Advanced Studies in Pure Mathematics 18-I, 1990
Recent Topics in Differential and Analytic Geometry
pp. 239–249

Compact Kähler Manifolds with Parallel Ricci Tensor

Ichiro Enoki

Introduction

Kähler manifolds with parallel Ricci tensor, which are called generalized Einsten Kähler manifolds in some literature, are locally direct products of Einstein Kähler manifolds. In this note we shall study global structures of a compact Kähler manifold locally decomposable as an isometric product of Ricci-positive, Ricci-flat and Ricci-negative parts:

Theorem. *Suppose that the universal covering space $\tilde{X}$ of a compact Kähler manifold X is (isometrically biholomorphic to) a Kählerian direct product $P \times F \times N$, where P, F, N are Kähler manifolds with positive, zero, negative Ricci curvature respectively. Then*

a) *X is a holomorphic fiber bundle $X \to Y$ with fiber P over a compact Kähler manifold Y, and*

b) *Y admits a holomorphic map $Y \to Z$ onto a compact Kähler V-manifold Z such that the universal covering space of a typical fiber is F.*

If P reduces to a point, namely $\tilde{X} \cong F \times N$, our theorem gives a partial affirmative answer to the Abundance Conjecture (see, e.g., a survey article [KMM]). For a compact Riemannian manifold with non-positive sectional curvature, an analogous result was obtained by Eberlein [E, Cor. 2].

The problem treated in this note can be regarded as an existence problem of compact leaves for the foliations on X and Y induced respectively by the projection $P \times F \times N \to F \times N$ and $F \times N \to N$. The compactness of leaves in P-directions, a) in Theorem, can be derived only from the positivity of Ricci curvature in these directions. On the other hand, the compactness of leaves in F-directions, b) in Theorem, depends on both tangential and transversal structures of the foliation. We illustrate this point with the following two examples.

Received April 5, 1990.

The first example is the foliation on X induced by the projection $P \times F \times N \to P$. Let $\tilde{X} = \mathbb{C} \times \mathbb{P}^1$ (the direct product of flat complex Euclidean line $\mathbb{C}$ and complex projective line $\mathbb{P}^1$ with the Fubini-Study metric). Note that our $\mathbb{P}^1$ is isometric to the standard 2-sphere S^2. Define holomorphic isometries σ and τ of $\tilde{X}$ by

$$\sigma\colon (z,p) \in \mathbb{C} \times \mathbb{P}^1 \mapsto (z+1,\, p),$$
$$\tau\colon (z,p) \in \mathbb{C} \times \mathbb{P}^1 \mapsto (z+\omega,\, \rho(p)),$$

where $\operatorname{Im}\omega > 0$, and $\rho\colon \mathbb{P}^1 \to \mathbb{P}^1$ is a rotation of $\mathbb{P}^1 = S^2$ by an irrational angle. Then σ and τ generate a group Γ whose action on $\tilde{X}$ is properly discontinuous and free of fixed points. Let $X := \tilde{X}/\Gamma$. Then every leaf of the foliation on X induced by the projection $\mathbb{C} \times \mathbb{P}^1 \to \mathbb{P}^1$ is isomorphic to $\mathbb{C}$.

The second example is a foliation on the Inoue surface S without curves [I]. This surface S has $\mathbb{C} \times \mathbb{H}$ as the universal covering space, where $\mathbb{H}$ is the upper half plane in $\mathbb{C}$. The Poincaré metric on $\mathbb{H}$ induces a d-closed (1, 1)-form on S. This form defines a Ricci-negative metric structure transverse to the foliation induced by the projection $\mathbb{C} \times \mathbb{H} \to \mathbb{H}$; each leaf is biholomorphic to a (non-compact) Ricci-flat Kähler manifold $\mathbb{C}$. Thus S satisfies the assumption of our theorem except that the Ricci-flat Kähler metrics on leaves are not induced by any Kähler metric on S. (In fact S is a non-Kähler manifold).

This note is organized as follows. In Section 1 we shall prove a) in Theorem. In Sections 2 and 3 we collect arguments which uses the negativity of the Ricci curvature in N-directions. We complete the proof in Section 4. Appendix contains a few propositions on properties of isometries, which may be known.

Acknowledgement. The author wishes to express his gratitude to Professors T. Mabuchi and M. Takeuchi, who pointed out gaps in the first version of this note.

