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Solving Models in Statistical Mechanics

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One of the main aims of statistical mechanics is to calculate the partition function Z. Here I shall discuss how this can be done for a certain class of two-dimensional lattice models (and one three-dimensional model). They are by definition "solvable". Most of them can also be related to one-dimensional integrable Hamiltonians, so in this sense they are also said to be "integrable".

Such models are made by placing spins σ_i on the N sites (or edges) of a planar lattice $\mathcal{L}$ (e.g. the square lattice). They have values $+1$ or -1; or $1, \ldots, q$; or indeed any set of values that is appropriate. Adjacent spins (i.e. those sharing an edge, or a face, or a vertex) interact. The partition function is

$$Z = \sum_{\sigma} \prod W(\sigma_i, \sigma_j, \ldots), \tag{1}$$

where the inner product is over all edges, faces or vertices of $\mathcal{L}$; $\sigma_i, \sigma_j, \ldots$ are the spins on each such edge, face or vertex; the sum is over all values of all the spins. If each spin takes q values, there are q^N terms in the summation. We want N to be large: at least 100, and of course q at least 2. Hence there are vastly many terms in the sum.

There are now a number of such solvable models. I list the ones I shall consider here in Table 1. There are of course many others, for instance the Izergin-Korepin [1], nested Bethe ansatz ([2] and refs. therein), and various colouring problems [3–6].

There are many relations between these models: for instance the Ising model [7] is a special case of both the 8-vertex [8] and chiral Potts [9–12] models. The 8-vertex model is equivalent to the 8-vertex solid-on-solid (SOS) model [13], in the sense that they both have the same partition function, even though they are formulated differently and have different order parameters. The hard hexagon model [14] is a special case of the 8-vertex SOS model, and further generalizations of these models

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have been discovered [15–18], culminating in the recent work here in Kyoto on the $A^{(1)}$ face models [19]. I indicate at least some of these relations in the "family tree" of Fig.1.

Model	*Solved by*	*When*	*Refs.*
Ising	Onsager	1944	7,22
Dimer	Kasteleyn, Fisher, Temperley	1961	28,56,57
Six-vertex	Lieb, Sutherland	1967	40,41
Eight-vertex	Baxter	1971	8
Three-spin	Baxter & Wu	1973	58
Self-dual Potts	Baxter	1973	52
Hard hexagon	Baxter	1980	14
Zamolodchikov 3D	Zamolodchikov, Baxter	1981/3	59,33
Fateev-Zamolodchikov	Fateev, Zamolodchikov	1982	60
8 Vertex SOS	Andrews, Baxter, Forrester	1973/82	46,13
Generalized hard hexagon	Kuniba, Akutsu, Wadati, Baxter, Andrews	1986	16-18
$A^{(1)}$ SOS models	Date, Jimbo, Kuniba, Miwa, Okado	1987	19
Chiral Potts	Albertini, Au-Yang, Baxter, McCoy, Perk , Tang	1988	9-12

Table 1

The simplest type of model is when the spins live on sites of $\mathcal{L}$, and interact only along edges. Then

$$Z = \sum_{\sigma} \prod_{\langle ij \rangle} W(\sigma_i, \sigma_j), \tag{2}$$

where now the product is over all edges $\langle ij \rangle$ of $\mathcal{L}$, $W(a, b)$ is the Boltzmann weight function associated with the edge $\langle ij \rangle$.

The Ising, Potts, chiral Potts and Fateev-Zamolodchikov models are all of this "edge-interaction" type. Some of the others can be put into this form: for instance the eight-vertex model is equivalent to the Ashkin-Teller model [20], which is an edge-interaction model. I shall focus attention on such models, though many of my remarks generalize quite easily to the "interaction-round-a-face" (IRF) or vertex models [21].

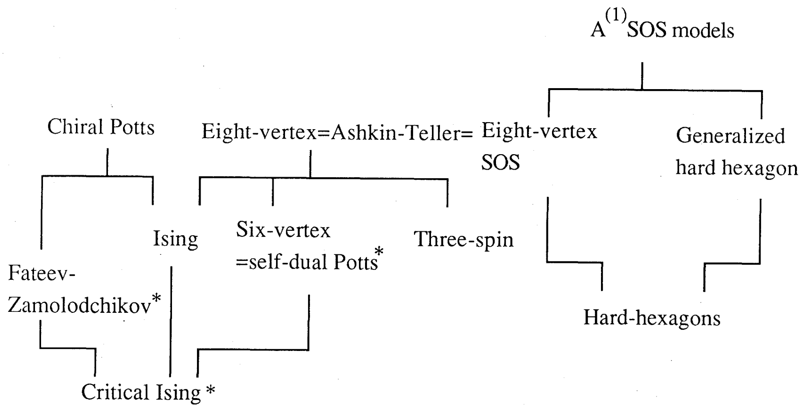

Fig. 1. **Relationship between various models; e.g. the Ising model is a special case of both the chiral Potts and eight-vertex models. Critical models are indicated by $*$.**

Star-triangle relation

As we know, the starting point for the solution of these models is the "star-triangle" or "Yang-Baxter" relation:

$$\sum_d \overline{W}_{qr}(b,d)W_{pr}(a,d)\overline{W}_{pq}(d,c) = R_{pqr}W_{pq}(a,b)\overline{W}_{pr}(b,c)W_{qr}(a,c). \tag{3}$$

(Here I am using the asymmetric notation that Perk, Au-Yang and myself used recently for the chiral Potts model [9].)

In this equation a, b, c, d are spins; the $W_{pq}(a,b)$, $\overline{W}_{pq}(a,b)$ are edge-interaction weight functions which also depend on certain other variables (complex numbers) p, q which are known as "rapidities". The normalization factor R_{pqr} must have the form [9]

$$R_{pqr} = F_{pq}F_{qr}/F_{pr} \ . \tag{4}$$

Note that if each spin takes q values, (3) represents q^3 equations in $6q^2+1$ unknowns (in fact there are really far fewer unknowns, because of homogeneity and other simple invariances). Thus in general (3) cannot be satisfied.

However, it (or its appropriate analogues) can be solved for the models I am considering. Onsager mentions it in his original and later papers on the Ising model [7,22].

The star-triangle relation is an invariance property. For example, suppose we consider the square lattice $\mathcal{L}$, drawn diagonally as in Fig. 2.

