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Homogeneous Einstein Metrics On Certain Kähler C-Spaces

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Dedicated to Professor Shingo Murakami on his 60th birthday

§0. Introduction

Most known non-standard examples of compact homogeneous Einstein manifolds are constructed via Riemannian submersions. Here the word "standard"means that the Einstein metric on a homogeneous manifold is constructed from the irreducible isotropy representation of the homogeneous manifold. However, such a method does not work effectively if the isotropy representation associated with the homogeneous manifold decomposes into more than two irreducible representations. In fact, only few examples (cf. Wang [8]) of such homogeneous Einstein manifolds are known so far.

Let $M = G/K$ be a Kähler C-space, where G is a compact connected simple Lie group. Then M carries a complex structure J and a Kähler metric g, with respect to J, such that the group $\mathrm{Aut}(M, J, g)$ of holomorphic isometries of the Kähler manifold $(M,\ J,\ g)$ acts transitively on M. Assuming now that the associated isotropy representation of K decomposes into non-equivalent three irreducible components, we construct in § 2 examples of such Kähler C-spaces . In § 3, in view of the method of Wang and Ziller (cf. § 1), we find all G-invariant Einstein metrics on the Kähler C-spaces G/K constructed in the preceding section. On the other hand, given a G-invariant complex structure on G/K, we have a unique G-invariant Einstein-Kähler metric on G/K up to homotheties (cf. § 2). Thus if a G-invariant Einstein metric on G/K found in § 3 is Kähler with respect to some G-invariant complex structure on G/K, then it is nothing but a known metric. Therefore, we check in § 4 whether the G-invariant Einstein metrics found in § 3

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are Kähler or not by taking suitable G-invariant complex structures on G/K. We now state our Main Theorem.

Main Theorem. *Let G/K be a Kähler C-space which is locally isomorphic to one of the following.*

$(1) E_6/U(2) \times SU(3) \times SU(3)$, $(2) E_7/U(3) \times SU(5)$,
$(3) E_7/U(2) \times SU(6)$, $(4) E_8/U(2) \times E_6$,
$(5) E_8/U(8)$, $(6) F_4/U(2) \times SU(3)$,
$(7)\ G_2/U(2)$, $(8) SU(\ell + m + n)/S(U(\ell) \times U(m) \times U(n))$,
$(9) SO(2\ell)/U(1) \times U(\ell - 1)$, $(10) E_6/U(1) \times U(1) \times \mathrm{Spin}(8)$,

where ℓ, m and n are positive integers in the case of (8), *and* $4 \leqq \ell \in \mathbb{Z}$ *in the case of* (9). *Then the isotropy representation of compact homogeneous space G/K is decomposed into non-equivalent three irreducible components* (cf. Proposition 2.6).

[I] *If G/K is either* (1), (2), (3), (4), (5), (6), *or* (7), *then G/K has exactly three G-invariant Einstein metrics up to homotheties* (cf. Theorem 3.2). *One of them is Kähler for a G-invariant complex structure on G/K and the other two are non-Kähler for any complex structure on G/K* (cf. Remark 4.2).

[II] *If G/K is either* (8), (9), *or* (10), *then G/K has exactly four G-invariant Einstein metrics, up to homotheties, which are written down very explicitly* (cf. Theorem 3.2). *Three of them are Kähler for suitable G-invariant complex structures on G/K and the rest is non-Kähler for any complex structure on G/K* (cf. Examples 4.3, 4.4, 4.5).

Note, in the above theorem, that the non-Kähler G-invariant Einstein metrics in the case of (8) with $\ell = m = n$ and (10) are known metrics of G/K, coming from the Killing form.

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§1. Preliminaries

In this section we recall some results of Wang and Ziller [10].

Let G be a compact connected simple Lie group, K a connected closed subgroup of G, and let $\mathfrak{g}$, $\mathfrak{k}$ be the Lie algebras of G, K respectively. For the compact connected homogeneous manifold $M = G/K$, we assume that the isotropy representation of G/K is decomposed into non-equivalent three irreducible components. Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the negative of the Killing form $-B$ of $\mathfrak{g}$ and let

$$\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3 \tag{1.1}$$

be the irreducible decomposition of $\mathfrak{m}$. Note that each G-invariant Riemannian metric on M can be represented by an inner product $x_1 B|_{\mathfrak{m}_1} + x_2 B|_{\mathfrak{m}_2} + x_3 B|_{\mathfrak{m}_3}$ (x_1, x_2, $x_3 > 0$) on $\mathfrak{m}$. From now on we identify G-invariant Riemannian metrics on M with inner products on $\mathfrak{m}$. Let $\mathcal{M}$ be the set of all G-invariant Riemannian metrics on M with volume 1. Then

$$\mathcal{M} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid {x_1}^{d_1} {x_2}^{d_2} {x_3}^{d_3} = 1/V^2,\ x_1, x_2, x_3 > 0\} \tag{1.2}$$

where $d_i = \dim \mathfrak{m}_i$ $(i = 1,2,3)$, $V = \mathrm{Vol}(M, B|_{\mathfrak{m}})$. Let $\{e_\alpha\}$ be a B-orthonormal basis of $\mathfrak{m}$ adapted to (1.1). We put

$$C_{ij}^k = \sum_{\substack{e_\alpha \in \mathfrak{m}_i \\ e_\beta \in \mathfrak{m}_j \\ e_\gamma \in \mathfrak{m}_k}} B([e_\alpha, e_\beta], e_\gamma)^2 \tag{1.3}$$

for $i, j, k = 1, 2, 3$. Note that C_{ij}^k is independent of the choice of B-orthonormal bases of $\mathfrak{m}$ adapted to (1.1) and symmetric in all three indices. We denote by $S(g)$ the scalar curvature of a Riemannian manifold (M, g). Then

$$S(g) = \frac{1}{2} \sum_i \frac{d_i}{x_i} - \frac{1}{4} \sum_{i,j,k} C_{ij}^k \frac{x_k}{x_i x_j} \tag{1.4}$$

for $g = x_1 B|_{\mathfrak{m}_1} + x_2 B|_{\mathfrak{m}_2} + x_3 B|_{\mathfrak{m}_3}$ (x_1, x_2, $x_3 > 0$) (cf. [10]). Now we have the following theorem.

Theorem 1.5 (Wang-Ziller [10]). *Let $M = G/K$ be as above and $\dim M \geqq 3$. Then $g \in \mathcal{M}$ is Einstein if and only if*

$$\frac{\partial S}{\partial u}(g) = \frac{\partial S}{\partial v}(g) = 0$$

where $u = x_2/x_1$, $v = x_3/x_1$ (cf. (1.4)).