§1. Ricci-positive directions

First we fix notations. For a Kähler manifold M, let $\mathrm{HIsom}(M)$ denote the group of holomorphic isometries of M. Let X be as in the theorem. We identify the fundamental group of X with the covering transformation group $\Gamma_{\tilde{X}} \subset \mathrm{HIsom}(\tilde{X})$ of $\tilde{X}$ over X. Note that $\mathrm{HIsom}(\tilde{X}) = \mathrm{HIsom}(P) \times \mathrm{HIsom}(F) \times \mathrm{HIsom}(N)$. Via the projections of the direct product $\tilde{X} = P \times F \times N$, the Kähler form ω_X of X induces Kähler forms ω_P, ω_F and ω_N on P, F and N respectively.

Now we prove a) in Theorem. By assumption, at each point of X the Ricci curvature of X is positive in the direction of P. Since X is compact, it follows that there is a positive constant c such that $\sqrt{-1}\,\mathrm{Ric}(\omega_P) \geq c\,\omega_P$ at each point of P, where $\mathrm{Ric}(\omega_P)$ is the Ricci tensor of ω_P. Therefore P is compact by the classical theorem of Myers [M]. Let $\mathrm{pr}\colon \Gamma_{\tilde{X}} \to \mathrm{HIsom}(F \times N)$ be a homomorphism induced by the projection $\mathrm{HIsom}(\tilde{X}) \to \mathrm{HIsom}(F \times N)$. Supposing now that $\mathrm{pr}(\Gamma_{\tilde{X}})$ has a fixed point $z \in F \times N$, let

$$\Gamma' = \{\gamma \in \Gamma_{\tilde{X}} \mid \mathrm{pr}(\gamma)(z) = z\}.$$

Since the action of $\Gamma_{\tilde{X}}$ is of fixed point free, so is the action of Γ' on $P \times \{z\}$. Hence the quotient space $(P \times \{z\})/\Gamma'$ is again a compact Ricci-positive Kähler manifold, which is simply connected by a theorem of Kobayashi [Kb1], in contradiction. Therefore the action of $\mathrm{pr}(\Gamma_{\tilde{X}})$ on $F \times N$ has no fixed points. Moreover, since P is compact, this action is properly discontinuous. Thus the projection $P \times F \times N \to F \times N$ defines a structure of a holomorphic fiber bundle $X \to Y$ over a compact Kähler manifold $Y := (F \times N)/\,\mathrm{pr}(\Gamma_{\tilde{X}})$.

§2. Bochner's vanishing theorem

For the proof of b) in Theorem we review first Bochner's vanishing theorem for holomorphic sections. Let M be an n-dimensional complex manifold with a Hermitian metric $g = \sum_{\alpha,\beta} g_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta$ (in holomorphic local coordinates $z^1, \ldots, z^n$ of M). Let $E \to M$ be a holomorphic vector bundle over M with a Hermitian metric h. Let

$$\Omega = \sum_{i,j,\alpha,\beta} \Omega^i_{j\alpha\bar{\beta}} s_i \otimes s^*_j \otimes dz^\alpha \wedge d\bar{z}^\beta$$

be the curvature of the Hermitian connection of (E, h), where $(s_1, \ldots, s_r)$ is a local holomorphic frame of E and $(s^*_1, \ldots, s^*_r)$ is its dual frame. The *mean curvature* of the Hermitian vector bundle (E, h) over (M, g) is defined by

$$K := \sum_{i,j,\alpha,\beta} g^{\alpha\bar{\beta}} \Omega^i_{j\alpha\bar{\beta}} s_i \otimes s^*_j,$$

where $(g^{\alpha\bar{\beta}})$ is the inverse matrix of $(g_{\alpha\bar{\beta}})$. Note that any eigen value of K is a real number everywhere.

Fact 2.1 (see [Kb2, p. 52, Th. (1.9)]). *Let (E, h) be a Hermitian vector bundle over a compact Hermitian manifold (M, g). Suppose all* *eigen values of the mean curvature K of (E,h) are non-positive at each point of M. Then every holomorphic section $\xi \in H^0(M,E)$ of E is parallel and $h(K(\xi),\xi)=0$ everywhere. In particular, if K is negative definite at each point, then $H^0(M,E)=0$.*

§3. Ricci-negative directions

Let W be a compact Kähler manifold whose universal covering space $\widetilde{W}$ is a Kählerian direct product $U \times N$. Suppose

$$\mathrm{HIsom}(U \times N) = \mathrm{HIsom}(U) \times \mathrm{HIsom}(N)$$

and let $\mathrm{pr}_N\colon \mathrm{HIsom}(U) \times \mathrm{HIsom}(N) \to \mathrm{HIsom}(N)$ be the projection homomorphism. Let $\Gamma_{\widetilde{W}}$ be the covering transformation group of $\widetilde{W}$ over W. Then $\Gamma_{\widetilde{W}} \subset \mathrm{HIsom}(U) \times \mathrm{HIsom}(N)$. Let $\Gamma_N = \mathrm{pr}_N(\Gamma_{\widetilde{W}})$.

