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Formal Groups and Conformal Field Theory over Z

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Abstract.

We introduce a formal group naturally associated with algebraic curves. The formal group is isomorphic to the one obtained from the universal Witt scheme. The charge zero sector of the boson Fock space is regarded as the coordinate ring of the formal group. Using this structure, we can give tau functions. We also define new operators f_n, v_n ($n \in \mathbf{Z}, n > 0$) on the fermion Fock space.

§0. Introduction

The conformal field theory of free fermions on compact Riemann surfaces has been investigated by many mathematicians and physicists (cf. [ABMNV], [AGR], [BMS], [BS],[EO], [IMO], and [KNTY]), and the bosonization rule (cf. [DJKM] for instance) plays the central role in the theory. In our previous paper [KSU], we define the new bosonization over the ring $\mathbf{Z}$ of integers (similar treatment can be also found in [CKK]), and constructed the conformal field theory over $\mathbf{Z}$. (In [KSU] we constructed the conformal field theory over $\mathbf{Z}[\frac{1}{2}]$ because of the complicated nature of spin bundles in characteristic 2, but our theory can be formulated over $\mathbf{Z}$ similarly.) In the theory, the coordinate ring of the universal Witt scheme plays an important role. In this paper, we introduce a formal group naturally associated with algebraic curves, in particular, Riemann surfaces. We show that the formal group is isomorphic to the one obtained from the universal Witt ring, and that the coordinate ring of the formal group is regarded as the charge zero sector of the boson Fock space (cf. Section 3). Then, using a theorem of Cartier

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(cf. Theorem 1.2), we define the tau function. The tau function coincides with the one in [KNTY], and gives its natural interpretation. On the universal Witt ring, we have two kinds of operators, i.e., Frobenius operators F_n and Verschiebungs V_n $(n \in \mathbf{Z}, n > 0)$. Using them and the new bosonization introduced in [KSU], we introduce operators f_n^* and v_n^* $(n \in \mathbf{Z}, n > 0)$ on the fermion Fock space and the dual fermion Fock space. We show that the operator f_n^* is adjoint to the operator v_n^* with respect to the natural pairing (cf. Theorem 4.1). The operators $T(n) = f_n^* \circ v_n^*$ (resp. $S(n) = \sum_{m|n} f_m^* \circ v_m^*$) $(n \in \mathbf{Z}, n > 0)$ satisfy the properties similar to the Hecke operators. Hence, we get systematically divisor functions (resp. the Riemann zeta function), using operators $T(n)$ (resp. $S(n)$) (cf. Theorem 5.5 (resp. Theorem 5.8)).

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§1. Formal groups

In the former part of this section, we give a brief survey of formal groups (for details, see [Hz]). For the sake of simplicity, we will explain finite dimensional cases. But we can easily generalize them to the infinite dimensional case. In the latter part of this section, we summarize the results on the universal Witt ring. Let A be a unitary commutative ring. We denote by $G_i(X,Y)$ $(i = 1, 2, \ldots, n)$ a power series in $2n$ indeterminates $X_1, \ldots, X_n; Y_1, \ldots, Y_n$ with coefficients in A. An n-tuple of power series

$$G(X,Y) = (G_1(X,Y), \ldots, G_n(X,Y))$$

is said to be an n-dimensional *formal group law* over a ring A, if it satisfies the following two conditions :

(i) $\quad G_i(X,Y) \equiv X_i + Y_i \mod (\text{degree } 2), \quad i = 1, 2, \ldots, n$,

(ii) $\quad G_i(G(X,Y), Z) = G_i(X, G(Y,Z)), \quad i = 1, 2, \ldots, n$.

An n-dimensional formal group law $G(X,Y)$ is said to be commutative if it satisfies the following condition:

(iii) $\quad G_i(X,Y) = G_i(Y,X), \quad i = 1, 2, \ldots, n$.

In this paper, we consider only commutative formal group laws, and so we mean by a formal group law a commutative formal group law. Formal group law gives a co-addition of the ring $A[[X_1, \ldots, X_n]]$ of formal power series over A. Therefore, it gives an addition on the formal scheme $\mathrm{Spf}(A[[X_1, \ldots, X_n]])$ over A. Conversely, if a formal scheme

$\mathrm{Spf}(A[[X_1, \ldots, X_n]])$ has a structure of abelian group, then this gives a formal group law (for details of formal schemes, see [Hz]). A formal scheme with addition is called a *formal group*. By abuse of language, we also call a formal group law a formal group.

Example 1.1. We denote by $G_a(A)$ (resp. $G_m(A)$) the additive group scheme (resp. the multiplicative group scheme) over A. Set-theoretically, we have $G_a(A) = A$ (resp. $G_a(A) = A^* =$ the unit group of A) with addition

$$\begin{array}{ccc} A \times A & \to & A \\ \Psi & & \Psi \\ (a,b) & \mapsto & a+b \end{array}$$

$$\left(\text{resp.} \quad \begin{array}{ccc} A^* \times A^* & \to & A^* \\ \Psi & & \Psi \\ (a,b) & \mapsto & ab \end{array} \right).$$

Therefore, using this group law, we get a formal group law given by

$$\widehat{\mathbf{G}}_a(X_1, Y_1) = X_1 + Y_1 \quad (\text{resp. } \widehat{\mathbf{G}}_m(X_1, Y_1) = X_1 + Y_1 + X_1 Y_1)$$

which is called the additive formal group law (resp. the multiplicative formal group law).

An n-tuple of power series $\gamma(\zeta) = (\gamma_1(\zeta), \gamma_2(\zeta), \ldots, \gamma_n(\zeta))$ in an indeterminate ζ such that $\gamma_i(\zeta) \equiv 0 \mod (\text{degree } 1)$ is said to be a *curve* in the formal group $G(X,Y)$. By $C(G;A)$ we denote the set of curves in the formal group $G(X,Y)$ over the ring A. For two curves $\beta(\zeta)$, $\gamma(\zeta)$ of $C(G;A)$, we define the addition $+_G$ as follows:

$$\beta(\zeta) +_G \gamma(\zeta) = G(\beta(\zeta), \gamma(\zeta)).$$

Then, by this addition, $C(G;A)$ becomes an abelian group.

