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Topics in Complex Differential Geometry

Ngaiming Mok

On December 21–24, 1987 the author gave a series of four talks entitled "PDE methods in Complex Differential Geometry" in Tokyo University on the occasion of a conference in Differential Geometry organized by Professor T. Ochiai. The present article is an expansion of these lectures. Since the lectures only covered recent results in Complex Differential Geometry related to the author's own research, I find it more appropriate to change the title to the present one. Moreover, as will be seen, PDE methods are only one of the major components of the methods to be described, the other component consists of algebro-geometric or complex-analytic techniques. The emphasis will very often be on the interplay between these two components of the methods employed. The four lectures are on harmonic mappings and holomorphic foliations, uniformization of compact Kähler manifolds of nonnegative curvature and compactification of complete Kähler manifolds – the case of positive curvature and the case of Kähler-Einstein manifolds of finite volume.

A common theme of these lectures is the study of Hermitian locally symmetric manifolds and/or their underlying complex manifolds. There are two different perspectives. First, Hermitian locally symmetric manifolds can be regarded as very special manifolds. To demonstrate that in a certain sense they are isolated phenomena one proves rigidity theorems on the complex structure and/or metric structure by imposing topological or geometric conditions. Classical examples of such theorems include the complex case of Mostow's Strong Rigidity Theorem [Mos1] and the theorems of Berger [Ber2] and Gray [Gray] characterizing compact Kähler manifolds of nonnegative sectional curvature. Another perspective is to view Hermitian locally symmetric manifolds as models of a rather general class of manifolds. For example, locally irreducible Hermitian locally symmetric manifolds can be regarded as special cases of

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Kähler-Einstein manifolds. From this perspective one can try to show that other manifolds of this general class enjoy properties similar to them.

The first two topics fall within the first perspective. Motivated by Mostow's Strong Rigidity Theorem and the local rigidity theorems of Calabi-Vesentini [CV] and Borel [Bo1] on the complex structure of compact quotients of bounded symmetric domains, Siu [Siu4, 6] studied the question of strong rigidity of complex structures of such manifolds. He used the method of harmonic mappings. One is then led to proving that certain harmonic mappings into such manifolds are holomorphic or anti-holomorphic. In the first lecture on "Harmonic mappings and holomorphic foliations" we give a general survey of results obtained using this method, placing emphasis on the special case when the target manifold is an irreducible quotient of the polydisc. This is the case when Bochner-Kodaira formulas for harmonic mappings into such manifolds are not strong enough to yield strong rigidity. In the compact case, as initiated by Jost-Yau [JY1, 2] and completed by Mok [Mok6], one has to use holomorphic foliations arising from the integral formula. We will cover the more general case when the target manifold is only assumed to be of finite volume with respect to the Bergman metric.

The second lecture, entitled "Uniformization of compact Kähler manifolds of nonnegative curvature", addresses in particular the problem of characterizing Hermitian symmetric manifolds of compact type by curvature conditions, including the Frankel Conjecture, as solved by Mori [Mo] and Siu-Yau [SY3]. We use the method of parabolic evolution equations as initiated by Hamilton [Ham1]. We adopt the characterization of Riemannian locally symmetric manifolds of Berger [Ber1] and Simon [Si] in terms of the holonomy group. Consider a compact Kähler manifold (X, g) of nonnegative holomorphic bisectional curvature and of positive Ricci curvature at some point. Replacing g by an evolved metric using Hamilton's flow we construct on the projectivized tangent bundle of X a subset S invariant under parallel transport. We use Mori's theory of rational curves and the deformation theory of complex submanifolds in the construction of S. This is an example in which one sets the stage by using methods of PDE and completes the proof by bringing in techniques of algebraic geometry.

The third and fourth lectures cover two aspects of the general question of compactifying complete Kähler manifolds. In the third lecture, we study the "Compactification of complete Kähler manifolds of positive curvature". The main motivation behind the study is the non-compact analogue of the Frankel Conjecture stating that a non-compact complete Kähler manifold of positive sectional curvature is biholomorpic to $\mathbf{C}^n$.

We can prove this conjecture only in the case of two complex dimensions with additional geometric conditions on curvature decay and volume growth. In higher dimensions and with the same geometric conditions we can prove that such manifolds are biholomorphic to affine-algebraic varieties. In particular, they can be compactified complex-analytically. The non-compact analogue of the Frankel conjecture can be regarded as a conjecture on the rigidity of the complex structure of the complex Euclidean space, on which one can construct complete Kähler metrics of positive sectional curvature. In this regard we are viewing the underlying complex structure of the simply-connected, flat Hermitian symmetric manifold $\mathbf{C}^n$ as a rigid structure. On the other hand, one can regard the method of proof as a general scheme of compactifying complete Kähler manifolds satisfying geometric conditions that are in a certain sense "parabolic". In this regard, $\mathbf{C}^n$ is considered as a model of this general class of parabolic manifolds. With regard to this second perspective, Demailly [De1] proved independently a theorem characterizing affine-algebraic varieties by properties of exhaustion functions and curvature conditions. A general approach to compactifying complete Kähler manifolds is to use L^2-estimate of $\bar{\partial}$ of Andreotti-Vesentini [AV] and Hörmander [Hör1] and to show that holomorphic functions or holomorphic sections of certain line bundles can be used to embed the manifold. To achieve the last point one proves certain finiteness theorems. The first step in this direction is to prove a version of Siegel's Theorem (on a field of meromorphic functions) with growth conditions. The formulation of such a Siegel's Theorem is motivated by the proof of Siegel's Theorem for compact Kähler manifolds.

The last lecture is on "Compactification of complete Kähler-Einstein manifolds of finite volume." This topic is an example in which arithmetic quotients (arising from torsion-free arithmetic subgroups) of finite volume in the Bergman metric of bounded symmetric domains are regarded as models of complete Kähler-Einstein manifolds of finite volume. There is an extensive algebraic theory of the compactification of such arithmetic varieties, due to Satake-Baily-Borel [Sat1, 2] [Bai] [BB] and Ash-Mumford-Rapoport-Tai [AMTR]. From the complex-analytic point of view, with a few exceptions, the arithmetic varieties are special cases of pseudoconcave manifolds admitting a complete Kähler-Einstein metric (or, more generally, a positive line bundle). This perspective was adopted in Andreotti-Grauert [AG1]. Their method can be used to show that such arithmetic varieties can be embedded as open subsets (in the complex topology) of projective-algebraic varieties. Very recently, Nadel-Tsuji [NT] adopted once more this point of view and proved a compactification theorem for certain pseudoconcave manifolds that implies the statement that arithmetic varieties can be compactified complex-analytically. From the differential-geometric point of view, arithmetic varieties are special cases of complete Kähler-Einstein manifolds of finite volume and bounded curvature. Siu-Yau [SY4] proved a compactification theorem for this latter class of complex manifolds when the sectional curvature is pinched between two negative constants without the Einstein assumption. Their theorem complements the theorems of Satake-Baily-Borel to include the rank-1 non-arithmetic case. In the second half of my last lecture I also sketch the proof of a theorem of Mok-Zhong [MZ2] showing in particular that complete Kähler-Einstein manifolds of finite volume and bounded curvature can be compactified complex-analytically provided that the underlying topological manifold is of finite topological type. In the proof of both Siu-Yau [SY4] and Nadel-Tsuji [NT] they used Siegel's Theorem for pseudoconcave manifold (Andreotti [An]). Nadel-Tsuji also used the existence of Kähler-Einstein metrics on bounded domains of holomorphy. In the proof of Mok-Zhong [MZ2] we used a Siegel's Theorem with growth, Bézout estimates, Oka's characterization of domains of holomorphy and the characterization of analytic sets locally as sets of density with respect to some plurisubharmonic potential functions, as given in Bombieri [Bom1]. Partly as a motivation for the proof of the compactification theorem of Mok-Zhong [MZ2] we also sketch a proof of a local compactification theorem for bounded domains of holomorphy in terms of the Kähler-Einstein volume ([Mok12]).

We give here a brief description of the background in PDE methods used in these lectures. In Lecture I we use the existence theorem for harmonic maps (Eells-Sampson [ES]) from compact domain manifolds into compact target manifolds with nonpositive sectional curvature, obtained by the method of the heat flow. In Lecture II we use the short time existence for the parabolic evolution equation of Hamilton ([Ham1]). While in general Hamilton resorted to the Nash-Moser Implicit Function Theorem in the Kähler case the short time existence is very simple since the Kähler condition allows one to reduce the parabolic evolution equation to a scalar equation (cf. Bando [Ban]). The principal PDE technique of Lectures III and IV is the L^2-estimate of $\bar{\partial}$ of Andreotti-Vesentini [AV] and Hörmander [Hör1]. In Lecture III we also use standard estimates on the Green kernels of certain complete Riemannian manifolds based on the iteration techniques of Nash-Moser. In Lecture IV in the formulation and proof of the local compactification theorem of [Mok12] and the compactification theorem for pseudoconcave manifolds of [NT] we need *a-priori* estimates and existence theorems for the complex Monge-Ampère equation. In addition to this, we also use Demailly's asymptotic

Weyl formula ([De2]) for the Dirichlet boundary condition.

The presentation of the various topics in these notes depends to a certain extent on the availability (or the lack) of reference materials. For example, in Lecture I on Harmonic Mappings and Holomorphic Foliations, since there is a general survey in [Siu8] on the application of harmonic mappings to problems of strong rigidity, more details will be given to the development after the survey. Summaries of the portion of results in Lectures II and III due to the author are available in [Mok8] and [Mok4] resp. [Mok4] is more generally a survey on complete Kähler manifolds of positive curvature. Since there is no systematic account of the materials covered in Lecture IV, we will be more thorough with this topic (on compactifying Kähler-Einstein manifolds). In general, we will give motivations to the approaches adopted to facilitate the reading of the original materials. A historical perspective and systematic outlines of the methods will be given but technical details will be omitted. It is my hope that the present notes will give a flavor of an aspect of the subject for which the key lies in understanding the interrelation between the differential-geometric, complex-analytic and algebro-geometric aspects of the problems.

I wish to thank Prof. Ochiai for inviting me to deliver this series of lectures in Tokyo University. Thanks are due to Shigetoshi Bando and Ryoichi Kobayashi for arranging my trips to Tokyo and Tohoku Universities and for their hospitality and unfailing help. I would also like to take this opportunity to thank T. Mabuchi for inviting me to Osaka University and to Koji Cho for making my stay in Tokyo a very pleasant one. Finally, my thanks to I-Hsun Tsai and Sai-Kee Yeung in Columbia University, who gave me a lot of help during the write-up and proof-reading of the manuscript.

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Lecture I. Harmonic Mappings and Holomorphic Foliations

(1.1) Some generalities about bounded symmetric domains

The Uniformization Theorem in one complex variable asserts that a simply-connected Riemann surface is conformally equivalent to the Riemann sphere $\mathbf{P}^1$, the complex plane $\mathbf{C}$ or the unit disc Δ. These Riemann surfaces carry Hermitian metrics of constant Gaussian curvature $+1, 0, -1$ resp. Accordingly, there is a trichotomy into the elliptic, parabolic and hyperbolic geometries.

These model manifolds with their canonical metrics of constant Gaussian curvature are precisely the Hermitian symmetric manifolds of complex dimension one. Their analogues in arbitrary finite dimensions are respectively the bounded symmetric domains with the Bergamn metric, the flat Euclidean space $\mathbf{C}^n$ and the Hermitian symmetric manifolds of compact type (e.g., the projective spaces $\mathbf{P}^n$ with Fubini-Study metrics, the hyperquadrics $\mathbf{Q}^n$ and the Grassmannians $G(r,n)$, with the induced metrics from standard embeddings into $\mathbf{P}^n$.)

The subject matter in my lectures will be related in one way or another to Hermitian locally symmetric manifolds or their underlying complex manifolds. We begin with the bounded symmetric domains Ω (cf. Hegalson [Hel], Chap.VIII, §3, p.364ff. for general reference). A bounded domain Ω is called a bounded symmetric domain if for each point $x \in \Omega$ there exists a holomorphic automorphism $\sigma_x \in \mathrm{Aut}(\Omega)$ which is a non-degenerate involution (symmetry) at x in the sense that $\sigma_x^2 = \mathrm{id}$ and that x is an isolated fixed point of σ_x. Any bounded domain D carries a Bergman metric ds^2_D. $\mathrm{Aut}(D)$ acts holomorphically and isometrically on (D, ds^2_D) as a real Lie group. By composing the involutions it follows that for a bounded symmetric domain $\Omega, (\Omega, ds^2_\Omega)$ is in particular homogeneous under the connected component $\mathrm{Aut}_0(\Omega) = G$ of $\mathrm{Aut}(\Omega)$. G is semi-simple and non-compact. Fix $o \in \Omega$ and let K be the isotropy subgroup of G at o. Then G/K with the Riemannian metric ds^2 induced by the Killing form of G is a Riemannian symmetric manifold. Under the identification of Ω with G/K as a homogeneous Riemannian manifold the Bergman metric ds^2_B agrees with ds^2 up to normalizing constants. In particular $(G/K, ds^2)$ is Kähler. There are canonical Harish-Chandra realization of G/K as bounded convex domains in $\mathbf{C}^n$ containing the origin o such that the symmetry at o is simply the reflection $z \to -z$.

As a first example consider $\Omega = \Delta$. In this case ds^2_Δ agrees with the Poincaré metric (up to a constant factor). Automorphisms of Δ are linear fractional transformation $F(z) = (az+b)/(cz+d)$ preserving the unit disc Δ. Identifying Φ with the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ G can be realized as the group of linear transformations preserving the indefinite quadratic from $x^2 - y^2$ in 2 variables up to $\{\pm 1\}$, 1 denoting the identity matrix. Thus, $G = SU(1,1)/\{\pm 1\}$. K is then the circle group $U(1)/\{\pm 1\}$. By abuse of notations we usually write $\Delta = SU(1,1)/U(1)$. (Δ, ds^2_Δ) carries constant negative Gaussian curvature. Immediate generalization to higher dimensions include the polydisc Δ^n and the unit ball B^n in $\mathbf{C}^n$. In case of Δ^n the sectional curvatures are nonpositive. In case of $B^n = \{z \in \mathbf{C}^n : \|z\| < 1\}$ the Bergman kernel function is determined by $K(z,w) = c_n/(1 - z \cdot \bar{w})^{n+1}$. The Bergman metric is defined by the Kähler potential $\log K(z,z)$. By direct computation we see that $(B^n, ds^2_{B^n})$ has negative sectional curvature. In both the case of Δ^n and B^n the Bergman metric is Kähler-Einstein in the sense that the Ricci form is proportional to the Kähler form (with a negative proportionality constant). This is in fact true for all homogeneous bounded domains D. In fact, denoting by $d\lambda$ the Euclidean volume form, both $K(z,z)\cdot d\lambda$ and the volume form $H \cdot d\lambda$ of the Bergman metric are, up to constants, the unique invariant (n,n)-forms under $\mathrm{Aut}(D)$, and we have Kähler form

$= i\partial\bar{\partial}\log K(z,z)$, Ricci form $= -i\partial\bar{\partial}\log H$.

From the general theory of Riemannian symmetric manifolds we know that for any bounded symmetric domain $\Omega, (\Omega, ds^2_\Omega)$ is of non-negative sectional curvature. Identify Ω with G/K and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. Let o denote the identity coset eK and identify the real tangent space T_o with $\mathfrak{p}$. ${T_o}^{1,0}$ and ${T_o}^{0,1}$ are then identified with complex linear subspaces $\mathfrak{p}^+$ and $\mathfrak{p}^-$ of $\mathfrak{p}_{\mathbf{C}} = \mathfrak{p} \otimes_{\mathbf{R}} \mathbf{C}$. Denote by R the curvature tensor and by Q the Hermitian bilinear form on ${T_o}^{1,0} \otimes {T_o}^{0,1}$ determined by $Q(\xi \otimes \bar{\zeta}; \xi \otimes \bar{\zeta}) = R(\xi, \bar{\xi}; \zeta, \bar{\zeta})$. Then Q is negative semi-definite on ${T_o}^{1,0} \otimes {T_o}^{0,1}$. In fact, if $v = \sum a_{ij}\xi_i \otimes \bar{\zeta}_j \in {T_o}^{1,0} \otimes {T_o}^{0,1}$, then

$$Q(v;v) = -\|\sum a_{ij}[\xi_i, \bar{\zeta}_j]\|^2.$$

Here $\sum a_{ij}[\xi_i, \bar{\zeta}_j] \in \mathfrak{k}_{\mathbf{C}} = \mathfrak{k} \otimes_{\mathbf{R}} \mathbf{C}$; $[\ \ ]$ denotes the Lie bracket on $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$; and $\|.\|$ denotes the Hermitian norm on $\mathfrak{k}_{\mathbf{C}}$ induced by the Killing form of $\mathfrak{g}$. This stronger notion of nonpositivity of curvature will be important for the application of harmonic maps to questions of strong rigidity. It corresponds to saying that the Hermitian holomorphic vector bundle T_Ω is negative in the dual sense of Nakano in Hermitian Differential Geometry (cf. Siu [Siu7]). Also, as remarked above, (Ω, ds^2_Ω) is Kähler-Einstein.

(1.2) Some remarks on quotients of bounded symmetric domains

Classically, quotients of Ω by torsion-free discrete groups of holomorphicautomorphisms $\mathrm{Aut}(\Omega)$ furnish a rich class of complex spaces. Of particular interest to number-theorists and algebraic-geometers are quotients of Ω by arithmetic lattices Γ. As a first example we map the unit disc Δ conformally to the upper half plane H so that $\mathrm{Aut}(H) \cong SL(2,R)/\{\pm 1\}$ and take $\Gamma_o = SL(2,\mathbf{Z})/\{\pm 1\}$. Then, Γ_o is an arithmetic lattice and the quotient space H/Γ_o parametrizes effectively isomorphism classes of elliptic curves. In general the arithmetic lattices Γ act properly discontinuously on the bounded symmetric domains Ω but they may have fixed points (as is the case of Γ_o). However, there always exists subgroups Γ' of Γ of finite index such that Γ acts on Ω without fixed points. (Equivalently, Γ' is torsion-free.) $X = \Omega/\Gamma'$ may be compact or non-compact, but it is always true that X is of finite volume with respect to the quotient Bergman metric (cf. Raghunathan [Rag]). In case of B^2 and B^3, there are also examples of lattices which are not arithmetic (Mostow [Mos2]).