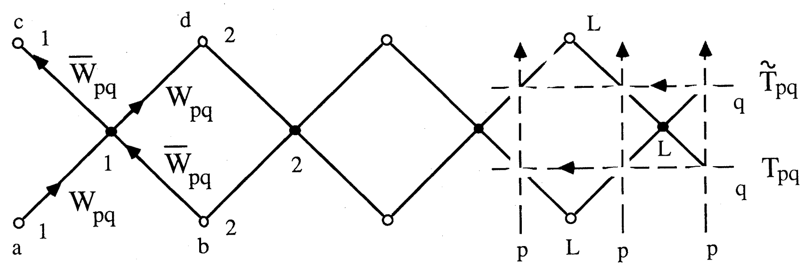

Fig. 2. Three rows of sites of the diagonal square lattice, showing the associated edge weight functions W_{pq}, $\bar{W}_{pq}$ and row-to-row transfer matrices $\mathbf{T}$, $\bar{\mathbf{T}}$. For instance, the weights of the (a, s) and (b, s) edges are $W_{pq}(a, s)$, $\bar{W}_{pq}(b, s)$. Also shown are some of the rapidy lines of the covering lattice (dotted lines). In (32)–(35) the rapidity of the upper row of the edges is changed from q to q'.

With toroidal boundary conditions, one can show in the usual way that

$$Z = \mathrm{Trace}(T\widetilde{T}T\widetilde{T}\ldots) = \mathrm{Trace}(T\widetilde{T})^{m/2}, \tag{5}$$

where m is the number of rows, $\mathcal{L}$ is the number of sites per row, and $T, \widetilde{T}$ are N^L by N^L transfer matrices with elements

$$\begin{aligned} T_{\sigma\sigma'} &= \prod_{i=1}^{L}[W_{pq}(\sigma_i, \sigma_i')\overline{W}_{pq}(\sigma_i, \sigma_{i-1}')], \\ \widetilde{T}_{\sigma\sigma'} &= \prod_{i=1}^{L}[W_{pq}(\sigma_i, \sigma_{i+1}')\overline{W}_{pq}(\sigma_i, \sigma_i')]. \end{aligned} \tag{6}$$

Here the rapidity p is associated with the vertical columns of $\mathcal{L}$, q with the horizontal rows. Obviously $T, \widetilde{T}$ depend on p, q so we can write them as $T_{pq}, \widetilde{T}_{pq}$. Then the star-triangle relation implies that transfer matrices with different q (but the same p) commute. More precisely

$$(7) \qquad T_{pq}\widetilde{T}_{pr} = T_{pr}\widetilde{T}_{pq}, \quad \widetilde{T}_{pq}T_{pr} = \widetilde{T}_{pr}T_{pq}$$

for all p, q, r.

This in turn implies that the eigenvectors of $T_{pq}\widetilde{T}_{pq}$ are independent of q, and that correlations between spins in the same row are independent of q. The full richness of these invariances is best seen by going to a very general Z-invariant model, where $\mathcal{L}$ is just an arbitrary collection of straight lines in the plane. I don't wish to pursue this any further here, but refer those interested to references 23 and 24.

Difference property

Until last year all the known solutions of the star-triangle relations possessed the "difference property", i.e. it was possible to choose the rapidities p, q, r so that $W_{pq}(a,b)$ and $\overline{W}_{pq}(a,b)$ depended on p and q only via their difference $p - q$:

$$(8) \qquad W_{pq}(a,b) = W_{p-q}(a,b), \qquad \overline{W}_{pq}(a,b) = \overline{W}_{p-q}(a,b).$$

This in turn means that for the square lattice, with vertical rapidity p and horizontal rapidity q, the partition function Z is also a function only of $p - q$:

$$(9) \qquad Z = Z_{p-q} \, .$$

Now, however, models have been discovered [9,25,26] that do *not* have this difference property. This is quite exciting, since it forces us to re-think many of our methods, where the difference property has been taken for granted.

Calculation of the free energy

A) Star-triangle + form of W + symmetry

The star-triangle relation thus gives a lot of information. Indeed, Enting and I used it to obtain what we slightly frivolously called the "399th solution" of the Ising model [27]. Since then I have performed a similar exercise for the three-dimensional Zamolodchikov model [33], and very recently for the chiral Potts model [10]. One uses the star-triangle relation together with the form of the dependence of $W_{pq}, \overline{W}_{pq}$

on p and q. From this and spatial symmetries one gets the bulk free energy

$$f = - \lim_{N\to\infty} \frac{1}{N} \ln Z \tag{10}$$

where N is the number of sites of $\mathcal{L}$.

Probably this technique could be pushed further, perhaps to the critical Potts and Ashkin-Teller = 8-vertex models. However, it is rather cumbersome and it doesn't give any information about finite-size properties. One would like to obtain these so as to obtain the correlation length ξ, interfacial tension s and conformal anomaly. Hence one does need some extra tricks. It is these I should like to discuss in the rest of this talk. I try to give an indication of the relationship between the various methods and models in Fig. 3.

B) Whole-lattice combinatorial methods

For the Ising and free-fermion models, one can write Z as a determinant or Pfaffian [28,29]. This is enormous progress, since an N by N matrix can be evaluated in about N^3 steps, instead of q^N. Further, for an homogeneous model the determinant is cyclic, so one ends up with a completely explicit product formula for Z for a finite lattice. Unfortunately this technique has so far resisted extension to other models, e.g. the 8-vertex model. It would be marvellous to so extend it.

C) The inversion relation trick

There is a simple trick for determining f [21,30,31]. For the square lattice edge interaction model, one can define a row-to-row transfer matrix T, working straight up the lattice, instead of diagonally as in (6):

$$T_{\sigma\sigma'} = \prod_{i=1}^{L} \left[W_{pq}(\sigma_i, \sigma_{i+1}) W_{pq}(\sigma'_i, \sigma'_{i+1})\right]^{1/2} \overline{W}_{pq}(\sigma_i, \sigma'_i). \tag{11}$$

The inverse of this matrix T is obtained by simply replacing $W_{pq}(a,b)$, $\overline{W}_{pq}(a,b)$ by $W^*_{pq}(a,b)$, where

$$W^*_{pq}(a,b) = 1/W_{pq}(a,b), \quad \sum_c \overline{W}_{pq}(a,c)\overline{W}^*_{pq}(c,b) = \delta_{a,b} \, . \tag{12}$$

(Thus W^*_{pq} is a "scalar" inverse; $\overline{W}^*_{pq}$ is a "matrix" inverse.) Suppose we take f to be defined by (10) only for the "physical domain", where the weights $W_{pq}(a,b), \overline{W}_{pq}(a,b)$ are positive. Then extend f to outside this

