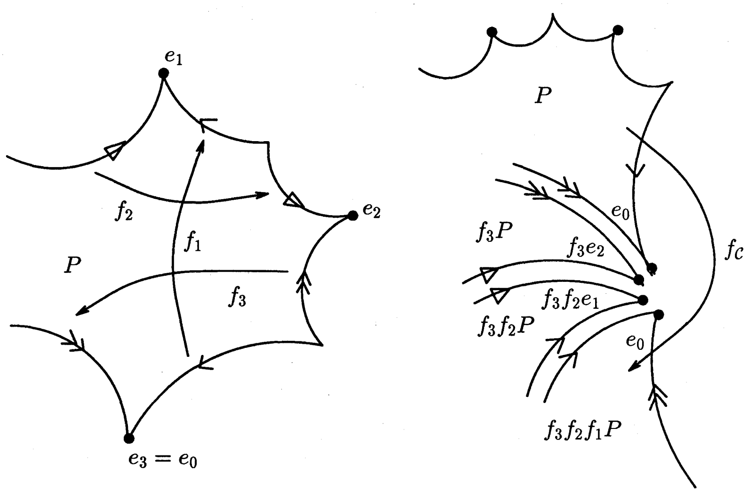

Figure (7.2)

For each edge $e \in \mathcal{E}$, the angle of P at e is denoted by $\theta(e)$. For a cycle C as above, define

$$\theta_C = \sum_{1 \le \nu \le p} \theta(e_\nu).$$

Our second hypothesis is;

(H.2). *For each cycle C, we have $\theta_C = 2\pi/q$ and $f_C^q = \mathrm{id}$ for some $q \ge 1$.*

The relation $f_C^q = \mathrm{id}$ is called a *cycle relation.* Denote by Γ the subgroup of $\mathcal{M}(S^n)$ generated by $\mathcal{F}$ and let Γ^* be the abstract group with generators the side pairing transformations and with relations the cycle relations. Clearly we have an epimorphism $\psi : \Gamma^* \to \Gamma$.

Definition (7.1). For a subgroup $G \subset \mathcal{M}(S^n)$, an open subset $R \subset S^n$ is called a *fundamental domain* of G if and only if the following two conditions are satisfied. (Ω_G denotes the domain of discontinuity of G.)

(FD.1) $$\Omega_G = \bigcup_{g \in G} g(\mathrm{Cl}R).$$

(FD.1) $$g(R) \cap R = \phi \ \text{ for any } \ g \in G \setminus \{1\}.$$

Theorem (7.2). *Assume* (H.1) *and* (H.2). *Then* $\psi : \Gamma^* \to \Gamma$ *is an isomorphism,* Γ *is a discrete subgroup of* $\mathcal{M}(S^n)$ *and* P *is a fundamental domain of* Γ.

Proof. Think of the family $\{\gamma P\}_{\gamma \in \Gamma}$ of domains. By a side pairing transformation f, fP is attached to P along a side of P. Next fgP $(g \in \mathcal{F}, g \neq f^{-1})$ is attached to fP. If this process is continued, then around an edge $e_0 \in \mathcal{E}$ which belongs to a cycle

$$\mathcal{C} = \{e_1 , \ldots , e_{p-1}, e_p = e_0\}$$

such that $f_\nu(e_{\nu-1}) = e_\nu$, there is a sequence of domains

$$\begin{gathered} P,\ f_pP,\ f_pf_{p-1}P,\ \ldots\ldots,\ f_p \cdots f_1P,\ \ldots \\ \ldots ,(f_p \cdots f_1)f_pP,\ \ldots\ldots,\ (f_p \cdots f_1)^{q-1}f_p \cdots f_2P. \end{gathered}$$

They surround the edge e_0. By virture of (H.2), the sum of their angles at e_0 is just 2π and the last domain

$$(f_p \cdots f_1)^{q-1}f_p \cdots f_2P$$

is attached to P by f_1. The essential part of the proof is to show that in this way the family $\{\gamma P\}$ forms a "tesselation" of S^n. However in order to be precise, we must argue in a formal way as follows. Start with abstract copies of P and attach them one by one by side pairing transformations, thus constructing a replica of the domain of discontinuity of Γ. Next we show the existence of an embedding of the replica into S^n. The abstract group Γ^* is convenient for this development. Let us embark upon the proof.

Define an equivalence relation $\sim$ in $\Gamma^* \times \mathrm{Cl}P$ generated by the following.

$$(\gamma, x) \sim (\gamma', x') \quad \text{if } \ \gamma' = \gamma f,\ x = f(x') \ \text{ for some } \ f \in \mathcal{F}.$$

Let

$$\Omega^* = \Gamma^* \times \mathrm{Cl}P/\sim .$$

The action of Γ^* on Ω^* is defined by

$$\gamma'(\gamma, x) = (\gamma'\gamma, x).$$

Claim 1. *Ω^* is a flat conformal manifold on which Γ^* acts conformally.*

Choose $(\gamma, x) \in \Gamma^* \times \mathrm{Cl}P$. Suppose first of all that $x \in P$. Then by (H.1), there is no point $x' \in \mathrm{Cl}P$ such that $x = f(x')$ $(f \in \mathcal{F})$. That is, (γ, x) is equivalent to no other point and therefore it certainly has a neighbourhood homeomorphic to an n-ball. Suppose next that $x \in \mathrm{Int}S$, where S is a side of P, with a side pairing map $f : S \to S'$. Then as is shown easily, the only point in $\Gamma^* \times \mathrm{Cl}P$ which is equivalent to (λ, x) is $(\lambda f^{-1}, f(x))$. Clearly one can construct a neighbourhood of the identified point $[(\lambda, x)]$, homeomorphic to an n-ball, in

$$\{\lambda\} \times \mathrm{Cl}P \cup \{\lambda f^{-1}\} \times \mathrm{Cl}P/\sim .$$

Finally consider the case where $x \in e_0$ $(e_0 \in \mathcal{E})$. Let

$$\mathcal{C} = \{e_1 , \dots , e_{p-1}, e_p = e_0\}$$

be a cycle such that $f_\nu(e_{\nu-1}) = e_\nu$. Then we have

$$\begin{aligned}(\lambda, x) &\sim (\lambda f_p^{-1}, f_p x) \sim \cdots\cdots \\ &\sim (\lambda f_2^{-1} \cdots f_p^{-1}(f_1^{-1} \cdots f_p^{-1})^{q-1}, (f_p \cdots f_1)^{q-1} f_p \cdots f_2 x).\end{aligned}$$

By the definition of the cycle, these are shown to be all the points that are equivalent to (λ, x). By (H.2), we can construct a desired neighbourhood of $[(\lambda, x)]$. This shows that Ω^* is a manifold. Since the side pairing transformations are Moebius transformations, it is easy to endow Ω^* a flat conformal structure. Also one can show without difficulty that the action of Γ^* is conformal.