From now on by a quotient of a bounded symmetric domain Ω we will always mean the quotient of Ω by a torsion-free group of holomorphic automorphisms. Γ will always stand for a torsion-free discrete subgroup of $\mathrm{Aut}(\Omega)$ such that $X = \Omega/\Gamma$ is of finite volume with respect to the Bergman metric. In case Γ is arithmetic and X is non-compact there is a well-developed theory of compactification due first to Satake-Baily-Borel (See Lecture IV for more details.) In particular, X is biholomorphic to a non-singular quasi-projective variety. Resolving singularities of the compactification if necessary, this is equivalent to saying that X is biholomorphic to $Z - W$ for some non-singular projective-algebraic variety and Z is a divisor with at worst normal crossings. By the Arithmeticity Theorem of Margulis (cf. Zimmer [Zim]) the non-arithmetic lattices Γ occur essentially only in the situation of rank-1 bounded symmetric domains (i.e. B^n). The compactification of such $X = B^n/\Gamma$ by methods of differential geometry was achieved by Siu-Yau [SY4]).

(1.3) Local rigidity for compact quotients of bounded symmetric domains

Let $X = \Omega/\Gamma$ be compact. In case $\Omega = \Delta, X$ is a compact Riemann surface of genus $g \geq 2$. From Teichmüler theory, it is well-known that the space of complex structures on the underlying smooth surface is effectively parametrized by a complex space of complex dimension $3g - 3 > 0$. In particular the complex structure of X can be locally deformed. That this is an exceptional phenomenon for general $X = \Omega/\Gamma$ was discovered by Calabi-Vensentini [CV] in 1960, followed by Borel [Bo1]. They showed in particular that for Ω irreducible and of complex dimension ≥ 2 (and X compact) the complex structure of X cannot be deformed. We say that the compact complex manifold X is locally rigid. They proved this among other things by showing that $H^1(X, T_X) = 0$ for such X and applying the theorem of Kuranishi [Ku] (cf. also Kodaira-Morrow [KM, Chap.IV, p.147ff.]).

In general if there is a finite covering X' of X such that $X' = \Omega_1/\Gamma_1 \times \Omega_2/\Gamma_2$ canonically we say that X is (globally) reducible. When Ω is reducible we will say that X is locally reducible. It may happen that X is locally reducible and but not (globally) reducible. The question of local rigidity for such X cannot be deduced from [CV] and [Bo1]. The essential case is given by $\Omega = \Delta^n, n \geq 1$, the polydisc. In 1963, Matsushima-Shimura [MS] considered irreducible compact quotients $X = \Delta^n/\Gamma$ of the polydisc. Their main aim was to compute the dimensions of certain spaces of automorphic forms associated to X. They used the method of differential geometry to prove vanishing theorems and showed in particular that $H^1(X, T_X) = 0$, so that X is locally rigid.

(1.4) Mostow's Strong Rigidity Theorem

Let (M, g) be a compact Riemannian locally symmetric manifold of nonpositive sectional curvature. In 1973, Mostow [Mos] studied the extent to which the geometry of M is determined by the fundamental group $\pi_1(M)$. In the case of negative Ricci curvature, i.e., removing the flat local de Rham factors, Mostow's Strong Rigidity Theorem asserts that $\pi_1(M)$ determines M isometrically up to normalizing constants except in the case when some finite covering of M admits a compact Riemann surface S as a direct factor isometrically. Specializing to the case of compact quotients X of bounded symmetric domains (by torsion-free lattices), this implies immediately that the complex structure of X is uniquely determined by $\pi_1(X)$ up to conjugations on the irreducible local components. In fact, on an irreducible bounded symmetric domain Ω equipped with the Bergman metric ds^2_Ω, the isometry group consists exactly of the holomorphic and the anti-holomorphic isometries. Mostow's proof of the Strong Rigidity Theorem makes use of quasi-conformal maps.

The local rigidity theorems of Calabi-Vesentini [CV] and Borel [Bo1] and Mostow's Strong Rigidity Theorem in the complex case suggest that the compact quotients X of bounded symmetric domains should exhibit a stronger rigidity property within the class of compact complex manifold. This prompted Siu [Siu4] to formulate and prove his Strong Rigidity Theorem for Kähler manifolds using harmonic mappings.

(1.5) Harmonic mappings into compact manifolds of nonpositive curvature

In the theory of one complex variable, harmonic functions are closely related to holomorphic (complex-analytic) functions since any real harmonic function is locally the real part of a holomorphic function. For instance, one can use harmonic functions to give a proof of the Uniformization Theorem, as was given in Ahlfors [Ahl3, Chap.10, p.136ff], by constructing holomorphic functions from harmonic functions. Harmonic functions are easier to construct since the Poisson equation $\Delta u = f$ is a determined elliptic equation for the Dirichlet boundary problem, whereas for a smooth (1, 0)-form ν on a plane domain with smooth boundary, one cannot always solve $\overline{\partial} u = \nu$ with prescribed boundary values.

In the theory of Several Complex Variables, the question naturally arises how one can construct holomorphic objects from harmonic objects. The gap between harmonic functions and holomorphic functions is very wide. The real part of holomorphic functions is given by pluriharmonic functions u characterized by the differential equation $\partial\bar{\partial}u = 0$, which is over-determined except in case of one complex variable.

Related to the question above Lelong ([Le1], 1960) studied on $\mathbf{C}^n$, the relation between the Poincaé-Lelong equation $\sqrt{-1}\partial\bar{\partial}u = \rho$ for a closed (1, 1)-form and the Poisson equation $\Delta u = f$ obtained by taking traces. He showed that by imposing growth conditions on ρ the minimal solutions of $\Delta u = f$ satisfies automatically the equation $\sqrt{-1}\partial\bar{\partial}u = \rho$. A second situation where a similar reduction occurs concerns harmonic mappings into compact Kähler manifolds, where the growth condition is replaced by curvature conditions. To explain this, we start with defining harmonic mappings.

On a bounded domain D with smooth boundary in $\mathbf{R}^n$, the harmonic functions f with given boundary values $f|_{\partial D} = \varphi$ are characterized by the fact that f is a critical point of the Dirichlet norm $\int_D \|du\|^2$ among functions u in a certain Sobolev space verifying $u|_{\partial D} = \varphi$. The harmonic function u can be regarded as a harmonic mapping from D to the real line $\mathbf{R}$ equipped with the Euclidean metric. In general, harmonic mappings from a domain Ω with smooth boundary on a Riemannian manifold (M, g) into a Riemannian manifold (N, h) is defined similarly where the Dirichlet norm is interpreted as the energy $e(f) = \int_\Omega \|du\|^2, \|du\|^2 = tr_g(u^*h)$. In case of harmonic functions f on Ω, f is necessarily an absolute minimum of the Dirichlet norm (with the Dirichlet boundary condition). When $\Omega = M$ is a compact Riemannian manifold, the boundary condition is void and one may hope to find minima of the energy functional within each homotopy class of maps $M \to N$. This is not always possible. It can however be attained by imposing curvature conditions on the target manifold as given by

Theorem (1.5) (Eells-Sampson [ES]). *Let (M, g) be a compact Riemannian manifold and (N, h) be a compact Riemannian manifold of nonpositive sectional curvature. Let $f_o: M \to N$ be a smooth map. Then f_o is homotopic via smooth maps to a harmonic mapping $f: M \to N$ which realizes the minimum of the energy functional $e(u)$ within the class of smooth maps u homotopic to f_o.*

In terms of local coordinates (x_α) on M and (y_i) on N, the Euler-

Lagrange equation on M for the energy functional $e(f)$ is given by

$$g^{\alpha\beta}\left[\frac{\partial^2 f^i}{\partial x_\alpha \partial x_\beta} - {}^M\Gamma^\gamma_{\alpha\beta}\frac{\partial f^i}{\partial x_\gamma} + {}^N\Gamma^i_{jk}\frac{\partial f^j}{\partial x_\alpha}\frac{\partial f^k}{\partial x_\beta}\right] = 0.$$

Here ${}^M\Gamma$ and ${}^N\Gamma$ stand for the Riemann-Christoffel symbols of the Riemannian connections on (M, g) and (N, h) resp., the latter being evaluated at the image under f, and we use the Einstein summation convention of summing over any index that occurs once as a subscript and once as a superscript. The Euler-Lagrange equation can be interpreted invariantly as follows. Let $df\colon T_M \to T_N$ denote the differential of f. We regard df as an f^*T_N−valued 1-form on M and write $df = \sum \partial_\alpha f \otimes dx^\alpha$. f^*T_N is equipped with the pulled-back connection. Let ∇ denote covariant differentiation on the bundle $f^*T_N \otimes T^*_M$ on M. Then, $\nabla_\alpha \partial_\beta f$ is an f^*T_N−valued section. We define the Laplace-Beltrami operator by $\Delta f = \sum g^{\alpha\beta}\nabla_\alpha\partial_\beta f$. The Euler-Lagrange equation for the energy functional is simply $\Delta f = 0$. We explain here the approach in proving the theorem of Eells-Sampson. For a smooth mapping $f_o\colon M \to N$ we will call $\tau(f_0) = \Delta f_o$ the tension field of f_o. It is an $f^*_o T_N$−valued smooth section over M. To deform f_o homotopically to a harmonic map one uses the heat flow

$$\begin{cases} \dfrac{\partial F}{\partial t} = \tau(F), \\ F(o; x) = f_o(x). \end{cases}$$

Here F is a mapping into N in the time variable t and the space variable x on M. Write $f_t(x) = F(t; x)$. Infinitesimally the heat flow moves $f_t(x)$ in the direction of $\tau(f_t)(x)$, interpreted now as a tangent vector to N at the point $f_t(x)$. In order to have short-time solvability (for small $t > 0$) of the heat flow, we need an equation which would linearize to a parabolic equation (of second order in the space variable). This makes our choice of the heat flow the natural one (instead of $\partial F/\partial t = -\tau(F)$). In order to get long time existence and exponential convergence the crucial point is that

$$(\partial/\partial t - \Delta)\|\tau\|^2 \geq 0$$

when the sectional curvatures of the target manifolds are nonpositive, so that one can apply the maximum principle for the heat equation on functions to show that the heat flow decreases the sup norm of the tension field.

(1.6) Siu's Strong Rigidity Theorem for Kähler manifolds

Motivated by the local rigidity theorems of Calabi-Vesentini [CV] and Borel [Bo1] on compact quotients of bounded symmetric domains and Mostow's Strong Rigidity Theorem on compact locally symmetric manifolds of nonpositive sectional curvature, Siu [Siu4, 6] studied the question of rigidity of complex structures of compact quotients of irreducible bounded symmetric domains within the class of Kähler manifolds. He used the method of harmonic mappings. A survey of results obtainable by this method and generalizations is available in Siu [Siu7]. We start here with some general remarks on harmonic mappings between Kähler manifolds.

Let now (M,g) and (N,h) be Kähler manifolds and $f\colon M \to N$ be a smooth harmonic mapping. We write everything in terms of holomorphic coordinates. Because of the Kähler condition, the canonical connections of the underlying Riemannian manifolds have the special property that in terms of complex coordinates, the only non-vanishing components are of the type Γ^c_{ab} where a,b and c are all unbarred indices or all barred indices. It follows that, in terms of local holomorphic coordinates (z_α) and (w_i) on M and N resp., the Euler-Lagrange equation for the energy functional is of the special form

$$g^{\alpha\bar{\beta}}\left[\frac{\partial^2 f^i}{\partial z_\alpha \partial \bar{z}_\beta} + \frac{\partial f^j}{\partial z_\alpha}\frac{\partial f^k}{\partial \bar{z}_\beta}{}^N\Gamma^i_{jk}\right] = 0.$$

In particular, any holomorphic or anti-holomorphic mapping from M to N is harmonic.

In [Siu4, 6], Siu proved

Theorem (1.6). *Let (N,h) be a compact quotient of an irreducible bounded symmetric domain Ω of complex dimension ≥ 2. Let (M,g) be a compact Kähler manifold homotopic to N. Then, M is either biholomorphic or conjugate-biholomorphic to N.*

Let $f_o\colon M \to N$ be a smooth homotopy equivalence. Applying the theorem of Eells-Sampson [ES] in (1.4), it follows that f_o is homotopic to a harmonic map $f\colon M \to N$. (It is not *a priori* clear that f is a diffeomorphism.) The key point of [Siu4, 6] is to show that f is necessarily either holomorphic or anti-holomorphic resp. in the cases of classical and exceptional bounded symmetric domains. To show that f is a diffeomorphism by conjugation on N we way assume without loss of generality that f is holomorphic. Since f is homotopic to a diffeomorphism and orientation-preserving, it is topologically of degree 1. To conclude that f is a biholomorphism it suffices to show that f does not blow down any positive-dimensional complex subvariety V. Since M is Kähler, by integrating the Kähler form over (the regular part of) V one knows that

V represents a non-trivial homology class. If V is mapped to a point this would contradict with the fact that f induces an isomorphism on homology groups.

To prove that f is holomorphic or anti-holomorphic, Siu [Siu4] developed a $\partial\bar{\partial}$-Bochner-Kodaira formula for harmonic mappings. To formulate this we examine first of all the Laplace-Beltrami operator Δf on mappings between Kähler manifolds. In case of smooth functions from a domain D in $\mathbf{C}^n$ to the complex plane $\mathbf{C}$ there is the notion of pluriharmonic functions u defined by the Poincaré-Lelong equation $\partial\bar{\partial}u = 0$. Contracting with the Euclidean metric tensor on D this implies that $\Delta u = 0$. In the general case of smooth maps between Kähler manifolds one can define the Poincaré-Lelong operator by using the Riemannian connections as follows. Because of the complex structures on M and N the complexified tangent bundles split into (1, 0) and (0, 1) components. The differential $df\colon M \to N$ splits after complexification into four components $\partial f, \partial\bar{f}, \bar{\partial} f, \bar{\partial}\bar{f}$. For example, $\partial\bar{f}$ is an $f^*T^{0,1}(N)$-valued $(1,0)$ form. Denote by ∇ the covariant differentiation on $f^*T_N \otimes_{\mathbf{R}} \mathbf{C}$ obtained by pulling back the Riemannian connection on N via f. Using this one can define the $f^*T^{1,0}(N)$-valued (1, 1) form $D\bar{\partial}f$ on M by the formula

$$D\bar{\partial}f = \sum \nabla_\alpha \partial_{\bar{\beta}} f dz^\alpha \wedge d\bar{z}^\beta.$$

A smooth mapping $h\colon M \to N$ is said to be pluriharmonic if and only if $D\bar{\partial}h = 0$. Equivalently, this means that for any local Riemann surface S on M, $h|S\colon S \to N$ is a harmonic map. Clearly a pluriharmonic map h is harmonic. In fact the Euler-Lagrange equation for the energy functional can be written in the form

$$g^{\alpha\bar{\beta}} \nabla_\alpha \partial_{\bar{\beta}} f = 0.$$

In other words, Δf is the contraction of $D\bar{\partial}f$ with the covariant metric tensor $(g^{i\bar{j}})$, so that $D\bar{\partial}f = 0$ implies $\Delta f = 0$.

We refer the reader to [Siu7] for details on the $\partial\bar{\partial}$-Bochner-Kodaira formula for harmonic maps and its various applications. For us, it suffices to state that as a consequence we have by the Stokes' Theorem the integral formula

$$\int_M H(\partial f \wedge \partial\bar{f}; \partial f \wedge \partial\bar{f}) + \|D\bar{\partial}f\|^2 = 0,$$

where $H(\cdot\,;\,\cdot)$ is a Hermitian bilinear form on $f^*T^{1,0}(N)$-valued (2, 0)-form on M. In terms of coordinates at a point $Q = f(P) \in M$ such that

$h_{ij}(P) = \delta_{ij}$ we have

$$H(\partial f \wedge \bar\partial \bar f; \partial f \wedge \bar\partial \bar f) = - \sum_{\substack{i,j,k,\ell \\ \alpha,\beta}} R_{i\bar j k\bar l} \left((\partial_{\bar\alpha} f^i)\overline{(\partial_\beta f^l)} - (\partial_{\bar\beta} f^i)\overline{\partial_\alpha f^l} \right) \times$$
$$\times \overline{\left((\partial_{\bar\alpha} f^j)\overline{(\partial_\beta f^k)} - (\partial_{\bar\beta} f^j)\overline{(\partial_\alpha f^k)} \right)}$$

Here and henceforth we use the sign convention so that on the Poincaré disc $R_{1\bar 1 1\bar 1} < 0$. Siu defined in [Siu4] a notion of strong (semi)negativity of the curvature tensor which would imply that H is positive (semi-) definite. It was later discovered in [Siu7] that this notion agrees with the more standard notion of negativity of the curvature tensor in the dual sense of Nakano. As explained in (1.3), when N is a compact quotient of a bounded symmetric domain and equipped with the Bergman metric h, the curvature operator Q is negative semi-definite so that the curvature tensor of (N, h) is negative semi-definite in the dual sense of Nakano. In particular, the preceding integral formula implies that any harmonic map from a compact Kähler manifold M into N is necessarily pluriharmonic. In [Siu4, 6] Siu made use of the curvature term to show that when N is a compact quotient of an irreducible bounded symmetric domain of complex dimension ≥ 2 and f is of maximal rank, in fact either $\partial f = 0$ or $\bar\partial f = 0$. Furthermore, in the case of classical bounded symmetric domains, Siu showed that it suffices to assume that the rank of df over $\mathbf{R}$ is at least $2p + 1$, for some integers $p = p(\Omega)$ that can be explicitly determined. For example, in case Ω is of rank 1, i.e., $\Omega = B^n, p(\Omega) = 1$. This was further completed to cover the exceptional case by Zhong [Zh]. We can relate $p(\Omega)$ with the exponents appearing in the vanishing theorems of Calabi-Vesentini [CV] and Borel [Bo1]. More precisely, they showed that $H^q(N, T_N) = 0$ for $0 \leq q \leq n - p(\Omega)$ for the same $p(\Omega)$. As an application of the more general result of Siu-Zhong (Siu [Siu6] and Zhong [Zh]), it was shown, using a result of Kalka [Kal] that for complex submanifolds S of N of complex dimension at least $p + 1$, the Kuranishi germ of deformation of S as an abstract complex manifold agrees with the Douady germ of deformation of S as a submanifold of N.