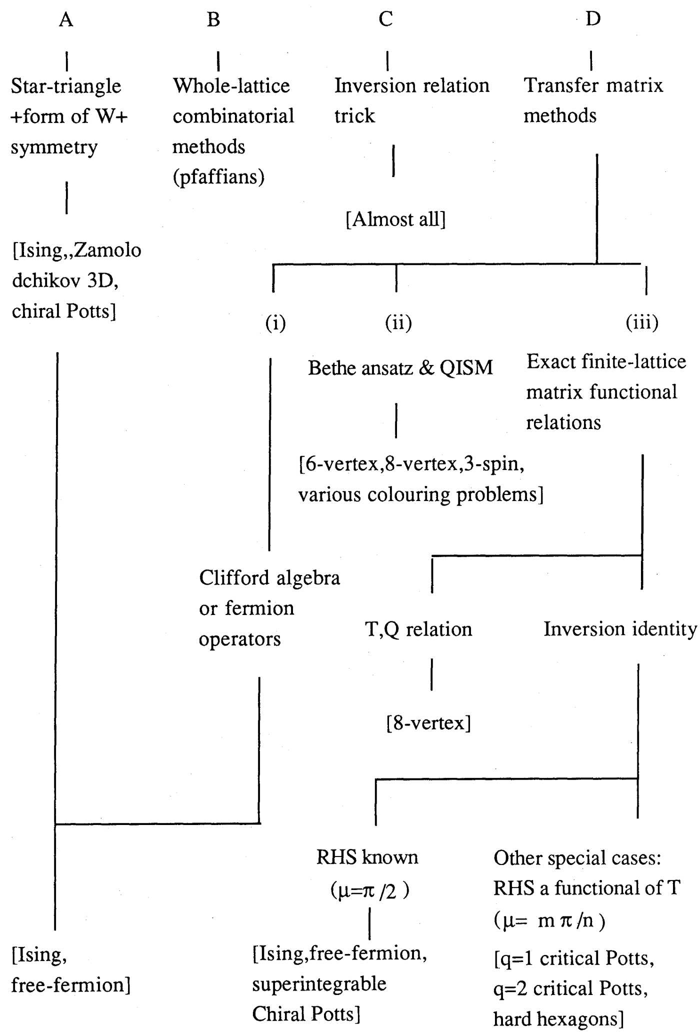

Fig. 3. Methods used for calculating the free energy f for solvable models. The square brackets contain the models solvable by a particular technique. Only methods A and D(iii) explicitly use the star-triangle relation, but it does seem to be intimately connected with D(ii), and with the analyticity properties needed in C. Methods B and D also give ξ and s.

domain by analytic continuation. Then one can argue [21] that inverting T should have the effect of negating f, so

$$f^* + f = 0 \; . \tag{13}$$

If we think of f as a function of the weights W, this relation (13) is a functional relation. In all the models, it turns out that replacing the W by the W^* is equivalent (to within a normalization factor, of which one can keep track) to interchanging p with q. Hence (13) becomes

$$f(q,p) + f(p,q) = D(p,q) \; , \tag{14}$$

where $D(p,q)$ is a known function.

By itself, this is just a definition of $f(q,p)$. However, if we <u>assume</u> that f is analytic (apart from known singularities due to poles of the weights $W, \overline{W}$) in some domain $\mathcal{D}$ enclosing both (p,q) and (q,p), then (14) does contain information on f. Usually there is also a rotation symmetry

$$\begin{gathered} W_{pq}(a,b) = \overline{W}_{q,Rp}(b,a), \qquad \overline{W}_{pq}(a,b) = W_{q,Rp}(a,b), \\ f(q,Rp) = f(p,q) \end{gathered} \tag{15}$$

corresponding to rotating the lattice through 90°. Here Rp is some function of p. (For the difference property models $Rp = p + \mu$, where μ is some constant; for the chiral Potts model Rp is defined in [9].)

If $\mathcal{D}$ is sufficiently large, in particular if it encloses also (q, Rp) and (Rp, q), then (14) and (15) actually define f .

This method can readily be extended to vertex and IRF models. (It is related to, but should not be confused with, the much more precise "inversion identity" method to be discussed later.) The trouble with it is that, like method A, it only works in the large-lattice limit, so it only gives f (not ξ or s). More seriously, it depends very much on making the correct analyticity assumption. Bazhanov and Stroganov [32] sought to solve the three-dimensional Zamolodchikov model this way, but it turned out that their analyticity assumption was wrong, so they obtained the wrong answer [33].

The importance of the analyticity assumption becomes obvious when one notes that many models (e.g. the two-dimensional Ising model in a magnetic field, the two-dimensional Potts model, and the three-dimensional Ising model) have inversion and rotation symmetries such as (14) and (15), but have nevertheless defied solution [34,35]. It seems that in general f has a complicated singularity at the "inversion point"

$p = q$, but when the star-triangle relation (3) is satisfied there are natural variables (the rapidities), in terms of which f is basically analytic at $p = q$.

D) Transfer matrix methods

The other methods all involve determining the eigenvalues Λ of the transfer matrix T. From (5), if m is large

$$(16) \qquad Z \sim \Lambda_0^m \ ,$$

where Λ_0^2 is the largest eigenvalue of $T\widetilde{T}$. Thus to obtain f it is sufficient to obtain Λ_0. To obtain ξ and s one needs the next-largest eigenvalues (section 7.10 of ref. 36). There are a few identifiable ways of doing this.

i) Clifford Algebra or fermion operators

Again the Ising and free-fermion models are simple, in that one can algebraically reduce T to a diagonal form, explicitly exhibiting all the eigenvalues [37,38].

In this form, T basically becomes a direct product of L two-by-two matrices, Again, this method has not been extended to the other models, though it has just been observed [11,12] that the superintegrable case of the chiral Potts model does have this structure for some of its eigenvalues.

It should be noted that this method, like method B, does not actually need the star-triangle relation. One can conceive of inhomogeneous solvable Ising models which are not Z-invariant. (But note that the checkerboard Ising model, which includes the homogeneous square, triangular and honeycomb Ising models, is still Z-invariant [24,39], so one would need a more complicated model than this.)

ii) Bethe ansatz and quantum inverse scattering methods

The application of the Bethe ansatz technique to two-dimensional lattice models was pioneered by Lieb [40] and Sutherland [41], using the work of Yang [42] for one-dimensional Hamiltonians. With this method, one explicitly obtains the eigenvectors of T: their elements have the general form

$$(17) \qquad \psi(x_1, \ldots, x_n) = \sum_P A_P \phi(k_{P1}, x_1)\phi(k_{P2}, x_2) \ldots \phi(k_{Pn}, x_n),$$

where the integers $x_1, \ldots, x_n$ are the positions of certain dislocations in a basic pattern (down arrows, negative spins, ...), ordered so that

$1 \le x_1 < x_2 < \ldots < x_n \le L$ (L being the number of columns of the lattice); $\phi(k,x)$ is a "single-particle function", often simply $\exp(ikx)$; $P = \{P1, \ldots, Pn\}$ is a permutation of the integers $1, \ldots, n$; A_P is some coefficient; and the sum is over all $n!$ permutations P.