Theorem 3.1. *If the Ricci curvature of N is negative definite everywhere, then every abelian normal subgroup of Γ_N is discrete.*

Proof. Suppose there is an abelian normal subgroup $H \subset \Gamma_N$ which is not discrete. Let $\bar{H}$ be the closure of H in $\mathrm{HIsom}(N)$, and $\bar{H}^0$ its identity component. $\bar{H}^0$ is a Lie subgroup of $\mathrm{HIsom}(N)$. Since H is a normal subgroup, $\bar{H}^0$ is stable under the action of $\mathrm{Ad}(\Gamma_N)$. Since $\bar{H}^0$ is abelian, generic orbits of $\bar{H}^0$ are covered by $\bar{H}^0$ (see Prop. A in Appendix). Let $\mathfrak{h}$ be the Lie algebra of $\bar{H}^0$ and $\tilde{\mathfrak{h}}$ the image of the following map:

$$V \in \mathfrak{h} \to V - \sqrt{-1}JV \in H^0(N,TN)$$

identifying $V \in \mathfrak{h}$ with a vector field on N, as usual. Then $\bigwedge^l \tilde{\mathfrak{h}} \neq 0$ in $H^0(N, \bigwedge^l TN)$ where $l = \dim \bar{H}^0$. Since $\tilde{\mathfrak{h}}$ is also $\mathrm{Ad}(\Gamma_N)$-stable, $\bigwedge^l \tilde{\mathfrak{h}}$ defines a flat line bundle L on W and a non-trivial holomorphic map $L \to \bigwedge^l TN$, i.e., $H^0(W, L^* \otimes \bigwedge^l T_N) \neq 0$, where L^* is the dual of L. Since W is Kähler, L^* admits a flat hermitian metric. Then the mean curvature of $L^* \otimes \bigwedge^l T_N$ with the induced metric is negative definte at each point. This contradicts to Bochner's vanishing theorem Fact 2.1.

Q.E.D.

§4. Ricci-flat directions

Let Y be as in b) of Theorem. Let $\tilde{Y} = F \times N$ denote the universal covering space of Y. Corresponding to the decomposition $\tilde{Y} = F \times N$, the holomorphic tangent bundle TY of Y decomposes into a direct sum $T_F \oplus T_N$ of holomorphic vector subbundles T_F, $T_N \subset T\tilde{Y}$. Note that the mean curvatures of T_F and T_N with induced metrics are zero and negative definite everywhere respectively. Let T_F^* and T_N^* be the dual bundles of T_F and T_N respectively. We first prove two basic propositions.

Proposition 4.1. *The following natural pairing is perfect:*

$$H^0(Y, T_F^*) \times H^0(Y, T_F) \to H^0(Y, \mathcal{O}) = \mathbb{C}.$$

Moreover we have $H^0(Y, T_N) = 0$ *and* $H^0(Y, T_F) \cong H^0(Y, TY)$.

Proof. Since the mean curvature of T_N is negative definite, it follows from Bochner's vanishing theorem Fact 2.1 that $H^0(Y, T_N) = 0$.

Let h be the Hermitian metric of T_F induced by the Kähler metric on Y. Let $\langle\ ,\ \rangle$ be the natural pairing of T_F^* and T_F. Define a linear map $i\colon T_F \to T_F^*$ by

$$\langle i(\xi), \eta\rangle = h(\xi, \eta) \quad \xi,\ \eta \in (T_F)_y,\ y \in Y.$$

By Bochner's vanishing theorem Fact 2.1, all holomorphic sections of T_F and T_F^* over Y are parallel. Therefore the linear map i induces a surjection $H^0(Y, T_F) \to H^0(Y, T_F^*)$, since $\langle i(\xi), \xi\rangle \neq 0$ for non-zero $\xi \in H^0(Y, T_F)$. Q.E.D.