(1.7) Irreducible compact quotients of the polydisc

As was explained in (1.3), local rigidity of complex structure holds for irreducible compact quotients N of polydiscs $\Delta^n, n \geq 2$, by a vanishing theorem due to Matsushima-Shimura [MS]. Moreover, as stated in (1.4) Mostow's Strong Rigidity Theorem also holds for this class of manifolds. It is therefore natural to expect that Siu's Strong Rigidity Theorem for Kähler manifolds can be strengthened to include such manifolds X.

The proofs of both the vanishing theorems of Calabi-Vesentini-Borel ([CV], [Bo1]) and Siu's Strong Rigidity Theorems for compact quotients of irreducible bounded symmetric domains ([Siu4, 6]) make use of pointwise computation in terms of the curvature tensor using Bochner-Kodaira formulas. When applied to compact quotients of the polydisc, such methods do not distinguish between the reducible and the irreducible cases. Global considerations based on the irreducibility of Γ have to enter. Because the proof of strong rigidity for $N = \Delta^n/\Gamma$ will be in some ways parallel to the proof of the vanishing theorem of Matsushima-Shimura [MS], as a motivation we sketch here a proof of the latter theorem. Without loss of generality we only consider irreducible compact quotients $N = \Delta^n/\Gamma$ with $\Gamma \subset (\mathrm{Aut}(\Delta))^n$. On Δ^n consider the holomorphic line bundles $L_k, 1 \leq k \leq n$ of complexified tangent vectors which are multiples of $\partial/\partial w_k$, where $(w_1, \ldots, w_n)$ is the system of Euclidean coordinates in $\mathbf{C}^n$. L_k is equipped with the Hermitian metric induced by the Poincaré metric on Δ^n. Since $\Gamma \subset (\mathrm{Aut}(\Delta))^n$, the Hermitian holomorphic line bundle L_k is invariant under Γ. By abuse of notations we also denote the induced Hermitian holomorphic line bundle on N by L_k.

Theorem (1.7.1) (Matsushima-Shimura [MS]). *Let $N = \Delta^n/\Gamma$ be an irreducible compact quotient of the polydisc $\Delta^n, n \geq 2$, such that $\Gamma \subset (\mathrm{Aut}(\Delta))^n$. Let E be the Hermitian holomorphic vector bundle on N of the form $L_{j_1}^{s_1} \oplus \cdots \oplus L_{j_m}^{s_m}$ with $s_k > 0$ for $1 \leq k \leq m \leq n, 1 \leq j_1 < \cdots < j_m \leq n$. Then, $H^p(N, E) = 0$ for $0 \leq p < n$.*

Remark. The holomorphic tangent bundle is given by $T_N = L_1 \oplus \cdots \oplus L_n$.

Sketch of proof of Theorem (1.7.1). It suffices to consider the case $E = L^s$, where $L = L_k(N)$ for some $k, 1 \leq k \leq n$, and $s > 0$. Equip L with the Hermitian metric induced by the Poincaré metric. The curvature form of L is negative semi-definite. Consider for $0 \leq p < n$ an L^s-valued harmonic $(0,p)$-form φ on N. The Bochner-Kodaira formula for the ∇-gradient (which yields vanishing theorems for negative line bundles) gives

$$0 = \int_N \|\bar{\partial}\varphi\|^2 + \|\bar{\partial}^*\varphi\|^2 = \int_N \|\nabla\varphi\|^2 + K_p(\varphi, \varphi),$$

where K_p stands for a Hermitian bilinear form which is positive semi-definite whenever the curvature form of the line bundle is negative semi-definite, as in our case. Thus, every term appearing in the integrands vanish identically on N. Since the curvature form of L_s is not identically zero, K_p is non-trivial for $0 \leq p < n$ $(K_n \equiv 0)$ so that the vanishing of the curvature yields non-trivial information. Let $\varphi = \sum \varphi_I$ be the decomposition of φ, where, in terms of Euclidean coordinates,

$$\varphi_I = f_I(w_1, \ldots, w_n) \left(\frac{\partial}{\partial w_k} \right)^s d\bar{w}^{i_1} \wedge \cdots \wedge d\bar{w}^{i_p}.$$

Then each φ_I is harmonic. The Bochner-Kodaira formula yields the information

$$\begin{cases} \bar{\partial}\varphi_I = \bar{\partial}^* \varphi_I = 0; \\ \nabla \varphi_I = 0; \\ \quad \varphi_I = 0 \quad \text{whenever } k \text{ does not appear in } I. \end{cases}$$

Permuting the Euclidean coordinates z_i we are led to consider a harmonic form φ such that the lifting φ to Δ^n is of the form

$$\varphi = f(w_1, \ldots, w_p) \left(\frac{\partial}{\partial w_i} \right)^s d\bar{w}^1 \wedge \cdots \wedge d\bar{w}^p.$$

The fact that f is independent of z_i for $p < i \leq n$ follows from the equation $\nabla\varphi = 0$ (which includes $\overline{\partial}^* \varphi = 0$) and $\overline{\partial}\varphi = 0$. The information obtained up to now is valid for products S of compact Riemann surfaces S_i of genus ≥ 2. In particular, the splitting phenomenon for harmonic forms representing $H^1(S, T_S)$ reflects the fact that local deformations of complex structures centered at S arise from local deformations of the complex structures of the individual factors S_i. For irreducible compact quotients N of the polydisc local rigidity of complex structure arises from global considerations based on the following density lemma on irreducible lattices in $(\mathrm{Aut}(\Delta))^n \cong (PSL(2, \mathbf{R}))^n, n \geq 2$.

Lemma (Borel-Matsushima-Shimura, [Bo3] & [MS]). *Let $N = \Delta^n/\Gamma$ be an irreducible quotient of Δ^n, $n \geq 2$, of finite volume in the Poincaré metric. Then, the canonical projection of Γ into any direct factor $(\mathrm{Aut}(\Delta))^r, 1 \leq r \leq n-1$, has dense image.*

To finish the proof of Theorem (1.7.1) consider $k = 1$ and φ of the form

$$\varphi = f(w_1, \ldots, w_p) \left(\frac{\partial}{\partial w_i} \right)^s d\bar{w}^1 \wedge \cdots \wedge d\bar{w}^p.$$

φ is invariant under the projection $\Gamma(1,\ldots,p)$ of Γ into $(\mathrm{Aut}(\Delta))^p$ of the first p factor Δ_i. By the density lemma φ is invariant under the full group $(\mathrm{Aut}(\Delta))^p$, so that φ has constant length on N. Recall that $\nabla\varphi = 0$, so that $\overline{\nabla}\bar{\varphi} = 0$. Write $(h_{i\bar{j}})$ for the Poincaré metric on Δ^n and let σ be the contraction of φ with $(h^{1\bar{1}})^s$ to obtain an L_1^{-s}-valued $(p,0)$ form. It follows from $\overline{\nabla}\sigma = 0$ that in fact $\bar{\partial}\sigma = 0$. In other words we have obtained a holomorphic section σ of $H = L_1^{-s} \otimes (L_1 \otimes \cdots \otimes L_p)^{-1}$ of constant length. If $\sigma \not\equiv 0$, this could only happen if H is flat, which is absurd since $s > 0$. This proves the vanishing of φ and hence Theorem (1.7.1).

The proof given above, which avoids using the maximum principle, is a slight modification of the proof given in Matsushima-Shimura [MS]. The modification applies immediately to proving vanishing theorem for L^2-cohomology when N is only assumed to be of finite volume (cf. Lai-Mok [LM]).

In order to strengthen Siu's Strong Rigidity Theorem for Kähler manifolds to cover also the case of compact and irreducible $N = \Delta/\Gamma$, there are two steps which essentially parallel the proof of the vanishing theorem of Matsushima-Shimura given above for $p = 1$. The first step is to make use of both the curvature term and the gradient term of Siu's $\partial\bar{\partial}$-Bochner-Kodaira formula to reduce the harmonic map under consideration to harmonic maps of one complex variable. The second step consists of using the density lemma to show that such a harmonic map is necessarily holomorphic or anti-holomorphic. The precise statement of Siu's Strong Rigidity Theorem for compact irreducible quotients of the polydisc is as follows:

Theorem (1.7.2) (Mok [Mok6], for $n = 2$ due to Jost-Yau [JY1, 2]). *Let M be a compact Kähler manifold. Suppose M is homotopic to an irreducible compact quotient N of the polydisc $\Delta^n, n \geq 2$. Then, there exists a diffeomorphism $f\colon M \to N$ such that, for the lifting $F\colon \widetilde{M} \to \widetilde{N} = \Delta^n$ to the universal coverings, all projections $F_i\colon \widetilde{M} \to \Delta\,(1 \leq i \leq n)$ are either holomorphic or anti-holomorphic.*

(1.8) Holomorphic foliations arising from harmonic maps into irreducible compact quotients of the polydisc

Let M be a compact Kähler manifold and $N = \Delta^n/\Gamma$ be an irreducible compact quotient of the polydisc. Without loss of generality we will assume that $\Gamma \subset (\mathrm{Aut}(\Delta)^n)$. Let $f_o\colon M \to N$ be a smooth homotopy equivalence and f be a harmonic mapping in the same homotopy class obtained using Theorem (1.4) (of Eells-Sampson). Recall the following integrated form of Siu's Bochner-Kodaira formula in (1.4)

$$\int_M H(\partial f \wedge \partial \bar{f}; \partial f \wedge \partial \bar{f}) + \|D\bar{\partial} f\|^2 = 0$$

where $H(\cdot\,;\,\cdot)$ is a Hermitian bilinear form on $f^*T^{1,1}(N)$-valued $(2,0)$-form on M. For $N = \Delta^n/\Gamma$, H is positive semi-definite so that both the curvature term and the gradient term vanish identically on M. Let $F\colon \widetilde{M} \to \widetilde{N} = \Delta^n$ be the lifting of f to the universal covering spaces and write $F = (F_1, \ldots, F_n)$ for the individual factors $F_i\colon \widetilde{M} \to \Delta_i$. By examining the curvature term and using the gradient term we obtain the equations

$$\begin{cases} \partial F_i \wedge \partial \bar{F}_i = 0\,; \\ D\bar{\partial} F_i = 0. \end{cases}$$

Since $f\colon M \to N$ is homotopic to a diffeomorphism, f is of degree ± 1 topologically so that F_i must be a submersion at some point P. Suppose $P \in \widetilde{M}$ is such that F_i is a submersion at P. Then, in a neighborhood U of P the level sets of F_i define a real codimension-2 foliation $\widetilde{\mathcal{F}}_i$ on U. Jost-Yau [JY1] made use of the above equations to deduce that the foliation is holomorphic on U. In other words, we can choose U to be biholomorphic to a polydisc Δ^n such that the level sets of F_i can be identified with $\{a\} \times \Delta^{n-1}$ and such that the harmonic map F_i depends only on the first complex variable. This allowed them to prove Theorem (1.7) for n =2 ([JY2]) by first showing that f is necessarily a diffeomorphism so that the holomorphic foliations $\widetilde{\mathcal{F}}_i$ can be globally defined. Their proof made use of Mostow's Strong Rigidity Theorem.

Let $\widetilde{\Theta}_i$ denote the holomorphic foliation on Δ^n with leaves $\Delta_{i-1} \times \{a_i\} \times \Delta^{n-i}$ and Θ_i denote the induced foliation on N. The density lemma of Borel-Matsushima-Shimura implies that every leaf of Θ_i is dense in N. Consider the harmonic map $F_i\colon \widetilde{M} \to \Delta$ on M. Since the Poincaré metric is real-analytic and f is pluriharmonic, hence harmonic with respect to any local choice of Kähler metric on M, by choosing a real-analytic Kähler metric on M locally we conclude from the regularity theory of elliptic equations that f is real-analytic. Let Y be the real-analytic subvariety on M where f fails to be a local diffeomorphism and let $\widetilde{Y}$ be the lifting to the universal covering M. Then, for any i, $F_i\colon \widetilde{M} \to \Delta$ is a submersion on $\widetilde{M} - \widetilde{Y}$. The holomorphic foliation $\widetilde{\mathcal{F}}_i$ can always be defined on $\widetilde{M} - \widetilde{Y}$. This induces a holomorphic foliation $\mathcal{F}_i$ defined on $M - Y$. Where f is locally a diffeomorphism we have

$f^*\Theta_i = \mathcal{F}_i$. The second step of the proof of Theorem (1.7) in general consists of using the density lemma to show that F_i is either holomorphic or anti-holomorphic. First we used the technique of extension of holomorphic objects to extend the holomorphic foliation $\widetilde{\mathcal{F}}_i$ on $\widetilde{M} - \widetilde{Y}$ meromorphically to the entirety of $\widetilde{M}$. We showed by using the maximum principle for harmonic maps (into negatively-curved manifolds) that unless F_i is either holomorphic or anti-holomorphic, there would be a closed leaf of the meromorphic foliation $\mathcal{F}_i$ on M. (Since the meromorphic foliation $\mathcal{F}_i$ is defined on M it is holomorphic on $M - Z$ for some complex-analytic subvariety Z of M of complex codimension ≥ 2. By a closed leaf L of $\mathcal{F}_i$ on M we mean a closed leaf of $\mathcal{F}_i|M - Z$. The closure of L in M is then necessarily a complex-analytic subvariety by a theorem of Remmert-Stein ([RS]).) We argued that this would contradict the density lemma (intuitively because of the fact that every leaf of Θ_i is dense in N). There was a gap in the proof in [Mok6, p.211, Step 3] that the existence of a closed leaf of $\widetilde{\mathcal{F}}_i$ on M contradicts the density lemma. A corrected proof is given in [Mok10, Appendix]. Another proof was given by Siu [Siu7]. For a general discussion of foliation techniques in Complex Differential Geometry we refer the reader to Mok [Mok5].

In [Siu7] Siu also gave a more conceptual proof of the existence and meromorphic extendability of $\mathcal{F}_i$ and the existence of a closed leaf. We give here a sketch of the proof.

Method of studying the foliations $\mathcal{F}_i$

1) Existence and meromorphic extendability of $\widetilde{\mathcal{F}}_i$

On $\overline{M} - \overline{Y}$, the level sets of F_i define a real-analytic foliation $\widetilde{\mathcal{F}}_i$. We are going to show that $\widetilde{\mathcal{F}}_i$ is holomorphic on $\overline{M} - \overline{Y}$ in such a way that the meromorphic extendability of $\widetilde{\mathcal{F}}_i$ to M is transparent. The level sets of F_i are real-analytic submanifolds of codimension 2. The foliation $\widetilde{\mathcal{F}}_i$ on $M - Y$ is defined by the distribution $x \to \mathrm{Ker}(dF_i)$ of real codimension 2. Let $W, W' \subset \overline{M} - \overline{Y}$ be such that ∂F_i resp. $\partial \bar{F}_i$ is non-zero. $\overline{M} - \overline{Y}$ is the union of W and W'. Without loss of generality we may assume that W is non-empty (and hence dense in M). Consider the distribution of complexified tangent vectors $x \to \mathrm{Ker}(\partial F_i)$. On W this defines a distribution of complex codimension 1 in $T^{1,0}(W)$. The two distributions $x \to \mathrm{Ker}(\partial F_i)$ and $x \to \mathrm{Ker}(dF_i)$ correspond to each other under the identification $v \leftrightarrow 2Re(v)$. To show that the foliation $\widetilde{\mathcal{F}}_i$ is holomorphic on W we show that the distribution $x \to \mathrm{Ker}(\partial F_i)$ is holomorphic. To do this, it suffices to give F^*L_i a holomorphic structure such that ∂F_i becomes a holomorphic section of F^*L_i over M. (Here

$T_{\Delta^n} = L_1 \oplus \cdots \oplus L_n$ as given in (1.7).) Recall that as a consequence of Siu's $\partial\bar{\partial}$-Bochner-Kodaira formula we have

$$\begin{cases} \partial F_i \wedge \partial \bar{F}_i = 0; \\ D\bar{\partial} F_i = 0. \end{cases}$$

In terms of the Riemann-Christoffel symbols (Γ^k_{ij}) and $(\overline{\Gamma^k_{ij}})$ on Δ^n using Euclidean coordinates the equation $D\bar{\partial}F_i = 0$ means that for any $\alpha, \beta, 1 \le \alpha, \beta \le n$, we have, for Euclidean coordinates w_i on Δ^n, for any $\alpha, \beta, 1 \le \alpha, \beta \le n$, and for $e^i = f^*(\partial/\partial w^i)$

$$\begin{aligned} 0 = \nabla_\alpha \partial_{\bar{\beta}} F_i &= \nabla_{\frac{\partial}{\partial z_\alpha}} \left(\frac{\partial F_i}{\partial \bar{z}_\beta} e^i \right) \\ &= \frac{\partial^2 F_i}{\partial z_\alpha \partial \bar{z}_\beta} e^i + \frac{\partial F_i}{\partial \bar{z}_\beta} \nabla_{\frac{\partial}{\partial z_\alpha}} (e^i) \\ &= \frac{\partial^2 F_i}{\partial z_\alpha \partial \bar{z}_\beta} e^i + \frac{\partial F_i}{\partial \bar{z}_\beta} \frac{\partial F_i}{\partial z_\alpha} \Gamma^i_{ii} e^i. \end{aligned}$$

(Note that i is a fixed index and there is no summation over i.) Similar calculation shows that

$$\nabla_{\bar{\beta}} \partial_\alpha F_i = \frac{\partial^2 F_i}{\partial \bar{z}_\beta \partial z_\alpha} e^i + \frac{\partial F_i}{\partial z_\alpha} \frac{\partial F_i}{\partial \bar{z}_\beta} \Gamma^i_{ii} e^i.$$

Thus, the equation $D\bar{\partial}F_i = 0$ means the same as $\bar{D}\partial F_i = 0$. (Alternatively, use complex geodesic coordinates to see this.)