Let $F(X,Y)$ (resp. $G(X,Y)$) be an m-dimensional formal group (resp. an n-dimensional formal group) over A, and we let $\alpha(X) = (\alpha_1(X), \ldots, \alpha_n(X))$ be an n-tuple of power series in m indeterminates such that $\alpha_i(X) \equiv 0 \mod (\text{degree } 1)$, $i = 1, \ldots, n$. Then $\alpha(X)$ is said to be a *homomorphism* over A from $F(X,Y)$ to $G(X,Y)$, if

$$\alpha(F(X,Y)) = G(\alpha(X), \alpha(Y)).$$

A homomorphism $\alpha(X)$ is said to be an isomorphism, if there exists a homomorphism $\beta(X)$ from $G(X,Y)$ to $F(X,Y)$ such that $\alpha(\beta(X)) = X$

and $\beta(\alpha(X)) = X$. We denote by $\mathrm{Hom}(F,G)$ the set of all homomorphisms from $F(X,Y)$ to $G(X,Y)$. It has naturally a structure of abelian group. A homomorphism $\alpha(X)$ induces a homomorphism

$$\begin{array}{ccc} C(F;A) & \xrightarrow{\alpha_*} & C(G;A) \\ (\gamma_1(\zeta),..,\gamma_m(\zeta)) & \mapsto & (\alpha_1(\gamma_1(\zeta),..,\gamma_m(\zeta)),..,\alpha_n(\gamma_1(\zeta),..,\gamma_m(\zeta))) . \end{array}$$

Now, let $W(A)$ be the set of A-valued points of the universal Witt scheme W_A over a ring A (cf. [KSU] and [Hz]). In the following we sometimes use the notation $W(A)$ as the meaning of the Witt scheme W_A over A. For an indeterminate T, we set

$$\Lambda(A) = \{1 + a_1T + a_2T^2 + \cdots \mid a_i \in A\}.$$

An addition on $\Lambda(A)$ is defined by multiplication of power series. We have the following homomorphisms:

$$(1.1)\qquad \begin{array}{ccccc} \mathbf{G}_a^\infty(A) & \xleftarrow{w} & W(A) & \xrightarrow{\lambda} & \Lambda(A), \\ (t_1,t_2,\dots) & \leftarrow & (x_1,x_2,\dots) & \mapsto & 1 + s_1T + s_2T^2 + \cdots , \end{array}$$

where w and λ are defined by

$$(1.2)\qquad nt_n = \sum_{d|n} dx_d^{n/d},$$

$$\prod_{i=1}^{\infty}(1 - x_iT^i) = 1 + s_1T + s_2T^2 + \cdots .$$

respectively (cf. [KSU] and [Hz]). It is well known that λ is an isomorphism. In case A contains the field $\mathbf{Q}$ of rational numbers, w is also an isomorphism. These homomorphisms induce the following homomorphisms:

$$(1.3)\qquad A[t_1,t_2,\dots] \xrightarrow{w^*} A[x_1,x_2,\dots] \xleftarrow{\lambda^*} A[s_1,s_2,\dots].$$

Using addition laws of abelian groups $\mathbf{G}_a^\infty(A)$, $W(A)$ and $\Lambda(A)$, we get the following formal group and homomorphisms :

$$(1.4)\qquad \begin{array}{ccccc} \widehat{\mathbf{G}}_a^\infty(A) & \xleftarrow{W} & \widehat{W}(A) & \xrightarrow{\Lambda} & \widehat{\Lambda}(A) \\ \| & & \| & & \| \\ \mathrm{Spf}(A[[t_1,t_2,\dots]]) & & \mathrm{Spf}(A[[x_1,x_2,\dots]]) & & \mathrm{Spf}(A[[s_1,s_2,\dots]]) \end{array}$$

where W (resp. Λ) is the homomorphism (resp. the isomorphism) induced from w (resp. λ). Here, $A[[t_1, t_2, \ldots]]$ (resp. $A[[x_1, x_2, \ldots]]$, resp. $A[[s_1, x_2, \ldots]]$) is the completion of the polynomial ring $A[t_1, t_2, \ldots]$ (resp. $A[x_1, x_2, \ldots]$, resp. $A[s_1, s_2, \ldots]$) of indeterminates $t_1, t_2, \ldots$ (resp. $x_1, x_2, \ldots$, resp. $s_1, s_2, \ldots$) with $\deg t_i = i$ (resp. $\deg x_i = i$, resp. $\deg s_i = i$). The homomorphisms in (1.4) induce homomorphisms of coordinate rings defined by (1.2) and (1.3):

$$A[[t_1, t_2, \ldots]] \overset{W^*}{\to} A[[x_1, x_2, \ldots]] \overset{\Lambda^*}{\leftarrow} A[[s_1, s_2, \ldots]]. \tag{1.5}$$

We set $\mathbf{t} = (t_1, t_2, \ldots)$ and $\mathbf{x} = (x_1, x_2, \ldots)$.

Let us define Frobenius operators F_n^Λ and Verschiebung operators V_n^Λ $(n = 1, 2, 3, \ldots)$ on $\Lambda(A)$. For this purpose, we prepare additional variables $\xi_1, \xi_2, \ldots$ such that

$$1 + s_1 T + s_2 T^2 + \cdots = \prod_{i=1}^{\infty} (1 - \xi_i T) \tag{1.6}$$

Then, we define F_n^Λ by

$$F_n^\Lambda (1 + s_1 T + s_2 T^2 + \ldots) = \prod_{i=1}^{\infty} (1 - \xi_i^n T), \tag{1.7}$$

where the coefficients of the right-hand side are expressed by $s_1, s_2, \ldots$ by using (1.6). As for V_n^Λ's, they are defined by

$$V_n^\Lambda (1 + s_1 T + s_2 T^2 + \ldots) = 1 + s_1 T^n + s_2 T^{2n} + \cdots. \tag{1.8}$$