Proposition 4.2. *A certain finite unramified covering Y' of Y is a Kählerian direct product $Y' = F_1 \times Y_1$ such that*

a) *F_1 is a compact simply connected Ricci-flat Kähler manifold;*

b) *the universal covering space $\tilde{Y}_1$ of Y_1 is the Kählerian direct product $\tilde{Y}_1 = \mathbb{C}^m \times N$ of the flat complex Euclidean space $\mathbb{C}^m$, $m \geq 0$, and N.*

Proof. In view of the de Rham decomposition theorem, F can be written as a Riemannian direct product $F = \mathbb{R}^k \times F_1$, where $\mathbb{R}^k$, $k \geq 0$, denote the flat (real) Euclidean space and F_1 has no Euclidean factors. Since F is Kählerian, this decomposition is Kählerian; in particular $\mathbb{R}^k$ is in fact a complex Euclidean space $\mathbb{C}^m$, $2m = k$. Assume now that F_1 is compact. Then, since F_1 is simply connected and Kählerian, we have $b_1(F_1) = 0$. It follows by Proposition 4.1 that F_1 has no non-zero holomorphic vector fields, and hence $\mathrm{HIsom}(F_1)$ is finite. Therefore a certain finite covering of Y has the desired properties.

Suppose therefore F_1 is non-compact. Let $\mathrm{pr}_1 : \tilde{Y} = F_1 \times (\mathbb{C}^m \times N) \to F_1$ denote the projection and fix a point $w_o \in \mathbb{C}^m \times N$. Then there is a sequence of points $y_j \in F_1 \times \{w_o\}$ such that $d(\mathrm{pr}_1(y_0), \mathrm{pr}_1(y_j)) \to \infty$ as $j \to \infty$, where $d(\ ,\ )$ is the Riemannian distance on F_1. Set $d_j := d(\mathrm{pr}_1(y_0), \mathrm{pr}_1(y_j))$. Note that d_j equals to the distance between y_0 and y_j. Let $c_j\colon [0, d_j] \to \tilde{Y}$ be a shortest geodesic from y_0 to y_j. Take a compact subset $D \subset \tilde{Y}$ such that D is mapped surjectively to Y by the natural projection $\tilde{Y} \to Y$. Then for each j there is $\gamma_j \in \Gamma$ such that $p_j := \gamma_j(c_j(d_j/2)) \in D$. Let $v_j := \gamma_{j*}\dot{c}_j(d_j/2)$. Taking a subsequence if necessary, the sequence v_j converges to a point, say, $v_\infty \in T\tilde{Y}|_D$; let $p_\infty \in \tilde{Y}$ be its base point. Note that the norms of v_j and hence the norms of v_∞, $\mathrm{pr}_{1*}(v_\infty)$ are 1. Then $\mathrm{pr}_1 \gamma_j c_j$ is a shortest geodesic joining $\mathrm{pr}_1 \gamma_j(y_0)$ and $\mathrm{pr}_1 \gamma_j(y_j)$ with $d(\mathrm{pr}_1 \gamma_j(y_0), \mathrm{pr}_1 \gamma_j(y_j)) = d_j$. Since $d_j \to \infty$ as $j \to \infty$, the geodesic c through $\mathrm{pr}_1(p_\infty)$ with $\dot{c}(\mathrm{pr}_1 p_\infty) = \mathrm{pr}_{1*} v_\infty$ is a ray on F_1. Consequently, by the splitting theorem of Cheeger-Gromoll [CG], F_1 has a non-trivial Euclidean factor. This contradicts to our assumption on F_1 and thus F_1 is compact. Q.E.D.

Let $\Gamma_{\tilde{Y}}$ be the covering transformation group of $\tilde{Y}$ over Y. Then $\Gamma_{\tilde{Y}} \subset \mathrm{HIsom}(F \times N) = \mathrm{HIsom}(F) \times \mathrm{HIsom}(N)$. Let $\mathrm{pr}_F\colon \Gamma_{\tilde{Y}} \to \mathrm{HIsom}(F)$ and $\mathrm{pr}_N\colon \Gamma_{\tilde{Y}} \to \mathrm{HIsom}(N)$ be the homomorphisms induced by the projections. The remaining part of our proof will be treated separately according to the cases as $b_1(Y) = 0$ or > 0.

Case: $b_1(Y) = 0$

Let $\Gamma_N = \mathrm{pr}_N(\Gamma_{\tilde{Y}})$. Assume $b_1(Y) = 0$. By Proposition 4.2 we may assume that $F = \mathbb{C}^m$ with a flat metric. Let $\Gamma^F := \mathrm{pr}_F(\mathrm{Ker}\,\mathrm{pr}_N)$ so that we have an isomorphism $i\colon \Gamma_N \to \Gamma_{\tilde{Y}}/\Gamma^F$. Let $\lambda\colon \mathrm{HIsom}(F) \to \mathrm{U}(m)$ be the projection to the linear part of an affine transformation of $F = \mathbb{C}^m$, and let $L := \lambda(\Gamma^F)$. Moreover, let $T = \bar{L}^0$ be the identity component of the closure of L in $\mathrm{U}(m)$. Then, since $\mathrm{U}(m)$ is compact, $L/(T \cap L)$ is finite. It follows that $\lambda^{-1}(T) \cap \Gamma^F$ has a finite index in Γ^F. Therefore the normalizer of $\lambda^{-1}(T) \cap \Gamma^F$ in $\Gamma_{\tilde{Y}}$ has a finite index in $\Gamma_{\tilde{Y}}$. Thus, taking a finite covering of Y, we may assume $L \subset T$. Let $N(L)$ and $N(T)$ be the normalizers of L and T respectively in $U(m)$.

