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# Yang-Baxter Algebras, Conformal Invariant Models and Quantum Groups

**H. J. de Vega**

**Abstract.**

The Yang-Baxter algebras (YBA) are introduced and formulated in a general way stressing graphical methods. Their various physical applications are then exposed: lattice statistical models, integrable field theories and factorizable S-matrices. The Bethe Ansatz (BA) and its generalizations provide the explicit solutions of all these models using the appropiate YBA. The six-vertex model solution is exposed. YB algebras and their associated physical models are classified in terms of simple Lie algebras.

It is exposed how these lattice models yield both solvable massive QFT and conformal models in appropiated scaling (continuous) limits within the lattice light-cone approach.

The method of finite-size calculations from the BA is exposed as well as its applications to derive the conformal properties of integrable lattice models. It is conjectured that all integrable QFT and conformal models follow in a scaling limit from these YB algebras.

To conclude braid and quantum groups are derived from the YBA in the limit of infinite spectral parameter.

## §1. Yang-Baxter algebras

A Yang-Baxter (YB) algebra consists of a set of operators $T(\theta)$ called generators. They depend on the complex variable $\theta$ (the spectral parameter). Each operator $T(\theta)$ acts on two vector spaces $\mathcal{A}$ and $\mathcal{V}$. The best way to work with Yang-Baxter algebras is to use graphical notation. It is defined as follows:

a) a line of different type is associated to each vector space (see Fig.1)

b) The intersection of two lines is associated to a generator $T(\theta)$ where $\theta$ is the angle between the two lines.(see Fig.2).

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Fig. 1. To each type of line is associated a vector space.

$$\left[T^{(A,V)}_{ab}(\theta)\right]_{\alpha\beta} =$$

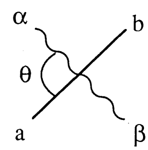
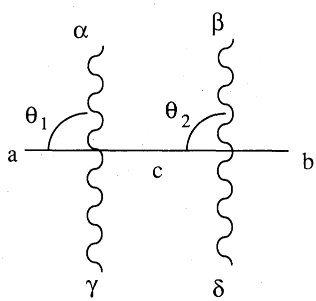



Fig. 2. A YB generator is associated to the intersection of two lines.




$$= \sum_c \left[T_{ac}(\theta_1)\right]_{\alpha\gamma}\left[T_{cb}(\theta_2)\right]_{\beta\delta}$$

Fig. 3. There is a summation over the states of internal lines.

c) There is summation over all states in the vector spaces associated to the lines between two vertices ["internal lines"] (see Fig.3)

Let us call $\mathcal{I}$ the set of all vector spaces where the YB algebra (YBA) generators act

$$\mathcal{I} = \{V^I\}.$$

$\mathcal{I}$ is also the set of different types of lines. The basic equation that characterizes the YBA is

$$
\begin{aligned}
&T^{(K,I)}(\theta-\theta')T^{(K,J)}(\theta)T^{(I,J)}(\theta') \\
&\quad = T^{(I,J)}(\theta')T^{(K,J)}(\theta)T^{(K,I)}(\theta-\theta')
\end{aligned}
\tag{1.1}
$$

for all spaces $V^I, V^J, V^K \in \mathcal{I}$. Eq.(1.1) is called Yang-Baxter equation (YBE) or triangular relation or factorization equation. It can be represented graphically as follows (Fig.4)

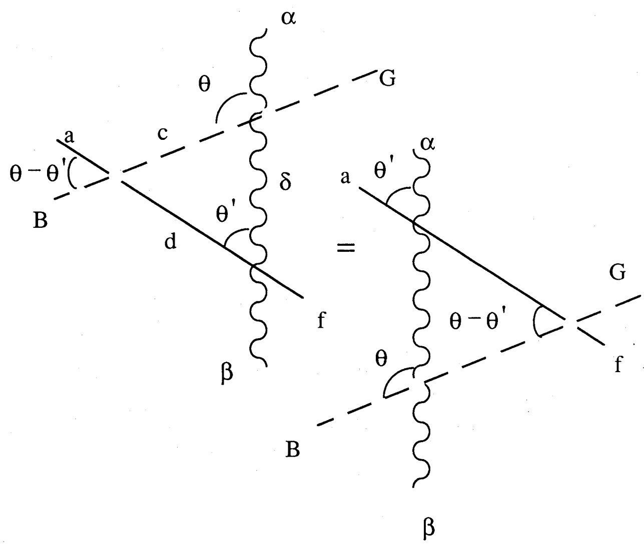



Fig. 4. The Yang–Baxter equation (general form).

Here eq.(1.1) writes putting all indices explicitly

$$
\begin{aligned}
&\left[T^{(K,I)}_{BC}(\theta-\theta')\right]_{ad}\left[T^{(K,J)}_{CG}(\theta)\right]_{\alpha\delta}\left[T^{(I,J)}_{df}(\theta')\right]_{\delta\beta} \\
&\quad = \left[T^{(I,J)}_{ag}(\theta')\right]_{\alpha\gamma}\left[T^{(K,J)}_{BD}(\theta)\right]_{\gamma\beta}\left[T^{(K,I)}_{DG}(\theta-\theta')\right]_{gf}.
\end{aligned}
\tag{1.2}
$$

As one sees in Fig.4 the YBE says that one can push any line through the intersection of other two. This is called sometimes $Z$-invariance since it leaves the partition function unchanged [see below] [1]. Eqs.(1.1)–(1.2) [or Fig.4] shows the general YBE.

The YB generator associated to the intersection of two lines of the same type is called an $R$-matrix

$$R^1(\theta) \equiv T^{(1,1)}(\theta).$$

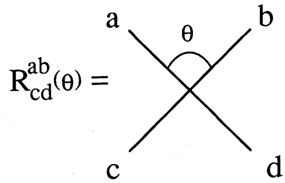



Fig. 5. The $R$-matrix.

In the particular case when two of the vector spaces are identical, say $V^I = V^K = \mathcal{A}$ and $V^J = \mathcal{V}$, one can rewrite eqs. (1.1)–(1.2) as [2]

$$R(\theta - \theta')[T(\theta) \otimes T(\theta')] = [T(\theta') \otimes T(\theta)]R(\theta - \theta') \tag{1.3}$$

where

$$R(\theta) = R^I(\theta) \qquad \text{and} \qquad T(\theta) = T^{(I,J)}(\theta).$$

In eq.(1.3) an operator product in the space $\mathcal{V}$ is understood. The $\otimes$ means tensor product of the space $\mathcal{A}$ multiplied by itself. $R$ acts in $\mathcal{A}\otimes\mathcal{A}$ as a matrix. The $R$-matrix associated to the space $V_0$ of lowest dimensionality in $\mathcal{I}$ as called the fundamental $R$-matrix. The fundamental $R$-matrix characterizes the YB algebra.

Let us see why YB algebras are connected deeply with integrable theory. Eq.(1.1) can be written as

$$\begin{aligned} &T^{(K,J)}(\theta)T^{(I,J)}(\theta') \\ &\quad = T^{(K,I)-1}(\theta - \theta')T^{(I,J)}(\theta')T^{(K,J)}(\theta)T^{(K,I)}(\theta - \theta') \end{aligned} \tag{1.4}$$

Taking the trace of eq.(1.4) in the space $V^K \otimes V^I$ yields

$$\tau_K(\theta)\tau_I(\theta') = \tau_I(\theta')\tau_K(\theta) \tag{1.5}$$

where we use the cyclic property of the trace and

$$\mathrm{Tr}_{V^I \otimes V^K}(T^{(I,J)} \otimes T^{(K,J)}) = \mathrm{Tr}_{V^I}(T^{(I,J)})\mathrm{Tr}_{V^K}(T^{(K,J)}). \tag{1.6}$$

Here an operatorial product in the space $V^J$ is understood. We denote by $\tau_I(\theta)$ and $\tau_K(\theta)$ the transfer matrices

$$\tau_I(\theta) = \mathrm{Tr}_{V^I}\left[T(\theta)^{(I,J)}\right] = \sum_a T^{(I,J)}_{aa}(\theta), \tag{1.7}$$

$$\tau_K(\theta) = \mathrm{Tr}_{V^K}\left[T(\theta)^{(K,J)}\right] = \sum_A T^{(K,J)}_{AA}(\theta). \tag{1.8}$$

$\tau_I(\theta)$ and $\tau_K(\theta)$ are operators acting on $V^J$. They form a set of families of commuting transfer matrices

$$(1.9) \qquad [\tau_I(\theta), \tau_K(\theta')] = 0, \qquad \forall \theta, \theta' \in \mathbf{C}, \qquad \forall I, K \in \mathcal{I}.$$

Moreover, series expanding in $\theta$ yields an infinite number of commuting operators acting on $V^J$.

$$(1.10) \qquad [C_n^I, C_m^K] = 0 \qquad \forall n, m \geq 0, \quad \forall I, K \in \mathcal{I}.$$

Here $C_n^I$ are the expansion coefficients of $\tau_I(\theta)$ or $\log \tau_I(\theta)$ in powers of $\theta$. The existence of an infinite number of commuting operators is the necessary condition to have a quantum integrable system with an infinite number of degrees of freedom. Actually, only in the thermodynamic limit this number of degrees of freedom is attained.

Since the operators $\tau_I(\theta)$ are mutually commuting for all $\theta$ and $V^I$, one can expect to be able to diagonalize all of them simultaneously. This is actually possible. Moreover, the eigenvectors and eigenvalues can be constructed using the YB algebra itself. This is probably the main application of YBA. They permit to built eigenvectors and eigenvalues of all $\tau_I(\theta)$ and operators $C_n^I$ derived from them in a purely algebraic framework.

A specially important YB equation follows when the three vector spaces in eq.(1.1) are equal: $V^I = V^J = V^K = \mathcal{A}$. One finds

$$(1.11) \qquad \begin{aligned} &[1 \otimes R(\theta - \theta')][R(\theta) \otimes 1][1 \otimes R(\theta')] \\ &\quad = [R(\theta') \otimes 1][1 \otimes R(\theta)][R(\theta - \theta') \otimes 1]. \end{aligned}$$

In explicit notation this reads

$$(1.12) \quad R_{a_1 a_2}^{cd}(\theta - \theta') R_{a_3 c}^{b_1 c}(\theta) R_{ed}^{b_1 b_2}(\theta') = R_{a_3 a_2}^{mn}(\theta') R_{n a_1}^{p b_2}(\theta) R_{mp}^{b_1 b_2}(\theta - \theta').$$

This equation can be depicted as

We see that eq.(1.11) or (1.12) is a system of $q^6$ equations ($q = \dim \mathcal{A}$) with $q^4$ unknowns (the functions $R_{cd}^{ab}(\theta)$, $1 \leq a,\ b,\ c,\ d \leq q$). That is, one finds a heavily over-determined set of equations. The existence of a solution is clearly a necessary condition to have a YBA. Actually it is also a sufficient condition since one can define a YB generator acting on $\mathcal{A} \otimes \mathcal{A}$ as

$$(1.13) \qquad \left[t_{ab\,(\theta)}^{(A,A)}\right]_{cd} = R_{ca}^{bd}(\theta).$$

It obeys

$$(1.14) \qquad R(\theta - \theta')[t_{(\theta)}^{(A,A)} \otimes t_{(\theta')}^{(A,A)}] = [t_{(\theta')}^{(A,A)} \otimes t_{(\theta)}^{(A,A)}] R(\theta - \theta'),$$

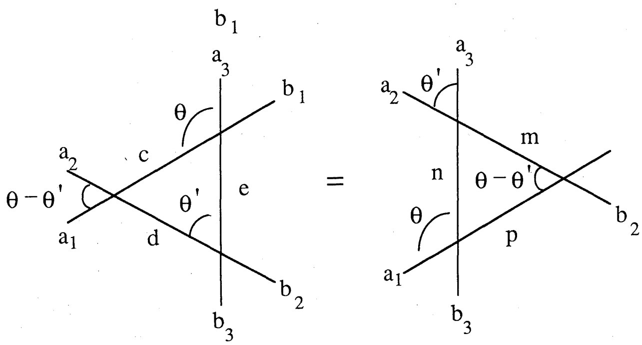



Fig. 6. The YBE for the $R$-matrix.

which just follows by rewriting eq.(1.12) with the help of eq.(1.13).

The most remarkable fact in integrable theories is that eqs.(1.11) or (1.12) do admit a rich set of non-trivial solutions. Actually each solution exhibits some invariance which probably explains its very existence. That is, thanks to the presence of an invariance the number of actual independent equations is largely reduce from $q^6$.

A YB algebra is invariant [see eq.(1.3)] under a transformation $g \in \mathcal{G}$ in $\mathcal{A}$

$$T_{ab}(\theta) \longrightarrow g_{ac}T_{cb}(\theta) \tag{1.15}$$

provided [16]

$$[g \otimes g, R(\theta)] = 0, \qquad \forall\theta \in \mathbf{C}, \quad \forall g \in \mathcal{G}. \tag{1.16}$$

More generally

$$[g_I \otimes g_J, T^{(I,J)}(\theta)] = 0, \qquad \forall\theta \in \mathbf{C}, \quad \forall g \in \mathcal{G} \tag{1.17}$$

where $g_I$ and $g_J$ are the representation of $g \in \mathcal{G}$ acting on the vector spaces $V^I$ and $V^J$ respectively. For an infinitesimal transformation

$$g_I = 1 + i\varepsilon S_I, \qquad g_J = 1 + i\varepsilon S_J$$

where $\varepsilon \ll 1$ and $S_I$ and $S_J$ are the generators representation in $V^I$ and $V^J$ respectively. Hence eq.(1.17) yields

$$[S_I, T^{(I,J)}(\theta)] + [S_J, T^{(I,J)}(\theta)] = 0. \tag{1.18}$$

There exists a direct connecion between the kind of symmetry group $\mathcal{G}$ of the YBA and the functional dependence on $\theta$. This connection is displayed in table I.

**TABLE I**

| $\mathfrak{g}$ : symmetry group | $\theta$ – dependence in $R_{ab}{}^{cd}(\theta)$ |
|---|---|
| discrete : $Z_q$ | elliptic |
| continuous abelian : $U(1)^q$ | trigonometric or hyperbolic |
| continuous non-abelian : $U(q), O(q)$ | rational |

Table I Correspondence between the symmetry group of the Yang-Baxter algebras and the functional dependence of their generators on the spectral parameter $\theta$.

Another important invariance of YB algebras is the shift invariance. That is, if $T(\theta)$ is a YB generator, so it is

$$T(\theta - \alpha)$$

with fixed $\alpha$. A look to eq.(1.3) shows that this is true since $R$ depends on the difference $\theta - \theta'$, $\alpha$ must be the same so it drops.

Let us now discuss the most important property of YBA: the reproduction property. It can be stated as follows: if $t(\theta)$ obeys a YBA [as eq.(1.3)] with horizontal space $\mathcal{A}$ and vertical $\mathcal{V}$, so does

$$T_{ab}^{[N]}(\theta) = \sum_{a_1 \cdots a_{N-1}=1}^{q} t_{aa_1}(\theta) \otimes t_{a_1a_2}(\theta) \otimes \cdots \otimes t_{a_{N-1}b}(\theta), \qquad q = \dim \mathcal{A} \tag{1.19}$$

with the same $R$-matrix. The auxiliary space for $T^{(N)}$ is also $\mathcal{A}$, the vertical one being

$$\mathcal{V}_N \equiv \bigotimes_{j=1}^{N} \mathcal{V}^j .$$

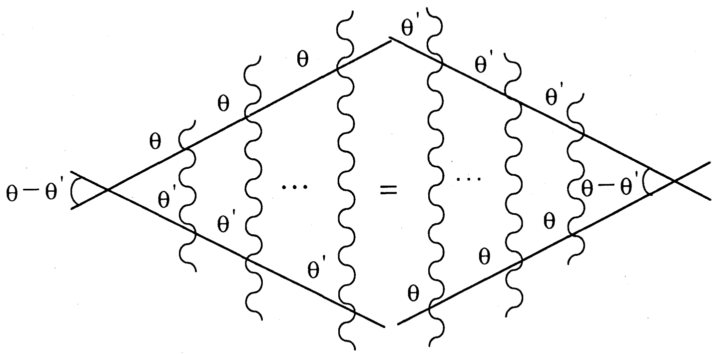



Fig. 7. The YBE for the YB generator (1.19).

The easiest way to prove that (1.19) fulfils eq.(1.3) is graphically. As depicted in Fig.7, eq.(1.19) follows by pushing a solid line $N$ times through Fig.2-type vertices

For $N = 2$ eq.(1.19) can be considered as a way of multiplying YB generators yielding new YB generators. This can be called a coproduct and shows that we have a Hopf algebra. More generally, if the YB generators are invariant under a group $\mathcal{G}$ [eq.(1.15)–(1.17)] we have as generator in $\mathcal{A} \otimes \mathcal{V}^N$,

(1.20)
$$T_{ab}^{[N]}(\theta, \alpha, g) = \sum_{a_1 \cdots a_{N-1}=1}^{q}$$
$$[g_1 t(\theta - \alpha_1)]_{aa_1} \otimes [g_2 t(\theta - \alpha_2)]_{a_1 a_2} \otimes \cdots \otimes [g_N t(\theta - \alpha_N)]_{a_{N-1} b}.$$

It obeys the YB eq.(1.3) for any fixed transformations $\underset{\sim}{g} = (g_1, \dots, g_N)$ and $\underset{\sim}{\alpha} = (\alpha_1, \dots, \alpha_N)$.

There exists in addition, another coproduct multiplying the generators from right to left

$$T_{ab}^{[N]}(\theta, \alpha, h) =$$
$$\sum_{a_1 \cdots a_{N-1}=1}^{q} [h_1 t(\theta - \alpha_1)]_{a_1 b} | \otimes [h_2 t(\theta - \alpha_2)]_{a_2 a_1} \otimes \tag{1.21}$$
$$\cdots \otimes [h_N t(\theta - \alpha_N)]_{a a_{N-1}}.$$

That is, $T(\theta, \underset{\sim}{\alpha}, \underset{\sim}{h})$ obeys the same YBA [eq.(1.3)] as $t(\theta)$ does.

As we see, YBA are not Lie algebras since the sum of two generators $T(\theta)$ is not a YB generator. However, one finds for the YBA the analogous for most of the features of Lie algebras. The YBE (1.12) plays the role of the Jacobi identity in Lie algebras. The fundamental $R$-matrix being the analogue of the structure constants. There exists for YBA an "adjoint representation" [eq.(1.13)] provided by the $R$-matrix. We also have a "Cartan algebra" formed by the commuting transfer matrices $\tau_I(\theta)$ [eq.(1.7)]. A representation theory for YBA has been developed. That is, the construction of $T(\theta)$ for different spaces $(\mathcal{A},\mathcal{V})$ given a fundamental $R$-matrix [2].

Actually there exist more general commuting transfer matrices than (1.7). It follows from eqs. (1.1) and (1.17) that the following operators on $V^J$:

$$\tau_{g_I}(\theta) = \mathrm{Tr}_{V_I}[g, T^{(I,J)}(\theta)], \qquad \tau_{g_K}(\theta) = \mathrm{Tr}_{V_K}[g, T^{(K,J)}(\theta)] \tag{1.22}$$

commute

$$[\tau_{g_I}(\theta), \tau_{g_K}(\theta')] = 0, \qquad \forall \theta, \theta' \in \mathbf{C}, \quad \forall g \in \mathcal{G}. \tag{1.23}$$

Notice in eq.(1.23) that the transformation $g \in \mathcal{G}$ is the same in both transfer matrices.

It is legitimate to call $T(\theta, \underset{\sim}{g}, \underset{\sim}{\alpha})$ [eq.(1.20)] a gauge transformation of $T(\theta)$ [eq.(1.19)]. We apply in eq.(1.20) a group symmetry transformation $(g_i, \alpha_i)$ that depends upon the site. This is a one-dimensional local gauge transformation on the lattice. Actually, a YB generator gauge transformed under $\mathcal{G}$ can be related with the untransformed one as follows [3]

$$T_{ab}^{[N]}(\theta, g) = \prod_{i=1}^{N} G_i^{-1} T_{ac}^{[N]}(\theta) \prod_{i=1}^{N} G_i J_{cb} \tag{1.24}$$

where $h_i^{-1}$ and $h_i$ here act on the $i$-th vertical space with

$$G_i = \prod_{j=1}^{i} g_j \qquad \text{and} \qquad J = \prod_{e=1}^{N} g_e .$$

Let us show that the YB algebra (1.1) is invariant under the replacement

$$T^{(I,J)}(\theta) \longrightarrow T^{(I,J)}(\theta)^T$$

where $T$ means transpose in both $\mathcal{V}$ and $\mathcal{A}$ spaces. That is

$$[T_{ab}^{(I,J)}(\theta)^T]_{\alpha\gamma} = [T_{ba}^{(I,J)}(\theta)]_{\gamma\alpha}$$

Taking the transpose of eq.(1.1) on $V^I, V^J$ and $V^K$ yields

$$
\begin{aligned}
(1.25)\qquad & [T^{(I,J)}(\theta')]^T[T^{(K,J)}(\theta)]^T[T^{(I,J)}(\theta-\theta')]^T \\
&= [T^{(K,I)}(\theta-\theta')]^T[T^{(K,J)}(\theta)]^T[T^{(I,J)}(\theta')]^T.
\end{aligned}
$$

This coincides with eq.(1.1) with the left and right members exchanged. Therefore, if $T(\theta)$ obeys a YB algebra, so does $T(\theta)^T$. Very often

$$T(\theta)^T = T(\theta).$$

Let us now prove that the YBE (1.12) guarantees that the product of the operators $T_{ab}(\theta)$ acting in $\mathcal{V}$ is associative. That is, the constraints imposed by eq.(1.3) on the products of $T_{ab}(\theta)$ are not in general compatible with associativity unless eq.(1.12) holds. There are in general two inequivalent ways to relate

$$T(\theta_1)\otimes T(\theta_2)\otimes T(\theta_3)$$

with

$$T(\theta_3)\otimes T(\theta_2)\otimes T(\theta_1).$$

(Here $\otimes$ means tensor product in $\mathcal{A}$).