Next consider the conformal mapping

$$E : \Omega^* \to S^n$$

defined by $E(\lambda, x) = \psi(\lambda)x$ where $\psi : \Gamma^* \to \Gamma$ is the canonical projection. E is well defined and ψ-equivariant, that is,

$$E(\lambda'(\lambda, x)) = \psi(\lambda')E(\lambda, x).$$

Claim 2. *E is an embedding onto a connected open subet $\Omega \subset S^n$.*

For the proof we need hyperbolic geometry of D^{n+1}. Let us extend first of all $(n-1)$-spheres T_i and T_i' used to define P to half n-spheres $\overline{T}_i, \overline{T}_i' \subset D^{n+1}$ orthogonal to S^n. These are totally geodesic hyperplanes in (D^{n+1}, g_H). Using $\overline{T}_i$ and $\overline{T}_i'$ we can extend the domain P to a domain $\overline{P} \subset D^{n+1}$. Define as before

$$\overline{\Omega}^* = \Gamma^* \times \overline{P}/\sim,$$
$$\overline{E} : \overline{\Omega}^* \to D^{n+1}.$$

The argument of Claim 1 shows that $\overline{\Omega}^*$ is a hyperbolic manifold and that $\overline{E}$ is an isometric immersion. Furthermore one can show that there exists $\varepsilon > 0$ such that any point in $\overline{\Omega}^*$ has a neighbourhood isometric to hyperbolic ε-ball. Therefore $\overline{\Omega}^*$ is complete and thus $\overline{E}$ is a covering map. That is, $\overline{E}$ is a bijective isometry. Since

$$E \cup \overline{E} : \Omega^* \cup \overline{\Omega}^* \to S^n \cup D^{n+1}$$

is continuous, we obtain Claim 2.

Claim 2 implies that $\psi : \Gamma^* \to \Gamma$ is an isomorphism and that Γ is a discrete subgroup of $\mathcal{M}(S^n)$ which acts discontinuously on $\Omega = E(\Omega^*)$. What is left is to show that P is a fundamental domain. This is equivalent to the following.

Claim 3. *Ω is precisely the domain of discontinuity Ω_Γ of Γ.*

We already had $\Omega \subset \Omega_\Gamma$. To show the converse, it suffices by (5.15) to show that $S^n \setminus \Omega \subset L_\omega(\Gamma)$. Take a point $x \in S^n \setminus \Omega$. Then for any small neighbourhood U of x in $S^n \cup D^{n+1}$, we have $\gamma_k \overline{P} \cap U \neq \phi$ for infinitely many $\gamma_k \in \Gamma$. By (5.8), we have $CI(\gamma_k) \cap \overline{P} = \phi$ for large k, since $\mathrm{Cl}P \cap L_\omega(\Gamma) = \phi$. This implies

$$\mathrm{diam}\gamma_k \overline{P} \le \mathrm{diam} CI(\gamma_k^{-1}) \to 0.$$

That is, U contains infinitely many $\gamma_k \overline{P}$, showing that $x \in L_\omega(\Gamma)$.
Q.E.D.

§8. Wild Cantor set as limit set

In this section we shall construct an example of type 1 compact flat conformal 3-manifold whose limit set is a wild Cantor set. Such an example was first obtained by Bestvina-Cooper [4] for an open 3-manifold. Our example is a variant of what they constructed. We shall follow [4] rather closely.

For a while we adopt the coordinates of $\widehat{\mathbf{R}}^3$ instead of S^3. First of all let K be a graph embedded in $\mathbf{R}^3$, depicted in Figure (8.1).

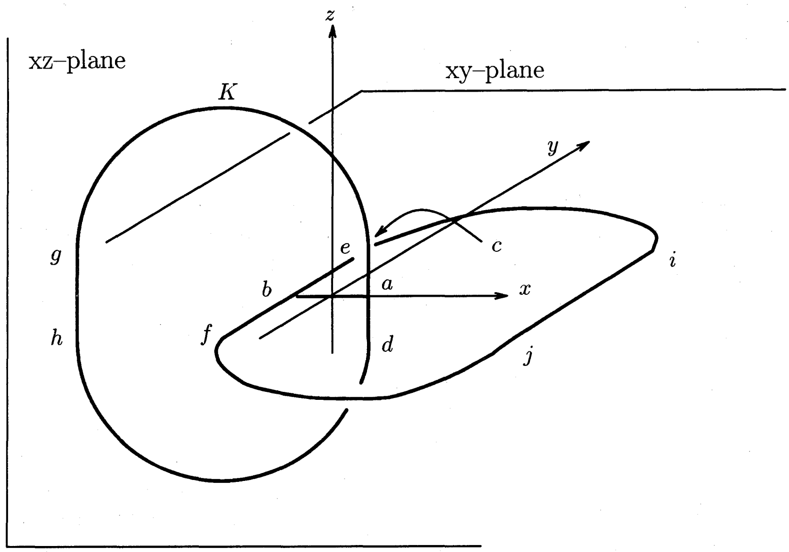

Figure (8.1)

The segment ab, cd, ef, gh and ij are straight lines and the other parts are circular arcs of the same radius. Choose a family of 2-spheres

$$A_1, \ldots, A_n,\ A'_1, \ldots, A'_n\ B_1, \ldots, B_n$$
$$B'_1, \ldots, B'_n\ C,\ C'\ D,\ D',\ E,\ E',$$

as in Figure (8.2).

We assume the followings.

(P.1) All the spheres have the same radius and have centers in K.

(P.2) The union of balls they bound covers K.

(P.3) The centers of C, E', D', C', E, D are in the x-axis. A_n, C, A'_n and A_1, E, A'_1 have centeres in straight lines parallel to z-axis. B_1, E', B'_1 and B_n, D, B'_n have centers in straight lines parallel to y-axis.

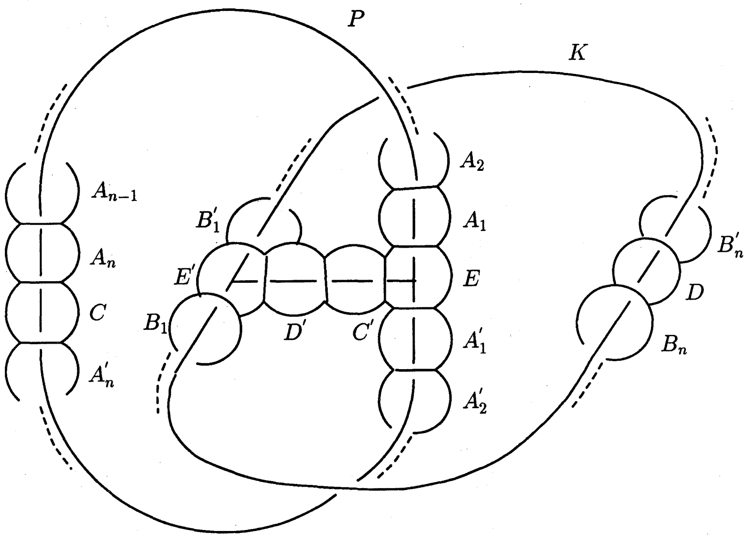

Figure (8.2)

(P.4) Adjacent two spheres intersect at angle $2\pi/28$.

Let P be the complement in $\widehat{\mathbf{R}}^3$ of the union of all the balls. Next we shall define side pairing transformations for P.

(S.1) $\alpha_j = I_{xy} \circ I_{A_j}$, where I_{xy} is the reflexion at the xy-plane and I_{A_j} is the inversion at the sphere A_j.

(S.2) $\beta_j = I_{xz} \circ I_{B_j}$.