Since Γ^F acts on $F = \mathbb{C}^m$ discretely, T is a torus (cf. [KN], p. 216, Lemma 5). Let $H := (\lambda \circ \mathrm{pr}_F)^{-1}(L) \subset \Gamma_{\tilde{Y}}$. Then, since L is abelian, $\mathrm{pr}_F[H, H]$ consists only of translations. It follows $\mathrm{pr}_F[[H, H], [H, H]] = 0$, and hence the action of $\mathrm{pr}_N[[H, H], [H, H]]$ on N is discrete since $\Gamma_{\tilde{Y}}$ acts on $\tilde{Y}$ discretely. Thus the identity component $\overline{\mathrm{pr}_N[H, H]}^0$ of the closure of $\mathrm{pr}_N[H, H]$ in $\mathrm{HIsom}(N)$ is abelian. Therefore $\mathrm{pr}_N[H, H] \cap \overline{\mathrm{pr}_N[H, H]}^0$ is an abelian normal subgroup of Γ_N, which is discrete by

Theorem 3.1. Thus $\mathrm{pr}_N H \cap \overline{\mathrm{pr}_N H}^0$ is an abelian normal subgroup of Γ_N, which is discrete by Theorem 3.1. Consequently, the kernel of $\lambda \circ \mathrm{pr}_F \circ i \colon \Gamma_N \to N(L)/L$ is discrete. Note that $N(L) = N(T)$ since L is dense in T. Since T is abelian, we have a decomposition $F = \mathbb{C}^{l_0} \times \mathbb{C}^{l_1} \times \cdots \mathbb{C}^{l_k}$ so that T acts each factor as scalors and that $N(T) \subset \mathrm{CU}(l_0) \times \mathrm{CU}(l_1) \times \cdots \mathrm{CU}(l_k)$, where $\mathrm{CU}(l) = \mathbb{C}^* \times \mathrm{U}(l)$. Let L_j denote the projection L to the j-th factor. Then, by taking projection and then determinants, we obtain homomorphisms

$$\Gamma_{\tilde{Y}} (\to \Gamma_N) \to \mathrm{CU}(l_j)/L_j \to \mathbb{C}^*/L_j, \quad j = 0, \ldots, k,$$

where we identify each element of L_j with its eigen value. Since $b_1(Y) = 0$, every abelian representation of $\Gamma_{\tilde{Y}}$ is finite. Hence, by taking a finite covering of Y, we may assume that the homomorphism above is trivial for every j. Consequently, $\lambda \circ \mathrm{pr}_F \circ i \colon \Gamma_N \to N(T)/T$ has a discrete kernel and its image is in $\mathrm{SU}(l_0) \times \mathrm{SU}(l_1) \times \cdots \times \mathrm{SU}(l_k)$. Letting $\bar{\Gamma}_N^0$ be the identity component of the closure of Γ_N in $\mathrm{HIsom}(N)$, we obtain thus that $\lambda \circ \mathrm{pr}_F \circ i$ induces an injective Lie group homomorphism $j \colon \bar{\Gamma}_N^0 \to \mathrm{SU}(l_0) \times \mathrm{SU}(l_1) \times \cdots \times \mathrm{SU}(l_k)$.

Suppose $\dim \bar{\Gamma}_N^0 > 0$. Let $\mathfrak{h}$ be the Lie algebra of $\bar{\Gamma}_N^0$ and $\tilde{\mathfrak{h}}$ the image of the following homomorphism:

$$V \in \mathfrak{h} \to V - \sqrt{-1}JV \in H^0(N, TN)$$

where J is the complex structure of N and $V \in \mathfrak{h}$ is identified with a vector field on N as usual. Since Γ_N consists of holomorphic isometries, the adjoint action of Γ_N on $\mathfrak{h}$ induces an action on $\tilde{\mathfrak{h}}$. The Killing form B of $\mathrm{SU}(l_0) \times \mathrm{SU}(l_1) \times \cdots \times \mathrm{SU}(l_k)$ defines a Hermitian metric $-j^*B$ on $\tilde{\mathfrak{h}}$ which is $\mathrm{Ad}(\Gamma_N)$-invariant. Thus $\tilde{\mathfrak{h}}$ induces a flat holomorphic vector bundle $E \to Y$ over Y with a holomorphic map $E \to T_N$ induced by the identification $\tilde{\mathfrak{h}} \subset H^0(N, TN)$, i.e., $H^0(Y, E^* \otimes T_N) \neq 0$, where E^* is the dual bundle of E. On the other hand, the mean curvature of $E^* \otimes T_N$ with the induced metric is equal to that of T_N, which is negative everywhere. This contradicts to Bochner's vanishing theorem Fact 2.1.

Thus we obtain that Γ_N is discrete, and hence the action on N is properly discontinous, since Γ_N consists of isometries (see Prop. B in Appendix). Therefore the projection $F \times N \to N$ induces a holomorphic map $Y \to Z$ of Y to a Kähler V-manifold $Z := N/\Gamma_N$.

Case: $b_1(Y) > 0$

In this case, the proof is reduced to lower dimensional cases by the following

Proposition 4.3. *Let Y_1 be a smooth fiber of the Albanese map of Y. Let $\pi\colon \tilde{Y} = F \times N \to Y$ be the covering projection and $\varpi\colon \tilde{Y}_1 \to Y_1$ the universal covering of Y_1. Then F decomposes into a Kählerian direct product $F = F_0 \times F_1$ such that*

a) *$\pi(F_0 \times \{(z_1,p)\})$, $(z_1,p) \in F_1 \times N$, is a compact submanifold of Y;*

b) *Y_1 has a Kähler metric such that the universal covering space $\tilde{Y}_1$ with the induced metric is the Kählerian direct product $\tilde{Y}_1 = F' \times N'$ of Kähler manifolds F' and N' whose Ricci tensors are respectively zero and negative definite everywhere;*

c) *if $\pi(\{z_0\}\times F_1 \times \{p\}) \cap Y_1 \neq \emptyset$ then $\pi(\{z_0\}\times F_1\times\{p\}) = \varpi(F'\times\{p'\})$ for some $p' \in N'$.*

Proof of b) *in Theorem.* Assuming now Proposition 4.3 above, we complete the proof for b) in Theorem, which is an induction on $\dim Y$. The one dimensional case is trivial, and we have already treated the case: $b_1(Y) = 0$. Suppose $b_1(Y) > 0$. We use the notations in Proposition 4.3 above. Then, since Y is Kähler, the Albanese map of Y is non-trivial and hence $\dim Y_1 < \dim Y$. By b) of Proposition 4.3 and by the induction assumption $\varpi(F' \times \{p'\})$ is compact. It follows by a) and c) of Proposition 4.3 that $\pi(F \times \{p\})$ is compact for every $p \in N$. Let Γ^F be the kernel of the projection $\mathrm{pr}_N\colon \Gamma_{\tilde{Y}} \to \mathrm{HIsom}(N)$ and $\Gamma_N := \mathrm{pr}_N(\Gamma_{\tilde{Y}})$. Then F/Γ^F is compact and hence the action of Γ_N on N is properly discontinuous. Thus we have a holomorphic map $Y \to Z$ of Y to a Kähler V-manifold $Z := N/\Gamma_N$ as desired.

Proof of Proposition 4.3. a) Let $\alpha\colon Y \to T$ be the Albanese map of Y. In view of Proposition 4.2, we may assume $F = \mathbb{C}^m$. Then $\mathrm{HIsom}(F)$ is a subgroup of the affine transformation group. Let λ: $\mathrm{HIsom}(F) \to U(m)$ be the projection to the linear part. Then T_F and T_F^* are flat vector bundles with holonomy group isomorphic to $H = (\lambda \circ \mathrm{pr}_F)(\Gamma) \subset U(m)$. By Bochner's vanishing theorem Fact 2.1 every holomorphic section of T_F and T_F^* over Y is parallel. Since the action of the holonomy group to the fiber of T_F is semi-simple, T_F decomposes into a holomorphic and isometric direct sum $T_F = E_0 \oplus E_1$ of a flat vector bundle E_1 with $H^0(Y, E_1) = 0$ and a trivial bundle E_0 with