Using the isomorphism Λ, we define Frobenius operators F_n and Verschiebung operators V_n $(n = 1, 2, 3, \ldots)$ on $\widehat{W}(A)$ by

$$F_n = \Lambda^{-1} \circ F_n^\Lambda \circ \Lambda \quad (n = 1, 2, \ldots)$$

and

$$V_n = \Lambda^{-1} \circ V_n^\Lambda \circ \Lambda \quad (n = 1, 2, \ldots).$$

These operators induce endomorphisms F_n^* and V_n^* on the coordinate ring $A[[x_1, x_2, \ldots]]$. In case A contains $\mathbf{Q}$, we have operators $\widetilde{F}_n = W^{-1} \circ F_n \circ W$ and $\widetilde{V}_n = W^{-1} \circ V_n \circ W$. They induce endomorphisms $\widetilde{F}_n^*$ and $\widetilde{V}_n^*$ on $A[[t_1, t_2, \ldots]]$ which are given by

$$\widetilde{F}_n^*(t_i) = n t_{ni}, \quad \widetilde{V}_n^*(t_i) = \begin{cases} t_{i/n} & \text{if } n | i \\ 0 & \text{otherwise} \end{cases} \tag{1.9}$$

(for details, see [Hz] and [M]). We denote by $\gamma_W(\zeta)$ the curve $(\zeta, 0, 0, \dots)$ of the formal group $\widehat{W}(A)$. Finally, we quote here a theorem of P. Cartier.

Theorem 1.2 [Cartier [C]]. *Let G be a formal group over A. Then, there exists the following isomorphism of abelian groups:*

$$\begin{array}{ccc} \mathrm{Hom}(\widehat{W}(A), G) & \simeq & C(G; A) \\ \alpha & \mapsto & \alpha_*(\gamma_W(\zeta)) . \end{array}$$

§2. Construction of a formal group

Let $A[[\zeta]]$ be the ring of formal power series with coefficients in A. We denote by ζ_i $(i = 1, 2, \dots)$ and ξ_i $(i = 1, 2, \dots)$ the copies of ζ. We have the following natural isomorphism:

$$\begin{array}{ccc} A[[\zeta_1, \dots, \zeta_\alpha, \xi_1, \dots, \xi_\beta]] & \to & A[[\zeta_1, \dots, \zeta_\alpha]] \otimes_A A[[\xi_1, \dots, \xi_\beta]] . \\ \zeta_i & \mapsto & \zeta_i \otimes 1 \\ \xi_j & \mapsto & 1 \otimes \xi_j \end{array} \tag{2.1}$$

Let s_i $(i = 1, \dots, \alpha+\beta)$ (resp. s'_i $(i = 1, \dots, \alpha)$, resp. s''_i $(i = 1, \dots, \beta)$) be the elementary symmetric functions of $\zeta_1, \dots, \zeta_\alpha, \xi_1, \dots, \xi_\beta$ (resp. $\zeta_1, \dots, \zeta_\alpha$, resp. $\xi_1, \dots, \xi_\beta$) of degree i. The symmetric group $\mathfrak{S}_{\alpha+\beta}$ (resp. $\mathfrak{S}_\alpha$, resp. $\mathfrak{S}_\beta$) acts on $A[[\zeta_1, \dots, \zeta_\alpha, \xi_1, \dots, \xi_\beta]]$ (resp. $A[[\zeta_1, \dots, \zeta_\alpha]]$, resp. $A[[\xi_1, \dots, \xi_\beta]]$) as the permutations of $\zeta_1, \dots, \zeta_\alpha, \xi_1, \dots, \xi_\beta$ (resp. $\zeta_1, \dots, \zeta_\alpha$, resp. $\xi_1, \dots, \xi_\beta$). Taking the invariants of these rings of formal power series, we have the following homomorphism induced by the isomorphism (2.1):

(2.2)

$$\begin{array}{ccc} A[[s_1, s_2, \dots, s_{\alpha+\beta}]] & \overset{m^*_{\alpha,\beta}}{\to} & A[[s'_1, \dots, s'_\alpha]] \otimes_A A[[s''_1, \dots, s''_\beta]], \\ s_i & \mapsto & s'_i \otimes 1 + s'_{i-1} \otimes s''_1 + \dots + s'_1 \otimes s''_{i-1} + 1 \otimes s''_i \\ & & (i = 1, 2, \dots, \alpha+\beta) \end{array}$$

where $s'_i = 0$ (resp. $s_i'' = 0$) if $i > \alpha$ (resp. $i > \beta$). Now, consider the projective system $\{A[[s_1, s_2, \dots, s_n]], f^*_{n-1,n}\}$ defined by

$$
(2.3)\qquad \begin{array}{rcl} f^*_{n-1,n} : A[[s_1, \dots, s_n]] & \to & A[[s_1, \dots, s_{n-1}]]\,. \\ s_i & \mapsto & s_i \quad \text{for } 1 \le i \le n-1, \\ s_n & \mapsto & 0 \end{array}
$$

Then, we have

$$A[[s_1, s_2, \dots]] = \varprojlim A[[s_1, \dots, s_n]]$$

which is isomorphic to the completion of $A[s_1, s_2, \dots]$ with $\deg s_i = i$. The homomorphism (2.2) induces the co-addition

$$
(2.4)\qquad \begin{array}{rcl} m^* : A[[s_1, s_2, \dots]] & \to & A[[s_1, s_2, \dots]] \otimes_A A[[s_1, s_2, \dots]]\,. \\ s_i & \mapsto & s_i \otimes 1 + s_{i-1} \otimes s_1 + \dots + s_1 \otimes s_{i-1} + 1 \otimes s_i \end{array}
$$

This co-addition gives a formal group $\widehat{U}$ of infinite dimension which coincides with $\widehat{\Lambda}(A)$. We have the isomorphism

$$
(2.5)\qquad \begin{array}{rcccl} \eta & : & \widehat{\Lambda}(A) & \to & \widehat{\Lambda}(A) \\ & & 1 + a_1 T + a_2 T^2 + \cdots & \mapsto & 1 + a_1(-T) + a_2(-T)^2 + \cdots . \end{array}
$$

Using (1.4) and (2.5), we have the following isomorphism

$$
(2.6)\qquad \theta = \eta \circ \Lambda : \widehat{W}(A) \xrightarrow{\Lambda} \widehat{\Lambda}(A) \xrightarrow{\eta} \widehat{\Lambda}(A) = \widehat{U}.
$$