The function $\phi(k,x)$ can be chosen so that (17) satisfies the eigenvalue/eigenvector equation, except for certain boundary conditions (chapter 8 of ref.36). These conditions give

$$A_P = \varepsilon_P \prod s(k_{Pj}, k_{Pi}), \tag{18}$$

where the product is over all i, j such that $1 \le i < j \le n$; $s(k,k')$ is a known function; and ε_P is the signature ($+1$ or -1) of the permutation P. They also give n simultaneous equations for the n unknowns $k_1, \ldots, k_n$. Once these are solved, the corresponding eigenvalue Λ is readily obtained.

The difficulty lies in solving for $k_1, \ldots, k_n$: usually the best one can do explicitly is obtain the largest (and next-largest) eigenvalues in the large-lattice limit (when n and L tend to infinity). One cannot write down tractable closed-form expressions for Z for a finite lattice, as one can with methods B and D(i). (One can write down an expression for Z, using the "perimeter Bethe ansatz" [43], that is of the form (17). So far it has defied further useful simplification, but I still have a lingering hope that it may be possible to do so by adapting the work of Gaudin et al [44] and Korepin [45].)

One interesting point is that in all such solutions there have been functions $\alpha(k), B(\alpha)$ such that

$$s(k',k)/s(k,k') = B[\alpha(k') - \alpha(k)] \, , \tag{19}$$

i.e. the LHS depends on k, k' only via $\alpha(k') - \alpha(k)$. This "transformation to a difference kernel" is closely related to the star-triangle difference property I mentioned earlier.

This method can only be used when the number of dislocations n is conserved for each row of the lattice. Then the transfer matrix breaks up into diagonal blocks, one for each value of n. At first this limited its use to the six-vertex model and various colouring problems [3–5], but it has been extended to the eight-vertex model by first making a transformation to the eight vertex solid-on-solid (SOS) model [46,13]. This corresponds to making a rather complicated change of basis of T.

The quantum inverse scattering method (QISM) has been pioneered by the Leningrad group [47,48], and is closely related to the Bethe ansatz method, and to the eight-vertex functional relation method of the following sub-section. It adopts an algebraic approach, while the Bethe anspatz method has a more combinatorial flavour.

iii) Exact finite-lattice matrix functional relations

This is the method on which I wish to focus in this talk. It is the method I originally used to solve the eight-vertex model, and very recently the superintegrable chiral Potts model. For the edge-models, one works with the $\mathbf{T}$ and $\widetilde{\mathbf{T}}$ of eqn (6), i.e. with the transfer matrices corresponding to going diagonally across the square lattice (this corresponds to going straight up the lattice in the vertex models, diagonally in the IRF models). The star-triangle relation (3) then ensures that these transfer matrices commute, as in (7), and one makes great use of this.

Eight-vertex model $\mathbf{T}(v), \mathbf{Q}(v)$ relations

For the eight-vertex model we regain the difference property, and there is no difference between $\mathbf{T}$ and $\widetilde{\mathbf{T}}$, so

$$\mathbf{T}_{pq} = \widetilde{\mathbf{T}}_{pq} = \mathbf{T}(p-q). \tag{20}$$

Thus the commutation relations (7) become

$$\mathbf{T}(u)\mathbf{T}(v) = \mathbf{T}(v)\mathbf{T}(u) \tag{21}$$

for all complex numbers u, v.

By just considering local "propagation through a vertex" properties, one can establish ([8,43] and chapter 10 of ref.36) that there exists another matrix function $\mathbf{Q}(u)$ such that

$$\mathbf{T}(u)\mathbf{Q}(v) = \mathbf{Q}(v)\mathbf{T}(u), \tag{22}$$

$$\mathbf{T}(v)\mathbf{Q}(v) = \phi(v-\mu)\mathbf{Q}(v+\mu) + \phi(v)\mathbf{Q}(v-\mu), \tag{23}$$

where μ is a real parameter such that $0 < \mu < \pi$ and $\phi(v)$ is a known scalar function. All the functions are entire and periodic (or antiperiodic) of period π :

$$\mathbf{T}(v+\pi) = \pm\mathbf{T}(v), \quad \mathbf{Q}(v+\pi) = \pm\mathbf{Q}(v), \quad \phi(v+\pi) = \pm\phi(v). \tag{24}$$

(This v is the $-i\pi(v+\eta)/K'$ of ref. 8 and we have removed exponential factors that cancel out of (23).)

Because the matrices all commute, one can make a similarity transformation (independent of v) which diagonalizes both $\mathbf{T}(v)$ and $\mathbf{Q}(v)$. Then (23) becomes

$$T(v)Q(v) = \phi(v-\mu)Q(v+\mu) + \phi(v)Q(v-\mu), \tag{25}$$

where now $T(v), Q(v)$ are scalars: particular eigenvalues of $\mathbf{T}(v), \mathbf{Q}(v)$.

In general $T(v), Q(v), \phi(v)$ are elliptic functions of v. It is illuminating to consider critical case, when the elliptic functions reduce to more familiar trigonometric ones. Then, to within a normalization constant

$$\phi(v) = (\sin v)^L, \tag{26}$$

so if we set $z = e^{2iv}$, then

$$\phi(v) = e^{-iLv}F(z), \tag{27}$$

where $F(z)$ is a known polynomial in z of degree L.

The functions $T(v), Q(v)$ are similarly proportional to polynomials in z of degree L, n (where $n = L/2$), respectively, except that we don't at first know the coefficients. If we substitute these forms into (25), each side is proportional to a polynomial of degree $L+n$, so we obtain $L+n+1$ equations. Because of homogeneity, we are free to choose one of the coefficients of $Q(v)$ arbitrarily, so we have $L+n+1$ unknowns. Hence we may hope that (25) contains enough information to determine $T(v), Q(v)$. It does. (There are many solutions, corresponding to the different eigenvalues.)

This argument generalizes readily to the non-critical case. One uses the fact that $T(v), Q(v)$ are entire functions with L, n distinct zeros (modulo the elliptic function periods), respectively. If one sets v in (25) equal successively to each of the zeros $v_1, \ldots, v_n$ of $Q(v)$, one obtains n equations for $v_1, \ldots, v_n$. These are the same as the equations for $k_1, \ldots, k_n$ in the Bethe ansatz.

There is one slight difficulty with this functional relation technique. While it gives all possibilities for the eigenvalues, it does not tell you if any particular possibility occurs, and if so how many times. In other words, it does not give the multiplicities of the eigenvalues. To determine these one needs further information, e.g. from special or limiting cases where $T(v)$ can be diagonalized explicitly.