§2. Kähler C-spaces

In this section we construct some examples of a Kähler C-space $M = G/K$ such that G is a compact connected simple Lie group and that the corresponding isotropy representation of K is decomposed into non-equivalent three irreducible components.

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{t}$ a maximal abelian subalgebra of $\mathfrak{g}$. We denote by $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{t}^{\mathbb{C}}$ the complexifications of $\mathfrak{g}$ and $\mathfrak{t}$ respectively. We identify an element of the root system Δ of $\mathfrak{g}^{\mathbb{C}}$ relative to the Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$ with an element of $\sqrt{-1}\mathfrak{t}$ by the duality defined by the Killing form (,) of $\mathfrak{g}^{\mathbb{C}}$. Let $\Pi = \{\alpha_1, \cdots, \alpha_\ell\}$ be a fundamental system of Δ and $\{\Lambda_1, \cdots, \Lambda_\ell\}$ the fundamental weights of $\mathfrak{g}^{\mathbb{C}}$ corresponding to Π; i.e.,

$$\frac{2(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} \qquad (1 \leqq i, j \leqq \ell).$$

Let Π_0 be a subset of Π and put

$$\Pi - \Pi_0 = \{\alpha_{i_1}, \cdots, \alpha_{i_r}\} \qquad (1 \leqq i_1 < \cdots < i_r \leqq \ell). \tag{2.1}$$

We put

$$[\Pi_0] = \Delta \cap \{\Pi_0\}_{\mathbb{Z}}$$

where $\{\Pi_0\}_{\mathbb{Z}}$ denotes the subgroup of $\sqrt{-1}\mathfrak{t}$ generated by Π_0. Consider the root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{t}^{\mathbb{C}}$:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in \Delta} \mathfrak{g}^{\mathbb{C}}_{\alpha}.$$

We define a parabolic subalgebra $\mathfrak{u}$ of $\mathfrak{g}^{\mathbb{C}}$ by

$$\mathfrak{u} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in [\Pi_0] \cup \Delta^+} \mathfrak{g}^{\mathbb{C}}_{\alpha}$$

where Δ^+ is the set of all positive roots relative to Π. Let $G^{\mathbb{C}}$ be a simply connected complex simple Lie group whose Lie algebra is $\mathfrak{g}^{\mathbb{C}}$ and U the parabolic subgroup of $G^{\mathbb{C}}$ generated by $\mathfrak{u}$. As is well known, the complex homogeneous manifold $M = G^{\mathbb{C}}/U$ is compact simply connected and G acts transitively on M. Note also that $K = G \cap U$ is a connected closed subgroup of G and $M = G/K$ as C^∞-manifold and M admits a G-invariant Kähler metric (cf. [4], [7]). Hence, M is a Kähler C-space.

We take a Weyl basis $E_\alpha \in \mathfrak{g}^{\mathbb{C}}_\alpha$ ($\alpha \in \Delta$) with

$$[E_\alpha, E_{-\alpha}] = -\alpha \qquad (\alpha \in \Delta)$$

$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha,\beta} E_{\alpha+\beta}, & \text{if } \alpha, \beta, \alpha+\beta \in \Delta \\ 0, & \text{if } \alpha, \beta \in \Delta, \alpha+\beta \notin \Delta. \end{cases}$$

where $0 \neq N_{\alpha,\beta} = N_{-\alpha,-\beta} \in \mathbb{R}$ $(\alpha, \beta, \alpha+\beta \in \Delta)$ such that

$$\mathfrak{g} = \mathfrak{t} + \sum_{\alpha \in \Delta} \{\mathbb{R}(E_\alpha + E_{-\alpha}) + \mathbb{R}\sqrt{-1}(E_\alpha - E_{-\alpha})\}.$$

Then the Lie algebra $\mathfrak{k}$ of K is given by

$$\mathfrak{k} = \mathfrak{t} + \sum_{\alpha \in [\Pi_0]} \{\mathbb{R}(E_\alpha + E_{-\alpha}) + \mathbb{R}\sqrt{-1}(E_\alpha - E_{-\alpha})\}.$$

For positive integers $k_1, \cdots, k_r$, we put

$$\Delta(k_1, \cdots, k_r) = \{\sum_{j=1}^{\ell} m_j \alpha_j \in \Delta^+ \mid m_{i_1} = k_1, \cdots, m_{i_r} = k_r\}.$$

For $\Delta(k_1, \cdots, k_r) \neq \emptyset$, we define an $\mathrm{Ad}_G(K)$-invariant subspace $\mathfrak{m}(k_1, \cdots, k_r)$ of $\mathfrak{g}$ by

$$\mathfrak{m}(k_1, \cdots, k_r) = \sum_{\alpha \in \Delta(k_1, \cdots, k_r)} \{\mathbb{R}(E_\alpha + E_{-\alpha}) + \mathbb{R}\sqrt{-1}(E_\alpha - E_{-\alpha})\}.$$

Denote by B the negative of the Killing form of $\mathfrak{g}$. Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B. Then

$$\mathfrak{m} = \sum_{k_1, \cdots, k_r} \mathfrak{m}(k_1, \cdots, k_r)$$

is a B-orthogonal decomposition of $\mathfrak{m}$. If $r = 1$, Omura [6] proved that each $\mathfrak{m}(k_1)$ is irreducible as $\mathrm{Ad}_G(K)$-module. We give a sufficient condition for the irreducibility of $\mathfrak{m}(k_1, \cdots, k_r)$ as $\mathrm{Ad}_G(K)$-module below (cf. [6]).

Let $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{m}(k_1, \cdots, k_r)^{\mathbb{C}}$ be the complexifications of $\mathfrak{k}$ and $\mathfrak{m}(k_1, \cdots, k_r)$ respectively. Then

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in [\Pi_0]} \mathfrak{g}_\alpha^{\mathbb{C}}$$

$$\mathfrak{m}(k_1, \cdots, k_r)^{\mathbb{C}} = \mathfrak{m}^+(k_1, \cdots, k_r) + \mathfrak{m}^-(k_1, \cdots, k_r)$$

where $\mathfrak{m}^{\pm}(k_1, \cdots, k_r) = \sum_{\alpha\in\Delta(k_1,\cdots,k_r)} \mathfrak{g}^{\mathbf{C}}_{\mp\alpha}$. Let $\mathfrak{k}'$ be the semi-simple part of the complex reductive Lie algebra $\mathfrak{k}^{\mathbf{C}}$; i.e,

$$\mathfrak{k}' = [\mathfrak{k}^{\mathbf{C}}, \mathfrak{k}^{\mathbf{C}}] = \mathfrak{h}' + \sum_{\alpha\in[\Pi_0]} \mathfrak{g}^{\mathbf{C}}_{\alpha}$$

where $\mathfrak{h}' = \sum_{\alpha\in\Pi_0} \mathbf{C}\alpha$. Note that each $\mathfrak{m}^{\pm}(k_1, \cdots, k_r)$ is an $\mathrm{ad}_{\mathfrak{g}^{\mathbf{C}}}(\mathfrak{k}')$-invariant subspace of $\mathfrak{g}^{\mathbf{C}}$. Now we have the following lemma.