By the construction of $\widehat{U}$ and (1.3), θ induces the homomorphism θ_* from $C(\widehat{W}(A); A)$ to $C(\widehat{U}; A)$ such that

$$
(2.7)\qquad \theta_*((\zeta, 0, 0, \dots)) = (\zeta, 0, 0, \dots).
$$

§3. Jacobian varieties and τ-functions

Let $f : C \to \mathrm{Spec}(A)$ be a curve of genus g over A (cf. [KSU]). We assume that $f : C \to \mathrm{Spec}(A)$ has a section $\sigma : \mathrm{Spec}(A) \to C$. We denote $\sigma(\mathrm{Spec}(A))$ by Q, and denote by I_Q the ideal sheaf of Q. There is a canonical $\mathcal{O}_A$-algebra isomorphism

$$u_0 : \mathcal{O}_C / I_Q \simeq \mathcal{O}_A.$$

Assume that the conormal bundle $N_Q^* = I_Q/I_Q^2$ of Q in C is a free $\mathcal{O}_A$-module. Then, as in [KSU, Lemma 4.1], we have

$$\mathcal{O}_C/I_Q^{n+1} \simeq \mathcal{O}_A[\zeta]/(\zeta^{n+1}).$$

Therefore, taking the completion of $\mathcal{O}_C$ with respect to the ideal sheaf I_Q, we have

$$\varprojlim \mathcal{O}_C/I_Q^{n+1} = \mathcal{O}_A[[\zeta]]. \tag{3.1}$$

We denote by $\widehat{\mathcal{O}}_Q$ the left-hand side of (3.1). The global sections on $\mathrm{Spec}(A)$ of $\mathcal{O}_A[[\zeta]]$ are given by $A[[\zeta]]$. We consider the triple $\{f : C \to \mathrm{Spec}(A), Q, u : \widehat{\mathcal{O}}_Q \simeq \mathcal{O}_A[[\zeta]]\}$. We set

$$C^n = \underbrace{C \times_{\mathrm{Spec}(A)} C \times \cdots \times_{\mathrm{Spec}(A)} C}_{n}.$$

The symmetric group $\mathfrak{S}_n$ of degree n acts on C^n over $\mathrm{Spec}(A)$ as permutations. We have a natural morphism defined by

$$\begin{array}{ccc} C^\alpha \times_{\mathrm{Spec}(A)} C^\beta & \to & C^{\alpha+\beta} \\ ((P_1, \ldots, P_\alpha), (P_1', \ldots, P'_\beta)) & \mapsto & (P_1, \ldots, P_\alpha, P'_1, \ldots, P'_\beta). \end{array} \tag{3.2}$$

This induces the following morphism:

$$m_{\alpha,\beta} : C^\alpha/\mathfrak{S}_\alpha \times_{\mathrm{Spec}(A)} C^\beta/\mathfrak{S}_\beta \to C^{\alpha+\beta}/\mathfrak{S}_{\alpha+\beta}. \tag{3.3}$$

We have a morphism

$$\begin{array}{ccc} C^{\alpha-1} & \to & C^\alpha \\ (P_1, \ldots, P_{\alpha-1}) & \mapsto & (P_1, \ldots, P_{\alpha-1}, Q). \end{array}$$

This induces the morphism

$$f_{\alpha,\alpha-1} : C^{\alpha-1}/\mathfrak{S}_{\alpha-1} \to C^\alpha/\mathfrak{S}_\alpha. \tag{3.4}$$

We consider the completion along Q in (3.3) and (3.4). Then, corresponding to (3.3), we have (2.2), and corresponding to (3.4), we have (2.3). Therefore, as in Section 2, taking the projective limit, we have the formal group $\widehat{U}$ with co-addition (2.4). As we explained in Section 2, $\widehat{U}$ coincides with $\widehat{\Lambda}(A)$.

Let $J(C)$ be the Jacobian variety of C over Spec(A). We denote by m_J the addition of $J(C)$, and by $\widehat{J}(C)$ the formal group over Spec(A) associated with $J(C)$. We have a morphism over Spec(A) :

$$\begin{array}{rcl} \varphi_\alpha : C^\alpha/\mathfrak{S}_\alpha & \to & J(C) \\ (P_1, \ldots, P_\alpha) & \mapsto & P_1 + \cdots + P_\alpha - \alpha Q\,, \end{array}$$

and a commutative diagram

$$\begin{array}{ccc} C^\alpha/\mathfrak{S}_\alpha \times C^\beta/\mathfrak{S}_\beta & \stackrel{m_{\alpha\beta}}{\longrightarrow} & C^{(\alpha+\beta)}/\mathfrak{S}_{\alpha+\beta} \\ \downarrow \varphi_\alpha \times \varphi_\beta & & \downarrow \varphi_{\alpha+\beta} \\ J(C) \times J(C) & \stackrel{m_J}{\longrightarrow} & J(C)\,. \end{array} \tag{3.5}$$

By the commutative diagram

(3.6)

$$\begin{array}{ccccccc} C & \stackrel{f_{1,2}}{\rightarrow} & C^2/\mathfrak{S}_2 & \stackrel{f_{2,3}}{\rightarrow} & \cdots & \stackrel{f_{g-1,g}}{\rightarrow} & C^g/\mathfrak{S}_g \\ \downarrow \varphi_1 & & \downarrow \varphi_2 & & & & \downarrow \varphi_g \\ J(C) & = & J(C) & = & \cdots & = & J(C) \end{array}$$

$$\begin{array}{cccccccc} \stackrel{f_{g,g+1}}{\rightarrow} & C^{g+1}/\mathfrak{S}_{g+1} & \stackrel{f_{g+1,g+2}}{\rightarrow} & \cdots & \stackrel{f_{n-1,n}}{\rightarrow} & C^n/\mathfrak{S}_n & \stackrel{f_{n,n+1}}{\rightarrow} & \cdots \\ & \downarrow \varphi_{g+1} & & & & \downarrow \varphi_n & & \\ = & J(C) & = & \cdots & = & J(C) & = & \cdots \end{array}$$

and by (3.5), we have a homomorphism

$$\varphi : \widehat{U} \to \widehat{J}(C).$$

Taking the completion along Q, we see that φ_1 (resp. $\cdots \circ f_{2,3} \circ f_{1,2}$) induces a morphism γ_J (resp. γ_U) from Spf$(A[[\,\zeta\,]])$ to $\widehat{J}(C)$ (resp. to $\widehat{U}$)

such that the following diagram is commutative:

$$\begin{tikzcd} \mathrm{Spf}(A[[\zeta]]) \arrow[r, "\gamma_U"] \arrow[dr, "\gamma_J"'] & \widehat{U} \arrow[d, "\varphi"] & \widehat{W}(A). \arrow[l, "\theta"'] \\ & \widehat{J}(C) & \end{tikzcd}$$

Here, we note that γ_U gives the curve $(\zeta, 0, 0, \dots)$ in $\widehat{U}$. Therefore, by (2.7), $\theta^{-1} \circ \gamma_U$ gives the curve $(\zeta, 0, 0, \dots)$ in $\widehat{W}(A)$. Hence, by Theorem 1.2, we have the following characterization of φ.