(S.3) $\gamma = R_y^{C'} \circ I_{\pi(C,C')} \circ I_C$, where $R_y^{C'}$ is the rotation by $+90$ degrees around the oriented line through the center of C' parallel to the positive direction of y-axis and $\pi(C, C')$ is the bisector of the centers of C and C'.

(S.4) $\delta = R_z^{D'} \circ I_{\pi(D,D')} \circ I_D$.

(S.5) $\epsilon = R_x^{E'} \circ I_{\pi(E,E')} \circ I_E$.

Denote the side of P by $A_j^* = A_j \cap \mathrm{Cl}P$, $B_j^* = B_j \cap \mathrm{Cl}P$ and so forth. They satisfy the condition (H.1) of §7. That is,

(H.1) $\alpha_j(A_j^*) = A_j^{'*}$, $\beta_j(B_j^*) = B_j^{'*}$, $\gamma(C^*) = C^{'*}$, $\delta(D^*) = D^{'*}$, $\epsilon(E^*) = E^{'*}$ and $f(P) \cap P = \phi$ $(f = \alpha_1, \ldots, \epsilon)$.

Next we shall verify the condition (H.2) of §7 by listing up the cycles.

First of all for any $1 \leq j \leq n-1$, we have the following cycle.

$$\text{(C.1)} \qquad A_j \cap A_{j+1} \xrightarrow{\alpha_j} A'_j \cap A'_{j+1} \xrightarrow{\alpha_{j+1}^{-1}} A_j \cap A_{j+1}$$

$\alpha_{j+1}^{-1}\alpha_j$ keeps points in $A_j \cap A_{j+1}$ fixed. By (P.4), it is a rotation by $2\pi/14$ around $A_j \cap A_{j+1}$. Therefore (H.2) is satisfied for $q = 14$. Likewise the following cycle satisfies (H.2).

$$\text{(C.2)} \qquad B_j \cap B_{j+1} \xrightarrow{\beta_j} B'_j \cap B'_{j+1} \xrightarrow{\beta_{j+1}^{-1}} B_j \cap B_{j+1}$$

There are two more cycles. The first one is;

$$\text{(C.3)} \qquad A_1 \cap E \xrightarrow{\alpha_1} A'_1 \cap E \xrightarrow{\epsilon} B'_1 \cap E' \xrightarrow{\beta_1^{-1}} B_1 \cap E' \xrightarrow{\epsilon^{-1}} A_1 \cap E$$

Computation shows that $\epsilon^{-1}\beta_1^{-1}\epsilon\alpha_1$ keeps $A_1 \cap E$ pointwise fixed. It is a rotation by $2\pi/7$ around $A_1 \cap E$ and thus (H.2) is satisfied for $q = 7$. Now the last cycle.

$$\text{(C.4)} \qquad E \cap C' \xrightarrow{\epsilon} E' \cap D' \xrightarrow{\delta^{-1}} B'_n \cap D \xrightarrow{\beta_n^{-1}} B_n \cap D \xrightarrow{\delta} C' \cap D'$$

$$\xrightarrow{\gamma^{-1}} A'_n \cap C \xrightarrow{\alpha_n^{-1}} A_n \cap C \xrightarrow{\gamma} E \cap C'$$

By studying Figure (8.2), we obtain that $\gamma\alpha_n^{-1}\gamma^{-1}\delta\beta_n^{-1}\delta^{-1}\epsilon$ yields the translation by $\pi/2$ on the circle $E \cap C'$. We also obtain (H.2) for $q = 4$.

Thus by (7.2), the group Γ generated by the side pairing transformations is discrete, with the domain of discontinuity Ω and P is the fundamental domain for Γ. That is, we have the followings.

$$\text{(FD.1)} \qquad \Omega = \bigcup_{\gamma \in \Gamma} \gamma(\mathrm{Cl}P).$$

$$\text{(FD.2)} \qquad \gamma(P) \cap P = \phi \quad \text{for any } \gamma \in \Gamma \setminus \{1\}.$$

Let Γ_0 be a torsion free subgroup of Γ of finite index. Then the quotient space $N = \Omega/\Gamma_0$ is a compact flat conformal manifold and we have

$$L = L(N) = L_P(\Gamma_0) = L_P(\Gamma) = \widehat{\mathbf{R}}^3 \setminus \Omega.$$

The rest of this section is devoted to the proof of the following theorem.

Theorem (8.1). *The limit set* L *is a wild Cantor set.*

The proof consists of a series of lemmas. The main part is to show that L is a Cantor set. First of all notice the following feature of our construction. See Figure (8.3). For any side T^* of P, let T be the 2-sphere which contains T^* and let $e \subset \partial T^*$ be an edge. Since all the translates of P which gather at e have angle $2\pi/28$ there, the part of T which is opposite to T^* w.r.t. e and near e is also a side of a translate of P. That is, the side "prolongs" in the tesselation.

So far in the construction of Γ, we have used the coordinates of $\widehat{\mathbf{R}}^3$. However in the rest, we change the coordinates from $\widehat{\mathbf{R}}^3$ to S^3. Thus, distance, radius, etc. are measured in the Euclidean metric of $\mathbf{R}^4$ which contains S^3 as a unit sphere $\{|x| = 1\}$.

Let $\{\gamma_k\} \subset \Gamma$ be an infinite sequence. Since Γ is discrete, we have $\gamma_k \to \infty$.

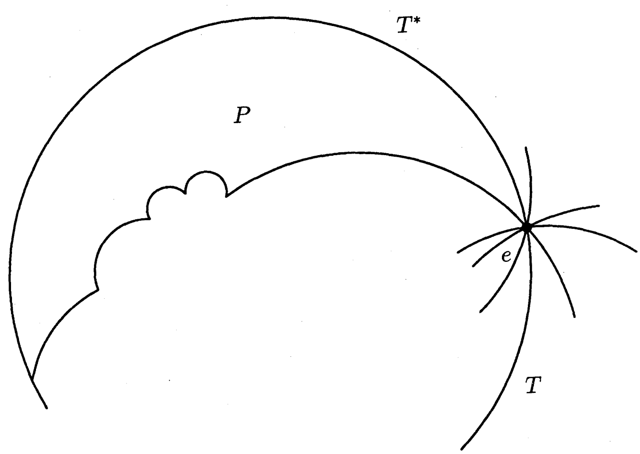

Figure (8.3)

Lemma (8.2). *For any edge* e *of* P, *we have* radius $\gamma_k(e) \to 0$.

Proof. This follows at once from (5.8), since we have $e \cap L = \phi$.
Q.E.D.

Let

$$\Sigma = \{\gamma(T) \mid \gamma \in \Gamma,\ T = A_1, \ldots, E'\},$$
$$\Lambda = \{\gamma(e) \mid \gamma \in \Gamma,\ e;\ \text{an edge of}\ P\}.$$

Notice that Λ consists of disjoint circles, while spheres in Σ may intersect. Furthermore at this point we do not know, for example, whether or not it so happens that two spheres in Σ are tangent. We have a control of Λ since any of its circle is contained in Ω. However this is not the case with Σ. We only have a rather weak grip on Σ.

For $S \in \Sigma$, let

$$\Lambda_S = \{l \in \Lambda \mid l \subset S\}.$$

Take a base point $x_0 \in S$ such that x_0 lies in a translate of the interior of a side of P.