Inversion identity

Replace v in (23) by $v+\mu$, then multiply this equation by the original and divide by $Q(v)Q(v+\mu)$. We get

$$\mathbf{T}(v)\mathbf{T}(v+\mu) = \phi(v-\mu)\phi(v+\mu)\mathbf{I} + \phi(v)\mathbf{P}(v) \tag{28}$$

where

(29)

$$\begin{aligned}\mathbf{P}(v) =&\phi(v)\mathbf{Q}(v-\mu)\mathbf{Q}(v+2\mu)/\mathbf{Q}(v)\mathbf{Q}(v+\mu)\\ &+\phi(v-\mu)\mathbf{Q}(v+2\mu)/\mathbf{Q}(v) \ + \ \phi(v+\mu)\mathbf{Q}(v-\mu)/\mathbf{Q}(v+\mu).\end{aligned}$$

As we went from (23) to (25), so we can replace (28) and (29) by scalar equations, $T(v)$, $P(v)$, $Q(v)$ being eigenvalues of $\mathbf{T}(v)$, $\mathbf{P}(v)$, $\mathbf{Q}(v)$, corresponding to some particular common eigenvector. In particular

$$T(v)T(v+\mu) = \phi(v-\mu)\phi(v+\mu) + \phi(v)P(v). \tag{30}$$

The function $P(v)$ is meromorphic: from (30) it can only have poles when $\phi(v)$ vanishes, while from (29) it can only have them when $Q(v)Q(v+\mu)$ vanishes. It follows that it is entire. It has L zeros.

Again it is instructive to consider the critical case. Then $\phi(v)$, $T(v)$, $P(v)$ are each proportional to polynomials in z of degree L, and each side of (30) is proportional to a polynomial of degree $2L$. Thus we have $2L+1$ equations for the $2L+2$ unknown coefficients of $T(v)$, $P(v)$. One only needs one more piece of information, perhaps obtainable from some special case (like $v=\mu$, when $\mathbf{T}(v)$ is proportional to a simple translational shift operator).

For the eight-vertex model there is no advantage in using (28) rather than (23). However, for other models (e.g. hard hexagons) it seems quite difficult to obtain analogues of (23), while it is quite easy to obtain analogues of (28). We can begin to see this for the edge-interaction models, using equations (12–15) above. These imply

$$W_{pq'}(a,b) = 1/\overline{W}_{pq}(b,a), \quad \sum_c W_{pq}(a,c)\overline{W}_{pq'}(c,b) = \eta_{pq}\delta_{a,b}\ , \tag{31}$$

where q' is a rapidity related to q by $q = Rq'$, and η_{pq} is some known function. Now consider the product $T_{pq}T_{pq'}$. From (6), this has elements

$$(T_{pq}T_{pq'})_{\sigma\sigma'} = \prod_i X(\sigma_i\sigma_{i+1} \mid \sigma'_i\sigma'_{i+1})\ , \tag{32}$$

where the product is from $i=1$ to L and

$$X(a,b \mid c,d) = \sum_s W_{pq}(a,s)\overline{W}_{pq}(b,s)\overline{W}_{pq'}(s,c)W_{pq'}(s,d). \tag{33}$$

This $X(a,b \mid c,d)$ is the weight of the four-pointed star shown in Fig. 2, summed over the centre spin s.

Suppose $b = d$. Then from (31)

$$X(a, b \mid c, b) = \eta_{pq}\delta_{a,c} \ . \tag{34}$$

Thus if $\sigma_{i+1}, \sigma'_{i+1}$ are equal, the RHS of (32) will vanish unless σ_i, σ'_i are also equal. If the elements of $X(a, b \mid c, d)$ with $a \neq c$ and $b \neq d$ all have some common factor ζ_{pq}, then

$$\mathbf{T}_{pq}\mathbf{T}_{pq'} = (\eta_{pq})^L\mathbf{I} + (\zeta_{pq})^L\mathbf{P}_{pq} \ , \tag{35}$$

where $\mathbf{P}_{pq}$ is a matrix whose elements σ, σ' are zero unless σ'_i, σ_i are different for all i. (For the two-state Ising model $\mathbf{P}_{pq}$ is proportional to the operator that reverses all spins: for higher N-state models it is more complicated.)

This equation (35) is the analogue of the eight-vertex model equation (28), and has been studied by Reshetikhin and Pearce [49–51]. The ordinary self-dual Potts model (which is equivalent to the six-vertex model [52], which in turn is a special case of the eight-vertex model) can be solved this way.

Although (35) appears to hold quite generally for solvable models, it only defines the eigenvalues of the transfer matrix if enough information is available on ζ_{pq} and $\mathbf{P}_{pq}$. We shall return to this point in the context of the chiral Potts model.

The Ising case: $\mu = \pi/2$ and P proportional to I

The simplest case is when $\mathbf{P}_{pq}$ is completely known. This happens for the Ising model, and for the corresponding case of the eight-vertex model, when $\mu = \pi/2$. Then from (24) $\mathbf{Q}(v + \mu) = \pm\mathbf{Q}(v - \mu)$, so the functions $\mathbf{Q}$ cancel out of (29), except for the $\pm$ signs, giving $\mathbf{P} \propto \mathbf{I}$ and

$$\mathbf{T}(v)\mathbf{T}(v + \mu) = [\phi(v) \pm \phi(v - \mu)]^2\mathbf{I} \ . \tag{36}$$

(Strictly this $\mathbf{I}$ is not the full identity matrix. The vector space breaks up into two parts, one in which $\mathbf{Q}(v)$ is periodic, the other in which $\mathbf{Q}(v)$ is anti-periodic. For the Ising model these are the spaces symmetric and anti-symmetric under spin-reversal. The above remarks apply separately within each of these symmetry blocks, with the appropriate choice of sign in (36).)

For $\mu = \pi/2$ the eight-vertex model is actually equivalent to *two* non-interacting Ising models - chapter 10 of ref.36 - which is why the RHS of (36) is a square. If one continues with the method of eqn (35) for the Ising model, one obtains (36) without the square: eqn (7.5.5) of ref.36.

It is very easy to solve this equation for all the eigenvalues: one determines the zeros of the RHS, which occur in pairs $v_i, v_i + \mu$, for $i = l, \ldots, L$. Then for each i the function has either v_i or $v_i + \mu$ as a zero. Thus there are 2^L distinct eigenvalues, and they have a direct product structure, as in the fermion method D(i).

In [12] some of the eigenvalues of the super-integrable chiral Potts model are obtained this way, which makes one wonder if this model can also be solved by fermion operators.

The use of the phrase "inversion identity" is obvious here: to within a scalar factor (and within each symmetry block), $\mathbf{T}(v+\mu)$ is the inverse of $\mathbf{T}(v)$.