That is,

$$(123)\longrightarrow(213)\longrightarrow(231)\longrightarrow(321)$$

or

$$(123)\longrightarrow(132)\longrightarrow(312)\longrightarrow(321).$$

Both must lead to the same result. One finds in this way the condition

$$
\begin{aligned}
(1.26)\qquad & S_{12}^{-1}S_{13}^{-1}S_{23}^{-1}[T(\theta_1)\otimes T(\theta_2)\otimes T(\theta_3)]S_{23}S_{13}S_{12} \\
&= S_{23}^{-1}S_{13}^{-1}S_{23}^{-1}[T(\theta_1)\otimes T(\theta_2)\otimes T(\theta_3)]S_{12}S_{13}S_{23},
\end{aligned}
$$

where the matrices $R_{ij}$ $(i,j=1,2,3, i\neq j)$ act in the space $\mathcal{A}_1\otimes\mathcal{A}_2\otimes\mathcal{A}_3$. $S_{ij}$ equals $PR(\theta_i-\theta_j)$ in the space $\mathcal{A}_i\otimes\mathcal{A}_j$ and it is the unit matrix in the space $\mathcal{A}_k$, $j\neq k\neq i$. That is $S_{12\,a_1a_2a_3}^{\;b_1b_2b_3} = R_{a_2a_1}^{b_1b_2}(\theta_1-\theta_2)\delta_{a_3}^{b_3}$. We denote by $P$ the matrix

$$(1.27)\qquad P_{ab}^{cd} = \delta_a^d\delta_b^c.$$

It follows from eq.(1.26) that

$$(1.28)\qquad [S_{23}S_{13}S_{12}S_{23}^{-1}S_{13}^{-1}S_{12}^{-1},\quad T(\theta_1)\otimes T(\theta_2)\otimes T(\theta_3)] = 0.$$

But, the YBE (1.12) implies for $S_{ij}$ that

$$S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12} \tag{1.29}$$

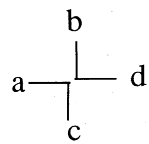
and hence eq.(1.28) is identically satisfied. Reciprocally one can derive the YBE (1.29) by requiring the associativity of the product of operators $T_{ab}$.

When $\theta = \theta'$ eq.(1.3) naturally suggests that $R(0)$ is a multiple of the unit matrix in $\mathcal{A}\otimes\mathcal{A}$. When this happens the corresponding $R$-matrix is called regular. That is

$$R(0) = c1 \tag{1.30}$$

where $c$ is a numerical constant and 1 the unit operator. This property can be represented graphically as follows (cf. Fig.5)

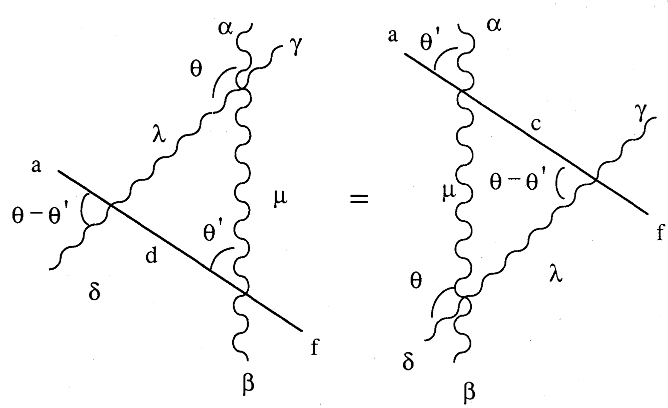
(1.31) $$R^{ab}_{cd}(\theta) = c\,\delta_{bd}\,\delta_{ac} =$$




This property plays a key role in the theory of integrable models. First it implies that the transfer matrices $\tau(\theta)$ built from $R$-matrices are generating functionals of local lattice operators. That is, those $\tau(\theta)$ following from eq.(1.19) when $\tau(\theta)$ is given by eq.(1.13) (see eqs.(2.12)–(2.13)).

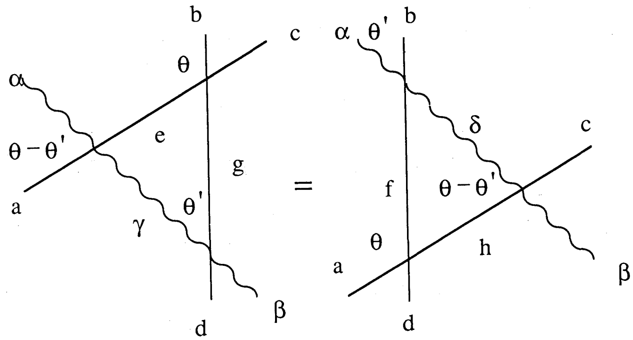
Secondly, the unitarity properties of $T(\theta)$ follows from eq.(1.30). Let us consider the YB equation (1.1) when a) $V^I = \mathcal{A}$, $V^J = V^K = \mathcal{V}$ and b) $V^I = \mathcal{V}$, $V^J = V^K = \mathcal{A}$. This gives respectively

(1.32)



and

(1.33)




Now, if we set $\theta = 0$ in (1.32) and (1.33), we find using eq.(1.30) for $R^A(\theta)$ and $R^{\mathcal{V}}(\theta)$,

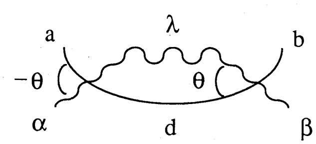
$$M_{af}^{\delta\beta}(\theta)\delta_{\alpha\gamma} = \delta_{\delta\beta}M_{af}^{\alpha\gamma}(-\theta)$$

(1.34)
$$M_{ad}^{\alpha\beta}(\theta)\delta_{bc} = \delta_{ad}M_{bc}^{\alpha\beta}(-\theta).$$

Here,

$$M_{ab}^{\alpha\beta}(\theta)\delta_{bc} = [T_{ad}(-\theta)]_{\alpha\lambda}[T_{db}(\theta)]_{\lambda\beta}$$

(1.35) $M_{ab}^{\alpha\beta}(\theta) =$




Eq.(1.34) shows that

(1.36)
$$M_{ab}^{\alpha\beta}(\theta) = \delta_{ab}\delta_{\alpha\beta}\rho(\theta)$$

where $\rho(-\theta) = \rho(\theta)$ is a $c$-number function. Eq.(1.35) is actually an operator product on two vector spaces $\mathcal{A}$ and $\mathcal{V}$. Keeping in mind this double matrix product, we find

(1.37)
$$T(\theta)T(-\theta) = \rho(\theta)1$$

where 1 stands for the unit operator in $\mathcal{V} \otimes \mathcal{A}$. We have found that all YB generators possess an inverse provided their $R$-matrix is regular in the same of eq.(1.30). That is

$$T^{-1}(\theta) = \frac{1}{\rho(\theta)} T(-\theta)$$

fulfils

$$T(\theta)T^{-1}(\theta) = T^{-1}(\theta)T(\theta) = 1. \tag{1.38}$$

The antipode generator is defined by

$$T^A(\theta) \equiv T^{-1}(\theta)^t \tag{1.39}$$

where $t$ means transpose in $\mathcal{A}$. That is

$$[T_{ab}(\theta)^t]_{\alpha\beta} = T_{ba}(\theta)_{\alpha\beta}. \tag{1.40}$$

The antipode is an automorphism of the YB algebra. It follows from eqs.(1.3) and (1.38) that

$$R(\theta - \theta')[T^A(\theta) \otimes T^A(\theta')] = [T^A(\theta') \otimes T^A(\theta)]R(\theta - \theta'). \tag{1.41}$$

The YB algebra in therefore a Hopf algebra with antipode. Since the coproduct [eq.(1.19) for $N = 2$] is non-commutative as well as the usual product of $T(\theta)$, we have a non-commutative and non-cocommutative Hopf algebra.

## §2. Physical realizations of Yang-Baxter algebras

In this section we shall describe YB algebras in two-dimensional statistical models, field theories and $S$-matrices. We associate in Section I a vector space $V^I$ to each type of lines and a YB generator $T^{(I,J)}(\theta)$ to a pair of lines $(I, J)$ intersecting with an angle $\theta$. This can be immediately applied to a two-dimensional lattice of lines [Fig.8] intersecting at the sites. The vector spaces describe the possible local states of the bonds and the $t(\theta)$ describe the statistical weights of the different link configurations.

That is the matrix element $[t_{ab}(\theta)]_{\alpha\beta}$ defines the probability for the local configuration depicted in Fig.9. The product of the local weights over all sites in the lattice yields the probability for such configuration of the whole system. Finally summing over all possible configurations gives the partition function $Z$. When periodic boundary conditions are used in both horizontal and vertical directions, $Z$ expresses as

$$Z = \mathrm{Tr}_{\mathcal{V}}[\tau(\theta)^M]. \tag{2.1}$$

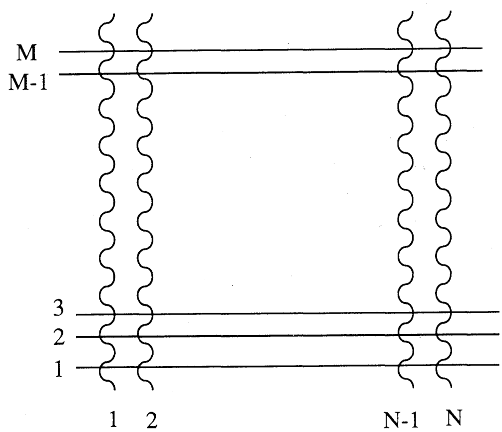



Fig. 8. A $N \times M$ two dimensional lattice. The local states of horizontal (vertical) bonds belong to the vector space $\mathcal{A}$ ($\mathcal{V}$).

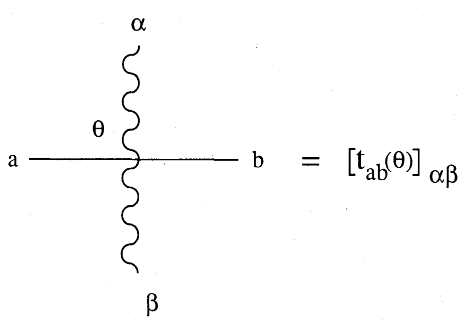



Fig. 9. The local statistical weights $w(\alpha\beta|ab)$ depend on the states of the four bonds joining at a vertex.

Actually eq.(2.1) holds irrespective of the YB equations.

The transfer matrices $\tau_{g_A(\theta)}$ [16] [eq.(1.22)] correspond to twisted boundary conditions. That is when the operators at sites $N+1$ and 1 are related by the transformation $g$:

$$S_{N+1} = g\ S_1\ g^{-1}, \quad \forall S. \tag{2.2}$$

Here $g$ acts in the appropriate representation of $\mathcal{G}$. Then $\tau_{g_A}(\theta)$ is the transfer matrix. If we also impose twisted b.c. in the vertical direction with a twist $h_{\mathcal{V}}$, $Z$ writes

$$A_{g,h} = \mathrm{Tr}[\tau_{g_{\mathcal{A}}}(\theta)^M h_{\mathcal{V}}]. \tag{2.3}$$

Eqs.(2.1) and (2.3) show how important is the knowledge of the eigenvalues of $\tau(\theta)$. Actually, just the largest eigenvalue $\Lambda^{[N]}_{MAX}(\theta)$ gives the free energy in the thermodynamic limit

$$\begin{aligned} f &= -\lim_{N,M\to\infty} \frac{1}{NM} \log Z \\ &= -\lim_{N\to\infty} \frac{1}{N} \log \Lambda^{[N]}_{MAX}. \end{aligned} \tag{2.4}$$

(The dependence on the b.c. drops in the $N = M = \infty$ limit).

The lattice model here described is called a vertex model. It is homogeneous but not isotropic since horizontal and vertical lines are of different nature. One can even generalize these integrable vertex models taking lines at arbitrary intersection angles [49]. Also taking inhomogeneous weights $g_x h_y t(\theta - \alpha_x - \beta_y)$ that depend upon the horizontal $x$ and the vertical $y$ coordinates. Moreover, one could take the lattice lines from all possible vector spaces $V^I \in \mathcal{I}$ at will. All these models are integrable and solvable although inhomogeneous and anisotropic.

Let us now study the transfer matrices $\tau(\theta)$ as generating functionals of commuting local operators on the lattice. This is the case for $R$-matrix models (where $\mathcal{A} = \mathcal{V}$) when $R$ is a regular $R$-matrix [eq.(1.30)]. We find from eqs.(1.13) and (1.30)–(1.31)

$$[t_{ab}(0)]_{cd} = c\delta^c_b \delta^d_a. \tag{2.5}$$

Then for a $N$-site transfer matrix as defined by (1.7) and (1.19)

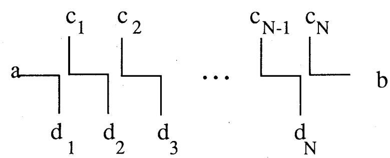
$$\begin{gathered} \tau(0)^{[N]}_{a|b} = c^N \prod_{i=1}^{N} \delta_{a_i b_{i+1}}, \\ a \equiv (a_1, a_2, \ldots, a_N), \quad b \equiv (b_1, b_2, \ldots, b_N) \end{gathered} \tag{2.6}$$

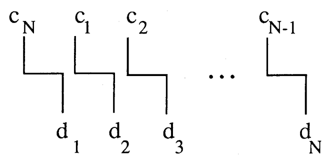
where $b_{N+1} \equiv b_1$. The operator in the rhs of (2.6) is just the lattice unit shift operator to the right. Therefore, we can write the momentum as

$$\mathcal{P} = i \log\left[c^{-N} \tau^{[N]}(0)\right]. \tag{2.7}$$

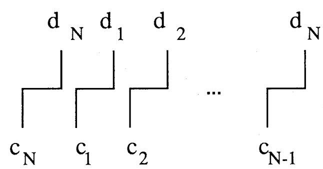
Let us now show that the logarithmic derivative of $\tau^{[N]}(\theta)$ at $\theta = 0$ gives an operator coupling nearest neighbors.

Using eq.(1.31), $T^{[N]}_{ab}(0)_{c|d}$ and $\tau^{[N]}(0)_{c|d}$ can be drawn as follows

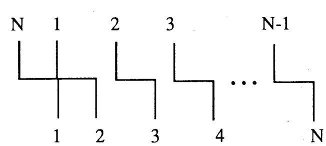
(2.8) $$T^{[N]}_{ab}(0)_{\underset{\sim}{c}|\underset{\sim}{d}} =$$ 

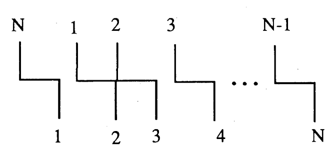

(2.9) $$\tau^{[N]}(0)_{\underset{\sim}{c}|\underset{\sim}{d}} =$$ 

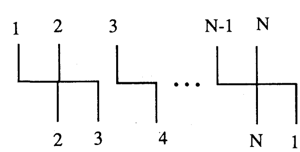

Similarly,

(2.10) $$\left(\tau^{[N]}(0)^{-1}\right)_{\underset{\sim}{c}|\underset{\sim}{d}} =$$ 

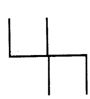

Now, if we compute $\frac{d}{dt}\tau^{[N]}(\theta)$ from eq.(1.19) we obtain $N$ terms, each one containing $\frac{d}{dt}t^{(h)}(\theta)$, $1 \leq h \leq N$ and the others $t^{(l)}$ $(l \neq h)$ not derived. Hence, setting $\theta = 0$ yields

(2.11) $$\dot{\tau}^{[N]}(0) =$$ 


$+$ 


$+\ldots+$ 

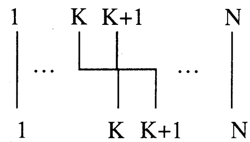

Here



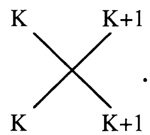
stands for $\dot{R}(0)$.

It is nów very simply to perform the product $\tau^{[N]}(0)^{-1}\dot{\tau}^{[N]}(0)$ just combining eqs.(2.10) and (2.11) with the result

$$\left(\tau^{[N]}{}_{(0)}{}^{-1}\right)_{\text{cld}}\dot{\tau}^{[N]}{}_{(0)} = \sum_{K=1}^{N}$$




$$= \sum_{K=1}^{N}$$




Therefore $\tau^{[N]}(0)^{-1}\dot{\tau}^{[N]}(0)$ is a sum of terms each one acting as an operator on two neighboring sites. Now, putting all factors

$$H = \frac{\partial}{\partial\theta}\log\tau^{[N]}(\theta)\big|_{\theta=0} = \sum_{n=1}^{N} h_{n,n+1}, \tag{2.12}$$

where the matrix elements of $h$ reads

$$\langle c_n d_n | h_{n,n+1} | c_{n+1} d_{n+1}\rangle = \frac{1}{c}\dot{R}(0)^{d_{n+1},c_{n+1}}_{d_n,c_n}. \tag{2.13}$$

More generally the $n$-th derivative of $\log\tau(\theta)$ at $\theta = 0$ is an operator that couple $n+1$ neighboring sites [47].

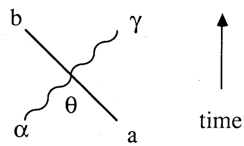
The operator $H$ can be interpreted as a one-dimensional quantum hamiltonian. It is an operator coupling neighboring q-component "spins". The word spins only applies, rigorously speaking, when the fundamental $R$-matrix corresponds to the six or eight vertex model. That is, the underlying Lie algebra in $A_1$ and we have true $SU(2)$ spins. Otherwise one finds $SU(q)$ spins, $O(q)$ spins, etc. Eq.(2.12) suggest that $\theta$ may be the imaginary time variale. This possibility has not been fully explored yet. Anyway it must be noticed that $\tau(\theta) \neq e^{\theta H}$.

The next physical application or interpretation of YB algebras is two-dimensional $S$-matrix theory [4]. In this context the lines in figs. 1-6 describe the world-lines of particles propagating in two-dimensional Minkowski space-time. The vector spaces $V^I$ associated to them describe the Hilbert space of internal states of the particles. The variable $\theta$ corresponds to the rapitidy (velocity $=$ tanh $\theta$). We recall that the energy and momentum of a particle of mass $m$ writes in two-dimensions as

$$E = m\cosh\theta, \qquad p = m\sinh\theta. \tag{2.14}$$

The matrix elements of the YB operators $T(\theta)$ correspond to the two-body $S$-matrix amplitudes as the intersection of two world-line trajectories suggests (cf. Fig.2)

$$S^{b\gamma}_{a\alpha}(\theta) = \left[T_{ab}(\theta)\right]_{\alpha\gamma} = \tag{2.15}$$




Here $\theta = \theta_1 - \theta_2 = \theta_1' - \theta_2'$ stands for the relative rapidity of the particles. $|\theta_1\alpha, \theta_2 a\rangle$ describes their initial state and $|\theta_1'\gamma, \theta_2' b\rangle$ their final one.

The $S$-matrix theories associated to YB algebras are those of factorizable scattering. That is,

I) There is no particle production. The number of particles of each type in the initial and final states coincide. The set of initial and final particle momenta coincide (particles can exchange their momenta during the collisions).

II) The $N$-particle $S$-matrix is expressed as a product of $N(N-1)/2$ two-particle $S$-matrices as if the process of $N$-particle scattering were reduced to a sequence of pair collisions.

Integrable quantum field theories in two dimensions provide explicit realizations of such S-matrices [3,4,6].

In those theories the basic object is the two-body scattering matrix, since all amplitudes write as appropriate products of two body amplitudes.

The YBE (1.2) ensures that one obtains the same $N$-body $S$-matrix irrespective of the temporal order of the pair interactions. This is immediately visualized graphically (Fig.10) since a change of the ordering means pushing lines through intersections keeping the angles (the kinematics) fixed.

The lack of particle production and the factorization can be understood as a consequence of the existence of extra conserved charges when a underlying QFT is available. That is, conservation laws besides the usual ones (energy, momentum, electric charge, isospin,...). Usually the

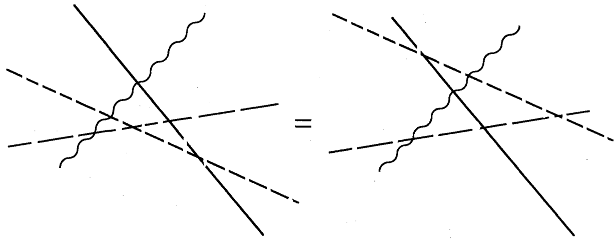


Fig. 10. The YBE allows to displace parallely any line.

presence of one extra law (local or non-local) is enough to forbid particle production and enforce factorization of two body amplitudes [5].

Although we are giving two different interpretations to YB operators the physical requirements are clearly different for vertex weights and for $S$-matrix amplitudes. Boltzmann weights are usually real and positive since they express probabilities. However, a vertex model where some weights are negative or complex may be also interesting.

$S$-matrix elements can be complex but they must be meromorphic functions of $\theta$ due to the general principles of scattering theory [4,48]. Moreover space-time symmetries impose the following requirements:

(2.16) Time-reversal invariance: $S(\theta) = S(\theta)^T$ or $S^{a\beta}_{c\delta}(\theta) = S^{c\delta}_{a\beta}(\theta)$,

(2.17) Parity inversion invariance: $S^{a\beta}_{c\delta}(\theta) = S^{\beta a}_{\delta c}(\theta)$,

$(S(\theta) = PS(\theta)P$ when $\mathcal{V} = \mathcal{A})$.