Lemma (8.3). *For $x \in S$, x lies in Ω if and only if there exists only finitely many circles in Λ_S which separate x from x_0.*

Proof. Notice first of all that circles in Λ_S are mutually disjoint. To show the if part, let p be a path in S from x_0 to x which meets circles in Λ_S at finitely many points. An induction on the number of points shows that p is contained in the union of translates of sides of P. In particular, we have $x \in \Omega$.

For the converse, suppose that for a fixed edge e of P, $\gamma_k(e)$ $(1 \leq k < \infty)$ separates x from x_0. By (8.2), we have that

$$\text{radius}\, \gamma_k(e) \to 0.$$

Notice that (FD.1) and (FD.2) implies that $\gamma_k(e)$ is disjoint from a small neighbourhood of x_0. Since $\gamma_k(e)$ are mutually disjoint, we obtain that $\gamma_k(e) \to x$. Therefore the family $\{\gamma_k^{-1}\}$ cannot be equicontinuous on any neighbourhood of x. That is, $x \in L_J(\Gamma) = L$. (See (5.15).) Q.E.D.

Corollary (8.4). *A connected component of $L \cap S$ is a singleton. In particular $\Omega \cap S$ is open and dense in S.*

Proof. This follows from the fact that $\gamma_k(e) \subset \Omega$. Q.E.D.

A word of caution. In the above corollary, we do not assert that the component of L at a point of S is a singleton.

In spirit we are going to show the total disconnectedness of L in a way similar to (8.4) using spheres in Σ instead of circles in Λ_S. However as we remarked earlier, we do not have yet a good grip on how Σ looks like. The main difficulty comes from the fact that $S \cap L \neq \phi$ for $S \in \Sigma$. In what follows we shall carry out study of Σ step by step.

Corollary (8.5). *We have* $S \cap \gamma(P) = \phi$ *for any* $\gamma \in \Gamma$ *and* $S \in \Sigma$. *(Recall that* P *is an open set.)*

Proof. For contradiction, take a point $x \in S \cap \gamma(P)$. By (FD.1), we have $x \in \Omega$. Therefore by (8.3), there exist finitely many circles in Λ_S separating x from x_0. Now the argument of the first part of (8.3) can be applied to show that x is contained in the union of sides of translates of P. That is, $x \notin \gamma(P)$. A contradiction. Q.E.D.

Lemma (8.6). *Let* $S,\ S' \in \Sigma$. *If* $l = S \cap S'$ *is a circle, then* $l \in \Lambda$. *In particular, we have* $l \subset \Omega$.

Proof. By (8.4), there exists a point $x \in l \setminus L$. By (8.5), we have that $x \notin \gamma(P)$ for any $\gamma \in \Gamma$. Likewise we obtain that x does not lie in a translate of the interior of a side of P, since otherwise either S or S' would intersect some $\gamma(P)$. Therefore we have $x \in l'$ for some $l' \in \Lambda$. Should l' not coincide with l, there would be another sphere $S'' \in \Sigma$ such that S, S' and S'' meet in general position at $x \in \gamma(P)$. Again one of the three spheres would intersect some $\gamma(P)$. A contradiction. Q.E.D.

Lemma (8.7). *Let* $S \in \Sigma$. *Suppose that for some* $\gamma_k \in \Gamma$, $\gamma_k(S)$ *are distinct spheres. Then we have* $\text{radius}\, \gamma_k(S) \to 0$.

Proof. Suppose the contrary. We may assume further that $\gamma_k(S)$ converges to a 2-sphere S_0. Let us show first that $S_0 \cap L \neq \phi$. Take a point $a \in S$ and assume that $\gamma_k(a) \to b \in S_0$. For any neighbourhood U of b, we have $\gamma_j \gamma_i^{-1}(U) \cap U \neq \phi$ for arbitrary $j \gg i \gg 1$. That is, $b \in L_P(\Gamma) = L$. (See (5.15).) On the other hand, we have $S_0 \not\subset L$. In fact, $S_0 \subset L$ would imply that Ω is not connected. However this is impossible since the fundamental domain P is connected. (Notice that by the minimality (5.6) of L, we have $\mathrm{Int} L = \phi$. Compare (5.22).)

Consider a path in S_0 which combines a point of $\Omega \cap S_0$ to a point of $L \cap S_0$. As in the proof of (8.3), one finds a sphere $S' \in \Sigma$ which separates these two points. Clearly $S' \cap S_0 = l$ is a circle. Since $\gamma_k(S) \to S_0$, we have that $\gamma_k(S) \cap S' \to l$. By (8.6), we have $\gamma_k(S) \cap S' \in \Lambda$. Since $\gamma_k(S)$ are all distinct, we may assume (passing to a subsequence if necessary) that $\gamma_k \cap S'$ are all distinct. This contradicts (8.2). Q.E.D.

Lemma (8.8). *Fix once and for all* $x_0 \in P$. *A point* $x \in S^3$ *belongs to* Ω *if and only if there exist only finitely many spheres in* Σ *which separate* x *from* x_0.

Proof. Suppose there exist infinitely many $S_k \in \Sigma$. Then by (8.7), $\operatorname{diam} S_k \to 0$. As in the proof of (8.3), we have $x \in L$. (In fact this part will not be used in the sequel.)

Let us embark upon the proof of the converse. Define a closed subset $Y_j \subset S^3$ inductively as follows.

$$Y_0 = \operatorname{Cl}P.$$
$$Y_j = \bigcup_{\gamma} \gamma \operatorname{Cl}P, \quad \text{where } \gamma \operatorname{Cl}P \cap Y_{j-1} \neq \phi, \quad \text{for } j > 0.$$

Define an open subset X_j by

$$X_j = S^3 \setminus Y_j.$$

The set theoretic frontier ∂X_j is an angular surface (possibly with singularities) composed of the translates of sides of P, which we call *sides* of X_j. We have a filtration

$$(*) \qquad \operatorname{Cl}X_0 \supset X_0 \supset \operatorname{Cl}X_1 \supset X_1 \supset \operatorname{Cl}X_2 \supset X_2 \cdots.$$

See Figure (8.4).

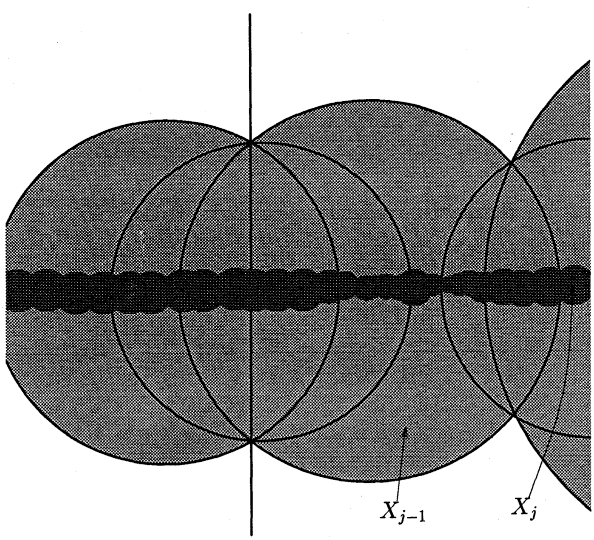

Figure (8.4)

Since P is the fundamental domain for Γ, we have

$$L = \bigcap_{j \geq 0} X_j.$$

For $S \in \Sigma$, the connected component of $S^3 \setminus S$ opposite to the base point x_0 is called the *inside* of S and is denoted by Inside(S). It is an open subset by definition. Let

$$\Sigma_j = \{S \in \Sigma \mid S \text{ contains a side of } X_j\}.$$

Once we establish the following sublemma, a point $x \in L$ can be shown to be inside infinitely many spheres in Σ, completing the proof of (8.8). Q.E.D.