The case $3\mu/\pi =$ *integer*

Other cases, special but not so simple, arise when μ is any rational fraction of π.

For instance, suppose that $\mu = \pi/3$ or $2\pi/3$. Then

$$Q(v + 3\mu) = rQ(v) \ , \tag{37}$$

where $r = \pm 1$. Replacing v in (25) by $v + \mu$ and $v + 2\mu$, we obtain a total of 3 equations, which can be written in the matrix form:

$$\begin{pmatrix} -T(v) & \phi(v-\mu) & r\phi(v) \\ \phi(v+\mu) & -T(v+\mu) & \phi(v) \\ r\phi(v+\mu) & \phi(v+2\mu) & -T(v+2\mu) \end{pmatrix} \begin{pmatrix} Q(v) \\ Q(v+\mu) \\ Q(v+2\mu) \end{pmatrix} = 0. \tag{38}$$

The determinant of the coefficient matrix must vanish and (for L even) $\phi(v + 2\mu) = \phi(v - \mu)$. Setting $t(v) = rT(v)/\phi(v + \mu)$, it follows that

$$t(v)t(v+\mu)t(v+2\mu) - t(v) - t(v+\mu) - t(v+2\mu) - 2 = 0. \tag{39}$$

This is a third-degree functional relation for $t(v)$. It can be written as

$$t(v)t(v+\mu) = 1 + [t(v) + t(v+\mu) + 2]/t(v+2\mu), \tag{40}$$

which is a form of the inversion identity (30). One simple solution is

$$t(v) = -1, \tag{41}$$

i.e $T(v) = -r\phi(v + \mu)$. Indeed, this solution does occur (section 8 of ref. 53). For the critical self-dual q-state Potts model [31,52], where $q^{1/2} = 2\cos\mu$, this is the solution for the trivial $q = 1$ case.

The case $4\mu/\pi =$ *integer*

Similarly, if $Q(v+4\mu) = rQ(v)$ and $\phi(v+4\mu) = \phi(v)$, we obtain a fourth degree functional relation for $T(v)$, which is satisfied if (but *not* only if)

$$T(v+2\mu) = -r\, T(v), \tag{42a}$$

$$T(v)T(v+\mu) = \phi(v-\mu)\phi(v+\mu) - r\phi(v)\phi(v+2\mu). \tag{42b}$$

In fact, the four-by four matrix of coefficients is then only of rank 2.

This equation (42b) is similar to the equation (36) for the Ising case of the eight-vertex model, which has $\mu = \pi/2$. This again fits, because if we come to these equations from the critical Potts model, the Ising case is when $q = 2$ and $\mu = \pi/4$. Thus there is a connection between the $\mu = \pi/4$ and $\mu = \pi/2$ cases.

The case $5\mu/\pi =$ *integer*

The final special case we shall look at is when $Q(v+5\mu) = r\, Q(v)$ and $\phi(v+5\mu) = \phi(v)$. Then the matrix of coefficients is five-by-five, and its rank reduces to three if $T(v+5\mu) = T(v)$ and

$$T(v)T(v+\mu) = \phi(v-\mu)\phi(v+\mu) - r\phi(v)T(v+3\mu). \tag{43}$$

Let me repeat that the utility of such special identities (all of which are of the "inversion identity" (30) type) is that for some models they can be derived even when no analogue of (25) has been found (i.e. no Q matrix is known). In particular, equation (43) is satisfied for the hard-hexagon model, and this is the way ξ and s were obtained for this model [54].

Superintegrable chiral Potts model

Let me conclude with a few remarks on some work of recent weeks. The chiral Potts model is an edge-interaction Z_N model, where each spin σ_i takes the N values $0, 1, \ldots, N-1$. The weights $W_{pq}(a,b), \overline{W}_{pq}(a,b)$ are spin translation invariant: they depend on a, b only via the difference $a-b$, and are periodic in $a-b$ of period N. One can derive the equation (35) as above. To use (35) one needs some information on ζ_{pq} and P_{pq}. In general it is not clear how to proceed, but for the "superintegrable" case, when the vertical rapidity p has a special value (in the notation of ref. 10, $v_p = -\pi/2$), some simplifications occur. In particular,

$$X(a,b \mid 0,0) = 0 \qquad \text{if} \quad 0 \le a < b \le N-1. \tag{44}$$

Together with the spin translation invariance that $X(a+j, b+j \mid c+j, d+j)$ is independent of j, this is a much stronger statement than (34). For $j = 0, \ldots, N-1$, let $\mathbf{u}_j$ be the N^L dimensional vector with entries

$$(\mathbf{u}_j)_\sigma = \delta(\sigma_1, j) \cdots \delta(\sigma_L, j). \tag{45}$$

Taking $\mathbf{u}_{j+N} = \mathbf{u}_j$, it follows from (32) that

$$\mathbf{T}_{pq}\mathbf{T}_{pq'}\mathbf{u}_j = \sum_k X(k, k|0, 0)^L \mathbf{u}_{j+k}, \tag{46}$$

the sum being from $k = 0$ to $N-1$. (One first establishes this equation for $j = 0$,using (32), (44) and the cyclic boundary condition $\sigma_{N+1} = \sigma_1$, $\sigma'_{N+1} = \sigma'_1$; then uses the spin translation invariance.) Define, for $Q = 0, 1, \ldots, N-1$, the Fourier transforms

$$\mathbf{v}_Q = \sum_j \omega^{-Qj} \mathbf{u}_j$$

$$g_Q = \sum_j \omega^{Qj} X(j, j|0, 0)^L, \tag{47}$$

where $\omega = \exp(2\pi i/N)$ and the sums are from $j = 0$ to $N-1$. It follows that

$$\mathbf{T}_{pq}\mathbf{T}_{pq'}\mathbf{v}_Q = g_Q \mathbf{v}_Q. \tag{48}$$

This equation is rather like the Ising case (36) of the inversion identity, in that the RHS is a known function of p and q. The big difference is that it is only a vector equation, rather than a full matrix one. Even so, one can still use it to obtain some of the eigenvalues of $\mathbf{T}_{pq}$.

Let V_Q be the vector space generated by pre-multiplying $\mathbf{v}_Q$ by any sum of products of matrices $\mathbf{T}_{pq}$, for fixed p but for any value of q. All such matrices commute (because of the star-triangle relation (3)), so (48) remains true if $\mathbf{v}_Q$ is replaced by any vector in V_Q. All such vectors are spatially translation invariant; and are eigenvectors of the spin translation operator $\mathbf{R}$ of [11], with eigenvalue ω^Q.