Lemma 2.2. *For each $\mathfrak{m}(k_1, \cdots, k_r)$, the following* (1) (2) (3) *are equivalent.*

(1) $(\mathrm{ad}_{\mathfrak{g}}|_{\mathfrak{k}}, \mathfrak{m}(k_1, \cdots, k_r))$ *is a real irreducible representation of $\mathfrak{k}$.*
(2) $(\mathrm{ad}_{\mathfrak{g}^{\mathbf{C}}}|_{\mathfrak{k}'}, \mathfrak{m}^{+}(k_1, \cdots, k_r))$ *is a complex irreducible representation of $\mathfrak{k}'$.*
(3) $(\mathrm{ad}_{\mathfrak{g}^{\mathbf{C}}}|_{\mathfrak{k}'}, \mathfrak{m}^{-}(k_1, \cdots, k_r))$ *is a complex irreducible representation of $\mathfrak{k}'$.*

Proof. We prove only the equivalence between (1) and (2). Since $(\mathrm{ad}_{\mathfrak{g}^{\mathbf{C}}}|_{\mathfrak{k}}, \mathfrak{m}^{+}(k_1, \cdots, k_r))$ is equivalent to $(\mathrm{ad}_{\mathfrak{g}}|_{\mathfrak{k}}, \mathfrak{m}(k_1, \cdots, k_r))$ as real representation of $\mathfrak{k}$, we get (1) $\Rightarrow$ (2). Conversely if $\mathfrak{m}_1$ is a non-trivial $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$-invariant subspace of $\mathfrak{m}(k_1, \cdots, k_r)$, then there exists a non-trivial subset Δ_1 of Δ such that

$$\mathfrak{m}_1 = \sum_{\alpha\in\Delta_1} \{\mathbf{R}(E_\alpha + E_{-\alpha}) + \mathbf{R}\sqrt{-1}(E_\alpha - E_{-\alpha})\}.$$

Since $\sum_{\alpha\in\Delta_1} \mathfrak{g}^{\mathbf{C}}_{-\alpha}$ is a non-trivial $\mathrm{ad}_{\mathfrak{g}^{\mathbf{C}}}(\mathfrak{k}')$-invariant subspace of $\mathfrak{m}^{+}(k_1, \cdots, k_r)$, hence we get (2) $\Rightarrow$ (1). Q.E.D.

We can consider that an element of $\Delta(k_1, \cdots, k_r)$ is a weight of the representation $(\mathrm{ad}_{\mathfrak{g}^{\mathbf{C}}}|_{\mathfrak{k}'}, \mathfrak{m}^{-}(k_1, \cdots, k_r))$ of $\mathfrak{k}'$ relative to $\mathfrak{h}'$. Thus $\mathfrak{m}^{-}(k_1, \cdots, k_r) = \sum_{\alpha\in\Delta(k_1,\cdots,k_r)} \mathfrak{g}^{\mathbf{C}}_{\alpha}$ is the decomposition into the weight spaces.

Lemma 2.3. *Suppose that there exists $\beta_0 \in \Delta(k_1, \cdots, k_r)$ satisfying the following properties:* (1) *$\beta_0 + \alpha_i \notin \Delta$ for any $\alpha_i \in \Pi_0$,* (2) *if α is an element of $\Delta(k_1, \cdots, k_r)$, either $\beta_0 - \alpha \in \Delta$ or there exist $\beta_1, \beta_2 \in [\Pi_0] \cap \Delta^{+}$ such that $\beta_0 - \alpha = \beta_1 + \beta_2$ and $\beta_0 - \beta_1 \in \Delta$. Then $\mathfrak{m}(k_1, \cdots, k_r)$ is $\mathrm{Ad}_G(K)$-irreducible.*

Proof. Since E_{β_0} is primitive, the $\mathrm{ad}_{\mathfrak{g}^{\mathbf{C}}}(\mathfrak{k}')$-submodule W of $\mathfrak{m}^{-}(k_1, \cdots, k_r)$ generated by E_{β_0} is irreducible. For $\alpha \in \Delta(k_1, \cdots, k_r)$, if $\beta_0 - \alpha \in$

Δ, $E_{\alpha-\beta_0} \in \mathfrak{k}^{\mathbb{C}}$ and thus $[E_{\alpha-\beta_0}, E_{\beta_0}] = \lambda E_\alpha \, (0 \neq \lambda \in \mathbb{C})$. Hence $E_\alpha \in W$. If $\beta_0 - \alpha \notin \Delta$, there are β_1, β_2 such that $E_{-\beta_1}, E_{-\beta_2} \in \mathfrak{k}^{\mathbb{C}}$ and $\beta_0 - \beta_1 \in \Delta$, and thus $[E_{-\beta_2}, [E_{-\beta_1}, E_{\beta_0}]] = \mu E_\alpha \, (0 \neq \mu \in \mathbb{C})$. Hence $E_\alpha \in W$. Thus $\mathfrak{m}(k_1, \cdots, k_r)$ is $\mathrm{Ad}_G(K)$-irreducible from Lemma 2.2. Q.E.D.

Remark 2.4. Note that $\mathfrak{m}(k_1, \cdots, k_r)$ are non-equivalent each other and $\overline{\mathfrak{m}^+(k_1, \cdots, k_r)} = \mathfrak{m}^-(k_1, \cdots, k_r)$.