Sublemma (8.8.1).

(1) *For any* $S \in \Sigma_j$, *we have* $S \subset \mathrm{Cl}X_j$.
(2) *We have*

$$X_j = \bigcup_{S \in \Sigma_j} \mathrm{Inside}(S).$$

Proof. The following properties of P, very easy to check, play a crucial part in th proof. Denote by T_α^* a side of P, by T_α the sphere containing T_α^* and by e_ν an edge of P.

(a) If $T_\alpha^* \cap T_\beta^* = \phi$, then we have $T_\alpha \cap T_\beta = \phi$.
(b) Suppose $T_\alpha^* \cap e_\nu = \phi$ and let $S \in \Sigma$ be an arbitrary sphere which passes through e_ν. If $S \cap T_\alpha \neq \phi$, then S contains a side S^* of P such that $S^* \cap T_\alpha^* \neq \phi$ and $e_\nu \subset \partial S^*$.
(c) If $T_\alpha \cap T_\beta \neq \phi$, $T_\beta \cap T_\gamma \neq \phi$ and $T_\gamma \cap T_\alpha \neq \phi$, then two of the three spheres T_α, T_β and T_γ must coincide.

The proof of (8.8.1) is by induction on j. For $j = 0$, this is clear by the construction of P. Let $j > 0$. Assume (8.8.1) for $j - 1$.

Proof of (1). For a given $S \in \Sigma_j$, let $S^* \subset S$ be a side of X_j. Choose a point x in the interior of S^*. By the filtration $(*)$, we have $x \in X_{j-1}$. The induction hypothesis implies that $x \in \mathrm{Inside}(T)$ for some $T \in \Sigma_{j-1}$. Since $S^* \subset \partial X_j$, there exists a translate $\gamma\mathrm{Cl}P$ having S^* as a side such that $\gamma\mathrm{Cl}P \cap X_j = \phi$. That is, $\gamma\mathrm{Cl}P \subset Y_j$. By the definition of Y_j, we have $\gamma\mathrm{Cl}P \cap \partial X_{j-1} \neq \phi$. Since $x \in \mathrm{Inside}(T)$, $\gamma\mathrm{Cl}P$ must lie in $T \cup \mathrm{Inside}(T)$. Therefore we have

$$\gamma\mathrm{Cl}P \cap T \cap \partial X_{j-1} \neq \phi.$$

Clearly $\gamma \mathrm{Cl}P \cap T \cap \partial X_{j-1}$ is either a side or an edge of $\gamma \mathrm{Cl}P$. Since $S^* \subset X_{j-1}$, we have $S^* \cap (\gamma \mathrm{Cl}P \cap T \cap \partial X_{j-1}) = \phi$.

If $\gamma \mathrm{Cl}P \cap T \cap \partial X_{j-1}$ is a side, then it follows from (a) that $S \cap T \neq \phi$. That is,

$$S \subset \mathrm{Inside}(T) \subset X_{j-1}.$$

Clearly this implies that $S \subset \mathrm{Cl}X_j$.

Suppose on the contrary that $\gamma \mathrm{Cl}P \cap T \cap \partial X_{j-1}$ is an edge, that is a circle $l \in \Lambda$. By (b), we obtain the same conclusion except in the case where T contains a side T^* of $\gamma \mathrm{Cl}P$ such that $T^* \cap S^* \neq \phi$ and $T^* \supset l$. See Figure (8.5). In this case choose a point $y \in S^* \cap T^*$. As before we obtain that $y \in \mathrm{Inside}(T')$ for some $T' \in \Sigma_{j-1}$ and that $l \subset T'$. If $S \cap T' \neq \phi$, then T' contains a side T'^* of $\gamma \mathrm{Cl}P$ such that $T'^* \cap S^* \neq \phi$ and $T'^* \supset l$. This contradicts (c). Therefore we have $S \cap T' = \phi$. As before we obtain $S \subset \mathrm{Cl}X_j$.

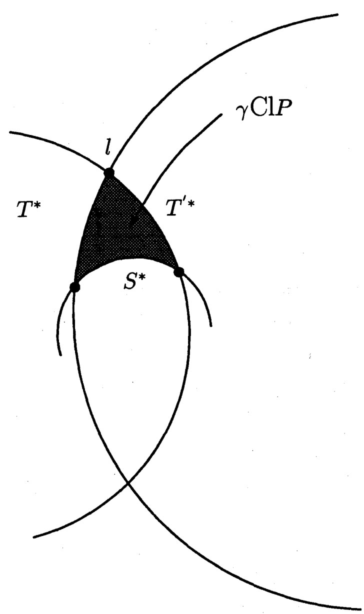

Figure (8.5)

Proof of (2). By the construction Y_j is connected. Therefore for

$S \in \Sigma_j$, $S \subset \mathrm{Cl}X_j$ implies that $\mathrm{Inside}(S) \subset X_j$. This shows

$$\bigcup_{S \in \Sigma_j} \mathrm{Inside}(S) \subset X_j.$$

For the converse, consider a path p combining the base point x_0 to a given point $x \in X_j$. Any such p must intersect ∂X_j and hence $\bigcap_{S \in \Sigma_j} S$. Choose p so that

(i) p does not pass through the intersection of two distinct spheres of Σ_j,

(ii) the sum

$$\sum_{S \in \Sigma_j} \mathrm{Card}(p \cap S)$$

is the minimal.

Then for each $S \in \Sigma_j$, we have $\mathrm{Card}(p \cap S) \leq 1$. In fact, if not, one can find a subarc q of p such that $\partial q \subset S$ and $q \setminus \partial q \subset \mathrm{Inside}(S)$. One can push q out of $\mathrm{Inside}(S)$ in such a way that the numbers of intersections of p with the other spheres do not change. This contradicts the minimality (ii).

We obtain that $\mathrm{Card}(p \cap S) = 1$ for some $S \in \Sigma_j$. That is, $x \in \mathrm{Inside}(S)$, as is required.

Q.E.D.

At this point we need a concrete picture how $\mathrm{Cl}X_1$ and $\mathrm{Cl}X_2$ look like. The picture of $\mathrm{Cl}X_1$ near $A_j \cap A_{j+1}$, $A_1 \cap E$ and $A_n \cap C$ are shown in Figures (8.6)$\sim$(8.8).

The point is that Figure (8.8) shows that there occurs a separation of components of $\mathrm{Cl}X_1$ near $A_n \cap C$. As a matter of fact, the same thing happens near any edge in the cycle of $A_n \cap C$. Furthermore we find a lot of separation of components of $\mathrm{Cl}X_2$. In particular in $\epsilon^{-1}(\mathrm{Cl}P)$ which is inside E, we observe that a component of $\mathrm{Cl}X_2 \cap \epsilon^{-1}(\mathrm{Cl}P)$ which intersets $\epsilon^{-1}B_1$ do not intersect $\epsilon^{-1}B_1'$. See Figure (8.9).