Replacing $\mathbf{v}_Q$ by an eigenvector in V_Q of $\mathbf{T}_{pq}$, with eigenvalue T_{pq}, it follows that

$$T_{pq}T_{pq'} = g_Q$$

This is a functional relation for T_{pq} of the form of the inversion identity (30), where the right-hand side is known, as in (36). We can solve it in a similar way, as is indicated in [12], by sharing out the zeros of the RHS between T_{pq} and $T_{pq'}$. Taking $W_{pq}(a, a), \overline{W}_{pq}(a, a) = 1$ and defining

k', x as in [12] (k' is a "temperature-like" constant, x is a variable that depends on q and can be used in place of q), we obtain

$$T_{pq} = \rho C_Q \prod_j [G \pm (1 + k'^2 - 2k'\cos\theta_j)^{1/2}], \tag{49}$$

where k', G are defined in [12] and

$$\rho = (x-1)/(x^N - 1)^{1/N} \tag{50}$$

$$C_Q = N^L x^{-Q} / [(2k')^m (1 - x^{-N})^{L-m-L/N}] \tag{51}$$

$$\cos\theta_j = (1 + z_j^N)/(1 - z_j^N), \tag{52}$$

the product is over $j = 1, \ldots, m$, where

$$m = \text{integer part of } [(NL - L - Q)/N] \tag{53}$$

and $z_1^N, \ldots, z_m^N$ are the zeros of the polynomial

$$P(z^N) = z^{-Q} \sum_n \omega^{(Q+L)n} [(z^N - 1)/(z - \omega^n)]^L, \tag{54}$$

this sum being from $n = 0$ to $N - 1$. (The RHS of (54) is a polynomial of degree m in z^N.) The sign choices in (49) can be made independently for each value of j, so there are 2^m different eigenvalues for each value of Q. It appears (from numerical studies for small lattices) that each eigenvalue occurs just once, so V_Q has dimension 2^m. The sum over Q of these dimensions is less than N^L, so we have only found some of the eigenvalues.

Even so, we have found all the eigenvalues that are relevant if we use cylindrical boundary conditions, with the the spins in the top row fixed to all have value 0, and those in the bottom row to all have value a, as in Fig.4.

In fact, we can evaluate the partition function of this lattice of $M+1$ rows exactly, for M either odd or even. It is

$$\begin{aligned} Z_a = N^{-1} \rho^{LM} \sum_Q \omega^{-Qa} (C_Q)^M \\ \prod_j \{[(G + \Delta_j)^M (\Delta_j + 1 - k' \cos\theta_j) \\ + (G - \Delta_j)^M (\Delta_j - 1 + k' \cos\theta_j)]/(2\Delta_j)\}, \end{aligned} \tag{55}$$

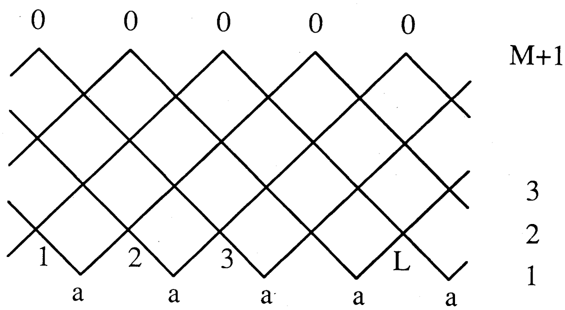

Fig. 4. The square lattice whose partition function, for the super integrable chiral Potts model, is given by (55) and (60). It has $M+1$ rows, each of L sites. The left-to-right cylindrical boundary conditions mean that site L is followed by site 1. The spins in the top (bottom) row are fixed to have value $0(a)$.

where the sum is over $Q = 0, \ldots, N-1$, the product over $j = 1, \ldots, m$, and

$$\Delta_j - (1 + k'^2 - 2k' \cos\theta_j)^{1/2} \tag{56}$$

Equivalently, let $r =$ integer part of $[(M+1)/2]$, and let $\gamma, c_1, \ldots, c_r$ be parameters (independent of θ_j) such that

$$\begin{aligned}[t] &[(G+\Delta_j)^M(\Delta_j + 1 - k'\cos\theta_j) \\ &\quad + (G-\Delta_j)^M(\Delta_j - 1 + k'\cos\theta_j)]/\Delta_j \\ &= \gamma(c_1 - \cos\theta_j)(c_2 - \cos\theta_j)\cdots(c_r - \cos\theta_j) \end{aligned} \tag{57}$$

for all complex numbers θ_j, Δ_j being given by (56). (The LHS is a polynomial of degree r in $\cos\theta_j$, so there must be such a factorization.) Also, set

$$\zeta_i = [(c_i - 1)/(c_i + 1)]^{1/N}, \tag{58}$$

$$g = N x^{N-1}(1-x)/(1-x^N). \tag{59}$$

Then we can choose $\zeta_1, \ldots, \zeta_r$ so that $\zeta_1 \cdots \zeta_r = x^{-M}$, and (55) can be transformed to

$$
\begin{aligned}
Z_a = {} & N^{-1} g^{LM} \sum_Q \omega^{-Qa} \\
& \times \prod_i N^{-L} \sum_n \omega^{(Q+L)n} [(\zeta_i^N - 1)/(\zeta_i - \omega^n)]^L,
\end{aligned}
\tag{60}
$$

the Q, n sums being from 0 to $N-1$, the product being from $i = 1$ to r. One curious feature of the superintegrable chiral Potts model is that standard arguments (e.g. section 7.10 of ref. 36) give the values

$$
\alpha = 1 - 2/N, \qquad \mu = 2/N, \qquad \nu = 1 \tag{61}
$$

for the critical exponents of the specific heat, interfacial tension and correlation length. These values satisfy the scaling relation $\mu + \nu = 2 - \alpha$, but violate the hyperscaling relation $d\nu = 2 - \alpha$ (with $d = 2$). M. N. Barber and P. A. Pearce have suggested to me that this may be due to the anisotropic nature of the model. (The vertical rapidity p having the special "superintegrable" value.) From the result (60) one can deduce all the eigenvalues of the column-to-column transfer matrix, but if I'm going to get to this meeting I'd better stop now. Let me just remark that I have only considered here the problem of calculating the partition functions of various solvable models. As many of you are very well aware, there is another whole field concerned with calculating local probabilities and order parameters, using corner transfer matrices ([55] and chapter 13 of ref. 36).

References

[1] Izergin, A. G. and Korepin, V. E., Commun. Math. Phys., **79** (1987), 303-316.

[2] Perk, J. H. H. and Schultz, C. L., Physica, **122A** (1983), 50-70.

[3] Baxter, R. J., J. Math. Phys., **11** (1970), 784-789.

[4] Baxter, R. J., J. Math. Phys., **11** (1970), 3116-3124.

[5] Baxter, R. J., J. Phys. A, **19** (1970), 2821-2839.

[6] Baxter, R. J., J. Phys. A, **20** (1987), 5241-5261.