We put $\delta_{\mathfrak{m}} = \frac{1}{2} \sum_{\alpha \in \Delta^+ - [\Pi_0]} \alpha$. Then

$$\delta_{\mathfrak{m}} = c_{i_1} \Lambda_{i_1} + \cdots + c_{i_r} \Lambda_{i_r} \tag{2.5}$$

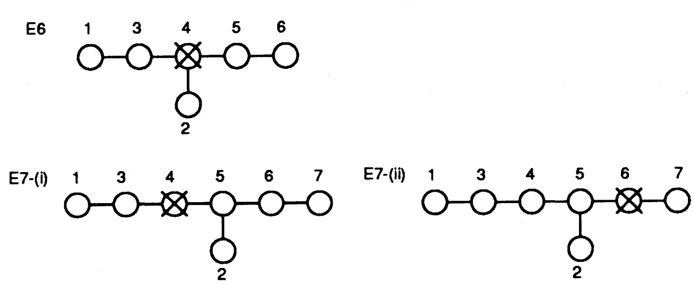
where $c_{i_1}, \cdots, c_{i_r} > 0$ (cf. Borel-Hirzebruch [2]). Let $\tilde{\alpha}$ be the highest root of Δ and

$$\tilde{\alpha} = \sum_{i=1}^{\ell} m_i \alpha_i \qquad (0 \leqq m_i \in \mathbb{Z}).$$

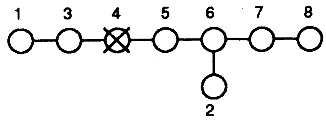
Now we construct our Kähler C-spaces $M = G/K$. If we regard M as the complex manifold $G^{\mathbb{C}}/U$, M is represented by the pair (Π, Π_0) of the Dynkin diagram. Our Kähler C-space $M = G/K$ is represented by the pair (Π, Π_0) such that either $\Pi - \Pi_0 = \{\alpha_p\}$ where $m_p = 3$ or $\Pi - \Pi_0 = \{\alpha_p, \alpha_q\}$ where $m_p = m_q = 1$. Next proposition can be easily checked by Lemma 2.3, Remark 2.4 and (2.5).

Proposition 2.6. *Let G be a compact connected simple Lie group corresponding to the following Dynkin diagram Π and $G^{\mathbb{C}}/U$ the complex manifold corresponding to the following pair (Π, Π_0). Put $K = G \cap U$. Then G/K is a Kähler C-space and the isotropy representation of G/K is decomposed into non-equivalent three irreducible components.*

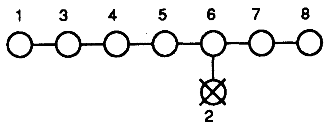
[I] (Π, Π_0) :



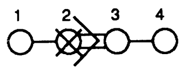
E8-(i)



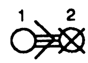
E8-(ii)



F4

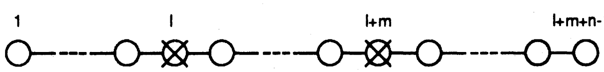


G2

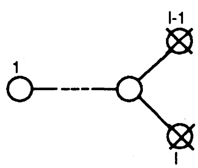


[II] (Π, Π_0) :

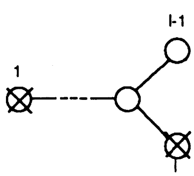
Al+m+n-1



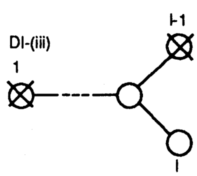
Dl-(i)



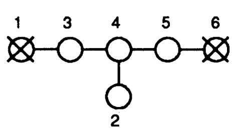
Dl-(ii)



Dl-(iii)



E6



where the vertices contained in $\Pi - \Pi_0$ *are denoted by* "$\times$".

Moreover, in the case [I], *the triple* $(|\Delta(1)|, |\Delta(2)|, |\Delta(3)|)$ *is*

(1) $(18, 9, 2)$, *for* G *of type* E_6,
(2) $(30, 15, 4)$, *for* G *of type* E_7-(i),
(3) $(30, 15, 2)$, *for* G *of type* E_7-(ii),
(4) $(54, 27, 2)$, *for* G *of type* E_8-(i),
(5) $(56, 28, 8)$, *for* G *of type* E_8-(ii),
(6) $(12, 6, 2)$, *for* G *of type* F_4,
(7) $(2, 1, 2)$, *for* G *of type* G_2,

and, in the case [II], *the quadruple* $(|\Delta(1,0)|, |\Delta(0,1)|, |\Delta(1,1)|, \delta_{\mathrm{m}})$ *is*

(1) $\left(\ell m,\ mn,\ \ell n,\ \frac{\ell+m}{2}\Lambda_\ell + \frac{m+n}{2}\Lambda_{\ell+m}\right)$,
for G *of type* $A_{\ell+m+n-1}$,
(2) $\left(\ell-1,\ \ell-1,\ \frac{(\ell-1)(\ell-2)}{2},\ \frac{\ell}{2}\Lambda_{\ell-1} + \frac{\ell}{2}\Lambda_\ell\right)$,
for G *of type* D_ℓ-(i),
(3) $\left(\ell-1,\ \frac{(\ell-1)(\ell-2)}{2},\ \ell-1,\ \frac{\ell}{2}\Lambda_1 + (\ell-2)\Lambda_\ell\right)$,
for G *of type* D_ℓ-(ii),
(4) $\left(\ell-1,\ \frac{(\ell-1)(\ell-2)}{2},\ \ell-1,\ \frac{\ell}{2}\Lambda_1 + (\ell-2)\Lambda_{\ell-1}\right)$,
for G *of type* D_ℓ-(iii),
(5) $(8,\ 8,\ 8,\ 4\Lambda_1 + 4\Lambda_6)$,
for G *of type* E_6,

where $1 \leqq \ell, m, n \in \mathbb{Z}$ *in the case of type* $A_{\ell+m+n-1}$ *and* $4 \leqq \ell \in \mathbb{Z}$ *in the case of type* D_ℓ.

§3. G-invariant Einstein metrics

In this section we find all G-invariant Einstein metrics on the Kähler C-spaces of Proposition 2.6. We will use the same notation as in § 2. The next theorem is well-known.

Theorem 3.1 (Borel-Hirzebruch [2], cf. [7]). *Let* $M = G/K$ *be* *the Kähler C-space in Proposition* 2.6. *We put*

$$g = \begin{cases} B|_{\mathfrak{m}(1)} + 2B|_{\mathfrak{m}(2)} + 3B|_{\mathfrak{m}(3)}, & \text{in the case [I]}, \\ c_p B|_{\mathfrak{m}(1,0)} + c_q B|_{\mathfrak{m}(0,1)} + (c_p + c_q) B|_{\mathfrak{m}(1,1)}, & \text{in the case [II]}, \end{cases}$$

where $\delta_{\mathfrak{m}} = c_p \Lambda_p + c_q \Lambda_q$ *in the case* [II]. *Then* g *is a unique* G*-invariant Einstein-Kähler metric on* $M = G^{\mathbb{C}}/U$ *up to homotheties, where we consider the natural complex structure on* $G^{\mathbb{C}}/U$.

We obtain the following theorem by Theorem 1.5.