The same thing happens inside E'. In summary we have the following.

Let T, T' be any adjacent pair of 2-spheres chosen from $A_1, \ldots, E'$. Then a component of $\mathrm{Cl}X_2 \setminus (\mathrm{Inside}(T) \cup \mathrm{Inside}(T'))$ which intersects T does not intersect T'.

As a matter of fact, much more can be said concerning the smallness of components of $\mathrm{Cl}X_2$. However this is all that we need.

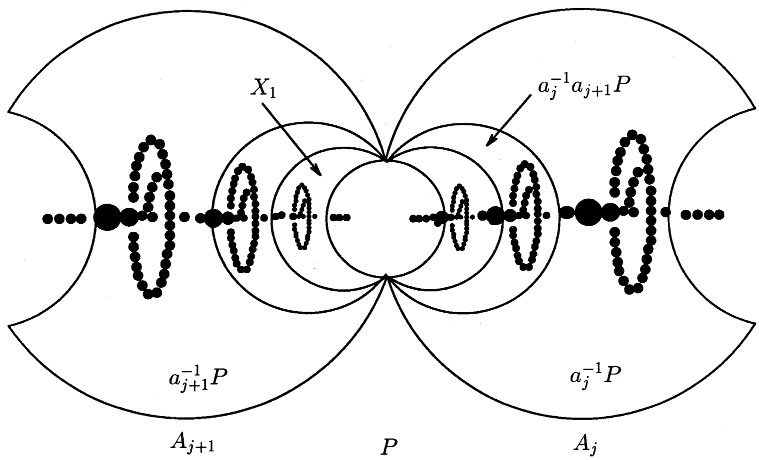

Figure (8.6)

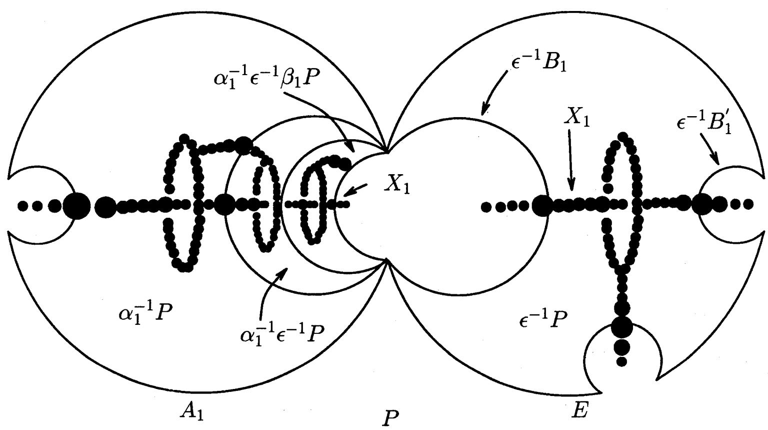

Figure (8.7)

Lemma (8.9). *For arbitrary spheres* S, $S' \in \Sigma$ *such that* $l = S \cap S'$ *is a circle, let* D *(resp.* D'*) be one of the disks in* S *(resp.* S'*) which is bounded by* l*. Suppose that the angle of* D *and* D' *at* l *is* $2\pi/28$*. Let* Q *be the closure of the component of* $S^3 \setminus (S \cup S')$ *bounded by* D *and* D'*. Then a component of* $L \cap Q$ *which intersects*

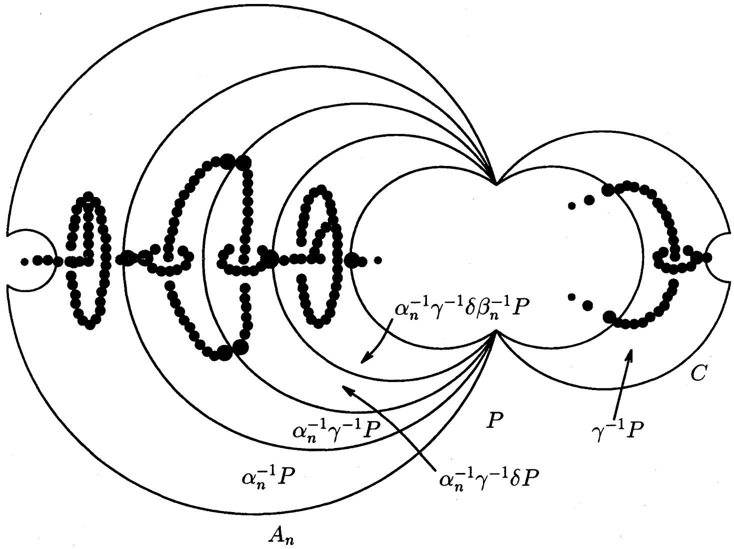

Figure (8.8)

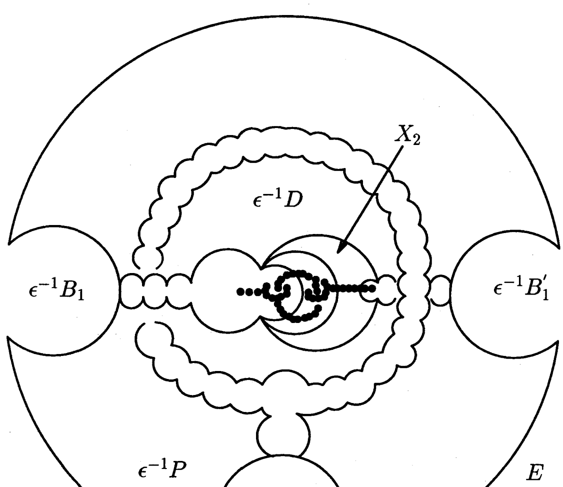

Figure (8.9)

D does not intersect D'.

Proof. Since we have $l \in \Lambda$ by (8.6), the proof reduces to the case where l is an edge of P and D and D' contains adjacent sides of P. Since $L \subset \mathrm{Cl}X_2$, (8.9) follows from the above observation. See Figure (8.10). Q.E.D.

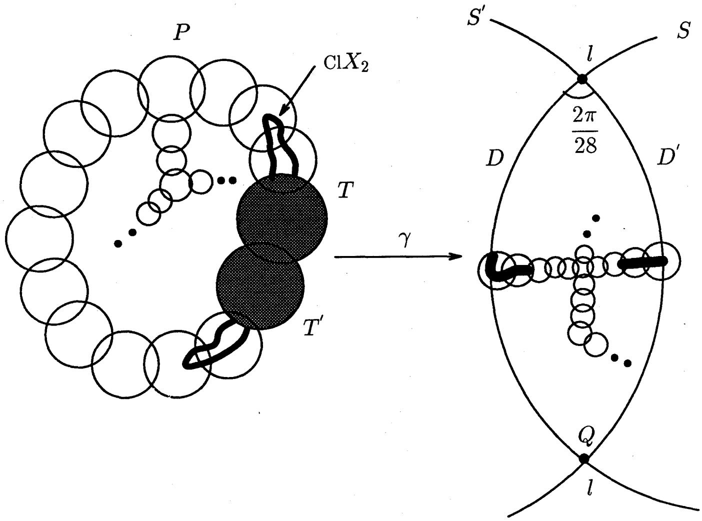

Figure (8.10)

Another way to put (8.9) is the following.