Theorem 3.1. *The homomorphism φ constructed above is characterized as the homomorphism which transforms the curve $(\zeta, 0, 0, \dots)$ in $\widehat{U}$ into the curve in $\widehat{J}(C)$ given by γ_J.*

Now, we assume $A = \mathbf{C}$. Then, ζ is a local parameter of C at the point Q. We fix a symplectic basis $\{\alpha_1, \alpha_2, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$ of the first homology group $H_1(C, \mathbf{Z})$. By definition, we have $(\alpha_i, \alpha_j) = 0$, $(\beta_i, \beta_j) = 0$ and $(\alpha_i, \beta_j) = \delta_{ij}$, where δ_{ij} is Kronecker's delta. We take a basis $\{\omega_1, \dots, \omega_g\}$ of the space $H^0(C, \Omega^1_C)$ of holomorphic one-forms on the curve C such that

$$\int_{\alpha_i} \omega_j = \delta_{ij} \quad \text{and} \quad \int_{\beta_i} \omega_j = \tau_{ij}.$$

Then, the $g \times 2g$ matrix (δ_{ij}, τ_{ij}) gives a lattice L of $\mathbf{C}^g$, and $J(C)$ is given by $\mathbf{C}^g/L$. We have the universal covering $\pi : \mathbf{C}^g \to J(C)$ and a commutative diagram

$$\begin{tikzcd} C \arrow[rr, "\psi"] \arrow[dr] & & \mathbf{C}^g \arrow[dl, "\pi"] \\ & \mathbf{C}^g/L. & \end{tikzcd}$$

The morphism ψ is given by

$$\psi(\zeta) = \left(\int_Q^{\zeta} \omega_i\right).$$

We consider the expansion of ω_i (cf. [KNTY]):

$$\omega_i = -d\left(\sum_{n=1}^{\infty} I_n^i \frac{\zeta^n}{n}\right).$$

Then, taking the completions, we have

$$\begin{tikzcd} \mathrm{Spf}(\mathbf{C}[[\zeta]]) \arrow[rr, "\widehat{\psi}"] \arrow[dr] & & \widehat{\mathbf{G}}_a^g, \arrow[dl, "\widehat{\pi}"] \\ & \widehat{J}(C) & \end{tikzcd} \tag{3.7}$$

where $\widehat{\psi}$ corresponds to the curve of $\widehat{\mathbf{G}}_a^g$ given by

$$\gamma = \Big(-\sum_{n=1}^{\infty} I_n^1 \frac{\zeta^n}{n}, \ldots, -\sum_{n=1}^{\infty} I_n^g \frac{\zeta^n}{n}\Big). \tag{3.8}$$

By Theorem 1.2 and (1.4), we have the morphism

$$\tilde{\varphi} : \widehat{U} \to \widehat{\mathbf{G}}_a^g \tag{3.9}$$

such that $\tilde{\varphi}_*((\zeta, 0, \ldots)) = \gamma$, where γ is the curve given by $\widehat{\psi}$ in (3.7). On the other hand, by [KNTY], we have the homomorphism

(3.10)
$$I : \widehat{\mathbf{G}}_a^\infty = \mathrm{Spf}(\mathbf{C}[[t_1, t_2, \ldots]]) \to \widehat{\mathbf{G}}_a^g = \mathrm{Spf}(\mathbf{C}[[z_1, \ldots, z_g]]),$$
$$I^*(z_i) = \sum_{n=1}^{\infty} I_n^i t_n.$$

Under the notations in (1.4), (2.6), (3.9) and (3.10) we have the following theorem.

Theorem 3.2. *The following diagram is commutative:*

$$\begin{tikzcd} & \widehat{W}(\mathbf{C}) \arrow[dl, "W"'] \arrow[r, "\theta=\eta\circ\Lambda"] & \widehat{U} \arrow[d, "\tilde{\varphi}"] \\ \widehat{\mathbf{G}}_a^\infty \arrow[dr, "I"'] & & \\ & \widehat{\mathbf{G}}_a^g & \widehat{\mathbf{G}}_a^g, \arrow[l, "\iota"'] \end{tikzcd}$$

where ι is the inversion of $\widehat{\mathbf{G}}_a^g$.

Proof. By Theorem 1.2, it suffices to prove

$$\iota_* \circ \tilde{\varphi}_* \circ \theta_*((\zeta, 0, 0, \ldots)) = I_* \circ W_*((\zeta, 0, 0, \ldots)).$$

We can check this by direct calculation. q.e.d.

Let $\{\omega_Q^{(n)}\}_{n=2,3,4,\ldots}$ be the set of abelian differentials of the second kind on C such that

$$\int_{\alpha_i} \omega_Q^{(n)} = 0, \quad \int_{\beta_i} \omega_Q^{(n)} = 2\pi\sqrt{-1}I_n^i,$$

$$\omega_Q^{(n)} = d(\zeta^{-n} - \sum_{m>0} q_{mn}(\zeta^m/m))$$

(cf. [KNTY]). It is a basis of the vector space of abelian differentials of the second kind with pole only at the point Q. We denote by $\Theta(z_1, \ldots, z_g)$ the Riemann theta function on $\mathbf{C}^g/L$. We can regard $\Theta(z_1, \ldots, z_g)$ as an element of $\mathbf{C}[[z_1, \ldots, z_g]]$. We define the tau function as follows.

Definition 3.3.

$$\tau(\mathbf{x}, C) = W^*\{\exp(\frac{1}{2} \sum_{n>0,m>0} q_{m,n}t_m t_n)\} \cdot (\iota \circ \tilde{\varphi} \circ \theta)^* \Theta(z_1, \ldots, z_g).$$

Theorem 3.4. *Let $\tau(\mathbf{t}, C)$ be the tau function defined in* [KNTY]. *Then, we have*

$$\tau(\mathbf{x}, C) = W^* \tau(\mathbf{t}, C).$$

Proof. This theorem follows from Theorem 3.2. q.e.d.