Recall that we want to give F^*L_i a complex structure such that ∂F_i becomes holomorphic. This would be the case if the $\bar{\partial}$-operator on F^*L_i agrees with $\bar{D}$, since $\bar{D}\partial F_i = 0$. In terms of the smooth section $e^i = F^*(\frac{\partial}{\partial w_i})$ over U we have the $(0,1)$-component of the connection ∇ defined by

$$\nabla_{\frac{\partial}{\partial \bar{z}_\alpha}} (e^i) = \frac{\partial F_i}{\partial \bar{z}_\alpha} \Gamma^i_{ii} e^i.$$

For s an F^*L_i-valued smooth section define $\overline{D}s$ as before by $\overline{D}s = \sum \nabla_{\bar{\alpha}} s \, d\bar{z}^\alpha$. $\overline{D}$ can be extended to exterior covariant differentiations on F^*L_i-valued $(0,p)$ forms. To endow F^*L_i with a holomorphic structure such that the $\bar{\partial}$-operator agrees with exterior covariant differentiaion $\overline{D}$ by the integrability condition of Newlander-Nirenberg [NN] it is equivalent to show that $\overline{D}^2 = 0$ at the level of F^*L_i-valued-sections. (This condition allows one to solve for non-vanishing local smooth sections $s^{(k)}$

such that $\overline{D}s^{(k)} = 0$. Different $s^{(k)}$'s are related holomorphically because $\overline{D}$ satisfies the product rule. One can then take such local sections as local holomorphic bases for F^*L_i to endow F^*L_i with the structure of a holomorphic line bundle.) The condition $\overline{D}^2 = 0$ is equivalent to the symmetry condition

$$\nabla_{\bar\alpha}\nabla_{\bar\beta}e^i = \nabla_{\bar\beta}\nabla_{\bar\alpha}e^i.$$

We are going to make use of the equation $\partial F_i \wedge \partial\overline{F}_i = 0$ to verify the symmetry condition. At a point $x \in \widetilde{M}$ using complex geodesic coordinates at $F_i(x)$ we obtain at x

$$\begin{aligned}\nabla_{\bar\beta}\nabla_{\bar\alpha}e^i &= \nabla_{\bar\beta}\left(\frac{\partial F_i}{\partial \bar z_\alpha}\Gamma^i_{ii}e^i\right)\\ &= \frac{\partial F_i}{\partial \bar z_\alpha}\frac{\partial \overline{F_i}}{\partial \bar z_\beta}(-R_{i\bar i i\bar i})e^i + \frac{\partial F_i}{\partial \bar z_\alpha}\frac{\partial F_i}{\partial \bar z_\beta}(\partial_i\Gamma^i_{ii})e^i. \quad (\sharp)\end{aligned}$$

Here since we use complex geodesic cordinates $\partial_{\bar i}\Gamma^i_{ii} = \partial_{\bar i}\Gamma_{ii,\bar i} = -R_{i\bar i i\bar i}$. The last term of $(\sharp)$ is visibly symmetric in α and β. The first term on the right hand side of $(\sharp)$ is symmetric in α and β because of the equation $\partial F_i \wedge \partial\overline{F}_i = 0$.

Since we have proved the integrability from the covariant exterior differentiation $\overline{D}$, which is already defined on f^*L_i, we have in fact shown that f^*L_i admits a complex structure such that ∂f_i is a holomorphic section of $f^*L_i \otimes T_M$ on M, where $f = (f_1, \ldots, f_n)$ is some local choice of lifting of f to Δ^n and $s_i = \partial f_i$ is independent of the choice of lifting. Here by abuse of notation L_i stands for the induced holomorphic line bundle on M, as in (1.7).

One can similarly define on $f^*\overline{L_i}$ the structure of a holomorphic line bundle such that $\partial\bar f_i$ is holomorphic. On the intersection of $W$and W' the holomorphic distributions $x \to \operatorname{Ker}(\partial F_i)$ and $x \to \operatorname{Ker}(\partial\overline{F}_i)$ define the same foliation. Both holomorphic distributions extend meromorphically to $\widetilde{M}$. We have therefore constructed a meromorphic extension of $\widetilde{\mathcal{F}}_i$ to $\widetilde{M}$ and hence of $\mathcal{F}_i$ to M. This induces a holomorphic extension of $\mathcal{F}_i$ to $M - Z$ for some complex-analytic subvariety Z of codimension ≥ 2.

2) Existence of a closed leaf on $M - Z$

Let $s_i = \partial f_i$ and $t_i = \partial\bar f_i$ be global holomorphic sections of $f^*\overline{L}_i$ and $f^*\overline{L}_i$ as above. To find a closed leaf of $\mathcal{F}_i|M - Z$ it suffices to show that either $\rho = s_i/t_i$ must have zeros or poles. Since f^*L_i and $f^*\overline{L}_i$ are equipped with Hermitian metrics we can measure the meromorphic section ρ of $(f^*L_i) \otimes (f^*\overline{L_i})^{-1}$. At a point P where f_i is a submersion we can choose a neighborhood G of P biholomorphic to a polydisc Δ^n such that in terms of coordinates on Δ^n, we have $f_i(z_1, \ldots, z_n) = h_i(z_1)$, where h_i is a harmonic map of one complex variable into (Δ, Poincaré metric). It then follows readily that the length of ρ on G is constant along every leaf of $\mathcal{F}_i$ on $M - Y$. By continuity this remains true on $M - Z$. Suppose the zero set D of ρ is non-empty, say. Then, $D - Z$ can be identified with some leaf of $\mathcal{F}_i|M - Z$. In particular, we have found a closed leaf of $\mathcal{F}_i|M - Z$.

We show by contradiction that ρ must either have zeros or poles. Otherwise ρ would be a nowhere-vanishing holomorphic section of $H = (f^*L_i) \otimes (f^*\overline{L}_i)^{-1}$ over M, so that H is holomorphically trivial on M. However, $H = f^*(L_i \otimes \overline{L_i}^{-1})$ as smooth vector bundles, and, since f is homotopic to a diffeomorphism, we can conclude that the smooth line bundle $L \otimes \overline{L}^{-1}$ over $N = \Delta^n/\Gamma$ is differentiably trivial. But $c_1(L_i \otimes \overline{L}^{-1}) = 2c_1(L_i) = c_1(L_i)$ and $c_1(L_i^2) \neq 0$ since the holomorphic line bundle L_i^2 admits a Hermitian metric with negative semi-definite curvature form which is not flat. This proves that ρ must have zeros or poles by contradiction and shows that $\mathcal{F}_i|M - Z$ must have a closed leaf.

We mention in passing the following application of the use of holomorphic foliations to finding holomorphic functions of Siu.

Theorem (1.8) (Siu [Siu7], Theorem (4.7), p. 138). *Let M be a compact Kähler manifold and N be a compact hyperbolic Riemann surface such that there exists a continuous map h_0 from M to N which is nonzero on the second homology group. Let h be the (unique) harmonic map homotopic to h_o. Then, there exists a holomorphic map g from M to a compact hyperbolic Riemann surface R and a harmonic map φ from R to N such that $h = \varphi \circ g$. As a consequence the lifting of g to the universal covers of M and R is a nonconstant bounded holomorphic function on the universal cover $\widetilde{M}$ of M.*

To prove Theorem (1.8) one considers the leaf space of the meromorphic foliation F on M defined by the harmonic map h. In this case all leaves are closed and the leaf space can be given the structure of a compact Riemann surface R. R is hyperbolic because there does not exist a harmonic map of real rank 2 from an elliptic curve or the Riemann sphere into a hyperbolic Riemann surface (cf., e.g., Eells-Sampson [ES]).

(1.9) Strong rigidity for quotients of the ball of finite volume

Siu [Siu7] and Jost-Yau [JY3, 4] studied the extension of the Strong

Rigidity Theorems (1.6) (1.7) to the non-compact situation. In particular, Theorem (1.6) was extended in the rank-1 case to quotients of finite volume in Jost-Yau [JY3] as

Theorem (1.9) (Jost-Yau [JY3]). *Let (N, h) be a quotient of the unit ball B^n of finite volume in the Poincaré metric, $n \geq 2$. Let M be a quasi-projective manifold diffeomorphic to N. Then, M is either biholomorphic or conjugate-biholomorphic to N.*

Recently, Jost-Yau [JY4] have proved extensions of Theorem (1.9) to the higher rank situation under some additional assumptions. For simplicity we restrict ourselves to Theorem (1.9). We explain here the necessary modifications in passing to the non-compact rank-1 situation. Since M is quasi-projective, we can choose a projective compactification $\overline{M}$ of M such that $\overline{M} - M = D$ is a union of smooth hypersurfaces of Z with normal crossings. One can construct on M a Kähler metric g of bounded sectional curvature and finite volume. For every point P on the divisor D there is a neighborhood W such that $U = W \cap M$ is biholomorphic to a domain $(\Delta^*)^k \times \Delta^{n-k}, 1 \leq k \leq n$, where Δ denotes the unit disc and Δ^* denotes the punctured unit disc. The Kähler metric g on U (which we call a punctured polydisc) is asymptotically equivalent to the Poincaré metric on U. We will say for short that g is asymptotically equivalent to the Poincaré metric. On the other hand N admits a smooth toroidal compactification $\overline{N}$ (unique because N is of rank-1) with an explicit asymptotic description of the Kähler-Einstein metric h. The metric h is asymptotically dominated by the Poincaré metric on punctured polydiscs (but not equivalent to it). As a quotient of the ball N can be decomposed into the union of a compact set K_N and a finite number of ends N_i such that K_N is homotopic to N and each N_i is diffeomorphic to $S_i \times [0, \infty)$ for some compact manifold S_i. By lifting N to a finite cover so that there are at least 2 ends, we can assume without loss of generalities that the $(2n-1)$-dimensional homology classes represented by the cycles S_i are non-trivial.

To simplify the presentation we make the following additional assumption: (A) There exists a projective compactification $\overline{M}$ of M and a diffeomorphism $f_o \colon M \to N$ which extends to a diffeomorphism of $\overline{M}$ onto $\overline{N}$.

In terms of the Kähler metrics h and g the diffeomorphism $f_o \colon M \to N$ is asymptotically distance-decreasing up to scaling factors so that the local energy $e(f_o) = tr_g(f_o^*(h))$ is uniformly bounded. Since (M, g) is of finite volume f_o is of finite Dirichlet energy. We can then deform f_o to a harmonic map f with finite Dirichlet energy by using the method of Schoen [Sch], which extends the results of Eells-Sampson [ES] to the case where the image manifold is non-compact and of non-positive Riemannian sectional curvature. More precisely, we take an exhaustion of M by domains Ω_ν with smooth boundary and construct harmonic maps $f_\nu\colon \Omega_\nu \to N$ such that $f_\nu|\partial\Omega_\nu = f_o|\partial\Omega_\nu$. f_ν extends to a homotopy equivalence from M to N. To extract a convergent subsequence by using the uniform Lipschitz estimate on compact sets of Schoen [Sch] it suffices to show that $\{f_\nu(x)\}$ is relatively compact at some point x. Otherwise for any compact subset K of M there would exist a ν such that $f_\nu(K)$ is contained in some end N_i. Taking $K = K_N$ this would contradict the fact that the $\pi_1(N_i)$ is nilpotent while $\pi_1(N)$ is not. (For facts on the topology of manifolds of negative curvature we refer the reader to Eberlein [Eber] or Siu-Yau [SY4].) To exploit the harmonic map f we need the extension of the $\partial\bar\partial$-Bochner-Kodaira formula to the non-compact case as given in Siu [Siu7, 6] and Jost-Yau [JY3, 3]. The key point is the following lemma:

Lemma. *Let M and N be Riemannian manifolds such that M is complete, the Riemannian sectional curvature of N is non-positive, and the Ricci curvature of M is bounded from below by a negative number $-k$. Let $f\colon M \to N$ be a harmonic map with finite Dirichlet energy. Then, the global L^2 norm over M of the covariant derivative ∇df of the differential df is finite.*

The above lemma can be regarded as a quasi-linear analogue of the following: Let M be a complete Riemannian manifold of Ricci curvature bounded from below by a negative number $-k$, then for any square-integrable harmonic 1-form $\varphi, \nabla\varphi$ is also square-integrable, as a consequence of which one shows that φ satisfies the integrated form of the Bochner formula. In the same vein we derive from the above lemma the following

Corollary. *If in the lemma M and N are Kähler, then the $\partial\bar\partial$-Bochner Kodaira formula of Siu holds for the harmonic map f.*

We return now to the situation of Theorem (1.9). We have a harmonic map $f\colon M \to N$ homotopic to a diffeomorphism. Recall that each end N_i of N is diffeomorphic to $S_i \times [0,\infty)$ where $S_i \times \{t\}$ represents a non-trivial $(2n-1)$-dimensional homology class. This forces f to have real rank $\geq 2n-1 \geq 3$ so that by (1.5) f is necessarily holomorphic or anti-holomorphic. Replacing N by its complex conjugate if necessary we may assume that f is holomorphic. It follows readily that f is of maximal rank at some point. To complete the proof of Theorem (1.9) one has to show that f is proper (so that $f: M \to N$ is a finite analytic cover) and that f is generically injective.

The holomorphic map $f: M \to N$ is a biholomorphism

Using the diffeomorphism $f_o: M \to N$ we decompose M into the union of a compact set K_M and a finite number of ends M_i such that each M_i is diffeomorphic to $T_i \times [0, \infty)$ for some compact manifold T_i.

We show first of all that $f: M \to N$ extends meromorphically to M. Let $P \in D$ and consider a neighborhood W of P in M such that W is biholomorphic to the polydisc Δ^n and $W \cap M$ is identified with $\Delta^* \times \Delta^{n-1}$. Consider the restriction of the holomorphic map f to $\Delta^* \times \{t\} = \Delta_t^*$. Since the Dirichlet energy of f is finite and since $E(f|\Delta_t^*) \leq E(f)$ at every point of $\Delta^* \times \Delta^{n-1}$ it follows from Fubini's theorem that $E(f|\Delta_t^*)$ is finite except at most for a set A of $\{t\}$ of zero Euclidean measure. To show that f extends meromorphically to W we make use the following theorem of Shiffman [Sh] which is a generalization of Hartogs' Theorem on separately meromorphic functions:

Theorem (special case of Shiffman [Sh]). *In the notations of the above $f|W \cap M$ extends meromorphically to W if $f|\Delta_t^*$ extends meromorphically except for a set A of $t \in \Delta^{n-1}$ of zero Euclidean measure.*

We now prove the meromorphic extension of $f|\Delta_t^*$ to $\Delta_t = \Delta \times \{t\}$ under the assumption that $E(f|\Delta_t^*)$ is finite. To do this we will use another compactification of N. By Satake [Sat1, 2], Baily [Bai] and Borel-Baily [BB] N admits a projective singular (minimal) compactification $\overline{N}_{\min}$ obtained by adjoining to N a finite number of isolated singular points. $\overline{N}_{\min}$ is obtained from the toroidal compactification N by blowing down the divisors at infinity. Each end N_i of N corresponds to an isolated singularity Q_i of $\overline{N}_{\min}$. Embed $\overline{N}_{\min}$ into some projective space $\mathbf{P}^N$ and let ω be the Kähler form of some Fubini-Study metric on $\mathbf{P}^N$. Consider the graph G of $f|{\Delta_t}^*$ as a subvariety of $\Delta_t^* \times \overline{N}_{\min} \subset \Delta_t^* \times \mathbf{P}^N$. Equip Δ_t with the Euclidean Kähler form β and denote by π_1, π_2 the projections of $\Delta_t \times \mathbf{P}^N$ onto the two factors resp. By Bishop's theorem to extend $f|\Delta_t^*$ meromorphically to Δ_t it suffices to show that Volume$(G, \pi_1^*\beta + \pi_2^*\omega)$ is finite. Denote by η the Kähler form of the Kähler-Einstein metric on N. The assumption that $E(f|\Delta_t^*) < \infty$ implies Volume$(G, \pi_1^*\beta + \pi_2^*\eta) < \infty$. It suffices therefore to show that there exists a positive constant c such that $\eta \geq c\omega$. For each isolated singularity Q_i of $\overline{N}_{\min}$ let B_i be a neighborhood of Q_i in $\mathbf{P}^N$ biholomorphic to the unit ball B^N. Let $B_i \Subset B_i^o, U_i^o = B_i^o \cap \overline{N}_{\min}$ and $U_i = B_i \cap \overline{N}_{\min}$. Assume that Q_i is the only isolated singular point on the closure of U_i^o

in $\overline{N}_{\min}$. Consider the inclusion mapping $\sigma: U_i^o \to B_i^o$. Equipping B_i with the Kähler-Einstein metric with Kähler form μ we obtain from the Ahlfors-Schwarz lemma that $\eta \geq c_1\mu$ on U_i for some positive constant c_1. Since $\mu \geq c_2\omega$ for some positive constant c_2 we have obtained the desired inequality $\eta \geq c\omega$ to show that $\text{Volume}(G, \pi_1^*\beta + \pi_2^*\omega) < \infty$. Since the point $P \in D = \overline{M} - M$ is arbitrary, applying Bishop's theorem and the theorem of Shiffman above we have proved that the holomorphic mapping $f: M \to N$ extends meromorphically to $f: \overline{M} \to \overline{N}_{\min}$.

Remarks. 1) The theorem of Shiffman quoted was proved actually under much weaker hypotheses. To give a simpler proof in the special case we need we can proceed as follows. Consider the closed (1, 1) form $f^*\omega$ defined on M. The boundedness of $\int_{\Delta_t^*} f^*\omega$ implies that $f^*\omega|\Delta_t^*$ extends as a closed positive (1, 1) current to Δ_t. One can apply the argument of Harvey-Polkings [HP] to show that the trivial extension of $f^*\omega$ to M defines a closed positive (1, 1) current on M. This implies that in fact $\int_{\Delta_t^*} f^*\omega$ is finite for any t. It follows then from Hartogs' Theorem on separately meromorphic functions to conclude that $f: M \to N$ extends meromorphically to $f: \overline{M} \to \overline{N}_{\min}$.