Theorem 3.2. *Let* $M = G/K$ *be the Kähler C-space in Proposition* 2.6. *In the case* [I], M *has three* G*-invariant Einstein metrics up to homotheties. In the case* [II], M *has four* G*-invariant Einstein metrics* g, *up to homotheties, expressed explicitly in the form*

$$g = x_1 B|_{\mathfrak{m}(1,0)} + x_2 B|_{\mathfrak{m}(0,1)} + x_3 B|_{\mathfrak{m}(1,1)}$$

where (x_1, x_2, x_3) *is given as follows:*
If G *is of type* $A_{\ell+m+n-1}$,
(1) $(\ell+m,\ m+n,\ \ell+2m+n)$, (2) $(\ell+m,\ m+n,\ \ell+n)$,
(3) $(\ell+m,\ 2\ell+m+n,\ \ell+n)$, (4) $(\ell+m+2n,\ m+n,\ \ell+n)$.
If G *is of type* D_ℓ-(i),
(1) $(1, 1, 2)$, (2) $(\ell, \ell, 2\ell-4)$,
(3) $(\ell, 3\ell-4, 2\ell-4)$, (4) $(3\ell-4, \ell, 2\ell-4)$.
If G *is of type* D_ℓ-(ii) *or* D_ℓ-(iii),
(1) $(\ell, 2\ell-4, 3\ell-4)$, (2) $(\ell, 2\ell-4, \ell)$,
(3) $(1, 2, 1)$, (4) $(3\ell-4, 2\ell-4, \ell)$.
If G *is of type* E_6,
(1) $(1, 1, 2)$, (2) $(1, 1, 1)$, (3) $(1, 2, 1)$, (4) $(2, 1, 1)$.
Moreover, in each type, the case (1) *is a Kähler metric on* $G^{\mathbb{C}}/U$.

Proof. First we consider the case [I]. We put $g = x_1 B|_{\mathfrak{m}(1)} + x_2 B|_{\mathfrak{m}(2)} + x_3 B|_{\mathfrak{m}(3)}$ $(x_1, x_2, x_3 > 0)$. Then we get the following from (1.4).

$$S(g) = \sum_i \frac{d_i}{x_i} - \frac{1}{4}\{C_{11}^2(\frac{x_2}{x_1{}^2} + \frac{2}{x_2}) + 2C_{12}^3(\frac{x_3}{x_1 x_2} + \frac{x_2}{x_1 x_3} + \frac{x_1}{x_2 x_3})\}$$

where $d_i = |\Delta(i)|$ $(i = 1,2,3)$. Note that d_i $(i = 1,2,3)$ are known by Proposition 2.6. We put $u = x_2/x_1$, $v = x_3/x_1$ and $N = d_1 + d_2 + d_3 = \dim_{\mathbb{C}} M$. By Theorem 1.5, g is Einstein if and only if

$$(3.3) \qquad d_1uv - (d_2 - \frac{1}{2}C_{11}^2)(\frac{N}{d_2} - 1)v + d_3u - \frac{1}{4}C_{11}^2(\frac{N}{d_2} + 1)u^2v$$

$$+\frac{1}{2}C_{12}^3(\frac{N}{d_2} - 1)v^2 - \frac{1}{2}C_{12}^3(\frac{N}{d_2} + 1)u^2 + \frac{1}{2}C_{12}^3(\frac{N}{d_2} - 1) = 0$$

$$(3.4) \qquad d_1uv + (d_2 - \frac{1}{2}C_{11}^2)v - d_3(\frac{N}{d_3} - 1)u - \frac{1}{4}C_{11}^2u^2v$$

$$-\frac{1}{2}C_{12}^3(\frac{N}{d_3} + 1)v^2 + \frac{1}{2}C_{12}^3(\frac{N}{d_3} - 1)u^2 + \frac{1}{2}C_{12}^3(\frac{N}{d_3} - 1) = 0.$$

Since $u = 2, v = 3$ is a common root of (3.3) and (3.4) by Theorem 3.1, we get C_{11}^2 and C_{12}^3. From (3.3) and (3.4), we see that

$$(3.5) \qquad v = \frac{c(u - u_1)(u - u_2)}{(u - u_3)(u - u_4)}$$

where $c > 0$, u_1, u_2, u_3, $u_4 \in \mathbb{R}$. Since $u > 0$, $v > 0$, we get the domain I of u from (3.5). Substitute (3.5) to (3.3) and multiply it by a constant multiple of $(u-u_3)^2(u-u_4)^2/(u-2)$. Then we have an equation $f(u) = 0$, where $f(u)$ is a polynomial of u with an integral coefficient. We have a one-to-one correspondence between the set $\{u = 2\} \cup \{u \in I | f(u) = 0\}$ and the set of G-invariant Einstein metrics on M up to homotheties. Consider the case of type E_6. In this case, we see that

$$C_{11}^2 = 6, C_{12}^3 = 3/2$$

and

$$u_1 = -2, u_2 = 10/11, u_3 = 11/7 + \sqrt{249}/21, u_4 = 11/7 - \sqrt{249}/21.$$

Hence

$$I = (0, u_4) \cup (u_2, u_3)$$

and

$$f(u) = 532u^5 - 3800u^4 + 8809u^3 - 9398u^2 - 4860u - 1000.$$

Now we obtain the following result from Strum's theorem.

$$|\{u \in (0, u_4) | f(u) = 0\}| = 1 \quad \text{and} \quad |\{u \in (u_2, u_3) | f(u) = 0\}| = 1.$$

Therefore M has three G-invariant Einstein metrics up to homotheties. Results for other types in the case [I] are obtained by the same method.