§4. Operators F_n and V_n

Let $\mathcal{M}_0$ be the set of Maya diagrams of charge zero, and let

$$\mathcal{F}_0(A) = \prod_{M \in \mathcal{M}_0} A|M\rangle \quad (\text{resp. } \overline{\mathcal{F}}_0(A) = \bigoplus_{M \in \mathcal{M}_0} A\langle M|\,)$$

be the fermion Fock space (resp. the dual fermion Fock space) of charge zero over a commutative ring A. We have the canonical pairing

$$(4.1) \qquad \begin{array}{ccc} \overline{\mathcal{F}}_0(A) \times \mathcal{F}_0(A) & \rightarrow & A \\ (\langle \Psi'|, |\Psi\rangle) & \mapsto & \langle \Psi' | \Psi \rangle \end{array}$$

(cf. [KNTY] and [KSU]). Let

$$\mathcal{H}_{T,0}(A) = A[[t_1, t_2, \ldots]] \quad (\text{resp. } \overline{\mathcal{H}}_{T,0}(A) = A[t_1, t_2, \ldots])$$

be the charge zero sector of boson Fock space (resp. the dual boson Fock space). We have the pairing

$$\overline{\mathcal{H}}_{T,0}(A) \times \mathcal{H}_{T,0}(A) \to A \tag{4.2}$$

defined by

$$(g(\mathbf{t}), h(\mathbf{t})) = g(\partial_t)h(\mathbf{t})|_{\mathbf{t}=0}\,, \tag{4.3}$$

where

$$\partial_t = \left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \dots, \frac{1}{n}\frac{\partial}{\partial t_n}, \dots\right).$$

We denote by J_m $(m \in \mathbf{Z})$ the current operators. If $A = \mathbf{Q}$, we have a bosonization

$$\begin{gathered} B : \mathcal{F}_0(\mathbf{Q}) \to \mathcal{H}_{T,0}(\mathbf{Q}) \\ (\text{resp. } \ \overline{B} : \overline{\mathcal{F}}_0(\mathbf{Q}) \to \overline{\mathcal{H}}_{T,0}(\mathbf{Q})) \end{gathered} \tag{4.4}$$

defined by

$$B|\Psi\rangle = \sum_{n\in\mathbf{Z}} \langle n| \exp(\sum_{m=1}^{\infty} J_m t_m)|\Psi\rangle \qquad \text{for } |\Psi\rangle \in \mathcal{F}_0(\mathbf{Q})$$

$$(\text{resp. } \ \overline{B}\langle\Psi'| = \sum_{n\in\mathbf{Z}} \langle\Psi'| \exp(\sum_{m=1}^{\infty} J_m t_m)|n\rangle \quad \text{for } \langle\Psi'| \in \overline{\mathcal{F}}_0(\mathbf{Q})).$$

By [DJKM], B (resp. $\overline{B}$) is an isomorphism as vector spaces. In [KSU], we introduced a new boson Fock space of charge zero

$$\mathcal{H}_0(A) = A[[x_1, x_2, \dots]]$$

and a new bosonization

$$\widetilde{B} : \mathcal{F}_0(A) \to \mathcal{H}_0(A). \tag{4.5}$$

We introduce a new dual boson Fock space of charge zero $\overline{\mathcal{H}}_0(A) = A[x_1, x_2, \dots]$. By the similar way to [KSU], we have a new bosonization

$$\widetilde{B}' : \overline{\mathcal{F}}_0(A) \to \overline{\mathcal{H}}_0(A). \tag{4.6}$$

$\widetilde{B}$ (resp. $\widetilde{B}'$) is an isomorphism as A-modules. If $A = \mathbf{Q}$, we have

$$\widetilde{B} = W^* \circ B \tag{4.7}$$

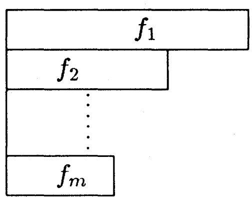
$$\text{(resp.}\quad \widetilde{B}' = W^* \circ \overline{B}), \tag{4.8}$$

where W^* is defined by (1.3). The Frobenius operators F_n^* and the Verschiebung operators V_n^* act on the coordinate ring of $\widehat{W}(A)$ (resp. on the coordinate ring of the Witt scheme W(A)) as in Section 1. Using the isomorphism (4.5) (resp. (4.6)), we set

$$\begin{aligned} f_n^* &= \widetilde{B}^{-1} \circ F_n^* \circ \widetilde{B} \quad \text{and} \quad v_n^* = \widetilde{B}^{-1} \circ V_n^* \circ \widetilde{B} \\ \text{(resp.}\quad f_n^* &= \widetilde{B}'^{-1} \circ F_n^* \circ \widetilde{B}' \quad \text{and} \quad v_n^* = \widetilde{B}'^{-1} \circ V_n^* \circ \widetilde{B}'). \end{aligned} \tag{4.9}$$

We denote by $p_i(t)$ $(i = 0, 1, 2, \dots)$ the Schur polynomials. For a Young diagram Y

(4.10)



the Schur function corresponding to Y is defined by

$$\chi_Y(t) = \det\left(p_{f_i - i + j}(t)\right)_{1 \le i,j \le m} . \tag{4.11}$$

The Young diagram Y in (4.10) is called the Young diagram of signature $(f_1, f_2, \dots, f_m)$.

Theorem 4.1. *Let* $|\Psi\rangle$ *(resp.* $\langle\Psi'|$*) be any element of* $\mathcal{F}_0(A)$ *(resp.* $\overline{\mathcal{F}}_0(A)$*). Then, with respect to the pairing* (4.1), *we have*

$$\begin{aligned} (\langle\Psi'|f_n^*)|\Psi\rangle &= \langle\Psi'|(v_n^*|\Psi\rangle) \\ (\langle\Psi'|v_n^*)|\Psi\rangle &= \langle\Psi'|(f_n^*|\Psi\rangle) \end{aligned} \tag{4.12}$$

for $\langle\Psi'| \in \overline{\mathcal{F}}_0(A)$ *and* $|\Psi\rangle \in \mathcal{F}_0(A)$.