In addition unless there are sources or sinks of particles $S$ must be unitary

$$S(\theta)S(\theta)^\dagger = 1. \tag{2.18}$$

Moreover, it must obey real analyticity

$$S(\theta^*)^* = S(-\theta). \tag{2.19}$$

The unitarity follows from the YBE if time reversal and real analicity hold. We find from eqs.(1.37) for $T(\theta) = R(\theta)$ [eq.(1.13)] and (2.15)

$$S(\theta)S(-\theta) = \rho(\theta)1,$$

where 1 is the unit operator in $\mathcal{A} \otimes \mathcal{V}$. Then using eqs.(2.16), (2.19) and (2.20) yields the unitarity relation (2.18) if one absorbs a factor $\sqrt{\rho(\theta)}$ in the definition of $S(\theta)$.

In particle theory one has in addition the crossing invariance. It says that the amplitudes of the process

$$a + \beta \to c + \delta \tag{2.21}$$

and

$$a + \bar{\delta} \to c + \bar{\beta} \tag{2.21}$$

(where $\bar{\beta}$, $\bar{\delta}$ means antiparticles of $\beta$ and $\delta$) are related by appropriate analytic continuations in $\theta$. Crossing yields using eqs.(2.14)–(2.15)

$$S_{a\beta}^{c\delta}(\theta) = S_{a\delta}^{c\beta}(i\pi - \theta) \tag{2.22}$$

in a real basis of particle states and with a special normalization of $\theta$ such that eq.(2.14) holds. In general crossing symmetry requires

$$S(\theta)^{t_1} = (1 \otimes W) S(-\theta - \eta)(1 \otimes W^{-1}), \tag{2.23}$$

where $t_1$ means transpose in the first horizontal space, $W$ is a constant matrix and $\eta$ a parameter that depends on the model.

The $P$ and $T$ symmetries (2.16)–(2.17) have a precise counterpart in the vertex language. $T$-invariance of the $S$-matrix implies invariance of the vertex weights under simultaneous up-down and left-right exchange. $P$-invariance means invariance of the weights under reflection over a line at $+45°$ from the horizontal axis. Crossing implies that a left-right exchange in the vertices is equivalent to make $\theta \to -\eta - \theta$ on the spectral parameter. Lack of any of these invariances can be interpreted as the presence of an external field in the vertex language.

The correspondence between an integrable vertex model and a factorizable $S$-matrix can be pushed even further as one sees from refs. [46]. That is, eqs.(1.2) and (2.15) define $S(\theta)$ up to a $c$-number normalization $\rho(\theta)$ that can be fixed by requiring unitarity [eq.(2.18)] and analyticity [eq.(2.19)]. It happens that these $S$-matrix requirements leads to a normalization that makes the free energy equal to zero for the corresponding vertex model in the thermodynamic limit. In other words they define a normalization $\rho(\theta)$ where $Z = 1$ at $N = \infty$. Therefore, if one starts from a given normalization of the weights, this factor $\rho(\theta)$ is just the partition function per site, or [46]

$$f = -\log \rho(\theta). \tag{2.24}$$

It may be noticed that the partition function [eq.(2.1)] in the $S$-matrix language will be the trace of the $S$-matrix describing the scattering of $N$ particles, all with rapidity $\theta$ by $M$ particles at rest.

## §3. The six vertex model and its descendants

The six vertex model corresponds to the trigonometric and hyperbolic solutions of the YBE (1.11) for $q = 2$; that is

$$R(\theta) = \begin{pmatrix} a(\theta,\gamma) & 0 & 0 & 0 \\ 0 & c(\gamma) & b(\theta) & 0 \\ 0 & b(\theta) & c(\gamma) & 0 \\ 0 & 0 & 0 & a(\theta,\gamma) \end{pmatrix} \tag{3.1}$$

We have here three different regimes,

I) $a(\theta,\gamma) = \sinh(\gamma - \theta)$, $b(\theta) = \sinh\theta$, $c(\gamma) = \sinh\gamma$, $\gamma > \theta > 0$ in the antiferroelectric regime.

II) $a(\theta,\gamma) = \sin(\gamma - \theta)$, $b(\theta) = \sin\theta$, $c(\gamma) = \sin\gamma$, $\pi > \gamma > \theta > 0$ in the trigonometric regime. This regime is critical (gapless).

III) $a(\theta,\gamma) = \sinh(\theta + \gamma)$, $b(\theta) = \sinh\theta$, $c(\gamma) = \sinh\gamma$, $\theta > 0, \gamma > 0$ in the ferroelectric regime.

The parameter $\gamma$ describes the anisotropy of the model. The character of regimes I, II and III will be clear from the ground state and excitations obtained below. This model enjoys the following symmetry group $\mathcal{G}$ [in the sense of eq.(1.15)]

$$\mathcal{G} = \{e^{i\alpha\sigma_z}, 0 \leq \alpha < 2\pi; \sigma_x\} \tag{3.2}$$

That is $\mathcal{G} = U(1) \otimes Z_2$. When $\gamma = 0$ this group enlarges to $SU(2)$. This point corresponds to a Kosterlitz-Thouless type transition as we will see below from the explicit solution.

It is called six vertex model, since the non-zero elements of the $R$–matrix, eq.(3.1) define six allowed configurations. The integrable eight-vertex model will not be considered here [1]. The state of a bond in the six-vertex (and eight-vertex) models is usually characterized by the sense of an arrow. This corresponds here to the values 1 or 2 of the vertical and horizontal indices. In Fig.11 the allowed configurations and their respective statistical weights are depicted.

It must be recalled that the trigonometric regime of the six-vertex model (II) describes the critical (zero gap) limit of the eight-vertex model [1]. As it will be clear from the solution one describes a critical line when $\gamma$ varies from 0 to $\pi$. As an $S$-matrix eq.(3.1) for regime II describes the scattering of a particle and its antiparticle with a conserved $U(1)$ charge [4]. The crossing symmetry (2.23) writes here

$$[PR(\theta)]^{t_1} = (1 \otimes \sigma)R(-\theta - \gamma)(1 \otimes \sigma)$$

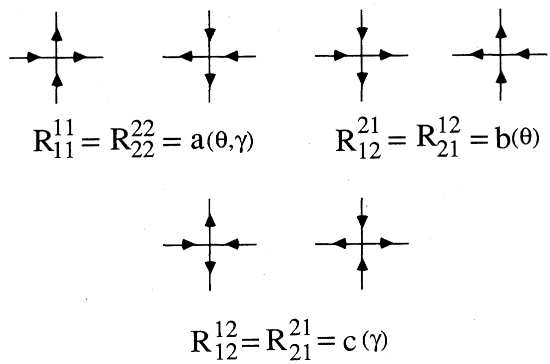



Fig. 11. Allowed configurations in the six-vertex model and their statistical weights (see eq.(3.1)).

where

$$\sigma \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_y$$

The one-dimensional quantum hamiltonian associated to the six-vertex model is the XXZ hamiltonian. One finds from eqs.(2.12)–(2.13) and (3.1)

$$h = \frac{1}{2\sin\gamma}[\cos\gamma + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y - \cos\gamma\sigma_z \otimes \sigma_z].$$

Then

$$\begin{aligned} H_{XXZ} =& \frac{N}{2}\mathrm{cotg}\gamma \\ &+ \frac{1}{2\sin\gamma}\sum_{a=1}^{N}(\sigma_x^a \otimes \sigma_x^{a+1} + \sigma_y^a \otimes \sigma_y^{a+1} - \cos\gamma\sigma_z^a \otimes \sigma_z^{a+1}). \end{aligned}$$

The YB generators read here (for one site)

$$(3.3) \quad t_{11}(\theta) = \begin{pmatrix} a(\theta,\gamma) & 0 \\ 0 & b(\theta) \end{pmatrix}, \qquad t_{22}(\theta) = \begin{pmatrix} b(\theta) & 0 \\ 0 & a(\theta,\gamma) \end{pmatrix}$$
$$t_{12} = c(\gamma)\sigma_-, \qquad t_{21} = c(\gamma)\sigma_+.$$

The YB generator $T_{ab}(\theta)$ follows from eq.(1.19) where one inserts the $t_{ab}(\theta)$ given by eq.(3.3). One can then set

$$T^{[N]}(\theta) = \begin{pmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{pmatrix} \tag{3.4}$$

The YB algebra defined by the $R$-matrix (3.1) yields some number of bilinear algebraic relations between the $T^{[N]}_{ab}(\theta)$. Let us just write down the more useful ones for the subsequent derivations

$$\begin{aligned} A(\theta)B(\theta') &= g(\theta' - \theta)B(\theta')A(\theta) - h(\theta' - \theta)B(\theta)A(\theta') \\ D(\theta)B(\theta') &= g(\theta - \theta')B(\theta')D(\theta) - h(\theta - \theta')B(\theta)D(\theta') \\ [C(\theta), B(\theta')] &= [A(\theta')D(\theta) - A(\theta)D(\theta')]h(\theta - \theta') \end{aligned} \tag{3.5}$$

$$B(\theta)B(\theta') = B(\theta')B(\theta), \tag{3.6}$$

where $g(\theta) = a(\theta,\gamma)/b(\theta)$ and $b(\theta) = c(\gamma)/b(\theta)$.

Let us now proceed to construct the exact eigenvectors and eigenvalues of

$$\tau^{[N]}(\theta) = \mathrm{Tr}_A T^{[N]}(\theta) = A(\theta) + D(\theta), \tag{3.7}$$

using the algebraic Bethe Ansatz [7]. We shall assume $N$ to be even. One notices that the ferromagnetic state

$$|\Omega\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_3 \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_N, \tag{3.8}$$

is an eigenvector of $A(\theta)$ and $D(\theta)$

$$\begin{aligned} A(\theta)|\Omega\rangle &= a(\theta,\gamma)^N|\Omega\rangle \\ D(\theta)|\Omega\rangle &= b(\theta)^N|\Omega\rangle. \end{aligned} \tag{3.9}$$

In addition

$$C(\theta)|\Omega\rangle = 0 \tag{3.10}$$

whereas $B(\theta)|\Omega\rangle$ is non-zero and not proportional to $|\Omega\rangle$. The algebraic Bethe ansatz consist in looking for eigenvectors of $\tau(\theta)$ with the form

$$\psi(\theta_1, \dots, \theta_r) = B(\theta_1)B(\theta_2)\cdots B(\theta_r)|\Omega\rangle. \tag{3.11}$$

Here, the complex number $\theta_1, \dots, \theta_r$ will be determined by requiring that $\psi(\theta_1, \dots, \theta_r)$ is an eigenvector of $\tau(\theta)$.

In order to do that one applies $A(\theta)+D(\theta)$ to the r.h.s. of eq.(3.11) and pushes $A(\theta)+D(\theta)$ through the $B(\theta_j)$ with the help of eqs.(3.5). After using eqs.(3.5) $r$ times, $A(\theta)$ and $D(\theta)$ reach $|\Omega\rangle$ where their action is known from eqs.(3.9). These operations produced a lot of terms. Let us first write down explicitely those generated by the first term in eqs.(3.5):

$$\begin{aligned} A(\theta)\psi(\underset{\sim}{\theta}) &= \prod_{j=1}^{r} g(\theta_j-\theta)a(\theta,\gamma)^N B(\theta_1)\cdots B(\theta_r)|\Omega\rangle \\ &\quad + \text{unwanted terms} \\ &= \Lambda_+(\theta)\psi(\theta_1,\ldots,\theta_r) + \text{unwanted terms} \\ &\quad \underset{\sim}{\theta}\equiv(\theta_1,\ldots,\theta_r), \end{aligned} \tag{3.12}$$

and an analogous formula for $D(\theta)\psi$.

The remaining terms are called "unwanted" since they are not proportional to $\psi(\theta)$ and hence they must finally cancel in order to get an eigenvector of $\tau(\theta)$.

Now, let us concentrate in terms containing the vector

$$B(\theta)B(\theta_2)B(\theta_3)\cdots B(\theta_r)|\Omega\rangle.$$

They originate when the second term in eq.(3.5) is used to express $A(\theta)B(\theta_1)$ and the first term for the rest when $A(\theta_1))$ is pushed through $B(\theta_j)(2\leq j\leq r)$. Hence, one finds

$$\begin{aligned} &-h(\theta_1-\theta)B(\theta)A(\theta_1)B(\theta_2)\cdots B(\theta_r)|\Omega\rangle \\ &\quad = -h(\theta_1-\theta)\prod_{j=2}^{r} g(\theta_j-\theta_1)a(\theta_1,\gamma)^N B(\theta)B(\theta_2)\cdots B(\theta_r)|\Omega\rangle \\ &\qquad +\text{other types of terms.} \end{aligned} \tag{3.13}$$

It is now very easy to determine the remaining coefficients since $\psi(\theta_1,\ldots,\theta_r)$ is a symmetric function of $\theta_1,\ldots,\theta_r$ due to eq.(3.6). Therefore one can permute $\theta_1$ by $\theta_j$ in eq.(3.13) with the result

$$A(\theta)\psi(\underset{\sim}{\theta}) = \Lambda_+(\theta,\underset{\sim}{\theta})\psi(\underset{\sim}{\theta}) + \prod_{k=1}^{r}\Lambda_k^+(\theta,\underset{\sim}{\theta})\psi_k(\theta,\underset{\sim}{\theta}) \tag{3.14}$$

where

$$\psi_k(\theta,\underset{\sim}{\theta}) \equiv B(\theta)\prod_{j=1}^{r} B(\theta_j)|\Omega\rangle \tag{3.15}$$

and

$$\Lambda_k^+(\theta,\underset{\sim}{\theta}) = h(\theta-\theta_d)a(\theta_k,\gamma)^N \prod_{\ell=1}^{r} g(\theta_\ell-\theta_k). \tag{3.16}$$

One analogously finds

$$D(\theta)\psi(\underset{\sim}{\theta}) = \Lambda_-(\theta,\underset{\sim}{\theta})\psi(\underset{\sim}{\theta}) + \prod_{k=1}^{r} \Lambda_k^-(\theta,\underset{\sim}{\theta}), \tag{3.17}$$

$$\Lambda_k^-(\theta,\underset{\sim}{\theta}) = h(\theta_k-\theta)b(\theta_k)^N \prod_{\ell=1}^{r} g(\theta_k-\theta_\ell). \tag{3.18}$$

Now, in order to get an eigenvector of $\tau(\theta)$ we must require

$$\Lambda_k^+(\theta,\underset{\sim}{\theta}) + \Lambda_k^-(\theta,\underset{\sim}{\theta}) = 0, \qquad 1 \le k \le r. \tag{3.19}$$

This yields a set of $r$ algebraic equation in $\theta_k$ $(1 \le k \le r)$ usually called Bethe Ansatz equations ($BAE$),

$$\begin{aligned} \left[\frac{\sinh(\lambda_j+i\gamma/2)}{\sinh(\lambda_j-i\gamma/2)}\right]^N &= -\prod_{k=1}^{r}\frac{\sinh(\lambda_j-\lambda_k+i\gamma)}{\sinh(\lambda_j-\lambda_k-i\gamma)} \\ &\qquad \text{regime II,} \\ \left[\frac{\sin(\lambda_j+i\gamma/2)}{\sin(\lambda_j-i\gamma/2)}\right]^N &= -\prod_{k=1}^{r}\frac{\sin(\lambda_j-\lambda_k+i\gamma)}{\sin(\lambda_j-\lambda_k-i\gamma)} \\ &\qquad \text{regime I and III.} \end{aligned} \tag{3.20}$$

Here we have introduced $\lambda_j \equiv i(\theta_j+\gamma/2)$, for regime III and $\lambda_j \equiv -i(\theta_j-\gamma/2)$, for regimes I and II, $1 \le j \le r$. It should be noticed that the $\theta$ dependence drops in the eigenvalue eqs.(3.20). This could be expected since the commutativity of $\tau(\theta)$ for different $\theta$ suggests that its eigenvectors can be chosen $\theta$-dependent. Once the $\lambda_j$ are found by solving eqs.(3.20) the eigenvalue $\Lambda(\theta)$ of $\tau(\theta)$ follows from eqs.(3.14) and (3.17) as

$$\begin{aligned} \Lambda_+(\theta,\underset{\sim}{\theta}) &= a(\theta,\gamma)\prod_{j=1}^{r} g(\theta_j-\theta), \\ \Lambda_-(\theta,\underset{\sim}{\theta}) &= b(\theta,\gamma)\prod_{j=1}^{r} g(\theta-\theta_j). \end{aligned} \tag{3.21}$$

We can assume $|\mathrm{Re}\lambda_j| \leq \pi/2$ in regimes I and III, whereas $-\infty < \mathrm{Re}\lambda_j < +\infty$ for regime II. The r.h.s. of eq.(3.21) would seem to have poles at $\theta = +i\lambda_j + \gamma/2$. However the corresponding residues identically vanish due to eqs.(3.20). Actually one can use this property as a short-cut to derive the $BAE$ when the construction of the explicit eigenvectors is more involved.

It is convenient to take logarithms of eqs.(3.20). One finds

$$N\phi(\lambda_i, \gamma/2) = \sum_{k=1}^{r} \phi(\lambda_i - \lambda_k, \gamma) + 2\pi I_i, \quad 1 \leq i \leq r, \tag{3.22}$$

where

$$\begin{aligned} \phi(\lambda, \alpha) &= i \log \frac{\sinh(\lambda + i\alpha)}{\sinh(\lambda + i\alpha)} \qquad \text{regime III} \\ \phi(\lambda, \alpha) &= i \log \frac{\sin(\lambda + i\alpha)}{\sin(\lambda + i\alpha)} \qquad \text{regime I and III} \end{aligned} \tag{3.23}$$

and $I_i \in \mathbf{Z} + 1/2$. The numbers $I_1, \ldots, I_r$ characterize the eigenstate. The cut of the logarithm in eq.(3.23) is taken such that $\phi(x, \lambda)$ is a continuous function for real s a monotonically increasing function and we choose $\phi(0, \alpha) = \pi$. For large $N$ and $|\theta| < \gamma/2$ the first term in eq.(3.21) dominates. Therefore, one can set

$$\begin{aligned} f_N(\theta, \gamma) \underset{N \gg 1}{=} & -\frac{1}{N} \log \Lambda(\theta, \gamma) \\ &= \frac{i}{N} \sum_{j=1}^{r} \phi(\lambda_j + i\theta, \gamma/2) + o(e^{-c_1 N}), \end{aligned} \tag{3.24}$$

with $c_1 > 0$ (Here we have normalized the weight $a(\theta, \gamma)$ to unit).

When $\gamma = \pi/2$ in regime II eqs.(3.22) decouple from each other. In this case the model reduces to free fermions [1].

Let us analyze in the different regimes which is the ground state. That is the eigenvector of $\tau(\theta)$ with maximum modulus eigenvalue. It follows from eq.(3.21) that $\Lambda_+$ dominates for large $N$ and fixed $r$ when $0 < \theta < \gamma/2$. It is then enough to compute $\lambda_q \equiv a(\theta, \gamma)^{-N} \Lambda_+(\theta, \gamma)$. Let us consider one pseudoparticle over $|\Omega\rangle$, that is $r = 1$. Eqs.(3.20)–(3.21) are easily solved with the result

$$\lambda_q = \frac{\sinh\theta - e^{iq}\sinh(\theta \pm \gamma)}{\sinh(\gamma \mp \theta) \pm e^{iq}\sinh\theta}$$

regime I: upper sign, regime III: lower sign

$$\lambda_q = \frac{\sin\theta - e^{iq}\sin(\theta + \gamma)}{\sin(\gamma - \theta) + e^{iq}\sin\theta}.$$

regime II

Here $q = 2\pi l/N$ $(1 \leq l \leq N)$ stands for the momentum of the pseudoparticle. A simple calculation shows that

$$|\lambda_q|^2 > 1 \text{ for } \theta > 0 \quad \text{regime I}$$
$$|\lambda_q|^2 > 1 \text{ for } \theta > 0 \text{ and } \gamma < \theta < 2\pi \quad \text{regime II}$$
$$|\lambda_q|^2 < 1 \text{ for } \theta > 0 \text{ and all } q \quad \text{regime III}$$

Since $\Lambda_+$ decreases in regime III by adding pseudoparticles, $|\Omega\rangle$ is the ground state [8]. This is indeed a ferroelectric regime and eq.(3.11) describes here states with $r$ spin waves interacting non-trivially. In regimes I and II we have the opposite behavior and the ground state follows by filling $|\Omega\rangle$ with pseudoparticles. The most regular filling is obtained for $r = N/2$ and

$$I_{j+1} - I_j = 1,$$

as follows analyzing eq.(3.22) (see Section IV for more details). Moreover, excitations around this antiferroelectric state decrease $|\Lambda_+|$ as it is shown below [eq.(3.57)]. Analogous conclusions for the states follows from the spectrum of the $XXZ$ Hamiltonian [eq.(2.9)] (See ref. [18]) for a rigorous discussion). For excited states the sequence $I_j$ exhibits jumps for some values of $j$

$$I_{j+1} - I_j = 1 + \sum_{h=1}^{N_h} \delta_{jj_h}. \tag{3.25}$$

The values of $\lambda$ associated with these missing half-integers are called holes and denoted $\theta_h$. The momentum can be defined in terms of the logarithm of $\tau^{[N]}(0)$ since this is the unit shift operator [see eq.(2.7)]

$$\mathcal{P}_N = i\log\left[c^{-N}\tau^{[N]}(0)\right]. \tag{3.26}$$

We get from eq.(3.21) for its eigenvalues

$$\mathcal{P}_N = \sum_{j=1}^{r} \phi(\lambda_j, \gamma/2), \tag{3.27}$$

where we choose $c$ such that $P_N$ vanishes for the reference state $|\Omega\rangle$. We find using eqs.(3.22) and (3.27) and the fact that $\phi(\lambda, \gamma)$ is an odd function of $\lambda$

$$\mathcal{P}_N = \frac{2\pi}{N} \sum_{j=1}^{r} I_j. \tag{3.28}$$

This formula allows to compute $P_N$ directly from the half-integers $I_j$ characterizing the state. It shows that the $P_N$ of excited states differs in multiples of $2\pi/N$ of that of the ground state which can be set equal to zero by appropriately choosing $I_1$.