2) Despite the fact that (U_i^o, η) is not complete, one can still apply the Ahlfors-Schwarz lemma on $U_i \Subset U_i^o$ using the argument of Cheng-Yau [CY3]. It suffices that there exists an $\epsilon > 0$ such that for any point $x \in U_i$, the geodesic ball $B(x; \epsilon)$ on (N, η) centered at x and of radius ϵ is contained in U_i^o.

We now proceed to show that f is proper. By the assumption (A) each component D_i of D is irreducible. The meromorphic mapping $f: M \to \overline{N}_{\min}$ establishes a correspondence of sets. The assertion that f is proper is equivalent to the assertion that $f(D_i) = Q_j$ for some j. Suppose otherwise. Then there exists a point $P \in D_i$ such that f is holomorphic at P and $f(P)$ is a point on N. Let $W \cong \Delta^n$ with $W \cap M \cong \Delta^* \times \Delta^{n-1}$ be as above such that P corresponds to the origin. Let γ be the loop defined by $\gamma(\theta) = (\epsilon e^{i\theta}, 0), 0 \leq \theta \leq 2\pi$, for an arbitrarily small $\epsilon > 0$. Taking $\epsilon > 0$ small enough the assumption that $f(P)$ is a point on N implies readily that $f_*(\gamma)$ is contractible. Since f is a homotopy equivalence and the initial diffeomorphism $f_o: M \to N$ extends smoothly to a diffeomorphism from M to N by the assumption (A) this contradicts with the following

Lemma. *Let T be an irreducible component of $\overline{N} - N$ and let γ' be a loop wrapping around T constructed in the same way as in the preceding construction on M. Then γ' is homotopically non-trivial on N.*

We remark that the lemma is not true in a general context. The lemma results from the explicit construction of ends N_i. For details of the construction of toroidal compactifications we refer the reader to Mumford [Mum] and Ash-Mumford-Rapoport-Tai [AMTR]. The construction can be extended to the non-arithmetic case by using descriptions of the ends as given in Siu-Yau [SY4].

Finally, we have to show that the proper holomorphic map $f: M \to N$ is a biholomorphism. Since f is proper for $x \in N$, $f^{-1}(x)$ is a compact complex subvariety of M. Since f is a homotopy equivalence it cannot map any positive-dimensional compact subvariety, which represents a non-trivial integral homology class, to a point. Thus, f is a finite analytic cover. It remains to show that there is only one sheet.

For M and N compact the lost assertion follows from the fact that f is of degree one. In the non-compact case we consider intersection (cap) products. Since $\overline{N}_{\min}$ is algebraic we can choose two non-singular algebraic subvarieties C, C' on $\overline{N}_{\min}$ of complementary dimension so that they avoid the isolated singularities Q_i and intersect non-trivially at normal crossings. The intersection product of C and C' on N is an element of $H_o(N, \mathbf{Z})$ which can be identified with the number $r \geq 1$ of intersection points. Since $f: M \to N$ is proper $f^{-1}(C)$ and $f^{-1}(C')$ are compact. By a generic choice of C and C' we may assume that $f^{-1}(C)$ and $f^{-1}(C')$ are reduced and intersect at normal crossings. Suppose the finite analytic cover $f: M \to N$ is λ-sheeted. The intersection product of $f^{-1}(C)$ and $f^{-1}(C')$ is then equal to the number of intersection points, which is λr. This would lead to a contradiction unless $\lambda = 1$ since f is a proper homotopy equivalence. It follows that f can induce isomorphisms on homology groups of M and N only if the sheeting number λ of f is equal to one. This proves that $f: M \to N$ is in fact a biholomorphism.

(1.10) Strong rigidity for irreducible quotients of the polydisc of finite volume

Considerations of (1.8) and (1.9) can be strengthened to yield strong rigidity theorems for irreducible quotients of the polydisc Δ^n of finite volume, $n \geq 2$. The key point missing is a theorem on the holomorphicity or conjugate-holomorphicity of the harmonic maps since the proof in (1.8) is based on finding a closed leaf of some holomorphic foliation, for which we need to use the maximum principle or a variation of it. This step was accomplished in Mok [Mok10]. We prove more specifically

Theorem (1.10.1) (Mok [Mok10]). *Let M be a complete Kähler manifold such that the Ricci curvature is bounded from below by a con-*

stant. Let N be an irreducible quotient of the polydisc $\Delta^n, n \geq 2$, of finite volume in the induced Poincaré metric. Let $f: M \to N$ be a harmonic map of finite Dirichlet energy. Suppose $f^*(\pi_1(M))$ is of finite index in $\pi_1(M) = G$ and that $\mathrm{Ker}(f_*)$ is finitely generated. Let $F: M \to N = \Delta^n$ be the lifting of F to the universal coverings, $F = (F_1, \ldots, F_n)$. Then each F_i is either holomorphic or conjugate-holomorphic.

We will use without further explanations the notations in (1.7) and (1.8). We explain the key points in the special case of Theorem (1.10.1) when f induces an isomorphism on fundamental groups, which yields strong rigidity. In this case it is clear that each F_i is a submersion at some point. Otherwise the image of the harmonic mapping F_i would lie on some geodesic and this would immediately violate the density lemma of Borel-Matsushima-Shimura in (1.7). To start with, recall that in (1.8) we knew in the compact case that there exist meromorphic foliations $\widetilde{\mathcal{F}}_i$ on the manifold $\widetilde{M}$ defined at dense set of points by level sets of F_i. The existence and meromorphicity of $\widetilde{\mathcal{F}}_i$ comes from the structural equations

$$\begin{cases} \partial F_i \wedge \partial \overline{F_i} = 0; \\ D\bar{\partial} F_i = 0 \end{cases}$$

coming from the $\partial\bar{\partial}$-Bochner-Kodaira formula of Siu. The justification of this formula as stated in (1.9) means that in the present non-compact case, we still have the meromorphic foliation F_i at our disposal. A key to proving the Strong Rigidity Theorem (1.8) is to find a closed leaf of the induced meromorphic foliation $\widetilde{\mathcal{F}}_i$ on M. In the proof of this step presented in (1.8) we endow (in the notation of (1.8)) $f^*L_i \otimes f^*\overline{L_i}^{-1}$ a holomorphic structure such that if neither ∂f_i nor $\partial \overline{f}_i$ vanish identically, then $\rho = \frac{\partial f_i}{\partial \overline{f}_i}$ can be interpreted as a meromorphic section of $f^*L_i \otimes f^*\overline{L_i}^{-1}$. The closed leaf we looked for corresponded to zeros or poles. If both sets are empty then $f^*L_i \otimes f^*\overline{L_i}^{-1}$ would be holomorphically trivial, giving rise to a contradiction by consideration of first Chern classes. A variation of this argument is to look at the length $\lambda = \|\rho\|$ and look for zeros or poles by considering $\Delta \log \lambda$ and using the maximum principle. (This was the approach taken originally in Mok [Mok6].) In the non-compact case neither argument works. However, intuitively the existence of λ on $M - Z$ should already lead to a contradiction. On $M - Z$ the level sets S_λ of the real analytic function $e^{-\lambda}$ define generically real-analytic hypersurfaces with the property that every leaf of $\mathcal{F}_i$ stays inside one of the sets S_λ. On the other hand on N as a consequence of the density lemma each of the corresponding foliation Θ_i is dense. The key point of Mok [Mok10] is to argue that this would contradict with the assumptions on the map $f_*: \pi_1(M) \to \pi_1(N) = \Gamma$.

Without loss of generality we assume as before that $\Gamma \subset (\mathrm{Aut}(\Delta))^n$. To simplify notations let $i = 1$ and $n = 2$. By the density lemma the two projections Γ_1 and Γ_2 of Γ are dense in $\mathrm{Aut}(\Delta)$. A further topological property that we need to know about N is the fact that $b_1(N) = 0$ due to Selberg (cf. Bass [Ba]). Equivalently, this means that $\Gamma/[\Gamma,\Gamma]$ is a finite group. The proof that F_1 is either holomorphic or anti-holomorphic breaks down into 2 cases: (i) when all leaves of $\mathcal{F}_1$ are closed and (ii) when there exists some leaf L of $\mathcal{F}_1$ which is not closed.

Case (i)

Unlike the compact situation, the existence of a single closed leaf of $\mathcal{F}_1$ is not enough to guarantee that F_1 is either holomorphic or anti-holomorphic. In Case (i) when all leaves are closed we can obtain a contradiction by considering the fundamental group. We give first a heuristic argument. Consider the equivalence relation on $\widetilde{M} - \widetilde{Z}$ of identifying points on the same leaves of $\widetilde{\mathcal{F}_i}$. Denote the equivalence relation by the same notation $\widetilde{\mathcal{F}}_i$. Suppose $(\widetilde{M} - \widetilde{Z})/\widetilde{\mathcal{F}}_i$ can be given the structure of a normal complex space Σ_i. Σ_i will then be a Riemann surface. Since all leaves of $\mathcal{F}_1$ are closed we can supposedly perform the same procedure to get a Riemann surface $(M - Z)/\mathcal{F}_1 = S$. We do not however assume that all leaves of $\mathcal{F}_2$ are closed so that we cannot talk about $(M - Z)/\mathcal{F}_2$. The fundamental group $\pi_1(M)$ acts on the product Riemann surface $\Sigma = \Sigma_1 \times \Sigma_2$. Identify thus $\pi_1(M)$ with a subgroup Φ of $\mathrm{Aut}(\Sigma_1) \times \mathrm{Aut}(\Sigma_2)$. Write $\varphi = (\varphi_1, \varphi_2)$ for $\varphi \in \Phi$. Then $\Phi_1 = \{\varphi_1\} \cong \Gamma_1$ acts on Σ_1 to get $S = \Sigma_1/\Phi_1$. Thus, Φ_1 can be identified with a subgroup of $\pi_1(S)$. With this identification Φ_1 is isomorphic to the fundamental group of some Riemann surface S', so that $\Phi_1/[\Phi_1,\Phi_1]$ is isomorphic to $H_1(S')$, a non-trivial free abelian group. On the other hand, since $\Gamma/[\Gamma,\Gamma]$ is finite, so is $\Gamma_1/[\Gamma_1,\Gamma_1]$. This leads to a contradiction, ruling out the possibility (i).

The difficulty in making the heuristic argument rigorous is in constructing the moduli space $(\widetilde{M} - \widetilde{Z})/\widetilde{\mathcal{F}}_i = \Sigma_i$. The first problem is that the topological quotient $(\widetilde{M} - \widetilde{Z})/\widetilde{\mathcal{F}_i}$ is not even Hausdorff in general. Even if it is, one may not be able to endow a complex structure on it so that the mapping $\widetilde{M} - Z \to (\widetilde{M} - \widetilde{Z})/\widetilde{\mathcal{F}}_i = \Sigma_i$ is holomorphic. In case of compact M one can use either the Chow space or the Douady space to construct a moduli space. Even in this case, it may happen that certain disjoint leaves have to be identified. In the general case we have the notion of holomorphic equivalence relation and moduli spaces constructed by Kaup [Kaup].

Let X be a complex manifold and $\mathfrak{R}$ be an equivalence relation on X. We also denote by $\mathfrak{R}$ the graph $\{(x,y) \in X \times X : x\mathfrak{R}y\}$. $\mathfrak{R}$ is said to be a holomorphic equivalence relation if $\mathfrak{R}$ is a complex-analytic subvariety of $X \times X$. Let $\nu: \mathfrak{R} \to X$ be the canonical projection onto the first factor. Then, we say that the holomorphic equivalence relation $\mathfrak{R}$ is open if ν is an open map. Let now R be a holomorphic equivalence relation on X and $X/\mathfrak{R}$ the topological space with the natural topology induced by the projection map $\theta: X \to X/\mathfrak{R}$. On the topological space $X/\mathfrak{R}$ one can introduce a structure sheaf of rings $\mathcal{O}_{X/\mathfrak{R}}$ by assigning to every open subset W of $X/\mathfrak{R}$ the set of holomorphic functions on $\theta^{-1}(W)$ invariant under $\mathfrak{R}$. The main result of Kaup [Kaup] in case of complex manifolds X asserts

Theorem (Kaup [Kaup]). *Let X be a complex manifold and $\mathfrak{R}$ be an open holomorphic equivalence relation on X. Then $(X/\mathfrak{R}, \mathcal{O}_{X/\mathfrak{R}})$ is isomorphic as a ringed space to a normal complex space.*

In general a holomorphic foliation does not define a holomorphic equivalence relation. In our situation we make use of the very special fact that at generic points the leaves of $\widetilde{\mathcal{F}}_i$ can be identified with connected components of level sets of the real-analytic function F_i. Identifying points on level sets of the harmonic mapping F_i introduces an equivalence relation which we denote by Ψ_i. The graph of Ψ is then a countable union of irreducible real-analytic subvarieties of $(\widetilde{M}-\widetilde{Z})\times(\widetilde{M}-\widetilde{Z})$. The irreducible component containing the diagonal is necessarily holomorphic since $\widetilde{\mathcal{F}}_i$ is a holomorphic foliation. We construct moduli spaces by introducing an equivalence relation $\widetilde{\mathfrak{R}}_i$ such that $\widetilde{\mathcal{F}}_i \subset \widetilde{\mathfrak{R}}_i \subset \Psi_i$ defined as the (non-empty) union of complex-analytic components of Ψ_i. We show that $\widetilde{\mathfrak{R}}_i$ is indeed an equivalence relation and open by making use of the fact that $\widetilde{\mathcal{F}}_i$ is a foliation of complex-analytic hypersurfaces. Given Kaup's result the heuristic argument can indeed be made rigorous to rule out Case (i).

Case (ii)

This is the situation when some leaf of $\mathcal{F}_1$ is not closed. The key idea of Mok [Mok10] is to exploit a recurrent point of the foliation to generate a one-parameter group G of automorphisms of the unit disc Δ_1 so that when one moves on a connected component of a level set $\widetilde{S}_\lambda$ of the function $\widetilde{\lambda}$ on $\widetilde{M} - \widetilde{Z}$ (obtained by lifting the function λ on

$M - Z$), the image under F_1 lies inside an orbit of the one-parameter group G. This is a serious restriction on the mapping F_1. On the one hand taking any point $x \in M$, the set $\{F_1(\varphi(x)) : \varphi \in \pi_1(M)\}$ is the orbit of $F_1(x)$ under Γ_1, which is dense in the unit disc Δ_1 by the density lemma. On the other hand the orbits under G are closed real-analytic curves on the unit disc Δ_1 which are permuted by the actions of Γ_1 since connected components of level sets $\widetilde{S}_\lambda$ are permuted under the action of the fundamental group $\pi_1(M)$, the function λ being lifted from M. The group of Möbius transformations ϑ which permute the orbits of G are precisely the normalizer of the group G. ϑ is necessarily a proper closed subgroup of $\mathrm{Aut}(\Delta_1)$, so that the orbit of $F_1(x)$ under Γ_1 lies in a proper closed subset of Δ_1, contradiction the density lemma.

The assumption that there exists a leaf of $\mathcal{F}_1$ which is not closed means that there is a recurrent point $x \in M - Z$ of $\mathcal{F}_1$. We argue that in fact every point of $M - Z$ is a recurrent point of $\mathcal{F}_1$. This will allow us to work at points $x \in M - Z$ at which F_1 is a submersion. To do this we first show that $\Sigma_1 = (M - Z)/\widetilde{\mathfrak{R}}_1$ is a hyperbolic Riemann surface. In fact, if Σ_1 were the Riemann sphere $\mathbf{P}^1$, we would obtain from F_1 a non-trivial harmonic mapping from $\mathbf{P}^1$ into Poincaré disc Δ_1, violating the maximum principle for harmonic maps into negatively-curved complete Riemannian manifolds. On the other hand if Σ_1 were the complex plane $\mathbf{C}$ or the punctured complex plane $\mathbf{C}^*$, we would have represented Γ_1 as a group of automorphisms of $\mathbf{C}$ or $\mathbf{C}^*$, so that Γ_1 is nilpotent, contradicting with the density of Γ_1 in $\mathrm{Aut}(\Delta_1)$. The upshot is that Σ_1 is equipped with the Poincaré metric. As $\pi_1(M)$ acts on Σ_1, the existence of a leaf of $\mathcal{F}_1$ which is not closed means that $\pi_1(M)$ does not act properly discontinuously on Σ_1. Denote the image of $\pi_1(M)$ on $\mathrm{Aut}(\Sigma_1)$ by Ξ. It follows by using the invariance property of the Poincaré metric that there exists a sequence $\{\xi_i\}$ in Ξ converging to the identity. From this one deduces that every point on Σ_1 is a point of accumulation of Ξ. Equivalently, every point of $M - Z$ is a recurrent point of $\mathcal{F}_1$.

Fix now a point $\tilde{x} \in \widetilde{M} - \widetilde{Z}$ at which F_1 is a submersion. Write x for the projection of $\tilde{x}$ into $M - Z$. Let $\widetilde{S}$ be a the local piece of the level set $\widetilde{S}_\lambda$ passing through $\tilde{x}$. $\widetilde{S}$ traces a smooth local curve C on Δ_1 passing through $F_1(\tilde{x})$, which we may take to be the origin o. Since x is a recurrent point on $M - Z$ for the foliation $\mathcal{F}_1$, the origin o is an accumulation point of $\gamma_{1,i}(o)$ for some sequence $\{\gamma_{1,i}\} \subset \Gamma_1 \subset \mathrm{Aut}(\Delta_1)$. Extracting a subsequence we may assume that $\{\gamma_{1,i}\}$ converges to a Möbius transformation fixing the origin, i.e., a rotation. Since F_1 is a submersion at $\tilde{x}$ by varying the point $\tilde{x}$ we see that a neighborhood of the origin is foliated by real-analytic curves $\{C_\lambda\}$ essentially invariant under $\{\gamma_{1,i}\}$. Thus, in fact $\{\gamma_{1,i}\}$ converges to the identity transformation. For i large enough write $\gamma_{1,i} = \exp(t_i v_i)$, where v_i is of unit length. Let v be an accumulation point of $\{v_i\}$ and G be the one-parameter subgroup $\exp(R_v)$. G is then a closed subgroup of $\mathrm{Aut}(\Delta_1)$ such that the local curves $\{C_\lambda\}$ traced by F_1 lie on orbits of G. (Any one-parameter group G of Möbius transformations is either conjugate to a rotation group, a group of hyperbolic translations, or a group of parabolic translations. In particular, G is closed.)