Proof. First, we consider the case $A = \mathbf{Q}$. Then, we have

$$(\overline{B}\langle\Psi'|\,,\, B|\Psi\rangle) = \langle\Psi'|\Psi\rangle \tag{4.13}$$

for $\langle\Psi'| \in \overline{\mathcal{F}}_0(\mathbf{Q})$ and $|\Psi\rangle \in \mathcal{F}_0(\mathbf{Q})$ (cf. [SN]). Let $i_1, \dots, i_k, j_1, \dots, j_\ell$ be positive integers such that $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_\ell$.

By (1.9), (4.13) and (4.3), we have

$$\begin{aligned}
&(F_n^* t_{i_1}^{\nu_1} t_{i_2}^{\nu_2} \dots t_{i_k}^{\nu_k}, t_{j_1}^{\mu_1} t_{j_2}^{\mu_2} \dots t_{j_\ell}^{\mu_\ell}) \\
&= ((nt_{ni_1})^{\nu_1}(nt_{i_2})^{\nu_2} \dots (t_{ni_k})^{\nu_k}, t_{j_1}^{\mu_1} t_{j_2}^{\mu_2} \dots t_{j_\ell}^{\mu_\ell}) \\
&= n^{\nu_1+\nu_2+\dots+\nu_k}(\frac{1}{ni_1}\frac{\partial}{\partial t_{ni_1}})^{\nu_1}(\frac{1}{ni_2}\frac{\partial}{\partial t_{ni_2}})^{\nu_2}\dots \\
&\qquad \dots(\frac{1}{ni_k}\frac{\partial}{\partial t_{ni_k}})^{\nu_k} t_{j_1}^{\mu_1} t_{j_2}^{\mu_2} \dots t_{j_\ell}^{\mu_\ell}|_{t=0} \\
&= \begin{cases} (\frac{1}{i_1})^{\nu_1}(\frac{1}{i_2})^{\nu_2}\dots(\frac{1}{i_\ell})^{\nu_\ell}\delta_{ni_1,j_1}\cdots\delta_{ni_\ell,j_\ell}\cdot\delta_{\nu_1,\mu_1} \\ \qquad \cdots\delta_{\nu_\ell,\mu_\ell}\cdot\nu_1!\cdots\nu_\ell! & \text{if } k=\ell \\ 0 & \text{if } k\neq\ell \end{cases} \\
&= \begin{cases} (\frac{1}{i_1}\frac{\partial}{\partial t_{i_1}})^{\nu_1}\cdots(\frac{1}{i_\ell}\frac{\partial}{\partial t_{i_\ell}})^{\nu_\ell}\{V_n(t_{j_1}^{\mu_1}\dots t_{j_\ell}^{\mu_\ell})\}|_{t=0} & \text{if } k=\ell \\ 0 & \text{if } k\neq\ell \end{cases} \\
&= (t_{i_1}^{\nu_1} t_{i_2}^{\nu_2} \dots t_{i_k}^{\nu_k}, V_n^* t_{j_1}^{\mu_1} t_{j_2}^{\mu_2} \dots t_{j_\ell}^{\mu_\ell}).
\end{aligned}$$

Therefore, for the Schur functions $\chi_Y(t)$, $\chi_{Y'}(t)$, we have

$$(F_n^*\chi_Y(t), \chi_{Y'}(t)) = (\chi_Y(t), V_n^*\chi_{Y'}(t)).$$

Therefore, by [KSU, Definition 2.1 and Lemma 3.3], we have

$$(\langle\Psi'|f_n^*)|\Psi\rangle = \langle\Psi'|(v_n^*|\Psi\rangle)$$

for $\langle\Psi'| \in \overline{\mathcal{F}}_0(\mathbf{Z})$ and $|\Psi\rangle \in \mathcal{F}_0(\mathbf{Z})$. Hence, over any ring A, we get the equality in the former part of (4.12). Since $\langle M|N\rangle = \langle N|M\rangle$ for $M, N \in \mathcal{M}_0$, the latter part of (4.12) follows from the former part.

q.e.d.

For a positive integer n and the Young diagram Y in (4.10), we denote by a (resp. b) the integral part of m/n (resp. $m - na$), and we set

$$S_i = \{f_j \mid f_j - j + i \equiv 0 \mod n\} \quad (1 \le i \le n).$$

We denote by α_i the number of elements of S_i. We consider the following condition for α_i's:

$$\text{Condition } (\alpha) \quad : \alpha_1 = \alpha_2 = \dots = \alpha_b = a+1, \quad \alpha_{b+1} = \alpha_{b+2} = \dots = \alpha_n = a.$$

Theorem 4.2. 1) *If* $\{\alpha_i\}_{i=1,\dots,n}$ *does not satisfy Condition* (α), *then the following holds:*

$$V_n^*(\chi_Y(t)) = 0.$$

2) *Assume that* $\{\alpha_i\}_{i=1,\dots,n}$ *satisfies Condition* (α). *For* $S_i = \{f_{j_1}, \dots, f_{j_{\alpha_i}}\} (j_1 < j_2 < \cdots < j_{\alpha_i})$ *let* Y_i *be the Young diagram of signature* $((f_{j_1} - j_1 + i)/n, \{(f_{j_2} - j_2 + i)/n\} + 1, \dots, \{(f_{j_{\alpha_i}} - j_{\alpha_i} + i)/n\} + \alpha_i - 1)$. *Then, the following equality holds.*

$$V_n^*(\chi_Y(t)) = \pm \chi_{Y_1}(t) \cdot \chi_{Y_2}(t) \cdot \dots \cdot \chi_{Y_n}(t).$$

Proof. By (1.9) and the definition of the Schur polynomials, we have

$$V_n^*(p_j(t)) = \begin{cases} p_{j/n}(t) & \text{if } n \mid j, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, by (4.11), we can calculate the action of V_n^*. We omit the details. q.e.d.