In the QFT associated to vertex models, the vacuum (ground state) corresponds precisely to the antiferroelectric ground state. Let us concentrate on this state and excitations around it from now on. The operators $B(\theta_j)$ play here the role of creation operators of excitations over the bare vacuum $|\Omega\rangle$. That is pseudo-particles or "bare" particles. The antiferroelectric ground state is the analog of the filled Dirac sea for free fermions. However, the pseudoparticles are here not free, they interact through two-body interactions. The functions $\phi(\lambda_i - \lambda_j, \gamma)$ describe the two-body phase-shift associated to such interactions.

The $BAE$ (3.20) can be rewritten as

$$\exp i\left[N\phi(\lambda_j, \gamma/2) - \sum_{k\neq j} \phi(\lambda_j - \lambda_k, \gamma)\right] = 1. \tag{3.29}$$

The first term in the exponent, $N\phi(\lambda_j, \gamma/2)$ is just the momentum of the $j$-th pseudoparticle times the number of sites. That is the phase for a free particle moving around this ring of length $N$. The second term can be interpreted as the phase shifts induced in the wave function of the $j$-th pseudoparticle by the (pair) interaction with the rest of them. In other words eq.(3.29) ensures the periodicity of the "wave function" when turning aroung the ring. This interpretation of the $BAE$ extends for more general models [3].

As it is clear, one can easily solve eqs.(3.20) analytically for small $r$ and $N$. For large $N$ the number of roots is very large but they become closer and closer in the real axis so one can define a continuous density in the $N = \infty$ limit

$$\rho_\infty(\lambda_j) = \lim_{N\to\infty} \frac{1}{N(\lambda_{j+1} - \lambda_j)}. \tag{3.30}$$

Once this function is calculated the different physical magnitudes can be computed by quadratures. That is

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{r} f(\lambda_j) = \int d\lambda f(\lambda) \rho_\infty(\lambda). \tag{3.31}$$

For example the free energy reads from eq.(3.24)

$$f(\theta, \gamma) = \lim_{N\to\infty} f_N(\theta, \gamma) = i \int d\lambda \rho_\infty(\lambda) \phi(\lambda + i\theta, \gamma/2). \tag{3.32}$$

It is useful to introduce the function [9]

$$z_N(\lambda) = \frac{1}{2\pi} \big[\phi(\lambda, \gamma/2) - \sum_{j=1}^{r} \phi(\lambda - \lambda_j, \gamma)\big]. \tag{3.33}$$

This function is continuous and monotonically increasing for real $\lambda$. At the real roots of the $BA$ eqs.(3.20)

$$z_N(\lambda_i) = \frac{I_i}{N}, \ 1 \le i \le r. \tag{3.34}$$

At the hole positions $\theta_h$

$$z_N(\theta_h) = \frac{I_{i_h} + 1}{N}. \tag{3.35}$$

For large $N$, neighboring $BA$ roots are very close and we have

$$\frac{dz_N}{d\lambda} \sim \frac{z_N(\lambda_{i+1}) - z_N(\lambda_i)}{\lambda_{i+1} - \lambda_i} = \frac{1 + \sum_{h=1}^{N_h} \delta_{i i_h}}{N(\lambda_{i+1} - \lambda_i)}, \tag{3.36}$$

where we used eq.(3.25). Now, in the $N = \infty$ limit using eq.(3.30)

$$\sigma_\infty(\lambda) \equiv \frac{dz_\infty}{d\lambda} = \rho_\infty(\lambda) + \frac{1}{N} \sum_{h=1}^{N_h} \delta(\lambda - \theta_h). \tag{3.37}$$

Also, using eq.(3.30) and (3.31)

$$\begin{aligned} z_\infty(\lambda) = \frac{1}{2\pi} \big[\phi(\lambda, \gamma/2) &- \int d\mu \phi(\lambda - \mu, \gamma) \rho_\infty(\mu) \\ &- \frac{1}{N} \sum_l \{\phi(\lambda - \xi_l, \gamma) + c.c.\}\big]. \end{aligned}$$

Now, combining this with eq.(3.36) yields a linear integral equation for $\sigma_\infty(\lambda)$

$$(3.38)\quad \sigma_\infty(\lambda) = \frac{1}{2\pi}\phi'(\lambda,\gamma/2) - \int \frac{d\mu}{2\pi}\phi'(\lambda-\mu,\gamma)\sigma_\infty(\mu) - \frac{1}{2\pi N}\sum_{h=1}^{N_h}\phi'(\lambda-\theta_h,\gamma) \times \frac{1}{2\pi N}\sum_l \left[\phi'(\lambda-\xi_l,\gamma)+\phi'(\lambda-\bar{\xi}_l,\gamma)\right].$$

We denoted in eq.(3.38) by $\xi_l$, $\bar{\xi}_l$ the complex roots ($\operatorname{Im}\xi_l > 0$). They always appear in conjugate pairs. In the limiting case $\gamma = 0$ (regime I or II) one has

$$(3.39)\qquad \phi(\lambda,\alpha) = i\log\frac{\lambda+i\alpha}{\lambda-i\alpha},\ \gamma = 0.$$

The linear integral equations (3.38) can be easily solved by Fourier integrals (or Fourier series for regime I). In order to do that one needs the following Fourier representations of $\phi(\lambda,\alpha)$:

$$(3.40)\qquad \phi(\lambda,\alpha) = \pi + 2\lambda - i\sum_{m=-\infty,m\neq 0}^{+\infty}\frac{e^{2im\lambda-2|m|\alpha}}{m},\ |\lambda|<\alpha$$

regimes I and II.

$$(3.41)\qquad \phi(\lambda,\alpha) = \pi + \int_{-\infty}^{+\infty}\frac{dk}{k}\sin(k\lambda)\frac{\sinh[(\pi/2-\alpha)k]}{\sinh[k\pi/2]},$$

regimes II.

$$(3.42)\qquad \phi(\lambda,\alpha) = \pi + \int_{-\infty}^{+\infty}\frac{dk}{k}\sin(k\lambda)e^{-|k\alpha|},\ (\gamma = 0 \text{ case}).$$

The solution of eq.(3.38) reads

$$(3.43)\qquad \sigma_\infty(\lambda) = \sigma_\infty^\vee(\lambda) + \frac{1}{N}[\sigma_h(\lambda)+\sigma_c(\lambda)]$$

$\sigma_\infty^\vee(\lambda)$ corresponds to the ground state. One finds

$$(3.44)\ \sigma_\infty^\vee(\lambda) = \frac{1}{2\pi}\sum_{m\in\mathbf{Z}}\frac{e^{2im\lambda}}{\cosh(m\gamma)} = \frac{K(k)}{\pi^2}\operatorname{dn}\left(\frac{2K\lambda}{\pi},k\right),\ \text{regime I}$$

Here $K'(k)/K(k) = \gamma/\pi$ .

$$(3.45) \quad \sigma_\infty^\vee(\lambda) = \frac{1}{2\cosh\pi\lambda}, \quad \gamma = 0$$

$$(3.46) \quad \sigma_\infty^\vee(\lambda) = \int_{-\infty}^{+\infty} \frac{dk}{4\pi} \frac{e^{ik\lambda}}{\cosh\frac{k\gamma}{2}} = \frac{1}{2\gamma\cosh(\frac{\pi\lambda}{\gamma})}, \quad \text{regime II.}$$

$\sigma_h(\lambda)$ in eq.(3.43) stands for the hole contribution to the density of real roots. One finds $\sigma_h(\lambda) = \frac{1}{\pi}\sum_{h=1}^{\infty} p(\lambda - \theta_h)$, where

$$(3.47)\ p(\lambda) = \frac{1}{2} + 2\sum_{m=1}^{\infty} \frac{\cos(2m\lambda)}{e^{2m\gamma}+1} \qquad \text{regime I,}$$

$$(3.48)\ p(\lambda) = \frac{1}{2}\int_0^\infty \frac{\cos(k\lambda)e^{-k/2}}{\cos(k/2)}dk, \quad \gamma = 0,$$

$$(3.49)\ p(\lambda) = \int_0^\infty \frac{\cos(k\lambda)\sinh[k(\pi/2-\gamma)]}{\cosh(k\gamma/2)\sinh[k(\pi-\gamma)/2]}\frac{dk}{2}, \qquad \text{regime II.}$$

The complex root contribution $\sigma_c(\lambda)$ can be found in refs. [9],[10].

It must be noticed that $\rho_h(\lambda)$ the hole contribution to $\rho(\lambda)$ is minus the resolvent kernel $R(\lambda)$ of the integral equation (3.38), defined by

$$(3.50) \qquad R(\lambda) + \int \frac{d\mu}{2\pi}\phi'(\lambda-\mu)R(\lambda) = \delta(\lambda).$$

That is,

$$\rho_h(\lambda) = \sum_h[-\delta(\lambda-\theta_h) + \frac{1}{\pi}p(\lambda-\theta_h)] = -\sum_{h=1}^{N_h} R(\lambda-\theta_h).$$

Therefore,

$$(3.51) \qquad z_\infty^\vee(\lambda) = -\int_{-\infty}^{+\infty}\frac{d\mu}{2\pi}R(\lambda-\mu)\phi\left(\mu,\frac{\gamma}{2}\right) + \frac{K_\gamma}{2\pi},$$

where $K_\gamma$ a $\lambda$-dependent constant. Eqs.(3.51) holds in regime II where

$$K_\gamma = \frac{\gamma}{2(1-\frac{\gamma}{\pi})}.$$

The densities (3.44)–(3.46) allow an easy calculation of the eigenvalues of $\tau(\theta)$ in the $N = \infty$ limit using eq.(3.31). One finds for example for the free energy per site

$$(3.52)\quad f(\theta,\gamma) = \theta + \sum_{m=1}^{\infty} \frac{e^{-m\gamma}}{m} \frac{\sinh(2m\theta)}{\cosh(m\gamma)}, \qquad \text{regime I,}$$

$$(3.53)\quad f(\theta) = \int_0^{\infty} \frac{dx}{x} e^{-x} \frac{\sinh(2x\theta)}{\cosh(x)}, \quad \gamma = 0,$$

$$(3.54)\quad f(\theta,\gamma) = \int_0^{\infty} \frac{dx}{x} \frac{\sinh(2x\theta)\sinh[x(\pi-\gamma)]}{\cosh(x\gamma)\sinh(x\pi)}, \qquad \text{regime II.}$$

It must be noticed that eq.(3.54) also gives $\log S^{sG}(i\pi\theta/\gamma)$ here $S^{sG}(\phi)$ is the soliton-soliton $S$-matrix in the sine-Gordon model as a function of the physical rapidity $\varphi$.

The excited states eigenvalues of $\tau(\ )$ have the following structure for large $N$ due to eq.(3.43)

$$(3.55)\qquad \Lambda_{\text{exc}}(\theta) = \lambda_0(\theta) e^{-ig(\theta)} \left[1 + O(e^{-c_1 N})\right],$$

where $\lambda_0(\theta)$ in the ground state eigenvalue and $g(\theta)$ is of order $N^0$ for $N \gg 1$. A look to eqs.(3.32) and (3.51) shows that the hole eigenvalues are given by

$$(3.56)\qquad g(\theta) = 2\pi \sum_{h=1}^{N_h} z_{\infty}^{\vee}(\theta_h + i\theta) = \sum_h g(\theta,\theta_h).$$

Using now eq.(3.37) and eq.(3.46) yields in regime II

$$(3.57)\qquad g(\theta,\theta_h) = 2\text{arctg}\left[\exp\frac{\pi}{\gamma}(\theta_h + i\theta)\right].$$

This is clearly a gapless regime since $g(\theta,-\infty) = 0$. Moreover $|e^{-ig(\theta)}| < 1$ for $0 < \theta < \gamma$. This shows that our identification of the ground state is correct since any deviation from it decreases the eigenvalue of $\tau(\theta)$.

Interesting complex roots appear for $\frac{\pi}{2} < \gamma < \pi$ in regime II. They appear in strings of length $n$, where $n$ may be $\leq \frac{\pi}{\pi-\gamma} - 1$ and $[x]$ stands for integer part of $x$,

$$(3.58)\qquad \lambda_r = \sigma + i\pi/2 - i(r+1/2)(\pi-\gamma),\ 0 \leq r \leq \frac{n-2}{2},$$

and

$$(3.59)\qquad \lambda_s = \sigma + i\pi/2 - is(\pi-\gamma),\ 0 \leq s \leq \frac{n-1}{2},$$

where $\sigma$ is the common real part of the roots. The associate eigenvalue of $\tau(\theta)$ can be shown to be [10]

$$g_n(\theta,\sigma) = 2\text{arctg}\,\frac{\sinh\frac{\pi}{\gamma}(\sigma+i\theta)}{\sin\frac{n\pi}{2\gamma}(\pi-\gamma)}. \tag{3.60}$$

Let us derive for future reference the asymptotic behaviour of $g(\theta,\theta_h)$ and $g_n(\theta,\sigma)$ for $i\tau \to -\infty$. Eqs.(3.57) and (3.60) yield in this limit

$$\begin{aligned} g(\theta,\theta_h) &\underset{i\theta\to-\infty}{=} 2\exp\frac{\pi}{\gamma}(i\theta+\theta_h) + o(e^{2\pi i\theta/\gamma}), \\ g_n(\theta,\sigma) &\underset{i\theta\to-\infty}{=} -\pi + 4\exp\left[\frac{\pi}{\gamma}(i\theta+\sigma)\right]\sin\left[\frac{n\pi}{2\gamma}(\pi-\gamma)\right] \\ &\qquad + o(e^{2\pi i\theta/\gamma}). \end{aligned} \tag{3.61}$$

The coefficients in this formulae give the mass spectrum of the QFT provided by the light-cone transfer matrix approach: the massive Thirring model [11,12] (see Section V).

For large $N$ the momentum of a hole excitation at $\theta_h$ writes

$$p(\theta_h) = 2\text{arctg}\left[e^{\pi\theta_h/\gamma}\right] \tag{3.62}$$

where we used eqs.(3.31), (3.27), (3.55) and (3.57).

The hole states eigenvalues of $\tau(\theta)$ write regime I

$$\begin{aligned} g(\theta,\theta_h) = 2\pi z_\infty^\vee(\theta_h+i\theta) &= \frac{\pi}{4} + \int_0^{\theta_h+\pi/2} d\lambda p(\lambda), \\ &-i\log\left\{\text{sn}\left[\frac{2K}{\pi}(\theta_h+i\theta)\right] - i\text{cn}\left[\frac{2K}{\pi}(\theta_h+i\theta)\right]\right\}. \end{aligned} \tag{3.63}$$

In this regime we have a non-zero gap given by the end-point excitations $\theta_h = \pm\pi/2$,

$$g(\theta,\pi/2) = \pi + i\log\frac{\text{dn}\left(\frac{2K\theta}{\pi},k'\right)}{1-k'\text{sn}\left(\frac{2K\theta}{\pi},k'\right)}. \tag{3.64}$$

This gap vanishes for $\gamma \to 0^+$ as

$$g(\theta,\pi/2) - \pi \underset{\gamma\to 0^+}{=} 4\pi\sin\theta e^{-\pi^2/2\gamma} + o\left(e^{-\pi^2/\gamma}\right). \tag{3.65}$$

Therefore the six-vertex model is critical (gapless) for regime II and massive for regime I.

Eq.(1.18) gives for the six-vertex model symmetry (3.2) (rotations around $z$)

$$(3.66) \qquad \begin{aligned} &[A(\theta), S_z] = [D(\theta), S_z] = 0, \\ &[S_z, B(\theta)] = -B(\theta), \quad [S_z, C(\theta)] = C(\theta), \end{aligned}$$

where $S_z = \frac{1}{2}\sum_{a=1}^{N} \sigma_z^a$ acts in the vertical space. Therefore $B(\theta)$ ($C(\theta)$) lowers (raises) the $z$-component of the spin in one unit.

In particular we find that the state (3.11) is an eigenvectors of $S_z$

$$(3.67) \qquad S_z \Phi(\underset{\sim}{\theta}) = (\frac{N}{2} - r)\Phi(\underset{\sim}{\theta}), \quad \underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_r).$$

This concludes our exposition of the Bethe Ansatz solution of the six-vertex model. A more general Bethe Ansatz construction provides the eigenvectors of the eight-vertex model [7,13]. It has been shown recently that these eight-vertex eigenvectors become the six-vertex eigenvectors in the limit where the eight-vertex weights become those of the six-vertex model [14].

The Bethe Ansatz has been also generalized for multi-state vertex models. That is when $\dim \mathcal{A}$ and/or $\dim \mathcal{V}$ is larger than two. The resulting construction is a set of nested Bethe Ansatz. It is reviewed in ref. [3].

## §4. Finite-size correctios from the Bethe-ansatz and conformal invariance

As we see in previous sections the Bethe Ansatz provides the exact eigenvalues and eigenvectors of an integrable model from the resolution of a system of coupled algebraic equations. A typical and relevant example is given by eqs.(3.20) (the six-vertex model). As we have seen, the explicit solution of the BAE in the $N = \infty$ limit is straightforward. The density of roots follows from a linear integral equation [eq.(3.38)] explicitly solvable by Fourier transformation. However, the analytic resolution of the BAE for a finite number of sites is a formidable task as soon as $N$ is not very small.

A systematic procedure for computing finite size corrections for integrable theories was proposed in ref. [9]. This method as well as subsequent improvements will be reported in this section.

We treat here the finite size corrections in the six-vertex model both in the zero gap and non-zero gap regimes. The generalization to multistate models can be find in ref. [15].

Let us consider the generalized transfer matrix $\tau_g(\theta)$ [16] considered in Section 2. For the six-vertex model we can take

$$g = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}. \tag{4.1}$$

Then,

$$\tau_\alpha(\theta) = e^{i\alpha} A(\theta) + e^{-i\alpha} D(\theta). \tag{4.2}$$

As discussed in Section 2, describes boundary conditions where the spins at the sites $N+1$ and 1 are related by a rotation $g$. That is

$$\sigma^{\pm}_{N+1} = e^{\pm i\alpha} \sigma^{\pm}_1. \tag{4.3}$$

The BA construction of the eigenvectors from Section 3 applies with minor changes to the BAE and eigenvalue expressions. We have now as BAE instead of eq.(3.22) [16]

$$N\phi(\lambda_i, \gamma/2) = 2\alpha + \sum_{k=1}^{r} \phi(\lambda_i - \lambda_k, \gamma) + 2\pi I_i. \tag{4.4}$$

The respective eigenvalue of expresses as

$$\Lambda_\alpha(\theta) = e^{i\alpha} \Lambda_+(\theta) + e^{-i\alpha} \Lambda_-(\theta), \tag{4.5}$$

where $\Lambda_\pm(\theta)$ are given by eqs.(3.21). As we know the first term dominates in eq.(4.5) for $|\theta| < \gamma/2$. The $N = \infty$ limit is $\alpha$ independent. Therefore the results of Section 3 hold also here. Let us consider the finite size corrections

$$L_N(\theta) = -\frac{1}{N} \log \Lambda_\alpha(\theta) + \lim_{N\to\infty} \frac{1}{N} \log \Lambda_\alpha(\theta). \tag{4.6}$$

Obviously $L_\infty(\theta) = 0$. Using eqs.(3.21) and (3.31) we can recast $L_N(\theta)$ in a more convenient form to study the large (but finite) $N$ regime

$$\begin{aligned} L_N(\theta) &= -i \int_{-A}^{A} \phi(\lambda + i\theta, \gamma/2) - \frac{i\alpha}{N} + \frac{i}{N} \sum_{j=1}^{r} \phi(\lambda_j + i\theta, \frac{\gamma}{2}) \\ &= -\frac{i\alpha}{N} + i \int_{-A}^{A} d\lambda \phi(\lambda + i\theta, \gamma/2) \big(\sigma_N(\lambda) - \sigma_\infty(\lambda)\big) \\ &\quad + \frac{i}{N} \int_{-A}^{A} d\lambda \phi(\lambda + i\theta, \gamma/2) \times \\ &\quad \times \Big\{ \frac{1}{N} \sum_{k=1}^{M} \delta(\lambda - \lambda_k) + \frac{1}{N} \sum_{h=1}^{N_h} \delta(\lambda - \theta_h) - \sigma_N(\lambda) \Big\}. \end{aligned} \tag{4.7}$$

Here $A = \pi/2$ for regime I and $A = +\infty$ for regime II, we have used eq.(3.37) and

$$\sigma_N(\lambda) = \frac{dz_N}{d\lambda}, \tag{4.8}$$

where

$$z_N(\lambda) = \frac{1}{2\pi}\left(\phi(\lambda, \gamma/2) - \frac{2\alpha}{N} - \frac{1}{N}\sum_{k=1}^{M}\phi(\lambda - \lambda_k, \gamma)\right) \tag{4.9}$$

The $\lambda_i (1 \leq i \leq M)$ are here the real roots of eq.(4.4). The function $z_N(\lambda)$ fulfils

$$z_N(\lambda_i) = \frac{I_i}{N}, \qquad z_N(\theta_h) = \frac{I_{i_h}+1}{N}, \tag{4.10}$$

as in Section III, eqs.(3.34)–(3.35). Notice that the phase $\alpha$ drops in the $N = \infty$ limit and hence $\sigma_\infty(\lambda)$ also obeys eq.(3.38). Let us now study the function $\sigma_N(\lambda) - \sigma_\infty(\lambda)$. Subtraction of eqs.(3.38) and the derivative of (4.9) yields

$$\begin{aligned} &\sigma_N(\lambda) - \sigma_\infty(\lambda) + \int_{-A}^{A} \frac{d\mu}{2\pi}\phi'(\lambda - \mu, \gamma)\big(\sigma_N(\mu) - \sigma_\infty(\mu)\big) \\ &= -\int_{-A}^{A} \frac{d\mu}{2\pi}\phi'(\lambda - \mu, \gamma) S_N(\mu), \end{aligned} \tag{4.11}$$

where

$$S_N(\mu) \equiv \frac{1}{N}\sum_{i=1}^{M}\delta(\mu - \lambda_i) + \frac{1}{N}\sum_{h=1}^{N_h}\delta(\mu - \theta_h) - \sigma_N(\mu). \tag{4.12}$$

eq.(4.11) is to be considered as a linear integral equation for $\sigma_N(\lambda) - \sigma_\infty(\lambda)$ with the r.h.s. as inhomogeneous term. Solving it with the help of the resolvent (3.50 ) yields

$$\sigma_N(\lambda) - \sigma_\infty(\lambda) = -\int_{-A}^{A} \frac{d\mu}{\pi} p(\lambda - \mu) S_N(\mu). \tag{4.13}$$

Inserting eq.(4.13) in eq.(4.7) and using eq.(3.51) gives

$$Ł_N(\theta) = i\int_{-A}^{A} d\lambda S_N(\mu)\big(2\pi z_N(\lambda + i\theta) + K_\gamma(\lambda)\big) - \frac{i\alpha}{N}. \tag{4.14}$$

The constant $K_\gamma(\lambda)$ is given by

$$K_\gamma(\lambda) = \frac{\gamma}{2(1-\gamma/\pi)}, \tag{4.15}$$

$$K_\gamma(\lambda) = \frac{\pi}{4} + \int_0^{\lambda+\frac{\pi}{2}} d\mu\, p(\mu),$$

where $p(\lambda)$ is given by eq.(3.47). The calculation of large but finite $N$ effects involve the evaluation for large $N$ of expressions like

$$I_N = \int_{-A}^{A} d\lambda f(\lambda) S_N(\lambda), \tag{4.16}$$

where $f(\lambda)$ is explicitly known. Notice that $I_\infty = 0$.