Next we consider the case [II]. We put $g = x_1 B|_{\mathfrak{m}(1,0)} + x_2 B|_{\mathfrak{m}(0,1)} + x_3 B|_{\mathfrak{m}(1,1)}$ $(x_1, x_2, x_3 > 0)$. Then by (1.4)

$$S(g) = \sum_i \frac{d_i}{x_i} - \frac{1}{2} C_{12}^3 \left(\frac{x_3}{x_1 x_2} + \frac{x_2}{x_1 x_3} + \frac{x_1}{x_2 x_3} \right)$$

where $d_1 = |\Delta(1,0)|$, $d_2 = |\Delta(0,1)|$, $d_3 = |\Delta(1,1)|$. By Theorem 1.5, g is Einstein if and only if

$$\begin{aligned} (3.6) \qquad & C_{12}^3 (d_1 + d_3) v^2 + 2 d_2 (d_1 u - d_1 - d_3) v \\ & - C_{12}^3 (d_1 + 2 d_2 + d_3) u^2 + 2 d_2 d_3 u + C_{12}^3 (d_1 + d_3) = 0 \end{aligned}$$

$$\begin{aligned} (3.7) \qquad & - C_{12}^3 (d_1 + d_2 + 2 d_3) v^2 + 2 d_3 (d_1 u + d_2) v \\ & + C_{12}^3 (d_1 + d_2) u^2 - 2 d_3 (d_1 + d_2) u + C_{12}^3 (d_1 + d_2) = 0 \end{aligned}$$

where $u = x_2/x_1$, $v = x_3/x_1$. We put $\delta_{\mathfrak{m}} = c_p \Lambda_p + c_q \Lambda_q$. Note that d_i $(i = 1, 2, 3)$, c_p and c_q are known by Proposition 2.6. Since $u = c_q/c_p$, $v = (c_p + c_q)/c_p$ is a common root of (3.6) and (3.7) by Theorem 3.1, we get C_{12}^3. Therefore we can get all positive common roots (u, v) of (3.6) and (3.7) for each type of the case [II] by the same method as in the case [I]. Q.E.D.

§4. G-invariant complex structures

Let $M = G/K$ be the Kähler C-space in Proposition 2.6. We have a one-to-one correspondence between the set $\mathcal{J}$ of G-invariant complex structures J on M and the set $\mathcal{P}$ of parabolic subgroups P of $G^{\mathbb{C}}$ with $G \cap P = K$. If a G-invariant Einstein metric g on M is Kähler for a complex structure J on M, J is G-invariant. Suppose that $J \in \mathcal{J}$ corresponds to $P \in \mathcal{P}$. Then (M, J) and $G^{\mathbb{C}}/P$ are biholomorphic, where we consider the natural complex structure on $G^{\mathbb{C}}/P$. Thus if we regard (M, J) as $G^{\mathbb{C}}/P$, g is the form of Theorem 3.1 up to homotheties. Hence if a G-invariant Einstein metric is Kähler, it is a known metric.

On the other hand we obtain the following results from Nishiyama [5]. There is a one-to-one correspondence between $\mathcal{J}$ and the set $\mathcal{W}'$ of elements σ of the Weyl group $\mathcal{W}$ with $\sigma(\Pi_0) \subset \Pi$. Suppose that $J \in \mathcal{J}$

corresponds to $\sigma \in \mathcal{W}'$. Then let U_σ be a parabolic subgroup of $G^{\mathbb{C}}$ whose Lie algebra $\mathfrak{u}_\sigma$ is

$$\mathfrak{u}_\sigma = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in [\sigma(\Pi_0)] \cup \Delta^+} \mathfrak{g}_\alpha^{\mathbb{C}}.$$

And let f be the diffeomorphism from M to $G^{\mathbb{C}}/U_\sigma$ induced from the automorphism of $\mathfrak{g}^{\mathbb{C}}$ defined by σ. Then f is a biholomorphic map from (M, J) to $G^{\mathbb{C}}/U_\sigma$. Moreover, $K_\sigma = G \cap U_\sigma$ is a connected closed subgroup of G, $M = G/K_\sigma$ as C^∞-manifold, and f defines a G-equivariant isometry from $(G/K, B|_{\mathfrak{m}})$ to $(G/K_\sigma, B|_{\mathfrak{m}^\sigma})$, where $\mathfrak{m}^\sigma$, $\Delta^\sigma(k_1, \cdots, k_r)$ and $\mathfrak{m}^\sigma(k_1, \cdots, k_r)$ for G/K_σ are corresponding to that of $\mathfrak{m}$, $\Delta(k_1, \cdots, k_r)$ and $\mathfrak{m}(k_1, \cdots, k_r)$ for G/K. G-invariant complex structures J and J' on M are said to be equivalent if the complex manifolds (M, J) and (M, J') are biholomorphic. Let J, J' be G-invariant complex structures on M and let σ, σ' be the elements of $\mathcal{W}'$ corresponding to J, J' respectively. Then J and J' are equivalent if and only if there exists a graph automorphism γ of the Dynkin diagram Π such that $\gamma(\sigma(\Pi_0)) = \sigma'(\Pi_0)$. Moreover, in this case the pairs $(\Pi, \sigma(\Pi_0))$, $(\Pi, \sigma'(\Pi_0))$ of the Dynkin diagrams are called equivalent.

Remark 4.1. Let $M = G/K$ be the Kähler C-space of Proposition 2.6. We put

$$\Delta_1 = \begin{cases} \Delta(1), & \text{if } M \text{ is in the case [I]}, \\ \Delta(1,0), & \text{if } M \text{ is in the case [II]}, \end{cases}$$

$$\Delta_2 = \begin{cases} \Delta(2), & \text{if } M \text{ is in the case [I]}, \\ \Delta(0,1), & \text{if } M \text{ is in the case [II]}, \end{cases}$$

$$\Delta_3 = \begin{cases} \Delta(3), & \text{if } M \text{ is in the case [I]}, \\ \Delta(1,1), & \text{if } M \text{ is in the case [II]}, \end{cases}$$

and we define $\mathfrak{m}_1$, $\mathfrak{m}_2$, $\mathfrak{m}_3$ similarly. Then we get the followings.

(1) Let σ be an element of $\mathcal{W}$ with $\sigma(\Pi_0) \subset \Pi$, and f the above G-equivariant diffeomorphism from G/K to G/K_σ induced by σ. Suppose that g_1 is a G-invariant Riemannian metric on G/K_σ. We put

$$g_1 = x_1 B|_{\mathfrak{m}_1^\sigma} + x_2 B|_{\mathfrak{m}_2^\sigma} + x_3 B|_{\mathfrak{m}_3^\sigma} \qquad (x_1, x_2, x_3 > 0).$$

Then

$$f^* g_1 = x_{\tau(1)} B|_{\mathfrak{m}_1} + x_{\tau(2)} B|_{\mathfrak{m}_2} + x_{\tau(3)} B|_{\mathfrak{m}_3}$$

where $\tau \in \mathfrak{S}_3$ such that $\sigma(\Delta_i) = \pm\Delta^\sigma_{\tau(i)}$ $(i = 1, 2, 3)$.