§5. Zeta functions

In this section, we assume $A = \mathbf{Q}$. By (1.9), we have

$$\widetilde{V}_n^* \circ \widetilde{F}_n^* : t_i \mapsto n t_i.$$

Therefore, by (4.4), $v_n^* \circ f_n^*$ is the endomorphism of $\mathcal{F}_0(\mathbf{Q})$ (resp. $\overline{\mathcal{F}}_0(\mathbf{Q})$) corresponding to the multiplication by n on $\mathbf{G}_a^\infty$. Now, we set

$$T(n) = f_n^* \circ v_n^* \quad (n = 1, 2, \dots). \tag{5.1}$$

Proposition 5.1. *The operators* $T(n)$*'s* $(n = 1, 2, \dots)$ *satisfy the following properties:*

(i) $(\langle\Psi'|T(n))|\Psi\rangle = \langle\Psi'|(T(n)|\Psi\rangle)$ *for* $\langle\Psi'| \in \overline{\mathcal{F}}_0(\mathbf{Q}), |\Psi\rangle \in \mathcal{F}_0(\mathbf{Q})$,
(ii) $T(m)T(n) = T(n)T(m)$,
(iii) *If* m *is prime to* n, *then* $T(mn) = T(m)T(n)$,
(iv) *If the greatest common divisor of* m *and* n *is equal to* d, *then*

$$T(m)T(n) = dT(\frac{mn}{d}).$$

Proof. These properties follow from (1.9), (4.12) and isomorphisms (4.4). q.e.d.

Definition 5.2. *We formally set*

$$z(s) = \sum_{n \geq 1} T(n) n^{-s}. \tag{5.2}$$

Using (4.4), we set

$$e(i_1, \ldots, i_\ell; \nu_1, \ldots, \nu_\ell) = B^{-1}(t_{i_1}^{\nu_1} t_{i_2}^{\nu_2} \ldots t_{i_\ell}^{\nu_\ell}), \tag{5.3}$$

where $i_1, \ldots, i_\ell, \nu_1, \ldots, \nu_\ell$ are positive integers. We denote by μ the greatest common divisor of $i_1, \ldots, i_\ell$, and we set $\nu = \nu_1 + \cdots + \nu_\ell$.

Definition 5.3. $Z_\mu^\nu(s) = \sum_{n|\mu} n^{\nu - s}.$

Remark 5.4. The functions $Z_\mu^\nu(s)$ are called divisor functions (cf. [A]). $Z_\mu^\nu(s)$ satisfies the following properties.

(i) Let $\mu = p_1^{n_1} \ldots p_r^{n_r}$ be the factorization into prime numbers. Then, we have an Euler product expansion

$$Z_\mu^\nu(s) = (1 + p_1^{(\nu - s)} + \cdots + p_1^{n_1(\nu - s)}) \cdots (1 + p_r^{\nu - s} + \cdots + p_r^{n_r(\nu - s)}).$$

(ii) We have a functional equation

$$Z_\mu^\nu(-s) = \mu^{-\nu + s} Z_\mu^\nu(s).$$

(iii) If μ_1 is prime to μ_2, we have the multiplicativity

$$Z_{\mu_1}^\nu(s) Z_{\mu_2}^\nu(s) = Z_{\mu_1 \mu_2}^\nu(s).$$

Theorem 5.5. *Under the above notations, we have*

$$z(s) e(i_1, \ldots, i_\ell; \nu_1, \ldots, \nu_\ell) = Z_\mu^\nu(s) e(i_1, \ldots, i_\ell; \nu_1, \ldots, \nu_\ell).$$

Proof. Since by (1.9) we have

(5.4)

$$\begin{aligned} &f_n^* \circ v_n^* e(i_1, \ldots, i_\ell; \nu_1, \ldots, \nu_\ell) \\ &= \begin{cases} n^\nu e(i_1, \ldots, i_\ell; \nu_1, \ldots, \nu_\ell) & \text{if } n \mid \mu, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence, the theorem follows from (4.4) and the definition of $z(s)$.

q.e.d.

Now, we set

$$S(n) = \sum_{m|n} f_m^* \circ v_m^* \quad (n = 1, 2, \dots). \tag{5.5}$$

Proposition 5.6. *$S(n)$'s ($n = 1, 2, \dots$) satisfy the following properties:*

(i) $(\langle\Psi'|S(n))|\Psi\rangle = \langle\Psi'|(S(n)|\Psi\rangle)$ *for* $\langle\Psi'| \in \overline{\mathcal{F}}_0(\mathbf{Q}), |\Psi\rangle \in \mathcal{F}_0(\mathbf{Q})$,
(ii) $S(m)S(n) = S(n)S(m)$,
(iii) $S(m)S(n) = S(mn)$ *if m is prime to n.*

Proof. This proposition follows from Proposition 5.1. q.e.d.

Definition 5.7. We formally set

$$Z(s) = \sum_{n=1}^{\infty} S(n)n^{-s}.$$

We denote by $\zeta(s)$ the Riemann zeta function.

Theorem 5.8. *Under the above notations, we have*

$$Z(s)e(i_1, \dots, i_\ell; \nu_1, \dots, \nu_\ell) = \zeta(s)Z_\mu^\nu(s)e(i_1, \dots, i_\ell; \nu_1, \dots, \nu_\ell).$$

Proof. By (5.4), we have

$$S(n)e(i_1, \dots, i_\ell; \nu_1, \dots, \nu_\ell) = \left(\sum_{m|(n,\mu)} m^\nu\right) e(i_1, \dots, i_\ell; \nu_1, \dots, \nu_\ell),$$

where (n, μ) is the greatest common divisor of n and μ. Therefore, we have

$$\begin{aligned} &Z(s)e(i_1, \dots, i_\ell; \nu_1, \dots, \nu_\ell) \\ &\qquad = \left\{\sum_{n=1}^{\infty} \left(\sum_{m|(n,\mu)} m^\nu\right) n^{-s}\right\} e(i_1, \dots, i_\ell; \nu_1, \dots, \nu_\ell). \end{aligned}$$

By direct calculation, we have

$$\zeta(s)Z_\mu^\nu(s) = \sum_{n=1}^{\infty} \left(\sum_{m|(n,\mu)} m^\nu\right) n^{-s}.$$

q.e.d.

Remark 5.9. By the property in Theorem 5.1 (iii) (resp. Theorem 5.6 (iii)), we see that the eigen-values of the operator $z(s)$ (resp. $Z(s)$) have Euler product expansions as in the case of the zeta functions associated with Hecke operators (cf. [S]).

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