It is convenient to change in eq.(4.16) to the integration variable $z_N(\lambda)$ as defined by eq.(4.9). Using eqs.(4.8) and (4.10) yields

$$I_N = \int_0^r dz f\big(\lambda_N(z)\big)\Big\{\frac{1}{N}\sum_{k=1}^{M+N_h} \delta(z-z_k) - 1\Big\}, \tag{4.17}$$

where

$$r = \int_{-A}^{A} d\lambda \sigma_N(\lambda) = \frac{M+N_h}{N}.$$

$\lambda_N(z)$ is the inverse function of the monotonous function $z_N(\lambda)$. We choose

$$z_k = \frac{k+1/2}{N}, \quad 1 \le k \le M+N_h.$$

By Fourier expanding the periodic $\delta(z)$ with period $p$, one gets

$$\frac{1}{N}\sum_{k=1}^{M+N_h} \delta(z-z_k) = \sum_{s=-\infty}^{+\infty} (-1)^s e^{2\pi i s N_z}.$$

Inserting this formula in eq.(4.17) gives [9]

$$I_N = \sum_{s\in\mathbf{Z}, s\neq 0} (-1)^s T_{Ns}, \tag{4.18}$$

where

$$T_n = \int_{-A}^{A} d\lambda f(\lambda)\sigma_N(\lambda) \exp\big(2\pi i n z_N(\lambda)\big). \tag{4.19}$$

Expressions (4.18) and (4.19) are *exact* for all values of $N$. Now we can proceed to obtain their asymptotic behavior for large $N$. The procedure is different depending if we are in the massive regime I or in the massless one (regime II). In the former case the roots $\lambda_i$ lie in a finite interval $[-\pi/2, \pi/2]$ for all $N$. When the gap vanishes the BAE roots are not anymore bounded. The root density (3.46) (valid for $N = \infty$ ) permits to estimate that the largest roots are at $\lambda = \pm\Lambda_\pm$ with

$$\Lambda_\pm \underset{N\to\infty}{\simeq} \frac{\gamma}{\pi} \log N + \beta_\pm, \tag{4.20}$$

where $\beta_\pm = O(1)$ for $N \to \infty$. The dominant finite size corrections to physical quantities depend on the value of $\beta_\pm$. Therefore, more information than that contained in the $N = \infty$ densities is needed.

Let us start by the massive case $A = \pi/2$. In eq.(4.18) we have a sum of $T_n$ with argument $n = Ns$ always much larger than one (in absolute value) since $|s|$g1. Therefore, we can try to evaluate $T_n$ from eq.(4.19) by stationary point methods since $n$ only appears in the exponent. We need to find the points $\lambda_0$ where

$$\sigma_N(\lambda_0) = \frac{dz_N}{d\lambda}(\lambda_0) = 0. \tag{4.21}$$

Moreover, the dominant large $N$ behavior follows by replacing $T_n$ by $T_n^{as}$ where

$$T_n^{as} \equiv \int_{-A}^{A} d\lambda f(\lambda)\sigma_\infty(\lambda)e^{2\pi i n z_\infty(\lambda)}, \tag{4.22}$$

and

$$\sigma_\infty(\lambda_0^{as}) = \frac{dz_\infty}{d\lambda}(\lambda_0^{as}) = 0. \tag{4.23}$$

The solutions of eq.(4.23) are exactly calculable from eq.(3.44) with the result

$$\lambda_0^{as} = \frac{\pi}{2} + \frac{i\gamma}{2} \quad \mod (\pi, i\gamma), \tag{4.24}$$
$$e^{2\pi i z_\infty(\frac{\pi+i\gamma}{2})} = \sqrt{k_1},$$

where $\sqrt{k_1} = (1-k')/k$ is the modulus associated with the nome $e^{-2\gamma}$. In particular [50]

$$\sqrt{k_1} = 2e^{-\gamma/2} \prod_{n=0}^{\infty} \left(\frac{1+e^{-2\gamma(2n+2)}}{1+e^{-2\gamma(2n+1)}}\right)^2. \tag{4.25}$$

We see that $0 < k_1 < 1$ for $\infty > \gamma > 0$. The same is true for the second derivative at $\lambda_0$ since

$$2\pi i z''_\infty(\frac{\pi + i\gamma}{2}) = -\frac{4k'K^2}{\pi}.$$

The infinite product (4.25) is slowly convergent for small $\gamma$. Applying the Poisson summation formul to $\log k_1$ yields

$$\sqrt{k_1} = \prod_{m=0}^{\infty} (\frac{1 - e^{-\frac{\pi^2}{\gamma}(m+\frac{1}{2})}}{1 + e^{-\frac{\pi^2}{\gamma}(m+\frac{1}{2})}})^2.$$

Then, for $\gamma \to 0^+$,

$$k_1 = 1 - 8e^{-\frac{\pi^2}{2\gamma}} + o\big(e^{-\frac{\pi^2}{\gamma}}\big), \tag{4.26}$$

we find a typical Kosterlitz-Thouless behavior. Hence, $T_n$ is exponentially small for large $n$

$$T_n = O\big(k_1^{n/2}\big) \quad n >> 1, \tag{4.27}$$

and the series (4.18) is dominated by the terms with $s = \pm 1$. This result holds for any expression with the form (4.16). We find for the finite size corrections to the free energy with $\alpha = 0$ [eq.(4.14)], calculations [9]

$$\begin{aligned}(4.28) L_N(\theta) = -\frac{2\pi^{\frac{3}{2}}}{K\sqrt{2k'N}} k_1^{N/2} \{ \frac{\gamma}{2} + \log k + \sum_{m=1}^{\infty} \frac{e^{-m\gamma}}{m} \tanh m\gamma \\ - \log \left[ \mathrm{dn}(\frac{2K\theta}{\pi}, k') - k'\mathrm{cn}(\frac{2K\theta}{\pi}, k') \right] \} \left[ 1 + O(\frac{1}{N}) \right] + o(k_1^N).\end{aligned}$$

Further results are reported in refs. [9]. In summary, finite size corrections appear as an asymptotic series in (positive) powers of $k_1^{N/2}$. When $\gamma$ (regime I) tends to $0+$, $k_1 \to 1^-$ and this expansion ceases to be useful. For small $\gamma$ these asymptotic formulae hold for regime I provided

$$N \gg \exp\frac{\pi^2}{2\gamma} \quad \text{or} \quad \log N \gtrsim \frac{\pi^2}{2\gamma}. \tag{4.29}$$

This is related to the vanishing of the mass gap [eq.(3.65)] when $\gamma \to 0^+$.

Let us now turn to the gapless regime. In this case, one can try to use the stationary point method as before. It follows that (4.21) only has infinite solutions (i.e. $\lambda_0 = \pm\infty$ ) for $\sigma_\infty(\lambda)$ given by eqs.(3.45)–(3.46). In other words the integrals in eq.(4.19) are dominated by their end-points of integration ($A = \infty$) for large $n$. It is possible to evaluate in this way the integral in eq.(4.19) with the result that the $T_n^{as}$ have for large $n$ an asymptotic expansion in powers of $1/n$. In this way the finite size corrections to the free energy result for $\alpha = 0$:

$$L_N(\theta) = -\frac{\pi}{6N^2}\sin\frac{\pi\theta}{\gamma} + o(\frac{1}{N^2}). \tag{4.30}$$

However for the study of excited states and the $\alpha \neq 0$ case is more effective to analyse $I_N$ in the following way [17]. Let $\pm\Lambda_\pm$ be the largest real roots and assume that there are no holes within the interval $[-\Lambda_-, \Lambda_+]$ . We assume also no complex roots for the moment. In this way their energy will be as small as possible. The motivation to study the finite size corrections to such lower states comes from conformal invariance. Since the model is here gapless one expects to find conformal invariant behavior in this regime.

As before, we study an expression of the form of eq.(4.16) (now with $A = \infty$ )

$$\int_{-A}^{A} d\lambda f(\lambda) S_N(\lambda) = \frac{1}{N}\sum_{k=1}^{M+N_h} f\big(\lambda_N(z_k)\big) - \int_{-A}^{A} d\lambda f(\lambda)\sigma_N(\lambda), \tag{4.31}$$

where eq.(4.17) was used. The sums in the r.h.s. of eq.(4.31) can be approximated for $N \gg 1$ using Euler-Maclaurin type formulae:

$$\begin{aligned} I_N = {} & \frac{1}{2N}\left[f(\Lambda_+) + f(-\Lambda_-)\right] + \frac{1}{12N^2}\left[\frac{f'(\Lambda_+)}{\sigma_N(\Lambda_+)} - \frac{f'(-\Lambda_-)}{\sigma_N(-\Lambda_-)}\right] \\ & - \int_{-A}^{A} d\lambda f(\lambda)\sigma_N(\lambda), \left[\theta(\lambda - \Lambda_+) + \theta(-\lambda - \Lambda_-)\right] + o(\frac{1}{N^2}). \end{aligned} \tag{4.32}$$

We shall apply this approximation both to eq.(4.13) determining $\sigma_N(\lambda)$ and to eq.(4.14) expressing $L_N(\theta)$. eq.(4.13) gives

$$\begin{aligned} \sigma_N(\lambda) - \sigma_\infty(\lambda) = {} & \int_{\Lambda_+<\mu<-\Lambda_-} d\mu \sigma_N(\mu)\frac{p(\lambda-\mu)}{\pi} - \\ & \frac{1}{2\pi N}\left[p(\lambda - \Lambda_+) + p(\lambda + \Lambda_-)\right] \\ & + \frac{1}{12N^2}\left[\frac{p'(\lambda - \Lambda_+)}{\sigma_N(\Lambda_+)} - \frac{p'(\lambda + \Lambda_-)}{\sigma_N(-\Lambda_-)}\right] + o(\frac{1}{N^2}). \end{aligned} \tag{4.33}$$

The relevant information about the finite size corrections to the lowest states comes from the regions around $\lambda = \Lambda_+$ and $\lambda = -\Lambda_-$ [eq.(4.20)].

It is then useful to define

$$\chi(t) = \sigma_N(t + \Lambda_+), \tag{4.34}$$

and the Fourier transforms

$$X_\pm(\omega) = \int_{-\infty}^{+\infty} dte^{i\omega t}\theta(\pm t)\chi(t),(4.35)$$

which are analytic functions of $\omega$ in $\pm$ Im $\omega > 0$. The contributions from the region around $\lambda = -\Lambda_-$ are treated analogously and added at the end.

Fourier transforming eq.(4.33) yields a Riemann-Hilbert (RH) problem for $X(\omega)$:

$$\begin{aligned} X_-(\omega) + \hat{R}(\omega)X_+(\omega) &= e^{-i\omega\Lambda_+}\hat{\sigma}(\omega) \\ &+ \frac{1}{2N}[-1 + \hat{R}(\omega)] - \frac{i\omega}{12N^2}\frac{1 - \hat{R}(\omega)}{\sigma_N(\Lambda_+)}. \end{aligned} \tag{4.36}$$

That is, we find an equation relating the functions $X_+(\omega)$ and $X_-(\omega)$ which are analytic for Im$\omega > 0$ and Im$\omega < 0$ respectively. Here $\hat{R}(\omega)$ and $\hat{\sigma}(\omega)$ are the Fourier transform of the resolvent (3.50) and the vacuum density of roots (3.46)

$$\hat{R}(\omega) = \frac{\sinh\frac{\omega\pi}{2}}{2\sinh\left[\frac{\omega(\pi-\gamma)}{2}\right]\cosh\frac{\omega\pi}{2}}, \quad \hat{\sigma}(\omega) = \frac{1}{2\cosh\frac{\omega\pi}{2}}. \tag{4.37}$$

In eq.(4.36) we only consider terms coming from $\lambda \sim \Lambda_+$, we also neglect contributions of order $e^{-2\pi\Lambda_+/\gamma}$ and smaller.

In order to solve the RH problem (6.36) one starts to factorize $R(\omega)$ as

$$\hat{R}(\omega)^{-1} = G_+(\omega)G_-(\omega), \tag{4.38}$$

where $G_\pm(\omega)$ are analytic functins in $\pm$ Im $\omega > 0$. The explicit form of $G_\pm(\omega)$ known in terms of $\Gamma$ functions [18] but we shall not need it here. It will be enough to notice that

$$G_+(\omega) = G_-(-\omega). \tag{4.39}$$

Therefore

$$G_+(0)^2 = \hat{R}(0)^{-1} = 2(1 - \frac{\gamma}{\pi}). \tag{4.40}$$

In addition, we have an integral representation that follows from eq.(4.38) and Cauchy theorem

$$\log G_{\pm}(z) = \mp \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} \frac{\log \hat{R}(\omega)}{\omega - z}, \quad \pm \mathrm{Im} z > 0. \tag{4.41}$$

Expanding eq.(4.41) for large $z$ yields

$$G_{+}(\omega)^{-1} = 1 - \frac{g}{\omega} + \frac{g^2}{2\omega^2} + + o(\frac{1}{\omega^3}) \quad (\omega \to \infty), \tag{4.42}$$

where $g$ is a numerical constant. As we shall see below it cancels in physical results. Now using eq.(4.38) the RH problem (4.36) can be written as

$$\begin{aligned} &G_{-}(\omega)X_{-}(\omega) - Q_{-}(\omega) + \frac{G_{-}(\omega)}{2N}\left[1 + \frac{i\omega}{6N\sigma_N(\Lambda_{+})}\right] \\ &= -G_{+}(\omega)^{-1}X_{+}(\omega) + \frac{G_{+}(\omega)^{-1}}{2N}\left[1 + \frac{i\omega}{6N\sigma_N(\Lambda_{+})}\right] + Q_{+}(\omega) \\ &\equiv -P(\omega), \end{aligned} \tag{4.43}$$

where the function $Q_{\pm}(\omega)$ are holomorphic in $\pm\mathrm{Im}\omega > 0$ and fulfil

$$Q_{+}(\omega) + Q_{-}(\omega) = e^{-i\omega\Lambda_{+}} G_{+}(\omega)\hat{\sigma}(\omega). \tag{4.44}$$

eq.(4.43) tells us that $p(\omega)$ is an entire function of $\omega$. It yields in addition the solution of our RH problem as

$$X_{+}(\omega) = G_{+}(\omega)\left[P(\omega) + Q_{+}(\omega)\right] + \frac{1}{2N}\left[1 + \frac{i\omega}{6N\sigma_N(\Lambda_{+})}\right]. \tag{4.45}$$

$P(\omega)$ is obtained by letting in eq.(4.43) and using $X_{+}(\infty) = 0$. We find

$$P(\omega) = -\frac{1}{2N}\left[1 + \frac{i(\omega - g)}{6N\sigma_N(\Lambda_{+})}\right]. \tag{4.46}$$

In addition, eqs.(4.37) and eq.(4.44) give

$$Q_{+}(\omega) = \frac{i}{\gamma}\frac{e^{-\pi\Lambda_{+}/\gamma}}{\omega + i\pi/\gamma} G_{+}(\frac{i\pi}{\gamma}) + o(e^{-\frac{2\pi\Lambda_{+}}{\gamma}}). \tag{4.47}$$

Contour integration and eq.(4.35) yield

$$\sigma_N(\Lambda_{+}) = \int_{-\infty}^{+\infty} \frac{d\omega}{\pi} X_{+}(\omega) = -i \lim_{\omega \to \infty} \omega X_{+}(\omega). \tag{4.48}$$

Combining eq.(4.43) and eq.(4.48), we find

$$N\sigma_N(\Lambda_+) = \frac{N}{\gamma}e^{-\frac{\pi\Lambda_+}{\gamma}}G_+(\frac{i\pi}{\gamma}) + \frac{ig}{2}\left[1 - \frac{ig}{12N\sigma_N(\Lambda_+)}\right]. \tag{4.49}$$

Up to now, we have not specified the physical state. That is, we must choose the integers $I_i$ (and $I_{j_h}$) in the BA eqs.(3.22) or equivalently in eq.(4.4). In the present case this information enters in the RH solution through the value of

$$X_+(0) = \int_{\Lambda_+}^{+\infty} \sigma_N(\lambda)d\lambda = z_N(\infty) - z_N(\Lambda_+), \tag{4.50}$$

and the analogous contribution from $\lambda \sim -\Lambda_-$. We find from eq.(4.9)

$$\begin{aligned} z_N(\infty) &= 1 - \frac{r}{N} - \frac{\gamma}{\pi}(\frac{1}{2} - \frac{r}{N}) - \frac{\alpha}{N\pi}, \\ z_N(-\infty) &= \frac{\gamma}{\pi}(\frac{1}{2} - \frac{r}{N}) - \frac{\alpha}{N\pi}, \end{aligned} \tag{4.51}$$

where we used [see eq.(3.23)] the formulae

$$\phi(+\infty, \alpha) = 2(\pi - \alpha), \qquad \phi(-\infty, \alpha) = 2\alpha, \tag{4.52}$$

valid for $0 < \alpha < \pi/2$.

The value of $z_N(\Lambda_+)$ is related to the half-integer associated to the last positive root. For the ground state we have a monotonous sequence of $N/2$ half-integers running from $1/2$ till $(N-1)/2$. Therefore

$$\begin{aligned} z_N(\Lambda_+) &= \frac{1}{2} - \frac{1}{2N}, \\ z_N(-\Lambda_-) &= \frac{1}{2N} \quad \text{(no holes)}. \end{aligned} \tag{4.53}$$

Now, if we put $h_+$ holes beyond $\Lambda_+$ and $h_-$ before $-\Lambda_-$, a shift in the sequence $I_i$ is produced and we find

$$\begin{aligned} z_N(\Lambda_+) &= 1 - \frac{r}{N} - \frac{1}{2N} - \frac{h_+}{N} \\ z_N(-\Lambda_-) &= \frac{1}{2N} + \frac{h_-}{N}. \end{aligned} \tag{4.54}$$

The total number of real roots is given by

$$r = 1 + N\left[z_N(\Lambda_+) - z_N(-\Lambda_-)\right].$$

Combining this with eq.(4.54) yields

$$r = (N - h_+ - h_-)/2. \tag{4.54a}$$

Since we assumed $N$ to be even, this shows that the total number of holes must also be even.

Then eqs.(4.50)–(4.51) and (4.54) yield

$$X_+(0) = \frac{1}{2N} - \frac{\gamma s}{\pi N} + \frac{h_+}{N} - \frac{\alpha}{N\pi} \tag{4.55}$$

where $S \equiv N/2 - r = (h_+ + h_-)/2$ is the spin of the state (the eigenvalue of $S_z = \frac{1}{2}\sum_{a=1}^{N} \sigma_a^z$). From eq.(4.45), (4.46), (4.47) and (4.55) we finally get

$$e^{-\frac{\pi\Lambda_+}{\gamma}} G_+(\frac{i\pi}{\gamma}) = \frac{\pi}{2N}\left[1 - \frac{ig}{6N\sigma_N(\Lambda_+)}\right] + \frac{\pi h_+ - \gamma s - \alpha}{N\sigma_+(0)}. \tag{4.56}$$

Let us apply this approximation scheme to the finite size corrections $L_N(\theta)$. We find from eqs.(4.14) and (4.32)

$$\begin{aligned} L_N(\theta) = 2\pi i \int_{\Lambda_+ < \mu < -\Lambda_-} & d\lambda \sigma_N(\lambda) z_\infty^\vee(\lambda + i\theta) - \\ & \frac{i\pi}{N}\left[z_\infty^\vee(\Lambda_+ + i\theta) + z_\infty^\vee(-\Lambda_- + i\theta)\right] \\ & - \frac{i\pi}{6N^2}\left[\frac{\sigma_\infty^\vee(\Lambda_+ + i\theta)}{\sigma_N(\Lambda_+)} - \frac{\sigma_\infty^\vee(-\Lambda_- + i\theta)}{\sigma_N(-\Lambda_-)}\right] - \frac{i\alpha}{N} + o(\frac{1}{N^2}). \end{aligned} \tag{4.57}$$

Now, we can approximate the integrals here as

$$\begin{aligned} 2\pi i \int_{\Lambda_+}^{+\infty} d\lambda \sigma_N(\lambda) z_\infty^\vee(\lambda + i\theta) &= i\pi X_+(0) \\ & -2ie^{-\frac{i\pi\theta}{\gamma} - \frac{\pi\Lambda_+}{\gamma}} X_+(\frac{i\pi}{\gamma}) + o\big(e^{-\frac{2\pi\Lambda_+}{\gamma}}\big), \end{aligned} \tag{4.58}$$

where eqs.(3.57), (4.34) and (4.35) were used. From eqs.(4.45)–(4.47), (4.49), (4.56) and (4.58) we derive the final expressions for $L_N(\theta)$ at large $N$. Let us first write the result for the ground state ($h_+ = h_- = S = 0$)

$$L_N^{G.S.}(\theta) = -\frac{\pi}{6N^2}\sin(\frac{\pi\theta}{\gamma})\left[1 - \frac{6\alpha^2}{\pi(\pi - \gamma)}\right] + o(\frac{1}{N^2}). \tag{4.59}$$

The contributions from contained in eq.(4.59) follow by a procedure analogous to eqs.(4.34)–(4.58).