(2) Let $\{J_1, \cdots, J_n\}$ be the set of all G-invariant complex structures on

M up to equivalence, and $\sigma_1, \cdots, \sigma_n$ the elements of $\mathcal{W}'$ corresponding to $J_1, \cdots, J_n$ respectively. Suppose that $g_1, \cdots, g_n$ are the G-invariant Einstein-Kähler metrics on $G/K_{\sigma_1}, \cdots, G/K_{\sigma_n}$, respectively. For each integer k $(1 \leqq k \leqq n)$, we put

$$g_k = x_1^k B|_{\mathfrak{m}_1^{\sigma_k}} + x_2^k B|_{\mathfrak{m}_2^{\sigma_k}} + x_3^k B|_{\mathfrak{m}_3^{\sigma_k}} \qquad (x_1^k, x_2^k, x_3^k > 0).$$

If g is a G-invariant Einstein-Kähler metric on M, there exist an integer k $(1 \leqq k \leqq n)$ and $\tau \in \mathfrak{S}_3$ such that

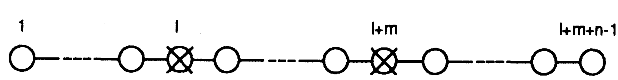
$$g = x_{\tau(1)}^k B|_{\mathfrak{m}_1} + x_{\tau(2)}^k B|_{\mathfrak{m}_2} + x_{\tau(3)}^k B|_{\mathfrak{m}_3}$$

up to homotheties.

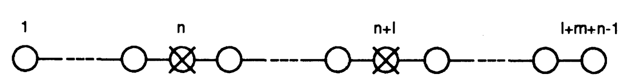
Remark 4.2. Let $M = G/K$ be the Kähler C-space in Proposition 2.6-[I]. Then M has one and only one G-invariant complex structure up to equivalent (cf. [2], [5]). Let $B|_{\mathfrak{m}(1)} + uB|_{\mathfrak{m}(2)} + vB|_{\mathfrak{m}(3)}$ be a G-invariant Einstein metric on M found newly in Theorem 3.2. Then u and v are irrational. Therefore they are not Kähler for any complex structure on M by Theorem 3.1 and Remark 4.1-(2).

When $M = G/K$ is a Kähler C-space of Proposition 2.6-[II], we construct the root system Δ in a subspace of the Euclidean space $\mathbb{R}^N$ of an appropriate dimension N as usual. Let $\{\varepsilon_1, \cdots, \varepsilon_N\}$ be the standard basis of $\mathbb{R}^N$.

Example 4.3. Let $M = G/K$ be the Kähler C-space of type $A_{\ell+m+n-1}$ of Proposition 2.6-[II]. Then $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leqq i \leqq \ell + m + n - 1)$. When we regard M as $G^{\mathbb{C}}/U$, M is represented by the following pair (Π, Π_0) of the Dynkin diagram.
(Π, Π_0) :



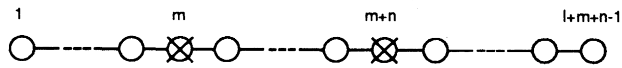
The pairs of the Dynkin diagrams corresponding to G-invariant complex structures on M up to equivalent are as follows
$(\Pi, \sigma(\Pi_0))$:



where $\sigma \in \mathcal{W}$ is defined by a permutation

$$\begin{pmatrix} \varepsilon_1 & \cdots & \varepsilon_{\ell+m} & \varepsilon_{\ell+m+1} & \cdots & \varepsilon_{\ell+m+n} \\ \varepsilon_{n+1} & \cdots & \varepsilon_{\ell+m+n} & \varepsilon_1 & \cdots & \varepsilon_n \end{pmatrix}.$$

$(\Pi, \sigma'(\Pi_0))$:



where $\sigma' \in \mathcal{W}$ is defined by a permutation

$$\begin{pmatrix} \varepsilon_1 & \cdots & \varepsilon_\ell & \varepsilon_{\ell+1} & \cdots & \varepsilon_{\ell+m+n} \\ \varepsilon_{m+n+1} & \cdots & \varepsilon_{\ell+m+n} & \varepsilon_1 & \cdots & \varepsilon_{m+n} \end{pmatrix}.$$

Note that if ℓ, m and n are all distinct, the above three pairs are not equivalent each other. Note also that if ℓ, m and n are not all distinct, there exist the equivalent pairs. By Theorem 3.2,

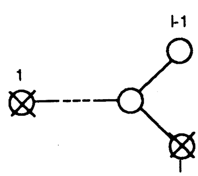
$$(n+\ell)B|_{\mathfrak{m}^\sigma(1,0)} + (\ell+m)B|_{\mathfrak{m}^\sigma(0,1)} + (n+2\ell+m)B|_{\mathfrak{m}^\sigma(1,1)}$$

is an Einstein-Kähler metric on $G^{\mathbb{C}}/U$. Moreover

$$\sigma(\Delta(1,0)) = \Delta^\sigma(0,1),\ \sigma(\Delta(0,1)) = -\Delta^\sigma(1,1),\ \sigma(\Delta(1,1)) = -\Delta^\sigma(1,0).$$

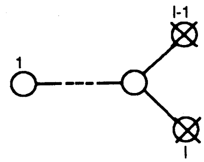
Hence the metric (3) of Theorem 3.2 is Kähler for the G-invariant complex structure corresponding to σ by Remark 4.1-(1). The metric (4) of Theorem 3.2 is Kähler for the G-invatiant complex structure corresponding to σ' similarly. On the other hand, the metric (2) of Theorem 3.2 is not Kähler for any complex structure on M from Theorem 3.2 and Remark 4.1-(2). If $\ell = m = n$, the metric (2) of Theorem 3.2 is the standard metric of G/K, in the sence that it comes from the negative of Killing form.

Example 4.4. Let $M = G/K$ be the Kähler C-space of type D_ℓ of Proposition 2.6-[II]. Then $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leqq i \leqq \ell - 1)$, $\alpha_\ell = \varepsilon_{\ell-1} + \varepsilon_\ell$. Since the Kähler C-spaces defined by the pairs (i), (ii) and (iii) of Proposition 2.6-[II] are isomorphic as G-manifold each other, we regard $G^{\mathbb{C}}/U$ as (i), i.e,
(Π, Π_0):



The pairs of the Dynkin diagrams corresponding to G-invariant complex structures on M up to equivalent are as follows:

$(\Pi, \sigma(\Pi_0))$:



where $\sigma \in \mathcal{W}$ is defined by a permutation

$$\begin{pmatrix} \varepsilon_1 & \cdots & \varepsilon_{\ell-1} & \varepsilon_\ell \\ \varepsilon_2 & \cdots & \varepsilon_\ell & \varepsilon_1 \end{pmatrix}.$$

Then

$$\sigma(\Delta(1,0)) = -\Delta^\sigma(1,0),\ \sigma(\Delta(0,1)) = \Delta^\sigma(1,1),\ \sigma(\Delta(1,1)) = \Delta^\sigma(0,1).$$