In the compact case, using the technique of generating a one-para meter subgroup by recurrent points of foliations, we obtained a proof of the following stronger theorem on the holomorphicity or conjugate-holo morphicity of harmonic mappings into irreducible compact quotients of the polydisc.

Theorem (1.10.2). *Let M be a compact Kähler manifold; N be an irreducible compact quotient of $\Delta^n, n \geq 2$. Suppose $f: M \to N$ is a harmonic map. Let $F: \widetilde{M} \to \widetilde{N} = \Delta_n$ be the lifting of f to universal coverings and write $F = (F_1, \ldots, F_n)$. Suppose $F_i, 1 \leq i \leq n$, are of rank* 2 *(i.e., a submersion) at some point and that f is of rank at least* 3 *at some point. Then, each F_i is either holomorphic or anti-holomorphic.*

Remark. Theorem (1.10.2) was already stated in Mok [Mok6]. There we proved the existence of a closed leaf L of $\mathcal{F}_i$ in $M - Z$, but as indicated above there was a gap in deducing that F_i is either holomorphic or anti-holomorphic. Let $\overline{L}$ be the closure of L in M. To complete the proof given in [Mok6], the key point is to show that $f_*(\pi_1(\overline{L})) \subset H \times (\mathrm{Aut}(\Delta))^{n-1}$ for some finite subgroup of $\mathrm{Aut}(\Delta_1)$. The difficulty is that we only know $f^*(\pi_1(\overline{L})) \subset S_1 \times (\mathrm{Aut}(\Delta))^{n-1}$ where S_1 is the circle group. If $f^*(\pi_1(\overline{L}))$ does not belong to $H \times (\mathrm{Aut}(\Delta))^{n-1}$ for any finite subgroup H of S_1, then we know from the technique above that the images of F_1 lie in the orbit of G which is the circle group S_1. This allowed us to complete the proof of the theorem.

Lecture II. Uniformization of Compact Kähler Manifolds of Nonnegative Curvature

(2.1) Hermitian symmetric manifolds of compact type

We now turn our attention to Hermitian symmetric spaces of compact type, which are in some sense duals of bounded symmetric domains. They are the higher dimensional generalization of the Riemann sphere $\mathbf{P}^1$. The first examples are the complex projective spaces $(\mathbf{P}^n, v)$ equipped with Fubini-Study metrics. $(\mathbf{P}^n, v)$ has positive Riemannian sectional (and hence holomorphic bisectional) curvatures and constant holomorphic sectional curvatures. In particular the tangent bundle $T_{\mathbf{P}^n}$ with the induced Hermitian metric is positive in the sense of Griffiths [Gri1], which implies that $T_{\mathbf{P}^n}$ is ample (cf. [Gri1] for the various notions of positivity). They are the rank-1 Hermitian symmetric manifolds of compact type. In terms of homogeneous coordinates we can take $v = \sqrt{-1}\partial\bar{\partial}\log\|z\|^2$. Further examples are provided by the hyperquadrics Q^n and the Grassmann manifolds $G(r,n)$ of complex r-planes in $\mathbf{C}^{r+n}$. The hyperquadric Q^n in $\mathbf{P}^{n+1}$ defined by $\sum z_i{}^2 = 0$ in homogeneous coordinates $[z_o, \ldots, z_{n+1}]$ inherits a Kähler metric from $v = \sqrt{-1}\partial\bar{\partial}\log\|z\|^2$ making $(Q^n, v|_{Q^n})$ a Hermitian symmetric space of compact type. Embedding the Grassmann manifold $G(r,n)$ into some $\mathbf{P}^N$ using the Plücker embedding, we obtain similarly the Hermitian symmet ric space $(G(r,n), v|_{G(r,n)})$ of compact type. In general, any irreducible Hermitian symmetric space (M,μ) of compact type can be embedded isometrically and holomorphically into some $(\mathbf{P}^N, v)$ using canonical embeddings (Nakagawa-Takagi [NaT]). The first canonical embedding σ induces an isomorphism $H^2(X,\mathbf{Z}) \to H^2(\mathbf{P}^N, \mathbf{Z}) \cong \mathbf{Z}$.

Let G be the identity component of the group of isometries of a Hermitian symmetric space (M,μ) of compact type. G is a compact semisimple Lie group. Fix $o \in M$ and let K be the isotropy subgroup of G at o. Then up to normalizing constants ω can be identified with the Riemannian metric ds^2 on G/K induced by the Killing form of G. Under this identification as in $(1,1)$ one computes the curvature tensor R using Lie brackets to show that the Hermitian form Q on $T_o{}^{1,0} \otimes T_o{}^{0,1}$ of (M,μ) defined by R is positive semi-definite. In the language of Hermitian Differential Geometry the tangent bundle T_M is nonnegative in the dual sense of Nakano (cf. Siu [Siu6]). This implies that Riemannian sectional curvatures and hence holomorphic bisectional curvatures of (M,μ) are nonnegative. In particular, the tangent bundle T_M is nonnegative in the sense of Griffiths [Gri1]. There is a duality between Hermitian symmetric spaces of non-compact type (as realized by bounded symmetric domains) and those of compact type. In this duality theory there are pairs (G_o, G) of semisimple Lie groups with G_o non-compact and G compact both containing K as a maximal compact subgroup such that the Lie algebras $\mathfrak{g} = \mathrm{Lie}(G_o)$ and $\mathfrak{g}_o = \mathrm{Lie}(G)$ are real forms of the same complex semisimple Lie algebra $\mathfrak{g}^{\mathbf{C}}$. The curvatures of $M_o = G_o/K$ and $M = G/K$ with the metrics induced by the Killing form are then opposite to each other at the identity coset $o = eK$. The Hermitian symmetric space (M, μ) is necessarily Kähler. They are simply-connected by Kobayashi [Kob] and admits hence a global de Rham decomposition as a product of irreducible Hermitian symmetric spaces (M_i, μ_i). Each (M_i, μ_i) is necessarily Einstein. (For a general reference see Helgason [Hel, Chap.VII, 4−6].)

From the Uniformization Theorem in one variable it follows that a compact Hermitian Riemann surface of positive Gauss curvature is conformal to the Riemann sphere $\mathbf{P}^1$. As a generalization to higher dimensions, Frankel [Fr, 1961] conjectured that an n-dimensional compact Kähler manifold X of positive sectional curvature is biholomorphic to a complex projective space $\mathbf{P}^n$. We refer to this as the Frankel conjecture. He also conjectured that in the Einstein case X is isometrically biholomorphic to the complex projective space.

In case of $n = 2$ the Frankel Conjecture was solved in the affirmative by Andreotti-Frankel [Fr]. Mabuchi [Ma, 1978] affirmed the case of $n = 3$ by studying $\mathbf{C}^*$-actions and using results of Kobayashi-Ochiai [KO1] on projective manifolds of positive tangent bundle. In 1979, Mori [M] and shortly afterwards Siu-Yau [SY3] solved independently the Frankel Conjecture in all dimensions. Mori's solution uses algebraic geometry in characteristic $p > 0$. His solution is based on finding rational curves on non-singular projective varieties X using deformation theory in characteristic p and applies to any algebraically closed field k. He affirmed at the same time the Hartshorne Conjecture that a non-singular projective variety over k with ample tangent bundle is biregular to the projective space. Siu-Yau [SY3] used on the other hand techniques of Kähler geometry. They used the characterization of complex projective spaces in terms of positive holomorphic line bundles of Kobayashi-Ochiai [KO1], which reduces the problem to finding a rational curve representing some generator of the torsion-free part of $H^2(X, \mathbf{Z})$ (of rank 1, cf. (2.2)). They used the method of stable harmonic mappings based on the existence theory of Sachs-Uhlenbeck [SU]).

As part of a program to generalize the Uniformization Theorem to higher dimensions, Siu-Yau [SY6, 1976] conjectured that a compact Kähler manifold (X, ω) of nonnegative holomorphic bisectional curvature is uniformized by a Hermitian symmetric manifold with bisectional curvature ≤ 0 (including possibly flat factors). We will call this the Generalized Frankel Conjecture. In case of 2 dimensions this was known to Howard-Smyth [HS, 1971]. Siu [Siu4, 1980] first used the technique of stable harmonic maps to give a curvature characterization of Q^n. Using Siu's result and the evolution equation of Hamilton [Ham1], Bando [Ban, 1983] affirmed the Generalized Frankel Conjecture in case of 3 dimensions. Mok-Zhong [MZ1, 1984] affirmed the conjecture under the additional assumption that X is Einstein. Cao-Chow [CC, 1986] affirmed the conjecture under the stronger assumption that the curvature operator of (M, μ) is nonnegative in the dual sense of Nakano. Finally, we prove

Theorem (2.1) (Mok [Mok9, 1988]). *Let (X, h) be an n-dimensional compact Kähler manifold of nonnegative holomorphic bisectional curvature and let (X, h) be its universal covering space. Then, there exist nonnegative integers $p, N_1, \ldots, N_q$ and irreducible Hermitian symmetric spaces of compact type $M_1, \ldots, M_p$ of rank ≤ 2 such that (X, ω) is isometrically biholomorphic to*

$$(\mathbf{C}^k, g_o) \times (\mathbf{P}^{N_1}, \theta_1) \times \cdots \times (\mathbf{P}^{N_q}, \theta_q) \times (M_1, g_1) \times \cdots \times (M_p, g_p),$$

where g_o denotes the Euclidean metric on $\mathbf{C}^k, g_1, \ldots, g_p$ are canonical metrics on $M_1, \ldots, M_p$, and $\theta_i, 1 \leq i \leq q$, is a Kähler metric on the complex projective space $\mathbf{P}^{N_i}$ carrying nonnegative holomorphic bisectional curvature.

In this lecture we will outline ideas that have been developed in the study of compact Kähler manifolds of nonnegative bisectional curvature and related topics, giving at the same time a sketch of the proof of Theorem (2.1). Our method of proof involves essentially three components: (i) evolution of Kähler metrics by the parabolic Einstein equation, (ii) existence theory of rational curves, (iii) the holonomy group. Our emphasis will be on the differential-geometric aspect of the problem.

On the algebro-geometric side Mori's theory of rational curves asserts in particular that there exist rational curves on any compact Kähler manifold (X, ω) of positive Ricci curvature. We will give a brief explanation of his approach in (2.3). We need the existence of rational curves under the additional assumptions that (X, ω) is of nonnegative holomorphic bisectional curvature and that the second Betti number $b_2(X)$ is equal to 1. Up to now there is no proof of this without using algebraic geometry of characteristic $p > 0$. We will describe nonetheless a strengthening of the method of stable harmonic maps due to Siu [Siu4] which guarantees the existence of a rational curve representing a positive generator of $H^2(X, \mathbf{Z})$ under an additional non-degeneracy assumption on the curvature tensor.

In addition to the results mentioned above, we will also discuss related results in the Einstein case, in particular Berger's theorem [Ber2] that a compact Kähler-Einstein manifold of positive sectional curvature is isometric to $(\mathbf{P}^n, v)$, Gray's theorem [Gray] characterizing Kähler-Einstein manifolds of nonnegative sectional curvature, and reductions of the Generalized Frankel Conjecture due to Bishop-Goldberg [BG] and Howard-Smyth-Wu [HSW].

(2.2) Bochner formulas and the maximum principle on tensors

A Bochner-formula of Bishop-Goldberg

We start by discussing results using the Laplace operator and the maximum principle. To start with we discuss

Theorem (2.2.1) (Bishop-Goldberg [BG], Goldberg-Kobayashi [GK]). *Let (X,ω) be a compact Kähler manifold of positive holomorphic bisectional curvature. Then, the second Betti number $b_2(X)$ is equal to* 1.

This result was first proved in [BG, 1965] under the assumption of positive sectional curvature. It was reformulated in [GK] in terms of bisectional curvatures.

Proof of Theorem (2.2.1). Since the Kähler form ω represents a non-trivial class in $H^2(X,\mathbf{R})$, the proposition asserts that any harmonic (1, 1) form is a scalar multiple of ω. Let ν be an arbitrary smooth harmonic (1, 1) form on X. From Hodge theory $b_2(X) = 1$ if and only if there exists a constant c and a smooth function u such that $\nu = c\omega + \sqrt{-1}\partial\bar{\partial}u$. If this is true then c is determined from the integral formula

$$\int_X \nu \wedge \omega^{n-1} = c\int_X \omega^n.$$

Write in local coordinates $\omega = \sqrt{-1}\sum g_{i\bar{j}}dz^i \wedge dz^{\bar{j}}, \nu = \sqrt{-1}\sum \nu_{i\bar{j}}dz^i \wedge dz^{\bar{j}}$ and define the trace $Tr_\omega(\nu)$ as $\sum g^{i\bar{j}}\nu_{i\bar{j}}$. Since a harmonic (1, 1) form has constant trace on X, replacing ν by $\nu - c\omega$ we may assume that $Tr_\omega(\nu) = 0$ everywhere. We compute $\Delta\|\nu\|^2$. Denote by $D\nu$ the full covariant derivative of ν. At each point x we use an orthonormal basis $\{e_i\}$ of $T_x{}^{(1,0)}(X)$ consisting of eigenvectors of $\nu(x)$ and write $\{\nu_i\}_{1\le i\le n}$ for the corresponding eigenvalues of ν. Then, we have

$$\Delta\|\nu\| = \|D\nu\|^2 + \sum_{i<j} R_{i\bar{i}j\bar{j}}(\nu_i - \nu_j)^2.$$

Here we used the commutation formula

$$\sum_k \nu_{i\bar{j},k\bar{k}} = \left(\sum_k \nu_{k\bar{k}}\right)_{\bar{j}i} + \text{commutation terms},$$

where here and henceforth indices appearing after a comma denotes covariant differentiation. (The comma is dropped in case of functions.) The harmonicity of ν is only used in showing that $\sum_k \nu_{k\bar{k}} = Tr_\omega(\nu) = 0$. fron the Bochner formula and the assumption that X is of positive holomorphic bisectional curvature it follows that all eigenvalues $\{\nu_i(x)\}$ at a given point x are equal. Since $Tr_\omega(\nu) = 0$ this forces ν to vanish identically, proving the assertion $b_2(X) = 1$.

A splitting theorem for manifolds of nonnegative bisectional curvature

Under the weaker assumption that (X, ω) is of nonnegative holomorphic bisectional curvature, the Bochner formula of Goldberg-Bishop also shows that every harmonic form on X is parallel. This observation was exploited by Howard-Smyth-Wu [HSW] to give the following splitting theorem

Theorem (2.2.2) (Howard-Smyth-Wu [HSW]). *Let (X, h) be a compact Kähler manifold of nonnegative holomorphic bisectional curvature. Then, (X, h) is uniformized by $(\mathbf{C}^N, g_o) \times (X_1, h_1) \times \cdots \times (X_k, h_k)$, where g_o is the* 2.2.3 *Euclidean metric on $\mathbf{C}^N$ and (X_i, h_i) is a simply-connected compact Kähler manifold of nonnegative holomorphic bisectional curvature such that the Ricci curvature is positive at some point and such that $b_2(X_i) = 1$.*

Proof. The assumption $\text{Bisect}(X, h) \geq 0$ implies $\text{Ric}(X, h) \geq 0$. To prove the theorem one first deduces from the splitting theorem of Cheeger-Gromoll [CG] for complete Riemannian manifolds of nonnegative Ricci curvature that (X, h) is uniformized by $(\mathbf{R}^m, g_o) \times (X', h')$, where g_o is the Euclidean metric on $\mathbf{R}^m$ and X' is compact and simply-connected. Furthermore, from the Kähler property of (X, h) one deduces that the parallel distribution given by the Euclidean de Rham factor $(\mathbf{R}^m, g_o)$ is invariant under the J-operator so that in fact $m = 2N, (\mathbf{R}^m, g_o) = (\mathbf{C}^N, g_o)$ and (X', h') is Kähler. Write w' for the Kähler form of (X', h'). Suppose $b_2(X') \geq 2$. By the Bochner formula of Bishop-Goldberg it follows that there exists a real harmonic $(1,1)$ form ν with $D\nu = 0$, i.e., ν is parallel. Replacing ν by $\nu + c\omega$ we may assume that $\det(\nu_{i\bar{j}}) = 0$ at some point. At each $x \in X'$ regard $\nu(x)$ as a Hermitian quadratic form on ${T_x}^{(1,0)}$ and let $E_x \subset {T_x}^{(1,0)}$ denote the kernel of $\nu(x)$. Since ν is parallel the assignment $x \to E_x$ defines a parallel distribution. The condition $\det(\nu_{i\bar{j}}) = 0$ at some point implies that $E_x \neq \{0\}$ at every point. Since X' is simply-connected the distribution $x \to E_x$ gives rise to a global de Rham decomposition of (X', h'). Repeating this procedure we obtain the decomposition $(X', h') = (X_1, h_1) \times \cdots \times (X_k, h_k)$ where $b_2(X_i) = 1$ for $1 \leq i \leq k$. Finally, if $\rho_i = \mathrm{Ric}(X_i, h_i)$ is not positive definite at any point and $\dim_{\mathbf{C}}(X_i) = n_i$ then ρ_i is not cohomologous to a constant multiple of the Kähler form ω_i of (X_i, h_i) since ${\rho_i}^{n_i}$ vanishes identically on X_i, while ${\omega_i}^{n_i}$ is positive everywhere. But this contradicts with the assertion $b_2(X_i) = 1$. In other words, $\mathrm{Ric}(X_i, h_i)$ is at least positive definite at one point.

Strictly speaking we do not need Theorem (2.2.1) for the Uniformization Theorem (1.1). One can in fact split off the Euclidean factor by using the heat equation method (cf. (2.5) and (2.9)) and the reduction to the case of $b_2(X) = 1$ is also inessential.