[7] Onsager, L., Phys. Rev., **65** (1944), 117-149.

[8] Baxter, R. J., Ann. Phys. (NY), **70** (1972), 193-228.

[9] Baxter, R. J., Perk, J. H. H. and Au-Yang, H., Phys. Lett A, **128** (1988), 138-142.

[10] Baxter, R. J., J. Statist. Phys., **52** (1988), 639-667.

[11] Albertini, G., McCoy, B. M., Perk, J. H. H. and Tang, S., Excitation spectrum and order parameter for the integrable N-state chiral Potts model, SUNY Stony Brook preprint ITP-SB-88-16 (1988).

[12] Baxter, R. J., Phys. Lett. A., **133** (1988), 185-189.

[13] Andrews, G. E., Baxter, R. J. and Forrester, P. J., J. Statist. Phys., **35** (1982), 193-266.

[14] Baxter, R. J. and Pearce, P. A., J. Phys. A, **15** (1982), 897-910.

[15] Jimbo, M. and Miwa, T., Physica, **15D** (1985), 335-353.

[16] Kuniba, A., Akutsu, Y. and Wadati, M., J. Phys. Soc. Japan, **55** (1986), 1092-1101; 1466-1474; 1880-1886; 2166-2176; 2605-2617; 3338-3353.

[17] Baxter, R. J. and Andrews, G. E., J. Statist. Phys., **44** (1986), 249-271; 713-728.

[18] Andrews, G. E. and Baxter, R. J., J. Statist. Phys., **47** (1987), 297-330.

[19] Date, E., Jimbo, M., Kuniba, A., Miwa, T. and Okado, M., Nucl. Phys., **B290** (1988), 231-273; Adv. Studies in Pure Math., **16** (1987), 17-22; Comm. Math. Phys., **119** (1988), 543-565; Lett. Math. Phys., **17** (1989), 51-54, 69-77.

[20] Wegner, F. J., J. Phys., **C 5** (1972), L131-132.

[21] Baxter, R. J., "Fundamental Problems in Statistical Mechanics V", ed. EGD Cohen, North-Holland, Amsterdam, 1980, pp. 109-141.

[22] Onsager, L., "Critical phenomena in alloys, magnets and superconductors", eds. RE Mills, E Ascher and RI Jaffee, McGraw-Hill, New York.

[23] Baxter, R.J., Phil. Trans. Roy. Soc. (London), **289** (1978), 315-346.

[24] Baxter, R. J., Proc. Roy. Soc. (London), **A404** (1986), 1-33.

[25] Au-Yang, H., McCoy, B. M., Perk, J. H. H., Tang, S. and Yan, M. L., Phys. Lett., **A 123** (1987), 219-223.

[26] McCoy, B. M., Perk, J. H. H., Tang, S. and Sah, C. H., Phys. Lett. A, **125** (1987), 9-14.

[27] Baxter, R. J. and Enting, I. G., J. Phys. A, **11** (1978), 2463-2473.

[28] Kasteleyn, P. W., J. Math. Phys., **4** (1963), 287-293.

[29] Montroll, E. W., Potts, R. B. and Ward, J. C., J. Math. Phys., **4** (1963), 308-322.

[30] Stroganov, Yu. G., Phys. Lett., **A 74**, 116-118.

[31] Baxter, R. J., J. Stat. Phys., **28** (1982), 1-41.

[32] Bazhanov, V. V. and Stroganov, Yu. G., Theor. and Math. Phys., **62** (1985), 253-260; **63** (1985 (USSR)), 519-527, 604-611.

[33] Baxter, R. J., Physica D, **18 321-347** (1986).

[34] Jaekel, M. T. and Maillard, J. M., J. Phys. A, **17** (1984), 2079-2094.

[35] Hansel, D., Maillard, J. M., Oitmaa, J. and Velgakis, M. J., J. Stat. Phys., **48** (1987), 69-80.

[36] Baxter, R. J., "Exactly solved models in statistical mechanics", Academic, London, 1982.

[37] Kaufman, B., Phys. Rev., **76** (1949), 1232-1243.

[38] Schultz, T. D., Mattis, D. C. and Lieb, E. H., Rev. Mod. Phys., **36** (1964), 856-871.

[39] Baxter, R. J., Local three-spin correlations in the free-fermion and planar Ising models, to appear in Proc. Roy. Soc. (London), (1989).

[40] Lieb, E. H., Phys. Rev., **162** (1967), 162-172 ; Phys. Rev. Lett., **18** (1967), 1046-1048 and **19** (1967), 108-110.
[41] Sutherland, B., Phys. Rev. Lett., **19** (1967), 103-104.
[42] Yang, C. N. and Yang, C. P., Phys. Rev., **150** (1966), 321-327, 327-329; **151** (1966), 258-264.
[43] Baxter, R. J., J. Phys. A, **20** (1987), 2557-2567.
[44] Gaudin, M., McCoy, B. M. and Wu, T. T., Phys. Rev., **D 23** (1981), 417-419.
[45] Korepin, V. E., Commun. Math. Phys., **86** (1982), 391-418.
[46] Baxter, R. J., Ann. Phys., **76** (1973), 1-24, 25-47, 48-71.
[47] Faddeev, L. D., Sov. Sci. Rev. Math. Phys., **C 1** (1981), 107-160.
[48] Izergin, A. G. and Korepin, V. E., Commun. Math. Phys., **94** (1984), 67-92.
[49] Reshetikhin, N. Yu., Lett. Math. Phys., **7** (1983), 205-213.
[50] Pearce, P. A., Phys. Rev. Lett., **58** (1987), 1502-1504.
[51] Pearce, P. A., J. Phys. A, **20** (1987), 6463-6469.
[52] Baxter, R. J., J. Phys., **C 6** (1973), L445-448.
[53] Baxter, R. J., Ann. Phys. (NY), **70** (1972), 323-337.
[54] Baxter, R. J. and Pearce, P. A., J. Phys. A, **15** (1982), 897-910.
[55] Baxter, R. J., Physica, **106A** (1981), 18-27.
[56] Fisher, M. E., Phys. Rev., **124** (1961), 1664-1672.
[57] Temperley, H. N. V. and Fisher, M. E., Phil. Mag., **6** (1961), 1061-1063.
[58] Baxter, R. J. and Wu, F. Y., Phys. Rev. Lett., **31** (1973), 1294-1297; Australian J. Phys., **27** (1974), 357-381.
[59] Zamolodchikov, A. B., Sov. Phys. JETP, **52** (1980), 325-336; Commun. Math. Phys., **79** (1981), 489-505.
[60] Fateev, V. A. and Zamolodchikov, A. B., Phys. Lett A, **92** (1982), 37-39.

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