We see that eq.(4.59) reproduces eq.(4.30) when $\alpha = 0$. Let us now compare with the conformal theory predictions. For periodic boundary conditions one expects a leading finite size correction equal to

$$-\frac{\pi c}{6N^2} \tag{4.60}$$

where $c$ is the central charge. However, one cannot blindly identify eqs.(4.59) and (4.60). For large distances one expects rotational invariance in the gapless regime. This invariance can be seen in the spectrum of low energy excitations is derived in Section III. In the present context the hamiltonian can be identified with

$$\mathcal{H} = -\mathrm{Re}\log\tau(\theta), \tag{4.61}$$

whereas the momentum is given by eq.(3.26). The low-lying eigenvalues of $H$ and $P$ follow from eqs.(3.57) and (3.62)

$$\begin{aligned} \epsilon &\simeq p\sin\frac{\pi\theta}{\gamma}, \\ p &= 2e^{\frac{\pi\theta_h}{\gamma}}, \qquad \theta_h \to -\infty. \end{aligned} \tag{4.62}$$

This shows that we must renormalize the energy by the "speed of sound", $\sin(\pi\theta/\gamma)$ in order to recover an ultra-relativistic dispersion law and hence rotational invariance for large distances (Large compared with the lattice spacing). After this renormalization

$$\tilde{L}_N^{G.S.}(\theta) = \frac{1}{\sin\frac{\pi\theta}{\gamma}} L_N^{G.S.}(\theta) = -\frac{\pi}{6N^2}\left[1 - \frac{6\alpha^2}{\pi(\pi-\gamma)}\right]. \tag{4.63}$$

It must be noticed that eq.(4.63) does not mean that $c \neq 1$. Eq.(4.60) only holds for periodic boundary conditions and not for twisted ones. The twisted b.c. affect the finite size corrections but not $c$ which is equal to one in this case. Actually eq.(4.63) is an example of an universal formula that can be derived by conformal theory methods [45]. Let us define the twist using a conformal operator $g_\theta(z)\big|_{z=0}$. That is

$$S_{N+1} = g_\theta(0) S_1 g_\theta(0)^{-1}.$$

If $\Delta\theta$ is the conformal dimension of $g_\theta(z)$, we can prove in general [45] that

$$\tilde{L}_N = -\frac{\pi}{6N^2}[1 - 24\Delta_\theta]. \tag{4.64}$$

In the case of eq.(4.63)

$$\Delta_\theta = \left(\frac{\alpha}{2\pi}\right)^2 \frac{1}{1-\gamma/\pi}$$

is the conformal dimension of the operator $\exp(i\alpha\sigma_3)$. This can be checked by a direct calculation in the six-vertex model.

Let us now consider the low lying excited states with $h_\pm$ holes near $\pm\infty$. We find after some computations from eqs.(4.55)–(4.58) and their analogous for the contributions around $\lambda = -\Lambda_-$

$$(4.65) \quad L_N^{exc}(\theta) = -\frac{\pi}{6N^2}\sin\frac{\pi\theta}{\gamma} + \frac{i\pi}{2N^2}\Big\{e^{\frac{-i\pi\theta}{\gamma}}\frac{\left(h_+ - \frac{\gamma s}{\pi} - \frac{\alpha}{\pi}\right)^2}{1-\frac{\gamma}{\pi}} - e^{\frac{i\pi\theta}{\gamma}}\frac{\left(h_- - \frac{\gamma s}{\pi} + \frac{\alpha}{\pi}\right)^2}{1-\frac{\gamma}{\pi}}\Big\} + \text{higher orders,}$$

where we disregard multiples of $2\pi i$.

For the six-vertex model ($\alpha = 0$), eqs.(4.59) and (4.65) can be recasted as [17,21]

$$(4.66) \; L_N^{exc}(\theta) - L_N^{G.S.}(\theta) = \frac{2\pi}{N^2}\left[(\Delta+\bar\Delta)\sin\frac{\pi\theta}{\gamma} + i(\Delta-\bar\Delta)\cos\frac{\pi\theta}{\gamma}\right],$$

where

$$(4.67) \qquad \Delta = \frac{1}{4}\frac{(h_+ - \frac{\gamma s}{\pi})^2}{1-\frac{\gamma}{\pi}}, \qquad \bar\Delta = \frac{1}{4}\frac{(h_- - \frac{\gamma s}{\pi})^2}{1-\frac{\gamma}{\pi}}.$$

Eq.(4.66) fits with conformal theory predictions [19]. It may be considered as a proof of the conformal invariance of the six-vertex model giving in addition the conformal weights $\Delta$ and $\bar\Delta$ of the low-lying states of it and central charge equal to one. The same results hold for the Heisenberg $XXZ$ chain [20].

For $\alpha = \gamma$ the critical Potts model properties follow from eqs.(4.64)–(4.65) [17,21].

Eqs.(4.67) give the conformal dimensions of primary fields. Actually some complex roots are present in these states besides the holes near $\pm\infty$. Otherwise one finds secondary fields with conformal dimensions

$$\Delta + K, \bar\Delta + \bar K,$$

where $K$ and $\bar K$ are positive integers [21].

The method exposed in this section has been applied to other models and other types of boundary conditions. We want to mention:

a) the $XXZ$ chain, the critical Ashkin-Teller model and the critical $q$-states Potts chain with free boundaries [22].

b) the critical $O(n)$ model

c) the half-filled Hubbard chain [24].

d) the four states Potts model. In this case the finite size corrections contain logarithms of the size [25].

e) the $XXZ$ chain and the one-dimensional Bose gas in external fields are treated in refs. [26].

Moreover, the present methods has been generalized to nested Bethe Ansatz system [15]. In this way the central charge and conformal dimensions of the model exposed in refs. [3,27] were found as well as for vertex models associated to all simply laced Lie algebras $\mathcal{G}$ [15]. We find

$$c = \operatorname{rank} \mathcal{G} \tag{4.68}$$

that gives $c = q - 1$ for the model $q(2q-1)$ vertex of ref. [3,27]. That is, each stage of the nested Bethe Ansatz contributes in one unit to $c$. The conformal weights turn to be [15]

$$\Delta = \frac{1}{8(1-\frac{\gamma}{\pi})} \sum_{l,l'=1}^{r} \left(h_l^+ - \frac{\gamma}{\pi} S_l\right) (M^{-1})_{l,l'} \left(h_l'^+ - \frac{\gamma}{\pi} S_{l'}\right), \tag{4.69}$$

and a similar formula for $\bar{\Delta}$ with $h_l^-$ instead of $h_l^+$. Here $h_l^\pm$ are the number of holes near $\pm\infty$ in the $l$-th branch, $S_l$ the $l$-th spin of the state [3], $M$ the Cartan matrix of the underlying Lie Algebra. For the simply laced Lie algebras

$$M_{jl} = \delta_{jl} + \operatorname{sign}\langle \alpha_j, \alpha_l \rangle. \tag{4.70}$$

In particular for $A_{q-1}$ [see ref. [3]]

$$M_{jl} = 2\delta_{jl} - \delta_{j,l+1} - \delta_{j,l-1}.$$

Once more $\Delta$ and $\bar{\Delta}$ vary continuoulsy with $\gamma$. In particular when $\gamma = \pi/(m+1)$, $m = q+1, q+2$, one recovers the conformal weights of theories possessing extended Virasoro invariance (W-algebras) [28]. More precisely one must consider the RSOS version of these multistate vertex models [29].

In this way the central charge takes the values

$$c = (q-1)\left[1 - \frac{q(q+1)}{m(m+1)}\right], \quad m\mathfrak{g}q + 1.$$

These integrable lattice models provide explicit realizations of the extended Virasoro algebra through their long-range behaviour. They may be a very useful framework to uncover the physical meaning of the extended conformal symmetries.

The fusion technique produces new YB generators from known ones. In the $S$-matrix language, one produces in this way the $S$-matrix of composite (bound states) particles from the $S$-matrices of their constituants. New integrable vertex models follow in this way. In ref. [11] we propose the following formula for the central charge of these models:

$$c = \frac{x \dim \mathcal{G}}{x + \tilde{g}}. \tag{4.71}$$

Here $x$ is the number of fundamental spaces $\mathcal{V} = \mathcal{A}$ fused and $\tilde{g}$ the dual Coxeter number of $\mathcal{G}$. It must be stressed that eq.(4.71) does not follow from any Sugawara type construction but just from the Bethe Ansatz solution. For $x = 1$ and simply laced $\mathcal{G}$ we recover eq.(4.68)

In the present review we only consider the dominant corrections for large $N$. From the subdominant ones one identifies irrelevant operators of the models [20]. In addition one sees that these subdominant powers of $N^{-1}$ coincide with the previously computed conformal dimensions plus positive integers. That is secondary conformal fields.

In the rational limit $\gamma \to 0$ besides powers corrections, logarithmic corrections emerge as one could expect. These logarithmic corrections has been also computed with the methods here exposed [20,22].

It must be remarked that all central charges and conformal weights are independent of the spectral parameter $\theta$. Moreover the phase transitions in vertex models are associated to changes in $\gamma$ or in the elliptic modulus $k$. Therefore $\theta$ plays the rôle of an irrelevant parameter in integrable statistical models.

## §5. The light-cone lattice approach

This approach starts by discretizing the two-dimensional Minkowski space-time in light-cone coordinates $x_{\pm} = x \pm t$. Space time is thus approximated by a diagonal lattice. This discretization scheme turns to be an useful regularization method for integrable quantum field theories since they become naturally connected with integrable vertex models in their scaling limit [11].

The sites in the light-cone lattice (Fig.12) are considered as world events. Each site (event) is joined by light-like links to its four nearest neighbours along $x_+$ and $x_-$. These diagonal links are possible world lines for the propagation upwards in time of "bare" massless particles. Particles on right-oriented (R) and left-oriented (L) links are called respectively right and left-movers.

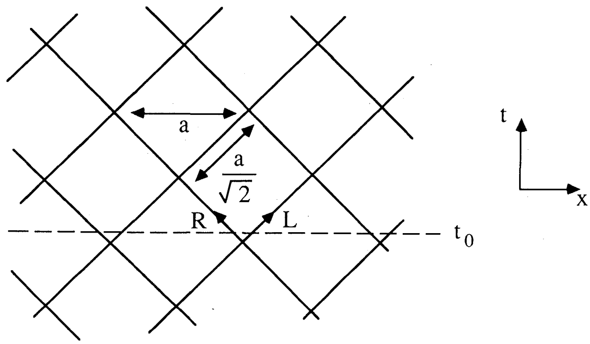



Fig. 12. Discretized Minkowski space-time. Sites are world events joined by world lines of the bare particle propagation.

One then associates microscopic amplitudes to each site (world event) where two oppositely oriented world lines cross. These amplitudes describe the different processes that can take place, and must verify general invariance properties like unitarity.

Let us start for the simplest case where each link describes only two different configurations. We assume that these two cases correspond to the presence or absence of a bare fermion without internal degrees of freedom. In general, there can be 16 different amplitudes per site corresponding to the 16 configurations (occupied/empty) of the four links joining there. Only $U(1)$ invariant microscopic amplitudes will be considered here such that the number of particles is conserved at each site. $U(1)$ transformations act on the link states by

$$|0\rangle \to e^{i\lambda}|0\rangle, \qquad |1\rangle \to e^{-i\lambda}|1\rangle, \tag{5.1}$$

where $|0\rangle$ = (empty) and $|1\rangle$ = (occupied). Therefore, there are only six non-zero amplitudes as depicted in Fig.13. The correspondence with the general (non-symmetric) six-vertex model is evident.

Of course, space-time translational invariance implies that the amplitudes are the same in all sites of the lattice. It is natural (and causes no loss of generality) to set the nothing-to-nothing amplitude to be 1. Unitarity then requires

$$\Omega\Omega^\dagger = 1, \qquad \Omega = \begin{pmatrix} \omega_3 & \omega_5 \\ \omega_6 & \omega_4 \end{pmatrix}, \qquad |\omega_2|^2 = 1. \tag{5.2}$$

While $\omega_3$ and $\omega_4$ are naturally interpreted as amplitudes for free prop-

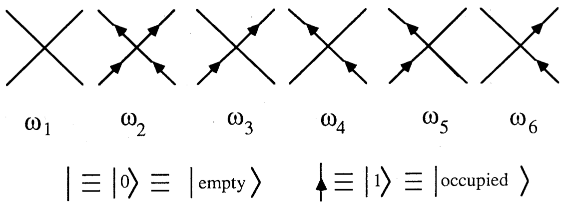



Fig. 13. The six non-zero microscopic transition amplitudes. They coincide with the weights of Fig.11.

agation (being therefore related to kinetic energies in the continuum limit), $\omega_5$ and $\omega_6$ play the role of mass terms since they couple right and left movers.

Symmetry under parity transformation holds if

$$\omega_3 = \omega_4 = b, \qquad \omega_5 = \omega_6 = c. \tag{5.3}$$

This corresponds now to an integrable six vertex model. Unitarity now reads

$$|b|^2 + |c|^2 = 1, \qquad b\bar{c} + \bar{b}c = 0. \tag{5.4}$$

One can organize these microscopic amplitudes at a site into a $4 \times 4$ unitarity "bare" $S$-matrix

$$R^{\alpha\beta}_{\alpha'\beta'} = {}^{\alpha}_{\alpha'}\!\times^{\beta}_{\beta'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \tag{5.5}$$

where $\omega \equiv \omega_2$ and $\alpha$, $\beta$, $\alpha'$, $\beta'$ take the values 0 or 1 for empty or occupied links like in eq.(5.1).

The amplitude for a global process, from a given state at $t = t_0$ to another given state at a later time, is obtained by summing over the amplitudes of all allowed vertex configurations compatible with initial and final conditions and with boundary conditions. Each of these is given by a product of microscopic amplitudes $\omega_i$. It clearly corresponds to the sum over all possible paths of an arbitrary, but constant in time number of particles. At any instant, a particle can move to the left or to the right at the speed of light. We are thus dealing with a discretization of Feynman path integral for fermions.

It is convenient to parametrize $b$ and $c$ following the constraints (5.4) as

$$b = b(\theta,\gamma) = \frac{\sinh\theta}{\sinh(\theta - i\gamma)},$$
$$c = c(\theta,\gamma) = \frac{\sinh\gamma}{\sinh(\theta - i\gamma)}, \tag{5.6}$$
$$0 < \theta < \infty, \qquad 0 < \gamma < \pi.$$

This makes (5.5) identical with the six-vertex model $R$-matrix (3.1) up to an overall factor $\mathrm{sh}(\theta - i\gamma)$ and a redefinition of $\theta \to i\theta$ when $\omega = 1$. Actually $\omega \neq 1$ corresponds to a six-vertex model in an external field.

Let us now describe the operator formalism for the light-cone approach [11]. The unit evolution operators in the light-cone direction (R or L) are given by simply juxtaposition of the microscopic $S$-matrices (5.5) at the same horizontal level. That is

$$U_R(\theta) = {}^{\beta_1}_{\alpha_N}\times^{\beta_2}_{\alpha_1}\,{}^{\beta_3}_{\alpha_2}\times^{\beta_4}_{\alpha_3}\cdots{}^{\beta_{N-1}}_{\alpha_{N-2}}\times^{\beta_N}_{\alpha_{N-1}}, \tag{5.7}$$

$$U_L(\theta) = {}^{\beta_1}_{\alpha_2}\times^{\beta_2}_{\alpha_3}\,{}^{\beta_3}_{\alpha_4}\times^{\beta_4}_{\alpha_5}\cdots{}^{\beta_{N-1}}_{\alpha_N}\times^{\beta_N}_{\alpha_1}. \tag{5.8}$$

Here $N$ is assumed to be even and $\alpha_{j+N} \equiv \alpha_j$. Notice that there are no summations in eqs.(5.7)–(5.8). One can now define the two light-cone lattice evolution generators as

$$H + P = \frac{2i}{a}\log U_R(\theta),$$
$$H - P = \frac{2i}{a}\log U_L(\theta), \tag{5.9}$$

where $H$ and $P$ stand for lattice hamiltonian and momentum and $a$ is the latttice spacing.

Eq.(5.9) is extremely suggestive since it provides a lattice version of field-theoretic $H$ and $P$ in terms of lattice vertex transfer matrices $U_R$ and $U_L$. The natural question is now to find the eigenvectors of them. It will be shown now that this is possible using the techniques of Sections III and IV (and their generalizations) provided $R(\theta)$ verifies the YB algebra (1.12) [11].

Let us consider the row-to-row transfer matrix $\tau^{[N]}(\theta, \underset{\sim}{a})$ of eq.(1.20) [with $g_1 = \ldots = g_N = 1$] with the particular choice of inhomogeneities (recall $N =$ even)

$$\alpha_j = (-1)^{j+1}\theta,$$
$$\underset{\sim}{\theta} = (\theta, -\theta, \ldots, (-1)^{j+1}\theta, \ldots, \theta, -\theta). \tag{5.10}$$

It then follow from eqs.(1.20) and (1.3) using eq.(1.12) that

$$\tau(\theta, \underset{\sim}{\theta}) = U_L(\theta), \tag{5.11}$$

$$\tau(-\theta, \underset{\sim}{\theta}) = U_R(\theta)^\dagger. \tag{5.12}$$

Let us check (5.12). Setting (5.10) in (1.13)–(1.20) yields

$$\begin{aligned}(5.13)\ \tau(\theta, \underset{\sim}{\theta})_{\underset{\sim}{\alpha}'|\underset{\sim}{\alpha}} &= \sum_{\beta_1,\ldots,\beta_N} \delta^{\alpha'_1}_{\beta_1}\delta^{\beta_2}_{\alpha_1} R(\theta)^{\alpha'_2\beta_3}_{\beta_2\alpha_2}\cdots\delta^{\alpha'_{N-1}}_{\beta_{N-1}}\delta^{\beta_N}_{\alpha_{N-1}} R(\theta)^{\alpha'_N\beta_1}_{\beta_N\alpha_N} \\ &= \prod_{j=1}^{N/2} R(\theta)^{\alpha'_{2j}\alpha'_{2j+1}}_{\alpha_{2j-1}\alpha_{2j}} = U_L(\theta)_{\underset{\sim}{\alpha}'|\underset{\sim}{\alpha}},\end{aligned}$$

after using eq.(1.12) repeatedly.

The key relations (5.9), (5.11)–(5.12) connect the lattice $H$ and $P$ with the row-to-row transfer matrices whose eigenvectors and eigenvalues can be constructed by the algebraic Bethe Ansatz developed in Sections III and IV. The light-cone or diagonal-to-diagonal transfer matrices resulted to be particular cases of the inhomogeneous row-to-row transfer matrices. The commutativity property (1.9) gives in addition

$$\begin{aligned}(5.14)\qquad &[\tau(\lambda, \underset{\sim}{\theta}), U_L(\theta)] = 0, \\ &[\tau(\lambda, \underset{\sim}{\theta}), U_R^\dagger(\theta)] = 0, \\ &[U_L(\theta), U_R^\dagger(\theta)] = 0.\end{aligned}$$

One can consider the infinite sequence of commuting operators ($1 \leq K < \infty$)

$$c_\kappa = \frac{\partial^\kappa}{\partial\lambda^\kappa} \log \tau(\lambda, \underset{\sim}{\theta})\big|_{\lambda=\theta}. \tag{5.15}$$

They all commute with $U_L(v)$, $U_R^\dagger(v)$ and with each other.

Let us now consider the continuum limit $a \to 0$ of the lattice models through eq.(5.9). The ground state of $\tau(\lambda, \theta)$ corresponds just to the physical vacuum (filled Dirac sea) of the QFT defined by $H$ and $P$. The particle states follow from the lowest excitations. Since a factor $a^{-1}$ appears in $H \pm P$ [see eq.(5.9)] only gapless models yield finite energy states in the scaling limit. Moreover, in order to compute the energy and momentum in the scaling limit it is enough to know the eigenvalues of $\tau(\pm\theta, \underset{\sim}{\theta})$ close to the bottom of the spectrum. The low-lying excitations are associated to holes and complex solutions with large (real) rapidity. Moreover, their eigenvalues normalized to the vacuum ones [as in eq.(3.55)] are independent of the inhomogeneities.