Kähler-Einstein manifolds of positive bisectional curvature

We consider now earlier results obtained from for Kähler-Einstein manifolds of nonnegative (bi-)sectional curvature. The first result in this direction is

Theorem (2.2.3) (Berger [Ber2, 1965], Goldberg-Kobayashi [GK]). *Let (X, h) be an n-dimensional compact Kähler-Einstein manifold of positive bisectional curvature. Then, up to a normalizing constant (X, h) is isometrically biholomorphic to the projective space $\mathbf{P}^n$ equipped with the Fubini-Study metric.*

Theorem (2.2.3) was first proved for the case of positive sectional curvature in [Ber2] and later reformulated to include the case of positive bisectional curvature in [GK].

Proof of Theorem (2.2.3). Hermitian locally symmetric manifolds of rank 1 are characterized by the fact that the Bochner-Weyl tensor vanishes identically. For n-dimensional Kähler-Einstein manifolds of constant positive Ricci curvature $+1$ the Bochner-Weyl tensor W is of the form $W_{\alpha\bar{\beta}\gamma\bar{\mu}} = R_{\alpha\bar{\beta}\gamma\bar{\mu}} - c_n(\delta_{\alpha\beta}\delta_{\gamma\mu} + \delta_{\alpha\mu}\delta_{\gamma\beta})$, where $(\delta_{\alpha\beta})$ denotes the Kronecker delta. The universal positive constant c_n is chosen such that $g^{\alpha\bar{\beta}} W_{\alpha\bar{\beta}\gamma\bar{\mu}} (= g^{\alpha\bar{\beta}} W_{\gamma\bar{\mu}\alpha\bar{\beta}}) = 0$. For a unit vector ξ of type $(1,0)$ we define $f(\xi) = W_{\xi\bar{\xi}\xi\bar{\xi}}$. f is then a smooth function on the unit sphere bundle S of $T^{1,0}(X)$. From $g^{\alpha\bar{\beta}} W_{\alpha\bar{\beta}\gamma\bar{\mu}} = 0$ it follows that the average of $f(\xi)$ over S_x is zero when S_x is equipped with the spherical measure. We consider a point $\alpha \in S$ (at $x \in X$) at which f attains its global maximum. Equivalently X attains the global maximum of holomorphic sectional curvatures at $\alpha \in S$. To prove the theorem it suffices to show that $f(\alpha) = W_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = 0$. This condition is equivalent to the assertion that for any $\beta \in S_x$ orthogonal to α we have $R_{\alpha\bar{\alpha}\beta\bar{\beta}} = \frac{1}{2}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$. To show this consider the Laplace operator Δ on tensors fields T. It is defined by the formula

$$\Delta T = \frac{1}{2}\sum g^{\mu\bar{\nu}}(\nabla_\mu \nabla_{\bar{\nu}} + \nabla_{\bar{\nu}}\nabla_\mu)T.$$

We have by interchanging the order of differentiation and the second Bianchi identity a commutation formula

$$\Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} - R_{\alpha\bar{\alpha},\alpha\bar{\alpha}} + F_o(R)_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = 0,$$

where $F_o(R)$ is a tensor (of the same type as the curvature tensor R) quadratic in R. Since (X, h) is Kähler-Einstein we have the vanishing of $R_{\alpha\bar{\alpha},\alpha\bar{\alpha}}$. To interpret $\Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$ we use the notion of parallel transport. In a small neighborhood U of x denote by $\alpha(y)$ the vector field of type $(1, 0)$ obtained by translating $\alpha = \alpha(x)$ in a parallel way along geodesics emanating from x. For a real vector η of unit length at x let $\{\gamma(t) : -\epsilon < t < \epsilon\}$ be a geodesic emanating from x parametrized by arc-length such that η is tangent ot γ at x. Denote $\alpha(\gamma(t))$ by $\alpha(t)$. Then, we have

$$\frac{\partial^2}{\partial t^2}R(\alpha(t), \overline{\alpha(t)}; \alpha(t), \overline{\alpha(t)}) = R_{\alpha\bar{\alpha}\alpha\bar{\alpha},\eta\eta}(x)$$

as $\alpha(t)$ is parallel along γ. Since α realizes the global maximum of holomorphic sectional curvatures of (X, h), integrating over the tangent unit sphere at x we have the inequality

$$\Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(x) \leq 0.$$

On the other hand in terms of an orthonormal basis we have the explicit formula

$$F_o(R)_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = \sum_{\mu\nu} |R_{\alpha\bar{\mu}\alpha\bar{\nu}}|^2 + R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}R_{\alpha\bar{\alpha}} - 2\sum_{\mu,\nu}|R_{\alpha\bar{\alpha}\mu\bar{\nu}}|^2.$$

Choose an orthonormal basis $\{e_\mu\}_{1\leq\nu\leq n}$ of $T_x{}^{(1,0)}$ at x consisting of eigenvectors of the Hermitian bilinear form H_α defined by $H_\alpha(\xi, \chi) = R_{\zeta\bar{\zeta}\xi\bar{\chi}}$. From the fact that at x the maximal holomorphic sectional curvature is attained at α we deduce readily that α is an eigenvector of H_α and that writing $\alpha = e_1$ we have the variational inequalities $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} \geq 2R_{\alpha\bar{\alpha}\beta\bar{\beta}}$ whenever $\beta \geq 2$. From this we obtain the inequality

$$\Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = -F_o(R)_{\alpha\bar{\alpha}\alpha\bar{\alpha}} \geq \sum_{\beta\geq 2} R_{\alpha\bar{\alpha}\beta\bar{\beta}}(R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} - 2R_{\alpha\bar{\alpha}\beta\bar{\beta}}) \geq 0.$$

Since $\Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} \leq 0$ this forces the equations $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = 2R_{\alpha\bar{\alpha}\beta\bar{\beta}}$ for $\beta \geq 2$. As said, this is equivalent to $W_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = 0$, proving therefore the identical vanishing of the Bochner-Weyl tensor W on X and hence that (X, h) is isometrically biholomorphic to $\mathbf{P}^n$ with a Fubini-Study metric.

As will be seen later on, a slight variation of the tensor $F_o(R)$ will play an important role in the heat equation method in the study of not necessarily Einstein Kähler manifolds of nonnegative bisectional curvature.

Kähler-Einstein manifolds of nonnegative sectional curvature

We mention in passing the theorem of Gray [Gray, 1977] on compact Kähler-Einstein manifold of nonnegative sectional curvature.

Theorem (2.2.4) (Gray [Gray]). *Let (X, g) be a compact Kähler-Einstein manifold of nonnegative Riemannian sectional curvature and constant scalar curvature. Then, (X, g) is isometric to a Hermitian locally symmetric manifold.*

To prove Theorem (2.2.4) Gray used the standard characterization of Riemannian locally symmetric manifolds by the identical vanishing of ∇R (cf. Kobayashi-Nomizu [KN]). The necessity comes from the fact that for a symmetry σ_x at $x \in X$, $\sigma_x^*(\nabla_\mu R_{\alpha\beta\gamma\delta}) = -\nabla_\mu R_{\alpha\beta\gamma\delta}$ since $\sigma_x^*(v) = -v$ for $v \in T_x$ and that $\sigma_x^*(\nabla_\mu R_{\alpha\beta\gamma\delta}) = \nabla_\mu R_{\alpha\beta\gamma\delta}$ since σ_x is an isometry. The sufficiency comes from expanding the metric in terms of normal geodesic coordinates using the curvature tensor and its covariant derivatives. The condition $\nabla R \equiv 0$ guarantees that the reflection $v \to -v$ on the tangent plane induces an isometry on (X, g).

Theorem (2.2.4) was obtained from an integrated Bochner formula associated to a second order semi-elliptic operator L on the unit sphere bundle S of the tangent bundle T_X. The definition of L itself involves the sectional curvature of (X, g) and the vanishing of ∇R is deduced from the vanishing of the gradient term in the Bochner formula. Attempts to generalize Gray's argument to the case of nonnegative holomorphic bisectional curvature have so far been unsuccessful. As will be seen in (2.6), the latter case can be dealt with by an algebraic method stemming from the maximum principle for tensors.

(2.3) Existence of rational curves and the Hartshorne conjecture

In 1979 Mori [Mo] solved the Hartshorne Conjecture [Har, 1970] in algebraic geometry, which implies the Frankel Conjecture. More precisely, he proved

Theorem (2.3) (Mori [Mo]). *Over an algebraically closed field k an n-dimensional non-singular projective-algebraic variety X with ample tangent bundle is biregular to the projective space $\mathbf{P}^n(k)$.*

Mori solved the Hartshorne Conjecture by developing a theory of rational curves on projective-algebraic varieties such that the canonical bundle K_X is not numerically effective (i.e., $C \cdot K_X < 0$ for some curve C). This includes in particular the case of Fano varieties, i.e., complete non-singular varieties with ample anti-canonical line bundle.

We give here a brief exposition of Mori's solution of the Hartshorne Conjecture. The exposition is an oversimplification meant only to give an idea of the proof. The projective space $M = \mathbf{P}^n$ (over an algebraically closed field k) can be recovered from the space of rational lines C passing through a base point $x \in M$. The rational lines C are dual to the hyperplane section line bundle $H = \mathcal{O}(1)$. Since a base point $x \in \mathbf{P}^n$ is effectively parametrized by $\mathbf{P}^{n-1}$. The totality of such curves $\{C\}$ can then be put together to form a $\mathbf{P}^1$-bundle E over $\mathbf{P}^{n-1}$ such that there is a distinguished section Z with normal bundle isomorphic to $\mathcal{O}(-1)$ over $\mathbf{P}^{n-1}$. The projective space $\mathbf{P}^n$ can be recovered from E by blowing down the section Z to a non-singular point. In the case of the Hartshorne Conjecture starting with X with ample tangent bundle the problem is reduced to finding a single rational curve C satisfying $C \cdot K_X^{-1} = n + 1$. Given this, one can fix a base point $x \in X$ and recover the space of all such rational curves passing through x by using the deformation theory of subvarieties of Grothendieck [Gro2].

Mori proved in general that if K_X is not numerically effective, there exists a rational curve C such that $C \cdot K_X^{-1} \leq n+1$. When X has ample tangent bundle by the splitting theorem of Grothendieck [Gro1] T_X splits over C into a direct sum of positive line bundles with at least one factor $\cong \mathcal{O}(a)$ with $a \geq 2$ since $T_C \cong \mathcal{O}(2)$. It follows that $C \cdot K_X^{-1} = n+1$ and one can recover $X \cong \mathbf{P}^n$ from the above procedure. To produce a single rational curve over any k one has to work with fields of characteristic $p > 0$. Consider as an illustration $k = \mathbf{C}$ and X is defined over the integers. One can then look at the reduction of X mod primes p to obtain varieties X_p over the algebraic closure $\overline{F_p}$ of the finite field F_p (with p elements). One can put X and all the X_p together to form a scheme over Spec$(\mathbf{Z}), X$ being the fiber over the zero ideal $\{0\}$. If one can construct for each p a rational curve C_p in X_p such that $C_p \cdot K_{X_p}^{-1} \leq n+1$ one can then use the powerful method of Hilbert schemes in algebraic geometry to conclude the existence of a rational curve C on X such that $C \cdot K_X^{-1} \leq n+1$. In simpler terms suppose $X \subset \mathbf{P}^N$ corresponds to an affine variety $X' \subset \mathbf{C}^{N+1}$. Thinking of C as given by a polynomial map $\mathbf{C} \to$ affine variety X' the existence of C can be interpreted in terms of the solvability of a finite system Σ of polynomial equations with integer coefficients. The existence of $\{C_p\}$ then implies the solvability in $\overline{F}_p$ of Σ mod p for all primes p, which implies the solvability of Σ over $\mathbf{C}$ (cf. Kollar [[Kol], §10.7]).

The advantage of working with fields of characteristic $p > 0$ is the Frobenius morphism. We continue with the assumption that there exists some curve irreducible D in X such that $D \cdot K_X < 0$. Let $\nu: S \to D$ be the normalization of D. To avoid discussing singularities we replace X by $S \times X = X'$ and D by $D' = \text{Graph}(\nu) \subset S \times X$. With this new notation we may assume that D' is a smooth curve in X' such that $\deg(N_{D'|X'}) = -D \cdot K_X > 0$ for the normal bundle $N_{D'|X'}$ of D' in X'. Let d be the dimension of the Chow space $\mathcal{C}$ of subvarieties of X' at the point $D' \in \mathcal{C}$ defined by D'. One-parameter deformations of D' in X' give rise to holomorphic sections of $N_{D'|X}$ over D'. One may not be able to realize all sections in $H^0(D', N_{D'|X'})$ as infinitesimal deformations because of obstructions given by $H^1(D', N_{D'|X'})$. In general one has the inequality

$$
\begin{aligned}
d &\geq \dim H^O(D', N_{D'|X'}) - \dim H^1(D', N_{D'|X'}) \\
&= \deg(N_{D'|X'}) - n.\,\text{genus}(D') + n
\end{aligned}
$$

by the deformation theory of subvarieties (Grothendieck [Gro2]) and by the Riemann-Roch Theorem. In case of characteristic 0 starting with such a D' on X' there is in general no way to deform. In case $\text{char}(k) = p > 0$ one can consider the graph D'_r of the composite of ν and r Frobenius morphisms $\Phi_p: X \to X$. Replace D' by D'_r and d by d_r. The key point is that for r large enough we can make d_r positive so that D'_r can be deformed, since $\text{genus}(D'_r) = \text{genus}(S)$ remains constant while $\deg(N_{D'|X'}) = p^r \deg(N_{D'|X'}) > p^r$. When $d_r \geq \dim_k(\text{Aut}(S))+1$ we obtain non-trivial deformations of $D_r = (\Phi_p)^r(D)$.

To find rational curves one considers deformations of D'_r which fixes some point $x \in D'_r$ corresponding to $P \in S$. The deformations are then described by the subsheaf $\mathcal{F}_r$ of the normal bundle $N_{D'_r|X'}$ defined by local holomorphic sections vanishing at x. Since $\deg(\mathcal{F}_r) = \deg(N_{D'_r|X'}) - n$ for r large enough it is always possible to obtain such a non-trivial deformation. If in so doing all the curves remained irreducible one would then have a projective-algebraic curve Γ and a morphism $F: S \times \Gamma \to X$ such that $F(S \times \Gamma)$ is a surface and such that $F(\{P\} \times \Gamma) = \{x\}$. $\{P\} \times \Gamma \subset X$ is thus an exceptional set so that the self-intersection is negative by the blowing-down criterion of Mumford [Mum1] and Grauert [Gra2]. We have thus proved by contradiction that no such Γ exists. Since D'_r can be deformed non-trivially we can nonetheless find Γ and a rational map $F: S \times \Gamma \to X$ which does not extend to a morphism on $S \times \Gamma$. Let Σ be a non-singular model of Graph(F). We have an induced morphism $\Sigma \to X$. Since $S \times \Gamma$ is non-singular Σ is obtained from $S \times \Gamma$ by a finite number of quadratic transforms (Hopf blow-ups). It cannot happen that all the rational curves thus adjoined to $S \times \Gamma$ are mapped to single points under the morphism $\Sigma \to X$, otherwise F would extend to a morphism on $S \times \Gamma$.

We have obtained at least one rational curve C_o on X when X is of characteristic $p > 0$. To cut down on $C_o \cdot K_X^{-1}$ we use again the blowing-down criterion of Mumford-Grauert. Suppose C is a rational curve such that $C \cdot K_X^{-1} > n+1$. We consider deformations of C fixing two distinguished points $P, Q \in C$. Let $f: \mathbf{P}^1 \to C$ be a normalization of C such that $f(o) = P$ and $f(\infty) = Q$. The infinitesimal deformations of $C' = \text{Graph}(f)$ fixing $(o, P), (\infty, Q)$ arise from the line bundle $N_{C'|X'} \otimes \mathcal{O}(-2)$. Since genus$(C') = 0$ the local deformations of $C' = \text{Graph}(f)$ is measured by at least d parameters with

$$d \geq C \cdot K_X^{-1} - 2n + n \geq 2.$$

Since the group of automorphisms $\mathbf{P}^1$ fixing $\{o, \infty\}$ is 1-dimensional we obtain thus non-trivial deformations of C. If in deforming C all the curves remained irreducible one would obtain a $\mathbf{P}^1$ bundle over a projective-algebraic curve Γ such that E admits two distinguished holomorphic sections σ_o, σ_∞ corresponding to P and Q. This contradicts with the blowing-down criterion of Mumford-Grauert since the self-intersection numbers of σ_o and σ_∞ are opposite to each other, so that they cannot be simultaneously negative. We have thus shown that C can be deformed to a reducible curve H. As a rational curve can only be deformed to a union of rational curves (cf. [BPV, p.142ff.]), by taking some irreducible component we have obtained a rational curve C_1 such that $C_1 \cdot K_X^{-1} < C \cdot K_X^{-1}$ if $C \cdot K_X^{-1} > n+1$.

(2.4) Stable harmonic mappings and the Frankel Conjecture

In [SY3, 1980] Siu-Yau solved the Frankel Conjecture by using techniques of Kähler geometry. More precisely they proved

Theorem (2.4.1) (Siu-Yau [SY3]). *Let (X, h) be a compact Kähler manifold of positive holomorphic bisectional curvature. Then, X is biholomorphic to the projective space $\mathbf{P}^n$.*

In their proof rational curves also played a crucial role. They used another characterization of the projective space $\mathbf{P}^n$ using positive line bundles due to

Theorem (2.4.2) (Kobayashi-Ochiai [KO1]). *Let X be an n-dimensional compact complex manifold such that for some positive holomorphic line bundle L we have $K_X^{-1} \cong L^q$ for some integer $q \geq n+1$. Then X is biholomorphic to the projective space $\mathbf{P}^n$.*

The proof of Theorem (2.4.2) makes use of the Riemann-Roch Theorem and the Kodaira Vanishing Theorem.