Hence the metric (3) of Theorem 3.2-(i) is Kähler for the G-invariant complex structure corresponding to $\sigma \in \mathcal{W}$ by Theorem 3.2-(ii) and Remark 4.1-(1). We define $\sigma' \in \mathcal{W}$ by a permutation

$$\begin{pmatrix} \varepsilon_1 & \cdots & \varepsilon_{\ell-1} & \varepsilon_\ell \\ -\varepsilon_\ell & \cdots & -\varepsilon_2 & \varepsilon_1 \end{pmatrix} \quad \text{if } \ell \text{ is odd,}$$

$$\begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_{\ell-1} & \varepsilon_\ell \\ \varepsilon_\ell & -\varepsilon_{\ell-1} & \cdots & -\varepsilon_2 & \varepsilon_1 \end{pmatrix} \quad \text{if } \ell \text{ is even.}$$

Then if ℓ is odd, the pair $(\Pi, \sigma'(\Pi_0))$ of the Dynkin diagram is the type (ii) of Proposition 2.6. And if ℓ is even, it is the type (iii) of Proposition 2.6. Moreover

$$\sigma'(\Delta(1,0)) = -\Delta^{\sigma'}(1,1), \qquad \sigma'(\Delta(0,1)) = \Delta^{\sigma'}(1,0),$$

$$\sigma'(\Delta(1,1)) = -\Delta^{\sigma'}(0,1).$$

The metric (4) of Theorem 3.2-(i) is Kähler for the complex structure corresponding to $\sigma' \in \mathcal{W}$ by Theorem 3.2-(ii),(iii) and Remark 4.1-(1). On the other hand the metric (2) of Theorem 3.2-(i) is not Kähler for any complex structure on M by Theorem 3.2 and Remark 4.1-(2).

Example 4.5. Let $M = G/K$ be the Kähler C-space of type E_6 of Proposition 2.6-[II]. Then M has one and only one G-invariant complex structure up to equivalent (cf. [5]). The metric (2) of Theorem 3.2 is not Kähler for any complex structure on M by Theorem 3.2 and Remark 4.1-(2). But it is the standard metric of G/K, in the sence that it comes from the negative of Killing form. Now we define automorphisms σ, σ' of Δ by the following:

$$\sigma(\alpha_1) = \alpha_6,\ \sigma(\alpha_2) = \alpha_3,\ \sigma(\alpha_3) = \alpha_5,\ \sigma(\alpha_4) = \alpha_4,\ \sigma(\alpha_5) = \alpha_2,$$

$$\sigma(\alpha_6) = -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6);$$

$$\sigma'(\alpha_1) = -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6),$$

$$\sigma'(\alpha_2) = \alpha_5,\ \sigma'(\alpha_3) = \alpha_2,\ \sigma'(\alpha_4) = \alpha_4,\ \sigma'(\alpha_5) = \alpha_3,\ \sigma'(\alpha_6) = \alpha_1.$$

Then

$$\sigma(\Delta(1,0)) = \Delta(0,1), \qquad \sigma(\Delta(0,1)) = -\Delta(1,1),$$
$$\sigma(\Delta(1,1)) = -\Delta(1,0)$$

and

$$\sigma'(\Delta(1,0)) = -\Delta(1,1), \qquad \sigma'(\Delta(0,1)) = \Delta(1,0),$$
$$\sigma'(\Delta(1,1)) = -\Delta(0,1).$$

We define parabolic subalgebras $\mathfrak{p}$, $\mathfrak{p}'$ of $\mathfrak{g}^{\mathbb{C}}$ by the followings:

$$\mathfrak{p} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in [\Pi_0] \cup [\sigma(\Pi)]^+} \mathfrak{g}_\alpha^{\mathbb{C}}$$

and

$$\mathfrak{p}' = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in [\Pi_0] \cup [\sigma'(\Pi)]^+} \mathfrak{g}_\alpha^{\mathbb{C}}$$

where $[\sigma(\Pi)]^+$ and $[\sigma'(\Pi)]^+$ are the sets of all positive roots relative to $\sigma(\Pi)$ and $\sigma'(\Pi)$ respectively. Let P, P' be the parabolic subgroups of $G^{\mathbb{C}}$ corresponding to $\mathfrak{p}$, $\mathfrak{p}'$ respectively, and let J, J_σ and $J_{\sigma'}$ be the G-invariant complex structures on M corresponding to the natural complex structures on $G^{\mathbb{C}}/U$, $G^{\mathbb{C}}/P$ and $G^{\mathbb{C}}/P'$ respectively (cf. [5]). Let f and f' be the G-equivariant diffeomorphisms on M defined by σ and σ' respectively. Then f and f' are biholomorphic maps from (M, J) to (M, J_σ) and $(M, J_{\sigma'})$ respectively. On the other hand, the pairs (Π, Π_0), $(\sigma(\Pi), \Pi_0)$ and $(\sigma'(\Pi), \Pi_0)$ of the Dynkin diagrams are all the same. Hence the metrics (3) and (4) of Theorem 3.2 are Kähler metrics on (M, J_σ) and $(M, J_{\sigma'})$ respectively by Theorem 3.2 (cf. Remark 4.1-(1)).

From above, we get our Main Theorem.

References

[1] A. L. Besse, "Einstein Manifolds", Springer Verlag, Berlin, 1987.

[2] A. Borel and F. Hirzebruch, Characteristic classes and homogeneous spaces I, Amer. J. Math., **80** (1958), 458–538.

[3] N. Bourbaki, "Groupes et algèbres de Lie, Chap. 4-6", Hermann, Paris, 1968.

[4] S. Murakami, "Compact complex homogeneous manifolds and induced representations", Lecture Notes, Inst. Math. Nat. Tsing Hua Univ., 1985.

[5] M. Nishiyama, Classification of invariant complex structures on irreducible compact simply connected coset spaces, Osaka J. Math., **21** (1984), 39–58.

[6] I. Omura, On Einstein metrics of certain homogeneous spaces, master thesis, Osaka Univ. (1987), in Japanese.

[7] M. Takeuchi, Homogeneous Kähler submanifolds in complex projective spaces, Japan. J. Math., **4** (1977), 171–219.

[8] M. Wang, Some examples of homogeneous Einstein manifolds in dimension seven, Duke Math. J., **49** (1982), 23–28.

[9] M. Wang and W. Ziller, On normal homogeneous Einstein manifolds, Ann. Sc. Ec. Norm. Sup., **18** (1985), 563–633.

[10] ———, Existence and non-existence of homogeneous Einstein metrics, Invent. Math., **84** (1986), 171–194.

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