Corollary (8.10). *For arbitrary spheres S, $S' \in \Sigma$ such that $l = S \cap S'$ is a circle, the component of L at a point $x \in S' \setminus S$ does not intersect S.*

Proof. Let Δ be the component of $S' \setminus S$ at x and let Ξ be the closure of either of the components of $S^3 \setminus (S \cup S')$ which contains Δ in its boundary. Then by (8.9), we obtain that for any $y \in \Delta \cap L$, the component of $\Xi \cap L$ at y does not intersect S. See Figure (8.11). It is easy to show that (8.10) follows from this. Q.E.D.

Lemma (8.11). *Let S be an arbitrary sphere in Σ. For any $x \in S \cap L$, the component of L at x is $\{x\}$ itself.*

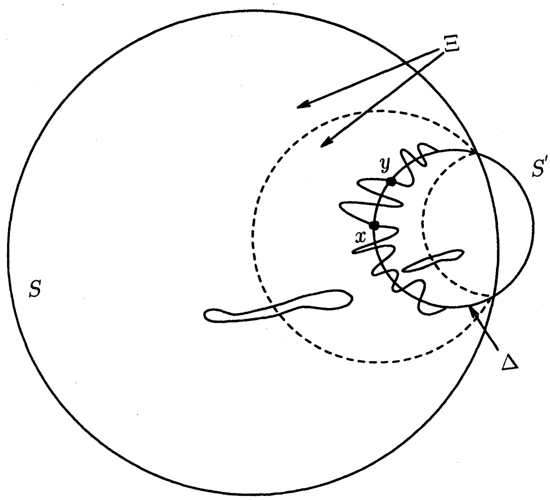

Figure (8.11)

Proof. By (8.8), there exists an infinite sequence $\{S_k\} \subset \Sigma$ such that S_k separates x from the base point $x_0 \in \Omega$. Note that $S_k \to x$. For large k, S_k intersects S at a circle. Therefore by (8.10), the component of L at x does not intersect S_k. This completes the proof. Q.E.D.

Corollary (8.12). *For any $S \in \Sigma$, the component of L at a point $x \in L \setminus S$ does not intersect S.*

Corollary (8.13). *L is totally disconnected.*

Proof. Let $x \in L$. If $x \in S$ for some $S \in \Sigma$, then we have already shown (See (8.11).) that the conponent of L at x is a singleton. So consider the other case. By (8.8), there exist infinitely many spheres $S_k \in \Sigma$ which separate x from a base point $x_0 \in P$. By (8.7), we have $S_k \to x$. Therefore (8.12) implies (8.13). Q.E.D.

By (6.5), this implies that L is a Cantor set. Thus we have finished the proof of the first part of Theorem (8.1). Let us show in the remainder that L is wild. First of all we have the following well known fact, which is easy to show.

Proposition (8.14). *If $\Upsilon \subset S^n$ is a tame Cantor set, then $S^3 \setminus \Upsilon$ is simply connected.*

Thus once we establish that the inclusion $i : P \to \Omega$ induces an injection on the fundamental groups, then the proof of Theorem (8.1) will be complete.

Some readers may have a feeling that this can be solved by looking at the homomorphism

$$\pi_1(P) \xrightarrow{i_*} \pi_1(\Omega) \xrightarrow{p_*} \pi_1(\Omega/\Gamma).$$

The computation of $\pi_1(\Omega/\Gamma)$, the fundamental group of an orbifold, is in fact easy. However in order to show the nontriviality of the homomorphism $p_* \circ i_*$, one is lead to the word problem of $\pi_1(\Omega/\Gamma)$, in which an approach geometric in nature is obviously indispensable. Instead of going to this direction, we employ the following argument which is totally geometric and is applicable also to the word problem of $\pi_1(\Omega/\Gamma)$.

The key fact is the following lemma.

Lemma (8.15). *Let T^* be a side of* ClP, *then the inclusion $T^* \to \mathrm{Cl}P$ induces an injection on π_1.*

Proof. By sliding handles of $S^3 \setminus P$, one obtains that ClP is a handlebody of genus 2. Therefore $\pi_1(\mathrm{Cl}P)$ is a free group freely generated by α and β. If $T^* \neq E^*$ or E'^*, then the lemma follows easily. For $T^* = E^*$, the image of $\pi_1(E^*)$ is generated by α and $\alpha\beta\alpha^{-1}\beta^{-1}$. It is well known, easy to show using the once puctured torus model, that they generate free subgroups. Q.E.D.

Now let us embark upon the proof. Let $\alpha : S^1 \to P$ be a loop such that $\alpha \not\simeq 1$ in P. Suppose on the contrary that $\alpha \simeq 1$ in Ω. Let $\beta : D^2 \to \Omega$ be the extension of α. By a small perturbation, one may assume that β is smooth and transverse to any circle in Λ and to any sphere in Σ. Their inverse images form a graph G in D^2. (G may contain smooth circles as connected components.) As a matter of fact, we have $G \neq \phi$ and $G \cap S^1 = \phi$. See Figure (8.12).

Let us choose β so

(M.1) the number of vertices of G is the minimal,
(M.2) the number of edges of G is the minimal among those which satisfy (M.1).

Let Δ be a connected component of $D^2 \setminus G$ which is homeomorphic to an open disk. Then $\beta(\Delta) \subset \gamma P$ for some $\gamma \in \Gamma$. Since β is transverse to $\gamma\partial P$, we have that $\partial\Delta$ is a simple closed curve.

Claim (8.16). *The number of vertices of $\partial\Delta$ is ≥ 3.*

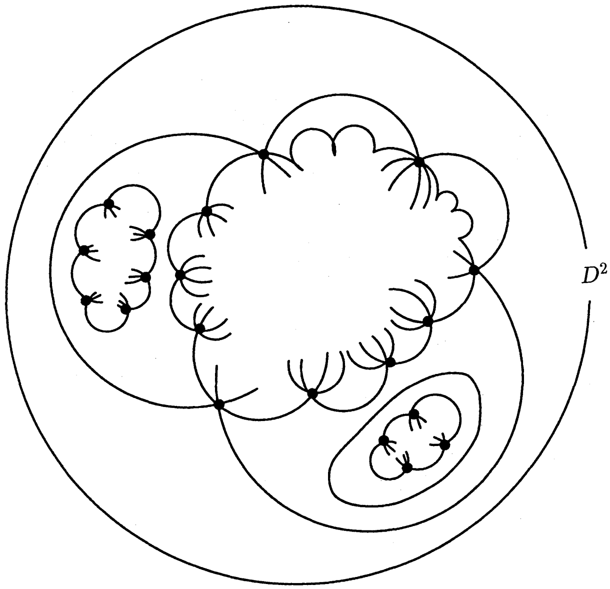

Figure (8.12)

Proof. If not, $\beta(\partial\Delta)$ is contained in the union $\mathcal{T}$ of adjacent two sides of $\gamma\partial P$ for some $\gamma \in \Gamma$. Clearly $\mathcal{T}$ is homotopic in $\gamma\mathrm{Cl}P$ to a single side. See Figure (8.2). Since $\beta(\partial\Delta)$ is null homotopic in $\gamma\mathrm{Cl}P$, we obtain by (8.15) that $\beta(\partial\Delta)$ is null homotopic in $\mathcal{T}$. But then we can alter the map β so that $\beta(\Delta) \subset \mathcal{T}$ and eventually push β out of $\gamma\mathrm{Cl}P$. This contradicts the minimality assumption (M.1) if $\partial\Delta$ has a vertex and (M.2) otherwise. Q.E.D.