$$H^0(Y, T_F) = H^0(Y, E_0) \cong H^0(Y, T_F^*).$$

This decompsition induces a Kählerian decomposition $F = F_0 \times F_1$ so that each element of $H^0(Y, E_0)$ induces a parallel vector field on F_0 and that $\Gamma_{\tilde{Y}} \subset \mathrm{HIsom}(F_0) \times \mathrm{HIsom}(F_1) \times \mathrm{HIsom}(N)$. Therefore the Lie algebra of the identity component $\mathrm{HIsom}(Y)^0$ of $\mathrm{HIsom}(Y)$ is identified with $H^0(Y, E_0)$ and each orbit of $\mathrm{HIsom}(Y)^0$ is of the form $\pi(F_0 \times (z_1, p))$ with $(z_1, p) \in F_1 \times N$, where $\pi\colon F_0 \times F_1 \times N \to Y$ is the projection. Consequently each $\pi(F_0 \times (z_1, p))$ is compact since $\mathrm{HIsom}(Y)^0$ is compact. We note here that $\pi(\{z_0\} \times F_1 \times \{p\})$ is contained in a fiber of α since $H^0(Y, E_1^*) = H^0(Y, E_1) = 0$.

b) Let $j\colon Y_1 \hookrightarrow Y$ be the inclusion map. Let ρ_Y be the Ricci form of Y, which represents $2\pi c_1(Y)_{\mathbb{R}}$. Then $j^*\rho_Y$ is negative semi-definite of constant rank since $(T_F)_y$ is just the zero eigen space of ρ_Y for each $y \in Y$. Since $j^*\rho_Y$ represents $2\pi c_1(Y_1)_{\mathbb{R}}$, according to the solution of the Calabi conjecture by Yau [Y], we have a Kähler form ω_1 on Y_1 whose Ricci form is equal to $j^*\rho_Y$. Let A be the second fundamental form of the Hermitian subbundle $T_F|_{Y_1}$ of (TY_1, ω_1). Let K be the mean curvature of the Hermitian vector bundle (TY_1, ω_0), which is equal to the Ricci operator of ω_1. Then $(T_F)_y$ is just the zero eigen space of K at each point $y \in Y_1$. Let K_F be the mean curvature of $T_F|_{Y_1}$ with the induced metric. It follows by the Gauss-Codazzi equation for holomorphic subbundles (see [Kb2], p. 23, (6.12)) that

$$\operatorname{tr} K_F + \|A\|^2 = \operatorname{tr}(\mathrm{pr}_F \circ K|_{T_F}) = 0,$$

where $\|A\|$ is the norm of A and $\mathrm{pr}_F\colon TY_1 \to T_F|_{Y_1}$ denotes the orthogonal projection. Let $n = \dim Y_1$. Then

$$\operatorname{tr} K_F \omega_1^n = 2\pi n \gamma_1(T_F|_{Y_1}) \wedge \omega_1^{n-1},$$

where $\gamma_1(T_F|_{Y_1})$ is the first Chern form of $T_F|_{Y_1}$ with respect to the induced metric. Thus we obtain

$$\int_{Y_1} \|A\|^2 \omega_1^n = -\int_{Y_1} 2\pi n c_1(T_F|_{Y_1}) \omega_1^{n-1}.$$

On the other hand, since the Ricci curvature of F is zero, we have $c_1(T_F)_{\mathbb{R}} = 0$. It follows $c_1(T_F|_{Y_1})_{\mathbb{R}} = j^*c_1(T_F)_{\mathbb{R}} = 0$. Consequently $A = 0$ on Y_1. This implies that the orthogonal complement Q of $T_F|_{Y_1}$ in TY_1 is in fact a holomorphic subbundle of TY_1 and that the orthogonal decomposition $TY_1 = T_F|_{Y_1} \oplus Q$ is holomorphic (see [Kb2], p. 23, Prop. (6.14)). Thus, by the de Rham decompositon theorem, the universal covering space $\tilde{Y}_1$ of Y_1 splits, $\tilde{Y}_1 = F' \times N'$ as desired.

c) If $\pi(\{z_0\} \times F_1 \times \{p\}) \subset Y_1$, then it is a leaf of the foliation defined by the zero eigen space of $j^*\rho_Y$, and hence $\pi(\{z_0\} \times F_1 \times \{p\}) = \varpi(F' \times \{p'\})$ for some $p' \in N'$. Q.E.D.

Appendix

Here we give proofs of propositions we have used.