Let us start by the fermion model (with $\omega = 1$) associated to the six-vertex models (eqs.(5.1)–(5.6)). The excitation spectrum is given in Section III in terms of the function $g(\theta)$ (eqs.(3.55)–(3.57)). Combining eq.(3.57) with (5.9) yields

$$\epsilon \pm p = \mp \frac{g(\mp i\nu)}{a}, \quad \nu \in \mathbf{R}. \tag{5.16}$$

Let us start by a hole excitation. One find for large $\nu$ from eq.(3.55) and (3.61)

$$g(\pm i\nu) = \pm 2 e^{\pm \frac{\pi\theta_h}{\gamma}} e^{-\frac{\pi\nu}{\gamma}} + o\left(e^{-\frac{2\pi\nu}{\gamma}}\right), \qquad \nu \to +\infty \tag{5.17}$$

after discarding an irrelevant $\pi$. (It does not contribute to the eigenvalue of $\tau(\theta)$ since the holes appear always by pairs). One finds a relativistic spectrum provided $\nu \to \infty$ when $a \to 0$ keeping fixed the renormalized mass

$$m = \frac{4}{a} e^{-\frac{\pi\nu}{\gamma}}. \tag{5.18}$$

The dispersion law results

$$\begin{aligned} \epsilon &= m \cosh \frac{\pi\theta_h}{\gamma}, \\ p &= m \sinh \frac{\pi\theta_h}{\gamma}. \end{aligned} \tag{5.19}$$

So, $\pi\theta_h/\gamma$ is the physical rapidity of the particle [see eq.(2.14)]. Besides these holes that are identified with the fermions of the massive Thirring model [11,12] one finds the string solutions (3.49)–(3.52). They provide relativistic particles in the *same* scaling limit (5.18) with masses

$$m_n = 2m \sin\left\{\frac{n\pi}{2\gamma}(\pi - \gamma)\right\}, \quad 1 \le n \le \left[\frac{\pi}{\pi - \gamma}\right] - 1, \tag{5.20}$$

as follows from eqs.(5.16) and (3.61). This set of particles are fermion-antifermion bound states. They relate semiclassically to the breather of sine-Gordon as the fermions (or holes) (5.19) correspond to sine-Gordon solitons.

The preceding exposition of the light-cone lattice method applies to all gapless vertex models. In ref. [15] the models with rational $R$-matrices associated to simple Lie algebras are analysed. The model of section IV is also considered in its gapless regime.

Within this light-cone approach it is possible to construct explicitly the canonical bare fields on the lattice and to show that in the scaling limit (5.18) the massive Thirring model emerges [11].

One introduces lattices fermion fields $\psi_{R,n}$ and $\psi_{L,n}$. They are associated to the links stemming upwards from each site at a fixed time (see Fig.14).

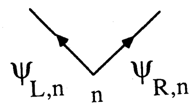



Fig. 14. Fermion lattice operators associated to the links stemming upwards from each site.

They satisfy usual anticommutation rules

$$\{\psi_{A,n}, \psi_{B,m}\} = 0, \quad A, B = R, L,$$
$$\{\psi_{A,n}, \psi^{\dagger}_{B,m}\} = \delta_{nm}\delta_{AB}, \quad 1 \leq n, m \leq N. \tag{5.21}$$

$\psi_{R,n}$ and $\psi_{L,n}$ can be assembled in a two-component spinors. In this representation obviously diagonal since chiral rotations act locally on the $\psi$'s.

$$\psi_{R,n} \to e^{i\lambda}\psi_{R,n}, \quad \psi_{L,n} \to e^{-i\lambda}\psi_{L,n}.$$

These lattice fermions are quite different from Kogut-Susskind fermions. In our case the species doubling is avoided thanks to the non-locality (on the lattice) of the hamiltonian (5.9). To simplify the notation we write

$$\psi_{R,n} = \psi_{2n}, \quad \psi_{L,n} = \psi_{2n-1}, \quad 1 \leq n \leq N, \tag{5.22}$$

so that eq.(5.21) is replaced by

$$\{\psi_n, \psi_m\} = 0, \quad \{\psi_n, \psi^{\dagger}_m\} = \delta_{nm}. \tag{5.23}$$

Consider now the bilocal, unitary, even operators

$$R_{nm} = 1 + bK_{nm} + (c-1)K^2_{nm} + (\omega - 1)\psi^{\dagger}_n\psi_n\psi^{\dagger}_m\psi_m, \tag{5.24}$$

where

$$K_{nm} \equiv \psi^{\dagger}_n\psi_m + \psi^{\dagger}_m\psi_n, \tag{5.25}$$

and $b$, $c$ and $\omega$ are given by eq.(5.2) and (5.6). The matrix elements $R_{nm}$ in the bare representation $|\,\underset{\sim}{\alpha}\rangle$ read

$$(5.26)\qquad \langle \underset{\sim}{\alpha}\,|R_{nm}|\,\underset{\sim}{\alpha}'\rangle = R^{\alpha_n \alpha_m}_{\alpha'_n \alpha'_m} \prod_{j=1, j\neq n,m}^{N} \delta^{\alpha_j}_{\alpha'_j},$$

where as usual

$$|0\cdots \overset{n_1}{1}\cdots \overset{n_2}{1}\cdots \quad \cdots \overset{n_M}{1}\cdots 0\rangle = \psi^\dagger_{n_1}\psi^\dagger_{n_2}\cdots\psi^\dagger_{n_M}|0\rangle,$$

with $n_1 < n_2 < \ldots < n_M$.

The second quantized representation (5.24)–(5.26) for $R$ allows to write the light-cone transfer matrix $U_R(\nu)$ in second-quantized language using eqs.(5.7)–(5.8). Now it is easy matter to derive the lattice equations of motion for the fermion operators $\psi_n$ and $\psi^\dagger_n$. One finds [11]

$$(5.27)\qquad \begin{aligned} U_R\psi_{2n-2}U_R^\dagger &= U_L\psi_{2n}U_L^\dagger = R_{2n-1,2n}\psi_{2n-1}R^\dagger_{2n-1,2n},\\ U_R\psi_{2n-1}U_R^\dagger &= U_L\psi_{2n+1}U_L^\dagger = R_{2n-1,2n}\psi_{2n}R^\dagger_{2n-1,2n}. \end{aligned}$$

This equation hold for any form of the $4\times 4$ $R$-matrix. Inserting now in (5.27) the explicit form (5.24)–(5.25) yields

$$(5.28)\qquad \begin{aligned} U_R\psi_{2n-2}U_R^\dagger &= U_L\psi_{2n}U_L^\dagger\\ &= \bar{b}\psi_{2n} + \bar{c}\psi_{2n-1} + \big(\frac{c}{\omega} - \bar{c}\big)\psi^\dagger_{2n}\psi_{2n}\psi_{2n-1}\\ &\qquad - \big(\frac{b}{\omega} + \bar{b}\big)\psi^\dagger_{2n-1}\psi_{2n-1}\psi_{2n},\\ U_L\psi_{2n-1}U_L^\dagger &= U_R\psi_{2n+1}U_R^\dagger\\ &= \bar{b}\psi_{2n-1} + \bar{c}\psi_{2n} + \big(\frac{c}{\omega} - \bar{c}\big)\psi^\dagger_{2n-1}\psi_{2n-1}\psi_{2n}\\ &\qquad - \big(\frac{b}{\omega} + \bar{b}\big)\psi^\dagger_{2n}\psi_{2n}\psi_{2n-1}. \end{aligned}$$

These second quantized field equations are perfectly defined on the lattice. The bare continuum limit $a \to 0$ is rather subtle. The detailed proof of ref. [11] shows that it leads to the continuum MTM provided the lattice parameters $b$ and $c$ scale as

$$(5.29)\qquad b \underset{a\to 0}{=} e^{i\mu}[1 + O(a^2)], \quad c \underset{a\to 0}{=} -im_0 a e^{i\mu}[1 + O(a^2)].$$

Notice that this bare limit is different from the renormalized one [eq.(5.18)]. Here $\mu$ and $m_0$ are fixed parameters that characterize the bare scaling limit. The lattice fermion operators leads to the continuum ones $\psi_R(x)$, $\psi_L(x)$ in the following way

$$(5.30) \qquad \psi_{R,n} = \sqrt{a}\psi_R(x+\xi a), \quad \psi_{L,n} = \sqrt{a}\psi_L(x-\xi a).$$

$x = na$ and $0 < \xi < 1/2$ is a fixed number whose precise value is irrelevant in the limit. Then the continuum hamiltonian and momentum follow from

$$(5.31) \qquad \begin{aligned} H + P &= \lim_{a\to 0} \frac{2i}{a}\left[e^{-i\mu Q}U_R - 1\right], \\ H - P &= \lim_{a\to 0} \frac{2i}{a}\left[e^{-i\mu Q}U_L - 1\right], \end{aligned}$$

where $Q$ is the bare $U(1)$ charge

$$(5.32) \qquad Q = \int_0^L dx(\psi_R^\dagger\psi_R + \psi_L^\dagger\psi_L)$$

where $L = Na$.

Notice that $\lim U_R = \lim U_L = e^{i\mu Q} \neq 1$. After some calculations [11] it can be shown that

$$P = -i\int_0^L dx\psi^\dagger\partial_x\psi,$$

and

$$(5.33) \qquad H = \int_0^L dx[-i\psi^\dagger(\gamma^5\partial_x + im_0\gamma^0)\psi + \frac{g}{2}(\bar\psi\gamma_\mu\psi)^2],$$

where

$$\psi = \begin{pmatrix}\psi_R \\ \psi_L\end{pmatrix}$$

and $g = -2\text{ctg}(\mu - \mu_0)$, $\omega = e^{2i\mu_0}$, $\gamma_1 = -i\sigma_y$, $\gamma_0 = \sigma_x$, $\gamma_5 = \sigma_z$.

That is we find the massive Thirring model (MTM) in the scaling limit (5.29). Eqs.(5.30)–(5.33) give the bare operators and eq.(5.29) defines the bare scaling limit. This is different to the renormalized scaling limit giving the physical sector of the Fock space [eq.(5.18)]. Both the particle spectrum and the physical S-matrices follow rigorously in the renormalized limit computed by this light-cone approach. The bare limit (5.30)–(5.33) tells us which model one is actually solving. We use the word "rigorous" since we solve in this approach a lattice model exactly, then we take the infinite volume limit and finally the $a \to 0$ (scaling) limit. In other words, here one solves (exactly) a model with both UV and volume cutoffs and then lets the cutoffs to infinity in a precise way. This is clearly much better than the coordinate Bethe-Ansatz (CBA) where the UV cutoff is introduced after the obtention of the solution. For the MTM and the chiral Gross-Neveu model the results of the CBA coincide with the light-cone approach for on-shell magnitudes. Hence the CBA works well in these cases. This is not the case for the multiflavor Chiral fermion model treated in ref. [30] by CBA. As it is shown in ref. [11] the results of ref. [30] are not correct.

Starting from richer vertex models than the six-vertex a large set of QFTs arises [11]. Let us first summarize the integrable vertex models classification in terms of simple Lie Algebras.

A deep connection exists between integrable theories and simple Lie algebras [31]. It is possible to associate an integrable vertex model to each representation of a simple Lie algebra. These rational models are invariant under the corresponding Lie group $\mathcal{G}$, since this $R$-matrix obeys

$$[R(\theta), g \otimes g] = 0. \tag{5.34}$$

Moreover, the structure of their BAE looks like the one of their respective Dynkin diagram. It must be noticed that a proof that these BAE lead to the eigenvectors and eigenvalues of the transfer matrix has been explicited only for a subset of models: those associated to $U(N)$ (see Section IV), $Sp(2N)$ [32c], and $SO(2N)$ [33] and some others. However, these statements are extremely likely to hold for all semisimple Lie algebras. Moreover, the whole structure of the BAE deforms in a very simple and suggestive way for the trigonometric/hyperbolic models where the symmetry contracts to the Cartan subalgebra of $\mathcal{G}$.

Let us describe the BAE for the trigonometric models. The derivation of these equations (for a subset of Lie algebras) are in Section IV and refs.[32,33]. The eigenvalues of the transfer matrix can be written as a sum of terms. The dominant one in the infinite volume limit ($N \to \infty$) is

$$\lambda_\omega\left(\theta\{\underset{\sim}{\lambda}^{(j)}\}\right) = \prod_{a=1}^{N} \prod_{k=1}^{r} \prod_{j_k=1}^{p_n} \frac{\sinh\left[i(\theta-\theta_a) + \lambda_{j_k}^{(k)} - i\gamma(\omega_a, \alpha_k)\right]}{\sinh\left[i(\theta-\theta_a) + \lambda_{j_k}^{(k)} + i\gamma(\omega_a, \alpha_k)\right]}, \tag{5.35}$$

for $\theta$ in the vicinity of $\theta = 0$, $|\theta| < \theta_0$. Here $\theta_1, \theta_2, \ldots, \theta_N$ are given numbers describing inhomogeneities of the lattice as discussed in Section II [1,16]. The $\omega_a$ are fundamental weights and $\alpha_k$ are the simple roots of $\mathcal{G}$ whose rank is $r$. $(\alpha, \beta)$ stands for the usual inner product in root space. The $\lambda_{j_k}(k)$ $(1 \leq j_k \leq p_j,\ 1 \leq k \leq r)$ are solutions of the nested BAE (NBAE):

$$\prod_{k=1}^{r} \prod_{j_k=1}^{p_k} \frac{\sinh\left[\lambda_{j_i}^{(i)} - \lambda_{j_k}^{(k)} + i\gamma(\alpha_i, \alpha_k)\right]}{\sinh\left[\lambda_{j_i}^{(i)} - \lambda_{j_k}^{(k)} - i\gamma(\alpha_i, \alpha_k)\right]}$$

$$= - \prod_{a=1}^{N} \frac{\sinh\left[\lambda_{j_i}^{(i)} - \theta_a + i\gamma(\omega_a, \alpha_i)\right]}{\sinh\left[\lambda_{j_i}^{(i)} - \theta_a - i\gamma(\omega_a, \alpha_i)\right]}, \tag{5.36}$$

$$1 \leq j_i \leq p_i, \quad 1 \leq i \leq r.$$

Here the upper indices $^{(i)}$ label the steps in the NBA. Each step is associated to a simple root $\alpha_i$. The structure of eq.(5.36) coincides with the respective Dynkin diagram: when two roots, say $\alpha_l$ and $\alpha_i$, are orthogonal, their associated parameters $\lambda_{j_l}^{(l)}$ and $\lambda_{j_i}^{(i)}$ $(1 \leq j_l \leq p_l$, $1 \leq j_i \leq p_i)$ are not directly coupled through (5.36) since $(\alpha_i, \alpha_l) = 0$. It must be noticed that due to the orthogonality of fundamental weights and simple roots [34]

$$(\omega_a, \alpha_i) = \frac{\delta_{ai}}{2}(\alpha_i, \alpha_i). \tag{5.37}$$

The normalization of the simple roots can be absorbed as a multiplicative factor on the $\lambda_{j_i}(i)$.

Taking the logarithm of eq.(5.36) leads, in the homogeneous case $(\theta_a = 0)$ to

$$N\phi(\lambda_{j_i}^{(i)}, \gamma(\omega_a, \alpha_i)) \tag{5.38}$$

$$= \sum_{k=1}^{r} \sum_{j_k=1}^{p_k} \phi(\lambda_{j_i}^{(i)} - \lambda_{j_k}^{(k)}, \gamma(\alpha_i, \alpha_k)) + 2\pi_{j_i}^{(i)},$$

$$1 \leq J_i \leq p_i, \quad 1 \leq i \leq r,$$

where the $I_{j_i}(i)$ are half-odd integers and $\phi(z, \alpha)$ is given by eq.(3.36). Actually eqs.(5.38) also hold in the rational and hyperbolic cases using the appropiate expression for $\phi(z, \alpha)$ [eq.(3.39) or (3.41) respectively]. In the thermodynamic limit eq.(5.38) yields for any $\mathcal{G}$ a system of linear integral equations for the root densities analogous to eq.(3.38) for the six-vertex model

$$
\begin{aligned}
\sigma_j(\lambda) - \sum_{l=1}^{r} \int_{-A}^{A} d\mu K_{jl}(\lambda-\mu)\sigma_l(\mu) &= \frac{1}{2\pi}\phi'\big(\lambda, \gamma(\omega_a, \alpha_j)\big) \\
&- \frac{1}{N}\sum_{l=1}^{r}\Big\{\sum_{h=1}^{N_h^{(l)}} K_{jl}(\lambda-\theta_h^{(l)}) \\
&\qquad - \sum_{s_l}\big[K_{jl}(\lambda-\xi_{s_l}) + K_{jl}(\lambda-\bar{\xi}_{s_l})\big]\Big\},
\end{aligned} \tag{5.39}
$$

where

$$
K_{jl}(\lambda) = \frac{1}{2\pi}\phi'\big(\lambda, \gamma(\alpha_i, \alpha_k)\big), \tag{5.40}
$$

or in Fourier space

$$
\hat{K}_{jl}(\omega) = -\mathrm{sgn}[(\alpha_j, \alpha_l)]e^{-|\omega(\alpha_j,\alpha_l)|}, \quad \text{(rational case)}, \tag{5.41}
$$

$$
\hat{K}_{jl}(\omega) = -\mathrm{sgn}[(\alpha_j, \alpha_l)]\frac{\sinh\left[\omega\left(\frac{\pi}{2} - \gamma(\alpha_j, \alpha_l)\right)\right]}{\sinh\left(\frac{\omega\pi}{2}\right)}, \quad \text{(trigonometric case)}, \tag{5.42}
$$

$$
\begin{aligned}
\hat{K}_{jl}(m) &= 2\,\mathrm{sgn}[(\alpha_j, \alpha_l)]\frac{e^{2|m\gamma(\alpha_j,\alpha_l)|}}{m}, \\
&\qquad m \in \mathbf{Z}, \quad m \neq 0, \\
\hat{K}_{jl}(0) &= 2, \\
&\qquad \text{(hyperbolic case)},
\end{aligned} \tag{5.43}
$$

where $\mathrm{sgn}(0) = 0$.

The resolvent of the integral equation (5.39) follows as the inverse of the $r \times r$ matrix

$$
\hat{R}_{jl}(\omega) = \left([1 - \hat{K}]^{-1}\right)_{jl}. \tag{5.44}
$$

This is not a formidable inversion problem since it is a sparse matrix[51] whose characteristic diagram is precisely the Dynkin diagram of $\mathcal{G}$. Explicit formulae for $R_{jl}(x)$ can be derived for each Lie algebra $\mathcal{G}$. We find for $A_{q-1}$ the result in ref. [3]. For $D_n$ see refs. [33] and [35]. For $E_6$, $E_7$ and $E_8$, $R_{jl}$ can be calculated explicitly by hand. For non-simply laced Lie algebras, the ground state is formed by complex roots and hence this treatment needs to be generalized. This is also the case for non-fundamental representations of $\mathcal{G}$. That is, the models obtained by fusion.

Eqs.(4.74)–(4.82) of ref. [3] valid for $A_{q-1}$, easily generalize for any simple-laced Lie algebra . In order to study the scaling limit we need the excitation eigenvalues $g_l(\theta, \theta_h)$. They write

$$g_l(\theta, \theta_h) = \sum_{k=1}^{r} \int_{-A}^{A} d\lambda R_{jl}(\lambda - \theta_h) \phi(\lambda + i\theta, \gamma\omega_k). \tag{5.45}$$

In the trigonometric (gapless) regime this can be recasted as [15]

$$g_l(\theta, \theta_h) = \sum_{k=1}^{r} \int_{-\infty}^{+\infty} \frac{d\omega}{i\omega} \frac{\sinh\left[\omega\left(\frac{\pi}{2} - \gamma\omega_k\right)\right]}{\sinh\left(\frac{\omega\pi}{2}\right)} e^{i\omega(\theta_h + i\theta)} \left(\left[1 - \hat{K}\right]^{-1}\right)_{lk}, \tag{5.46}$$

where we used eqs.(3.41), (5.44) and (4.72) of ref. [3]. As before [eqs.(5.17)–(5.19)] only the large $i\theta$ behavior is relevant for the scaling limit. Eq.(5.46) tells us that this behavior is determined by the zeros of det$[1 - K(x)]$ closer to the real axis. These values are clearly $l$ independent. One finds from eq.(5.46) by the residue method [31]

$$g(\theta, \theta_h) = \frac{m_l}{\gamma\pi} \exp\left[\mp\frac{\kappa}{\gamma}(\theta_h + i\theta)\right] \left\{1 + o(e^{-|\theta\delta|})\right\}, \quad i\theta \to \pm\infty, \tag{5.47}$$

where $\delta > 0$. The parameters $\kappa$ and $m_l$ are given in Table II.

### TABLE II

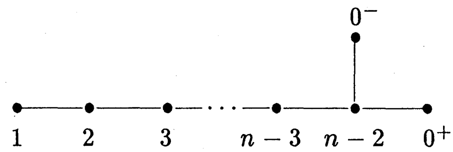
**Lie Algebra Dynkin's diagram**

$A_n$ ●——●——●——●—···—●——●
1 2 3 4 $n-1$ $n$

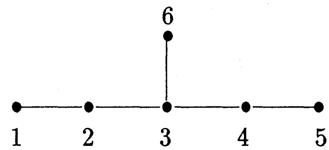
$B_n$ ●——●——●——●—···—●$\Rightarrow$●
1 2 3 4 $n-1$ $n$

$C_n$ ●——●——●——●—···—●$\Leftarrow$●
1 2 3 4 $n-1$ $n$

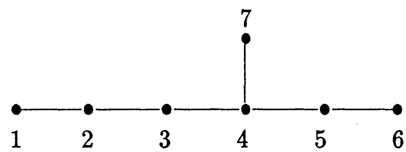
$D_n$



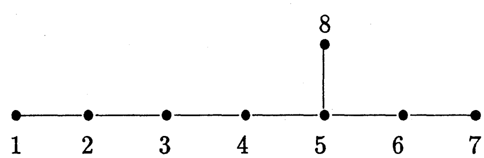

$E_6$



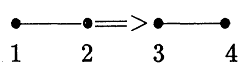

$E_7$



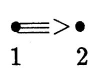

$E_8$




$F_4$




$G_2$




| | $\kappa$ | $m_k$ |
|---|---|---|
| $A_n$ | $2\pi/(n+1)$ | $\sin(\pi k/(n+1)),\ 1 \le k \le n$ |
| $B_n$ | $\pi/(2n-1)$ | $\sin(\pi k/(2n-1)),\ 1 \le k \le n-1$ |
| | | $m_n = 1/2$ |
| $C_n$ | $\pi/(n+1)$ | $\sin(\pi k/2(n+1)),\ 1 \le k \le n$ |
| $D_n$ | $\pi/(n-1)$ | $\sin(\pi k/2(n-1)),\ 1 \le k \le n-2$ |

| | | |
|---|---|---|
| | | $m_\pm = 1/2$ |
| $E_6$ | $\pi/6$ | $m_1 = m_5 = m_6/2 = \sqrt{3}/2$ |
| | | $m_2 = m_4 = (3+\sqrt{3})/2$ |
| | | $m_3 = (3+\sqrt{3})/\sqrt{2}$ |
| $E_7$ | $\pi/9$ | (*) |
| $E_8$ | $\pi/15$ | (*) |
| $F_4$ | $\pi/9$ | (*) |
| $G_2$ | $\pi/6$ | (*) |

(*) These values can be extracted from refs. (15) and (31).