Now consider the family of smooth circle components of G. Let l be the innermost one and let V be the open ball bounded by l. In case there is no smooth circles, let $V = \mathrm{Int}D^2$. By (8.16), there must exist components of G in V. Consider $G' = G \cap V$. G' has no longer a smooth circle component. Let $G'_1, \ldots, G'_r$ be the connected component of G'. Let E'_i be the component of $V \setminus G'_i$ which contains ∂V and let

$$H(G'_i) = V \setminus E'_i.$$

Notice that $\partial H(G'_i)$ is a simple closed curve since it is the inverse image by β of a surface $\gamma\partial P$ for some $\gamma \in \Gamma$. Therefore $H(G'_i)$ is a closed disk. Define a partial order $\prec$ in the set $\{G'_1, \ldots, G'_r\}$ by

$$G'_j \prec G'_i \Longleftrightarrow H(G'_j) \subset H(G'_i).$$

Let G'_j be the minimal element. Then any component of $H(G'_j)\backslash G'_j$ is an open disk. That is, G'_j gives a polyhedral decomposition of $H(G'_j)$. Let f, e and v be the number of faces, edges and vertices of the decomposition. By virtue of (8.16), we have

$$3f \leq 2e.$$

Notice that by (P.4), exactly 28 edges gather at each vertex of G'_j. Therefore we have

$$14v = e.$$

The computation of Euler number yields;

$$1 = f - e + v \leq \frac{2}{3}e - e + \frac{e}{14} < 0.$$

This contradiction shows that $i_* : \pi_1(P) \to \pi_1(\Omega)$ is an injection, as is requied.

Appendix End

The concept of end of a topological space and of a discrete group was first introduced in 1931 by Freudenthal ([11]) and was studied, among others, by Hopf ([23]). See also Freudenthal [12] and Epstein [8]. After almost 40 years, Stalling ([54],[55]) established a celebrated theorem concerning finitely generated groups with infinite ends. See Dunwoody [7] for related topics and a geometric proof of Stalling's theorem for finitely presented groups. All this has a wide range of applications. For the convenience of the reader, we collect here some parts of the theory, mostly without proof.

First of all we define the ends of a connected locally finite simplicial complex U.

Definition (A.1). A sequence $\{M_k\}$ of subsets of U is called *discrete* if for any compact subset C of U, we have $M_k \cap C = \phi$ for but finitely many k.

Definition (A.2). A point sequence $\{x_k\} \subset U$ is called *admissible* if for any $k > 0$, there exists a path $P_k \subset U$ combining x_k and x_{k+1} such that the family $\{P_k\}$ is discrete.

Definition (A.3). Two admissible sequence $\{x_k\}$ and $\{x'_k\}$ are said to be *equivalent,* (denoted by $\{x_k\} \sim \{x'_k\}$) if and only if there exists a path P_k $(k > 0)$ combining x_k and x'_k such that the family

$\{P_k\}$ is discrete. An equivalence class of admissible sequences are called an *end* of U. The set of the ends of U is denoted by $\mathcal{E}(U)$.

It is easy to show that the relation in (A.3) is in fact an equivalence relation. Notice that a subsequence of an admissible sequence is again admissible and in the same equivalence class.

For applications to flat conformal structures, we need the following.

Proposition (A.4). *For an open domain U of S^n $(n \geq 2)$, the set of ends $\mathcal{E}(U)$ is in one to one correspondence with the set of connected components of $Y = S^n \setminus U$.*

Proof. First of all let us define a correspondence of an end to a connected component. Let $\{x_k\}$ be an admissible sequence of U. Then we have $d(x_k, Y) \to 0$. Let us show furthermore that there exists a unique connected component Y_ν of Y such that $d(x_k, Y_\nu) \to 0$. Suppose the contrary. Then there exist subsequences $\{y_k\}$ and $\{z_k\}$ of $\{x_k\}$ such that $d(y_k, Y_\nu) \to 0$ and $d(z_k, Y_\mu) \to 0$ for disjoint components Y_ν and Y_μ of Y. Then there exists a compact neighbourhood B of Y_ν in S^n such that $B \cap Y_\mu = \phi$ and $\partial B \subset U$. Then any path P_k in U combining y_k and z_k must intersect the compact set ∂B. This contradicts the fact that $\{y_k\} \sim \{z_k\}$. The same argument shows that the component thus chosen is independent of the particular choice of an admissible sequence in the equivalence class. Thus an end corresponds to a connected component.

The converse correspondence is defined as follows. For any connected component Y_ν of Y, we can find a sequence $\{B_k\}$ of compact connected neighbourhoods of Y_ν in S^n such that $\partial B_k \subset U$ and that $\bigcap_k B_k = Y_\nu$. Furthermore one may assume that ∂B_k is a finite union of codimension one connected submanifold. Notice that any codimension one connected submanifold splits S^n $(n \geq 2)$ into two parts. Since U is connected, this shows that $\mathrm{Int}B_k \setminus Y$ is arcwise connected. Choose an arbitrary point $x_k \in \mathrm{Int}B_k$. Combine x_k and x_{k+1} by a path P_k in $\mathrm{Int}B_k \setminus Y$. This shows that $\{x_k\}$ is an admissible sequence. Q.E.D.

The ends of a group is defined by virtue of the following theorem.

Theorem (A.5). *Let Γ be a finitely generated group which acts on a connected locally finite simplicial complex U freely and discontinuously such that the quotient U/Γ is compact. Then the set of ends $\mathcal{E}(U)$ is determined (up to a bijection) only by the group Γ. It does not depends upon the particular choice of the space U.*

Definition (A.6). *The set of ends in (A.5) is called the* end set *of the group Γ and is denoted by $\mathcal{E}(\Gamma)$.*

Theorem (A.7). *The end set* $\mathcal{E}(\Gamma)$ *of a finitely generated group* Γ *is infinite (in fact uncountably infinite) if* $\mathrm{Card}\mathcal{E}(\Gamma) \geq 3$.

We have the following characterization of the group according to its end set.

Theorem (A.8). *Let* Γ *be a finitely generated group.*

(1) $\mathcal{E}(\Gamma) = \phi$ *if and only if* Γ *is a finite group.*
(2) $\mathcal{E}(\Gamma)$ *consists of two points if and only if* Γ *has the infinite cyclic group* $\mathbf{Z}$ *as a finite index subgroup.*

For a group with infinitely many ends, Stalling obtained a complete characterization. However for the sake of simplicity we only state the following partial result.

Theorem (A.9). *A finitely generated torsion free group* Γ *has infinite ends if and only if* Γ *has a nontrivial decomposition as a free product* $\Gamma = \Gamma_1 * \Gamma_2$.

As an application, if a torsion free group Γ in (A.9) acts on a domain $U \subset S^n$ freely and discontinuously and if the complement $S^n \setminus U$ has more than two components, then Γ is a nontrivial free product.

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