Proposition A. *Let G be a connected abelian subgroup of the isometry group of a Riemannian manifold M. Then there is an open subset $U \subset M$ such that* $\dim G = \dim G \cdot p$ *for every $p \in U$.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of G. As usual, each element of $\mathfrak{g}$ is identified with a Killing vector field on M. Let k be the maximal number of elements $X_1, \ldots, X_k \in \mathfrak{g}$ such that $X_1 \wedge \cdots \wedge X_k$ never vanishes as section of $\bigwedge^k TM$ on a certain open subset $U \subset M$. Then each $Y \in \mathfrak{g}$ can be written as $Y = \sum_{i=1}^k a^i X_i$ for some functions a^i defined on U. We claim that every a^i is constant. First, since $\mathfrak{g}$ is abelian, we have $X_j a^i = 0$ for all i, j. Fix $q \in U$ arbitralily and let Z be a vector field defined on a neighborhood of q. Let $W = Y - \sum_l a_l(q) X_l$ so that $W(q) = 0$. Then $W \in \mathfrak{g}$ and hence $[W, X_i] = 0$. Since W is a Killing vector field and $X_j a^i = 0$, we have at q

$$\begin{aligned} 0 &= W(q) g(X_i,\, Z) \\ &= g([W, X_i],\, Z)(q) + g(X_i,\, [W, Z])(q) \\ &= -\sum_l g(X_i,\, (Z a_l) X_l)(q). \end{aligned}$$

Since $X_1, \ldots X_k$ are linearly independent at q, we obtain thus $Z a_l = 0$ at q, i.e., each a_l is constant. Consequently $k = \dim \mathfrak{g} = \dim G \cdot p$ for $p \in U$. Q.E.D.

Proposition B. *Let M be a complete Riemannian manifold and* $\mathrm{Isom}(M)$ *the isometry group of M. Then a subgroup $\Gamma \subset \mathrm{Isom}(M)$ is discrete in* $\mathrm{Isom}(M)$ *if and only if the action of Γ on M is properly discontinuous.*

This follows from the following

Theorem (see [H], p. 202, Th. 2.2). *Let M be a Riemannian manifold and $\{\varphi_n\}$ a sequence of isometries of M. Suppose there is a point $p \in M$ such that $\{\varphi_n(p)\}$ converges to a point in M. Then a cetain subsequence of $\{\varphi_n\}$ converges to an isometry of M in the compact open topology.*

In fact, suppost Γ is discrete in $\mathrm{Isom}(M)$ but the action of Γ on M is not properly discontinuous. Then there are a compact subset

$D \subset M$ and a sequence $\{\varphi_n\}$ in Γ such that $\varphi_m \neq \varphi_n$ if $m \neq n$ and $\varphi_m(D) \cap D \neq \emptyset$ for all m. Let d be the diameter of D. Fix $p \in D$ and let $B_{2d}(p)$ be the closed geodesic ball of radius $2d$ with center p. Then $\varphi_m(p) \in B_{2d}(p)$ since $\varphi_n(D) \cap D \neq \emptyset$. Therefore a certain subsequence of $\{\varphi_n(p)\}$ converges to a point of M. Thus by the theorem above, a subsequence of $\{\varphi_n\}$ converges in $\mathrm{Isom}(M)$, which contradicts the discreteness of Γ in $\mathrm{Isom}(M)$.

References

[**CG**] Cheeger, J. and D. Gromoll, The splitting theorem for manifolds of nonnegative Ricci curvature, J. Diff. Geometry, **6** (1971), 119-128.

[**E**] Eberlein, P., Euclidean de Rham factor of a lattice of nonpositive curvature, J. Diff. Geometry, **18** (1983), 209-220.

[**H**] Helgason, S, "Differetatial Geometry, Lie Groups, and Symmetric Spaces", Academic Press, New York, London, 1978.

[**I**] Inoue, M., On surfaces of Class VII_0, Inventiones math., **24** (1974), 269–310.

[**KMM**] Kawamata, Y., K. Matsuda and K. Matsuki, Introduction to the minimal model, in "Algebraic Geometry, Sendai 1985", Advanced Studies in Pure Math. **10**, Kinokuniya and North-Holland, 1987, pp. 283-360.

[**Kb1**] Kobayashi, S., On compact Kähler manifolds with positive definite Ricci tensor, Ann. of Math., **74** (1961), 570–574.

[**Kb2**] ———, "Differential Geometry of Complex Vector Bundles", Iwanami Shoten, Tokyo, and Princeton Univ. Press, Princeton, 1987.

[**KN**] Kobayashi, S. and K. Nomizu, "Foundation of Differential Geometry, I", Wiley & Sons, New York, London, 1963.

[**M**] Myers, S. C., Riemannian manifolds in the large, Duke Math. J., **1** (1953), 39–49.

[**Y**] Yau, S.-T., On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I, Comm. Pure Appl. Math., **31** (1978), 339-411.

Department of Mathematics
College of General Education
Osaka University
Toyonaka, 560 Osaka
Japan