Table II Integrable QFT associated to trigonometric Yang-Baxter algebras in the light-cone approach. We indicate the underlying Lie algebra $\mathcal{G}$, the respective scale parameter $\kappa$ (it coincides with the one-loop beta function) and the corresponding mass spectrum.

$\kappa$ is just $2\pi$ times the length squared of the shortest simple root in the normalization where [34]

$$B(E_\alpha, E_{-\alpha}) = -1$$

and $B(x,y)$ is the Killing form.

Light-cone evolution operators can be defined through eqs.(5.7)–(5.9) for any $R$-matrix. Let us see that a relativistic dispersion law arises from any excitation spectrum as given by eq.(5.47). Let us call $E_l(\phi)$ and $p_l(\phi)$ the eigenvalues of $H$ and $P$, respectively. Eqs.(5.9) and (5.47) yield

$$\text{(5.48b)}\qquad E_l(\theta_h) \underset{i\theta\to\infty}{=} \frac{e^{-i\kappa\theta/\gamma}}{\gamma\pi a} m_l \cosh\left(\frac{\kappa\theta_h}{\gamma}\right) + o(e^{-2i\kappa\theta/\gamma}),$$
$$p_l(\theta_h) \underset{i\theta\to\infty}{=} \frac{e^{-i\kappa\theta/\gamma}}{\gamma\pi a} m_l \sinh\left(\frac{\kappa\theta_h}{\gamma}\right) + o(e^{-2i\kappa\theta/\gamma}).$$

It is then natural to define the scaling limit according to

$$\text{(5.49)}\qquad a \to 0, \quad i\theta \to \infty, \quad \mu = \frac{e^{-i\kappa\theta/\gamma}}{\pi a \gamma} = \text{fixed}$$

is the renormalised or physical mass scale and the particle mass spectrum of these integrable QFTs is given by

$$\text{(5.50)}\qquad M_l = \mu m_l.$$

We recognize in eq.(5.48) $\kappa\theta_h/\gamma$ as the physical particle rapidity.

This is a very general way of constructing integrable QFTs. The operators $H$ and $P$ given by eq.(5.9) are well defined on the lattice as well as all the higher conserved charges. In the continuum limit $a \to 0$, they provide the energy and momentum of a relativistic invariant QFT, as long as the spectrum of the initial vertex model is gapless. This is usually the case for rational or trigonometric weights. In addition to the particle spectrum , the $S$-matrix is exactly calculable from the BAE by standard methods [1,36].

As it was the case for the MTM, the evolution operators $U_R$ and $U_L$ are much simpler than $H$ and $P$ on the lattice. This was exploited before [eqs.(5.27)–(5.28)] to obtain the lattice field equations for the fermionic fields of the MTM regularized by the lattice. An analogous local construction would be very interesting to obtain in the general case of a Lie algebra $\mathcal{G}$. We present here a lattice construction for the current operators for all rational models discussed before. The $H$ and $P$ are always given by eqs.(5.7) and (5.9). The renormalized scaling limit (5.49) yields the mass spectrum (5.50) [see Table II]. Now the $R$-matrix for all rational models has the asymptotic behavior

$$R(\theta) \underset{\theta\to\infty}{=} P\left[1 + \frac{\pi+\lambda}{i\theta} + o(\frac{1}{\theta^2})\right], \tag{5.51}$$

where $\lambda$ is a numerical constant, the exchange operator $P$ was defined in eq.(1.27) and

$$\Pi = \sum_{\alpha=1}^{\dim G} T_\alpha \otimes T^\alpha. \tag{5.52}$$

We then introduce the lattice operator

$$T_n^\alpha = 1 \otimes \cdots \otimes \underset{n-th}{T^\alpha} \otimes \cdots \otimes 1. \tag{5.53}$$

Using eqs.(5.7)–(5.8),(5.51) and (5.52) and the Lie Algebra commutators

$$[T^\alpha, T^\beta] = i f^{\alpha\beta}{}_\gamma T^\gamma,$$

we can show that the $T_n^\alpha$ obey the local equations of motion on the lattice

$$
\begin{aligned}
U_R T^\alpha_{2n-2} U_R^\dagger &= U_L T^\alpha_{2n} U_L^\dagger \\
&= T^\alpha_{2n} + \frac{2i}{\theta} f^\alpha{}_{\beta\gamma} T^\beta_{2n-1} T^\gamma_{2n} + o(\frac{1}{\theta^2}), \\
U_R T^\alpha_{2n-1} U_R^\dagger &= U_L T^\alpha_{2n+1} U_L^\dagger \\
&= T^\alpha_{2n-1} - \frac{2i}{\theta} f^\alpha{}_{\beta\gamma} T^\beta_{2n-1} T^\gamma_{2n} + o(\frac{1}{\theta^2}).
\end{aligned} \tag{5.54}
$$

The bare scaling limit is now defined as $a \to 0$, $\theta \to 0$, $x = na$ fixed. We get

$$
\begin{aligned}
&\partial_\mu J^{\mu\alpha}(x) = 0, \\
&\partial_0 J_1^\alpha(x) - \partial_1 J_0^\alpha(x) + igf^\alpha{}_{\beta\gamma}[J_0^\beta, J_1^\alpha] = 0,
\end{aligned} \tag{5.55}
$$

where

$$
\begin{aligned}
J_R^\alpha(x) &= \frac{1}{ga\theta} T^\alpha_{2n}, \\
J_L^\alpha(x) &= \frac{1}{ga\theta} T^\alpha_{2n-1}.
\end{aligned} \tag{5.56}
$$

Therefore we have a lattice version of the $\mathcal{G}$-algebra currents $J_\mu^\alpha(x)$ associated to an exactly solvable discretization of the field theoretic models. These equations (5.55) characterize the currents in the non-abelian Thirring model associated to the group $\mathcal{G}$. This theory, also called Chiral Gross-Neveu model, has as Lagrangian,

$$
\mathcal{L} = i\bar{\psi}\ \partial\!\!\!/ \psi - \frac{g}{4}(\bar{\psi}\gamma_\mu T^\alpha \psi)(\bar{\psi}\gamma_\mu T^\beta \psi) K_{\alpha\beta}. \tag{5.57}
$$

Here $\psi$ transforms under the irreducible representation $\rho$ of $\mathcal{G}$, $T^\alpha$ are the $\mathcal{G}$-generators in that representation and $K_{\alpha\beta}$ is proportional to the inverse of the Killing form. Actually the $H$ and $P$ constructed from eqs.(5.7)–(5.9) with the $R$-matrix (5.57) describe the zero-chirality sector of the model (5.57) and we can identify

$$
J_\mu^\alpha(x) = \bar{\psi}\gamma^\mu T^\alpha \psi(x). \tag{5.58}
$$

The field theoretic models discussed up to here correspond to finite dimensional $\mathcal{V}$ and $\mathcal{A}$. Namely, a finite dimensional vector space at each link in the light-cone lattice. This is clearly appropiate for fermion or parafermion fields. Since there exists infinite dimensional representations of YB algebras, also bosonic QFTs may be described in this framework. The infinite spin representation of the $SU(2)$ invariant $R$-matrix (rational limit of the six-vertex model) relates to the $SU(2)$ principal chiral model (PCM) as it is investigated in ref. [39]. For arbitrary spin $S$, this $R$-matrix writes[37]

$$R_{12}(\theta) = \frac{\Gamma(2S+1+i\theta)\Gamma(\mathbf{J}+1-i\theta)}{\Gamma(2S+1-i\theta)\Gamma(\mathbf{J}+1+i\theta)}, \tag{5.59}$$

where the operator $\mathbf{J}$ is defined by

$$\mathbf{J}(\mathbf{J}+1) = 2S(S+1) + 2\bar{S}_1 \otimes \bar{S}_2. \tag{5.60}$$

$\vec{S}_1$ and $\vec{S}_2$ are spin $S$ operators acting on the spaces $\mathcal{A}$ and $\mathcal{V}$ respectively $[(\vec{S}_1)^2 = (\vec{S}_2)^2 = S(S+1)\ ]$. The light-cone hamiltonian (5.7)–(5.9) provides particle states that yield all particle masses and $S$-matrix amplitudes for the PCM letting $S = \infty$. However, this $H$ is not the full hamiltonian of the PCM as it is proven in refs. [38] and [39]. There is a very simple explanation for this, the physical particle states for this model transform under the group $SU(2)_L \otimes SU(2)_R$ and from the present construction only left or right operators can be obtained. Therefore all states obtained in this way are left (or right) singlets. The detailed counting of states in ref. [39] is confirmed by the simple proof of ref. [38].

The lattice current construction, eqs.(5.53)–(5.56) also applies to the PCM. For large $\theta$ the $R$-matrix (5.59) admits a semiclassical expansion of the type (5.51). Therefore, the whole constructin holds. It must be noticed that we get only one conserved and curvatureless current: either the $SU(2)_L$ or the $SU(2)_R$.

This whole cosntruction generalizes to the $SU(N)$ PCM. It also applies for Chiral fermion models and PCM with one anisotropy axis (trigonometric YB algebras) [52].

## §6. Braid groups and quantum groups from Yang-Baxter algebras

Let us see first how braid groups follow from trigonometric/ hyperbolic YBA. In the limit $\theta \rightarrow \pm\infty$ $(\pm i\infty)$ the hyperbolic/ trigonometric generators behave as $e^{\pm K\theta}$ $(e^{\mp Ki\theta})$ times a well defined operator ($K$ being a constant). Since such exponential factor can be absorbed in $T(\theta)$ respecting the YBE, we can in general assume that the limit

$$\lim_{\theta \to \mp\infty(\mp i\infty)} T(\theta) = T_\pm, \tag{6.1}$$

is finite and non-trivial for hyperbolic (trigonometric) YB generators. These limiting operators can be graphically represented as follows

$T_+$ $\qquad$ $T_-$

Letting $\theta \to \pm\infty, \theta' \to \pm\infty$ with $\theta - \theta' \to \pm\infty$ in the hyperbolic regime of eq.(1.1) yields

$$T_\pm^{(K,I)} T_\pm^{(K,J)} T_\pm^{(I,J)} = T_\pm^{(I,J)} T_\pm^{(K,J)} T_\pm^{(K,I)}. \tag{6.2}$$

In addition eq.(1.37) tells us that $T_+$ and $T_-$ are inverses of each other

$$T_+ T_- = T_- T_+ = 1, \quad T_\pm \equiv T_\pm^{(I,J)}. \tag{6.3}$$

A factor $\sqrt{\rho(\theta)}$ has been absorbed in the definition of $T(\theta)$ as in eq.(6.1).

If one considers $R$-matrices instead of general YB operators $T^{(I,J)}(\theta)$, eqs.(6.2)–(6.3) read

$$R_\pm^{23} R_\pm^{12} R_\pm^{23} = R_\pm^{12} R_\pm^{23} R_\pm^{12}, \tag{6.4}$$

$$R_+ R_- = R_- R_+ = 1, \tag{6.5}$$

where now $V^I = V^J = V^K = \mathcal{A}$, $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$ and $R_\pm = T_\pm^{(\mathcal{A},\mathcal{A})}$.

The matrices $R_+$ and $R_-$ give a representation of a braid group in the following way. Let us consider the operators $X_i(\theta)$ acting in the tensor product of $n$ auxiliary spaces $\mathcal{A}$ [40]

$$X_i(\theta) = \underset{1}{1} \otimes \cdots \otimes \underset{(i,i+1)}{R(\theta)} \otimes \cdots \otimes \underset{n}{1}, \quad 1 \leq i \leq n-1 \tag{6.6}$$

That is

$$(a_1 a_2 \cdots a_n | X_i(\theta) | b_1 b_2 \cdots b_n) = R_{a_i a_{i+1}}^{b_i b_{i+1}} \prod_{k=1, \neq i, i+1}^{n} \delta_{a_k}^{b_k}. \tag{6.7}$$

They fulfil the relations

$$\begin{aligned} &[X_i(\theta), X_j(\theta')] = 0 \quad \text{if } |i-j| g 2, \\ &X_i(\theta) X_{i+1}(\theta+\theta') X_i(\theta') = X_{i+1}(\theta') X_i(\theta+\theta') X_{i+1}(\theta), \\ &X_i(\theta) X_i(-\theta) = 1. \end{aligned} \tag{6.8}$$

that follow from eqs.(1.11) and (6.6)–(6.7). These operators are clearly of "light-cone" type. They are closely related to the light-cone evolution operators discussed in Section V. We find

$$U_+ = X_1 X_3 \cdots X_{N-1},$$
$$U_- = X_2 X_4 \cdots X_N,$$

where [11] (see eqs.(5.7)–(5.8))

$$U_R = V_+ V \quad = V U_-$$
$$U_L = U_+ V^+ = V^+ U_-.$$

Here $V$ $(V^\dagger)$ is the shift operator affecting one-half translation to the right (to the left) [11]. In the $\theta = \infty$ limit we get

$$b_i = \lim_{\theta \to +\infty} X_i(\theta), \qquad b_i^{-1} = \lim_{\theta \to -\infty} X_i(\theta) \tag{6.9}$$

These $b_i$ $(1 \leq i \leq n)$ precisely obey the relations of the $n$-braid group generators [41]

$$\begin{aligned} b_i b_{i+1} b_i &= b_{i+1} b_i b_{i+1}, \\ b_i b_j &= b_j b_i, \qquad |i-j| \text{g} 2. \end{aligned} \tag{6.10}$$

Let us briefly recall the notion of a braid group. Braids are formed when $n$ points in a straight line are connected by $n$ lines with other $n$ points on a parallel line as shown in Fig.15.

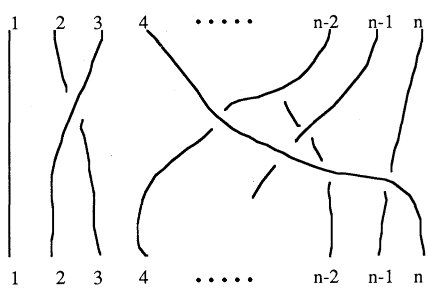



Fig. 15. A braid from $B_n$.

When the lines connecting the points have no intersections, the braid is called trivial. A general $n$-braid is obtained from the trivial one applying succesively the operations $b_i$ and/or the inverses $b_i^{-1}$ $(1 \leq i \leq n-1)$. The operations $b_i$ and $b_{i-1}$ are depicted in fig.16.

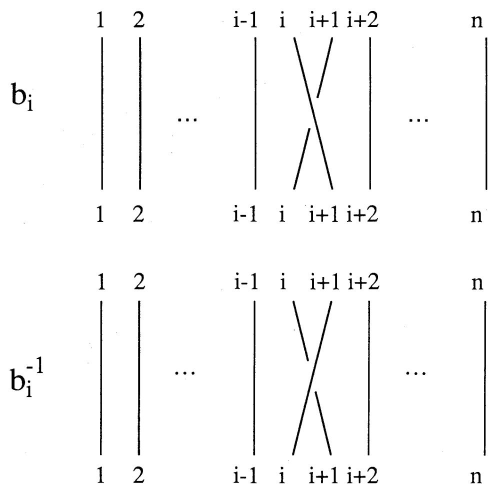



Fig. 16. The elementary operations $b_i$ and $b_i^{-1}$ from the braid group $B_n$.

Then each topologically equivalent class of braids is identified with an element in $B_n$. Eq.(6.10) shows that the $\theta = \infty$ limit of hyperbolic $R$-matrices provide a representation of $B_n$. This connection between YBZF algebras and braid groups revealed recently very fruitful to obtain knot invariants and link polynomials [42,43].

The exchange of points in the $n$-point conformal blocks forming the conformal invariant correlation functions yields a representation of a braid group (6.10) [44]. We want to remark that the $R$-matrix associated to such braid groups defines a lattice statistical model whose critical behavior is described precisely by the conformal theory yielding this braid group.

Let us now discuss the quantum groups. They are related to trigonometric/hyperbolic YBA in the $\theta = \infty$ limit (as the braid groups).

Let us start by the six-vertex case, where [eq.(3.4)]

$$T(\theta) = \begin{pmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{pmatrix} .$$

In the $\theta = \infty$ limit, these operators yield for regime I [16]

$$(6.11)\qquad \begin{aligned} A(\theta) &\underset{\theta\to\pm\infty}{=} y^N e^{\pm\gamma S_z}[1+o(y^{-2})],\\ D(\theta) &\underset{\theta\to\pm\infty}{=} y^N e^{\mp\gamma S_z}[1+o(y^{-2})],\\ B(\theta) &\underset{\theta\to-\infty}{=} y^{N-1}(\sinh\gamma)J_-[1+o(y^{-1})],\\ C(\theta) &\underset{\theta\to+\infty}{=} y^{N-1}(\sinh\gamma)J_+[1+o(y^{-1})], \end{aligned}$$

where $y = \pm\frac{1}{2}\exp[\pm(\theta+\gamma/2)]$ for $\theta\to\pm\infty$, $S_z = \frac{1}{2}\sum_a(\vec{\sigma}_a)_z$,

$$(6.12)\qquad J_\pm = \sum_{k=1}^{N}\prod_{1\le j<k} e^{\frac{\gamma}{2}(\vec{\sigma}_j)_z}(\sigma_\pm)_k\prod_{k<j\le N} e^{-\frac{\gamma}{2}(\vec{\sigma}_j)_z},$$

In addition,

$$\begin{aligned} B(\theta) &\underset{\theta\to+\infty}{=} y^{N-1}(\sinh\gamma)K_-[1+o(y^{-1})],\\ C(\theta) &\underset{\theta\to-\infty}{=} y^{N-1}(\sinh\gamma)K_+[1+o(y^{-1})]. \end{aligned}$$

$K_\pm$ follow from $J_\pm$ changing $\gamma\to-\gamma$. Inserting this limiting behavior in the YBA equations (3.5)–(3.6) yield

$$(6.13)\qquad e^{\gamma S_z}J_\pm = e^{\pm\gamma}J_\pm e^{\gamma S_z},$$

$$(6.14)\qquad [J_+, J_-] = \frac{\sinh(2\gamma S_z)}{\sinh\gamma}.$$

Eqs.(6.13) are actually equivalent to eq.(3.65) for $\theta = \pm\infty$

$$(6.15)\qquad [S_z, J_\pm] = \pm J_\pm.$$

Eqs.(6.14)–(6.15) for $J_+$, $J_-$ and $S_z$ define a deformation of the angular momentum $[SU(2)]$ algebra, since in the isotropic limit ($\gamma\to 0$) one recovers the usual $SU(2)$ commutators. This is called the $\widehat{SU(2)}_\gamma$ quantum group. Analogous results follow in regimes II and III. In regime II $\gamma$ is replaced by $i\gamma$ with $\gamma\in\mathbf{R}$. The operators $K_+$, $K_-$ and $S_z$ obey the same algebra with $J_+$, $J_-$ and $S_z$. In regime II these representations are complex conjugate of each other.

It is possible to relate the operators $J_\pm$ with the usual spin operators $S_\pm$, $S_3$ obeying

$$(6.16)\qquad [S_+, S_-] = 2S_3, \quad [S_3, S_\pm] = \pm S_\pm.$$

Inserting the ansatz

$$J_+ = S_+ f(\gamma, S_3, S), J_- = f(\gamma, S_3, S), S_-, \tag{6.17}$$

in eqs.(6.14)–(6.15) yields the recursion relation

$$\begin{aligned}
&[f(\gamma, S_3 - 1, S)]^2 (S_3 - 1)[S(S+1) - S_3(S_3 - 1)] \\
&- [f(\gamma, S_3, S)]^2 [S(S+1) - S_3(S_3 + 1)] = \frac{\sin(2\gamma S_3)}{\sin\gamma},
\end{aligned} \tag{6.18}$$

where $S(S+1) = (S_+S_- + S_-S_+)/2 + (S_3)^2$, as usual. This has as solution [54]

$$f(\mu, S_3, S) = \frac{1}{\sin\gamma}\sqrt{\frac{\sin[\gamma(S - S_3)]\sin[\gamma(S + S_3 + 1)]}{(S - S_3)(S - S_3 + 1)}}. \tag{6.19}$$

The quadratic ("Casimir") operator commuting with $J$ and $S_3$ writes here

$$\mathbf{C} = \frac{1}{2}(J_-J_+ + J_+J_-) + \frac{\cos\gamma}{\sin\gamma}\sin^2(\gamma S_3). \tag{6.20}$$

This deformation of $S^2$ has the value

$$\mathbf{C} = \frac{\sin[(S+1)\gamma]\sin(S\gamma)}{\sin^2\gamma}. \tag{6.21}$$

In summary eqs.(6.17)–(6.19) explicitly display the $\widehat{SU(2)}_\gamma$ quantum group generators in terms of the usual $SU(2)$ generators $S$, $S_3$.

Here we restricted ourselves to the "$\gamma$-deformation" of the $SU(2)$ algebra. The $\gamma$-deformations of all simple Lie algebra are known [53]. They are also connected with the $\theta = \infty$ limit of YBA[3].

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*LPTHE*
*Université Pierre et Marie Curie, Tour 16 1er étage*
*4, place Jussieu, 75252 Paris cedex 05, France*