Let now (X, h) be a compact Kähler manifold of positive holomorphic bisectional curvature. By the Bochner formula of Bishop-Goldberg (cf. (2.2)) we know that $b_2(X) = 1$. From the Universal Coefficient Theorem it follows that $H^2(X, \mathbf{Z}) \cong \mathbf{Z}$. Let L be a positive generator of $\mathrm{Pic}(X) \cong \mathbf{Z}$. By the Theorem of Bonnet-Myers (cf. [CE]) $\pi_1(X)$ is finite. Since $\mathbf{P}^n$ admits no non-trivial torsion-free finite group of automorphisms we may assume that X is simply-connected, so that by Hurwicz' Theorem $\pi_2(X) \cong H^2(X, \mathbf{Z})$. Let $\gamma: \mathbf{P}^1 \to X$ be a generator of the torsion-free part of $\pi_2(X)$ such that the first Chern class of $\gamma^* L$ is positive on $\mathbf{P}^1$ with respect to the standard orientation on $\mathbf{P}^1$. The key point of the proof of [SY3] consists of showing the existence of a rational curve C representing γ in $\pi_2(X)$. When this is the case from the ampleness of T_X that $C \cdot K_X^{-1} > 0$. Since $C \cdot L = 1$ and $\mathrm{Pic}(X) \cong \mathbf{Z}$ we see that $K_X^{-1} = L^q$ for some integer $q \geq n + 1$. It follows from the criterion of Kobayashi-Ochiai that X is biholomorphic to $\mathbf{P}^n$, proving the Frankel Conjecture.

In order to find the rational curve C Siu-Yau [SY3] used the existence theory of stable harmonic maps of Sacks-Uhlenbeck [SU]. Let $\mathbf{P}^1$ be the Riemann sphere equipped with some Hermitian metric and let $f_o: \mathbf{P}^1 \to X$ be a smooth mapping representing γ. Equip X with the Kähler metric h of positive holomorphic bisectional curvature and denote as in (1.5) by $E(f_o)$ the energy of f_o. Let $E(f_o)$ denote the infimum of the sums of $\mathcal{C}^1$ maps whose sums are homotopic to f. We have as a consequence of the method of Sacks-Uhlenbeck [SU]

Proposition (2.4.3) (Siu-Yau [SY3], p.195ff.). *There exists energy-minimizing $\mathcal{C}^1$ maps $f_i: \mathbf{P}^1 \to X$, $1 \leq i \leq m$, such that the sum of f_i is homotopic to f_o and $E(f_o) = \sum E(f_i)$.*

Since the energy functional $E(f_o) = \int_X \|df\|^2$ and in case of real dimension 2 the critical Sobolev exponent is precisely 2, Sacks-Uhlenbeck considered instead the modified energy functionals $E_\alpha(f) = \int_X (1 + \|df\|^2)^\alpha$ for $\alpha > 1$ to guarantee the existence of C^1 maps f_α representing the absolute minimum of E_α among C^1 maps homotopic to f. If $\sup_{\mathbf{P}^1} \|df_\alpha\|$ is uniformly bounded in α one can extract a convergent subsequence. Otherwise one considers points x_α at which the supremum of $\|df_\alpha\|$ is attained. By suitable dilations of $f_\alpha|D_\alpha$ for small discs D_α around x_α it was proved in [SU] that one can extract a convergent harmonic map $g: \mathbf{R}^2 \to X$ which extends to a harmonic map $g': \mathbf{P}^1 \to X$. In other words, the concentration of the energies $E(f_\alpha)$ near x_α gives rise to the bubbling $g': \mathbf{P}^1 \to X$. This way one shows that f_α is homotopic to the sum of g' and f'_α with $E(f'_\alpha)$ arbitrarily close to $E(f_\alpha) - E(g')$. In [SU] it was also proved that there exists $c > 0$ such that any homotopically non-trivial harmonic map $h: \mathbf{P}^1 \to X$ satisfies $E(h) \geq c$. Theorem (2.4.2) can then be proved inductively in k for k the largest integer such that $E(f) \geq \frac{kc}{2}$.

There are two key steps in the work of [SY3]. The most important step is to show that each $f_i: \mathbf{P}^1 \to X$ is either holomorphic or anti-holomorphic. The second step is to show that $m = 1$ in Theorem (2.4.2). This is done by moving the images $f_i(\mathbf{P}^1)$ so that two of them touch each other. One can then cut and paste these two Riemann spheres to obtain a sum of smooth maps homotopic (as a sum) to f of smaller total energy than f. This contradicts with Theorem (2.4.2) and shows that in fact $m = 1$. The first step is obtained by a second variation formula for the energy functional. The second step is achieved by using the deformation theory of complex-analytic submanifolds.

A second variation formula

Let (N, g) be a Hermitian compact Riemann surface and $f: N \to X$ be a harmonic mapping from (N, h) to the compact Kähler manifold (X, h) of positive holomorphic bisectional curvature. Let $\{f_t\}_{t \in \Delta}$ be a smooth real-2-parameter family of smooth maps $f_t: N \to X$ such that $f_o = f$. We consider the second variation of the energy functional $\frac{\partial^2}{\partial t \partial \bar{t}} \int_N \|df\|^2$. From the formula $\int_N \|\partial f\|^2 - \|\bar{\partial} f\|^2 = \int_N \sqrt{-1} h_{\alpha\bar{\beta}} df^\alpha \wedge \overline{df^\beta}$ one sees that $\int_N \|\bar{\partial} f\|^2 = \frac{1}{2} \int_N \|df\|^2 - \frac{1}{2}[\omega][f(N)]$, where the Kähler class of (X, h) (with Kähler form ω) and $\{f(N)\}$ is the cohomology class defined by $f(N)$. It follows that one needs only to compute $\frac{\partial^2}{\partial t \partial \bar{t}} \int_N \|\bar{\partial} f\|^2$. A straight forward computation and a simplification using the divergence theorem and the harmonicity of f yields the second variation formula

$$\begin{aligned}
&\frac{\partial^2}{\partial t\partial\bar t}\int_N \|df\|^2 \\
&= \int_N g^{i\bar j}\left(\frac{D}{\partial t}f^\alpha_{\bar j}\right)\overline{\left(\frac{D}{\partial t}f^\beta_{\bar i}\right)}h_{\alpha\bar\beta} \\
&+ \int_N g^{i\bar j}\left(\frac{D}{\partial \bar t}f^\alpha_{\bar j}\right)\overline{\left(\frac{D}{\partial \bar t}f^\beta_{\bar i}\right)}h_{\alpha\bar\beta} \\
&- \int_N g^{i\bar j} f^\alpha_{\bar j}\overline{f^\beta_{\bar j}} R_{\mu\bar\nu\alpha\bar\beta}\frac{\partial f^\mu}{\partial \bar t}\overline{\frac{\partial f^\nu}{\partial \bar t}} \\
&- \int_N g^{i\bar j} f^\alpha_{\bar j}\overline{f^\beta_{\bar j}} R_{\mu\bar\nu\alpha\bar\beta}\frac{\partial f^\mu}{\partial \bar t}\overline{\frac{\partial f^\nu}{\partial \bar t}} \\
&+2Re\int_N g^{i\bar j}\overline{f^\nu_{\bar j}}\,\overline{f^\beta_{\bar j}} R_{\mu\bar\nu\alpha\bar\beta}\frac{\partial f^\mu}{\partial \bar t}\frac{\partial f^\alpha}{\partial t}.
\end{aligned}$$

Here as before we use the sign convention such that $R_{1\bar 11\bar 1} > 0$ on the Riemann sphere equipped with the Fubini-Study metric. Furthermore, D denotes covariant differentiation using the Hermitian connection on (X,h), and to keep track of contractions we use indices for the single holomorphic local coordinate $w = w_1$ on N. The notation $f^\alpha_{\bar i}$ is a short form for $\frac{\partial f^\alpha}{\partial \overline{\omega_i}}$, etc. To handle the gradient and curvature terms one imitates holomorphic deformation by considering a real 2-parameter family $\{f_t\}$ such that at $t = 0$, $\frac{\partial f^\alpha}{\partial \bar t}$, $\frac{D}{\partial t}f^\alpha_{\bar i}$, $\frac{D}{\partial \bar t}f^\alpha_{\bar i} = 0$ and such that $\left\{\frac{\partial f^\alpha}{\partial t}\right\}_{1\le\alpha\le n}$ does not vanish identically on $N = \mathbf{P}^1$. Given this we have the second variation formula

$$\frac{\partial^2}{\partial t\partial\bar t}\int_N \|df\|^2 = \int_{\mathbf{P}^1}\frac{\partial f^\alpha}{\partial\bar w}\overline{\frac{\partial f^\beta}{\partial\bar w}}R_{\mu\bar\nu\alpha\bar\beta}\frac{\partial f^\mu}{\partial t}\overline{\frac{\partial f^\nu}{\partial t}}(\sqrt{-1}dw\wedge d\bar w).$$

Let now $f\colon \mathbf{P}^1 \to X$ be an energy-minimizing harmonic mapping so that $\partial_t\partial_{\bar t}\int_{\mathbf{P}^1}\|\partial f\|^2 \le 0$. From the fact that (X,h) carries positive holomorphic bisectional curvature we conclude that $\frac{\partial f^\alpha}{\partial\bar w} \equiv 0$ for $1 \le \alpha \le n$ on $\mathbf{P}^1$, i.e. f is holomorphic.

To obtain the particular deformation as stated we need an additional condition. We endow $f^*T^{1,0}(X)$ with a complex structure in the same way as in (1.8) by pulling back the Hermitian connection of $T^{1,0}(X)$. Since we are on the Riemann surface $\mathbf{P}^1$ there is no integrability condition. Recall that in Theorem (2.4.2) f_o is homotopic to the sum of m energy-minimizing harmonic maps $f_i: \mathbf{P}^1 \to X$. Let $f = f_i$ be such that $c_1(f^*(T^{1,0}(X))$ is nonnegative. Let $f^*T^{1,0}(X) \cong L_1 \oplus \ldots L_n$ be the Grothendieck splitting of the holomorphic vector bundle $f^*T^{1,0}(X)$ over $\mathbf{P}^1$. At least one of the components L_i must be of degree ≥ 0 over $\mathbf{P}^1$ so that there exists some holomorphic section s of $f^*T^{1,0}(X)$ over $\mathbf{P}^1$. In terms of local holomorphic coordinates $\{z_\alpha\}$ on X we write $s = \sum s_\alpha(\partial/\partial z_\alpha)$. We now prescribe the first order jet of the deformation $\{f_t\}_{t\in\Delta}$ by requiring that at $t = 0, \partial f^\alpha/\partial t \equiv 0$ and that $\partial f^\alpha/\partial t = s_\alpha$. The gradient terms vanish because

$$\frac{D}{\partial t}\left(\frac{\partial f^\alpha}{\partial \bar{w}}\right) = \frac{D}{\partial \bar{w}}\left(\frac{\partial f^\alpha}{\partial t}\right) = \frac{D}{\partial \bar{w}} s^\alpha = 0$$
$$\frac{D}{\partial \bar{t}}\left(\frac{\partial f^\alpha}{\partial \bar{w}}\right) = \frac{D}{\partial \bar{w}}\left(\frac{\partial f^\alpha}{\partial \bar{t}}\right) = 0.$$

We have proved that the energy-minimizing harmonic mapping $f: \mathbf{P}^1 \to X$ is holomorphic under the assumption $\deg(f^*T_X^{(1,0)}) \geq 0$. If on the other hand $\deg(f^*T_X^{(1,0)}) \leq 0$ by using the conjugate complex structure on $\mathbf{P}^1$ (thus changing the signs in the Gothendieck splitting of $f^*T_X^{(1,0)}$) one proves similarly that $f: \mathbf{P}^1 \to X$ is anti-holomorphic with respect to the standard complex structure on $\mathbf{P}^1$. This completes the first step of the argument of [SY3].

Deforming a rational curve

As explained above, the second step consists of deforming some rational curve $f_i(\mathbf{P}^1)$ so that it touches a slight perturbation of some other $f_j(\mathbf{P}^1)$. Since the topological line bundle $f_o^*T^{1,0}(X)$ is of degree > 0 so that the sum $\sum_{1\leq i\leq m} \deg(f_i^*T^{1,0}(X)) > 0$ we can certainly choose f_i, say f_1, to be holomorphic. If one can deform f_1 holomorphically and slightly perturb f_2 (smoothly) so that $f_1(\mathbf{P}^1)$ touches $f_2(\mathbf{P}^1)$ tangentially at some smooth point, one can cut and paste to get a sum of maps homotopic to f_o having smaller total energy than $\sum E(f_i) = E([f_o])$, contradicting the minimality of $\{f_i\}$.

We deform the map $f_1: \mathbf{P}^1 \to X$ holomorphically using the deformation theory of complex submanifolds. Write D for Graph$(f_1) \subset \mathbf{P}^1 \times X$. Since (X, h) has positive holomorphic bisectional curvature and $f_1: \mathbf{P}^1 \to X$ is non-trivial, the Grothendieck splitting of the normal bundle $N_{D|\mathbf{P}^1\times X}$ over $D \cong \mathbf{P}^1$ consists only of positive components $\mathcal{O}(a), a > 0$. In particular $H^1(D, N_{D|\mathbf{P}^1\times X}) = 0$ so that there are no obstruction to realizing $H^o(D, N_{D|\mathbf{P}^1\times X})$ as infinitesimal deformations of D. As $\mathcal{O}(a) \to D$ is generated by global sections it follows that in deforming $C_o = f_1(\mathbf{P}^1)$ one obtains rational curves covering at least a non-empty open subset of X. Let $\mathcal{D}$ be the component of the Douady space of X containing the curve $\{C_o\}$ (cf. [Dou]). Since (X,h) is Kähler, $\mathcal{D}$ is compact by Bishop's Theorem on convergence of analytic subvarieties (cf. Fujiki [Fu1] and Lieberman [Lie]). Let $\mathcal{E}$ be the subset of $\mathcal{D}\times X$ consisting of all points $([C],x)$ such that $x\in C$. From the Proper Mapping Theorem of Remmert [Rem] it follows that the natural projection $\mathcal{E}\to X$ is surjective. In other words, in deforming C_o holomorphically we can cover the entire manifold X.

We want in fact to deform C_o so that the tangent directions at smooth points of deformed rational curves C would cover a dense open subset of the projectivized tangent bundle $\mathbf{P}T_X$. To do this we consider deformations of D in $\mathbf{P}^1\times X$ fixing one point. The candidates for infinitesimal deformations are then given by $H^o(D, N_{D|\mathbf{P}^1\times X}\otimes\mathcal{O}(-1))$. As the Grothendieck splitting of $N_{D|\mathbf{P}^1\times X}\otimes\mathcal{O}(-1)$ consists only of nonnegative components, $H^1(D, N_{D|\mathbf{P}^1\times X}\otimes\mathcal{O}(-1))$ vanishes so that all of $H^o(D, N_{D|\mathbf{P}^1\times X}\otimes\mathcal{O}(-1))$ are realized as infinitesimal deformations. As a nonnegative line bundle over $\mathbf{P}^1$ is generated by global sections this means that at a smooth point x of C_o, by deforming C_o while fixing x the tangent directions cover an open subset of $PT_x(X)$. By a similar application of the Proper Mapping Theorem of Remmert one concludes that tangent directions to deformed rational curves C at smooth points fill up a Zariski-open subset of $\mathbf{P}T_X$. One can now replace $C_o = f_1(\mathbf{P}^1)$ by some C and smoothly perturb f_2 so that C and $f_2(\mathbf{P}^1)$ touches each other tangentially at some point, as desired.

We have thus proved that $m = 1$ and the proof of the Frankel conjecture in [SY3] is completed by using the criterion of Kobayashi-Ochiai (Theorem (2.4.1)).

In [Siu4] Siu continued the study of the characterization of Hermitian symmetric spaces by curvature conditions by using the method of stable harmonic maps. He obtained the characterization of hyperquadrics Q^n. First, he strengthened the argument of [SY3] to prove the existence of rational curves under the assumption that (X,h) carries nonnegative holomorphic bisectional curvature, with one additional nondegeneracy assumption on the curvature, called condition(C) in [Siu4]:

Condition (C)

Suppose (X,h) is a compact Kähler manifold of nonnegative holomorphic bisectional curvature. Then (X,h) is said to violate condition (C) at a point $x\in X$ if there exists an orthogonal direct sum decomposition of $T_x(X)$ into $V\oplus W$ such that for some non-zero vectors $\xi\in V$

and $\eta \in W$ we have

$$R_{\xi\bar{\xi}w\bar{w}} = R_{v\bar{v}\eta\bar{\eta}} = 0 \text{ for all } w \in W \text{ and } v \in V.$$

To obtain a curvature characterization of the hyperquadric Q^n Siu used the following criterion of Kobayashi-Ochiai [KO1]: In Theorem (2.4.1) if we have instead $K_X \cong L^q$ for some positive line bundle L and for some $q \geq n$, then X is either biholomorphic to $\mathbf{P}^n$ or Q^n. Using the method of stable harmonic maps Siu proved

Theorem (2.4.4) (Siu [Siu4]). *Let (X, h) be an n-dimensional compact Kähler manifold of nonnegative holomorphic bisectional curvature. Suppose X is not biholomorphic to the projective space $\mathbf{P}^n$ and $\wedge^2 T_X$ with the Hermitian metric h' induced by h is positive at some point $x \in X$ in the sense of Griffiths. Then, X is biholomorphic to the hyperquadric Q^n.*

For $n \geq 3$ using the splitting theorem of Howard-Smyth-Wu [HSW] (cf. Theorem 2.2.2) we may assume that $b_2(X) = 1$. Also, for $n \geq 3$ the positivity of $(\wedge^2 T_X, h')$ and the fact that (X, h) is of nonnegative bisectional curvature implies that X verifies condition (C). Siu deduced from this the existence of a rational curve representing some generator of the torsion-free part of $H^2(X, \mathbf{Z})$, which was the key point for the proof of Theorem (2.4.4).

(2.5) Evolution of Kähler metric by the parabolic Einstein equation

In [Ham1, 1982] Hamilton initiated the study of Riemannian manifolds using the parabolic Einstein equation

$$(*) \qquad \frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}}$$

on a compact Riemannian manifold (M, g). Using modified versions of this evolution equation obtained by rescaling, he proved

Theorem (2.5.1) (Hamilton [Ham1]). *Let (M, g) be a three-dimensional compact Riemannian manifold of positive Ricci curvature. Then, M is diffeomorphic to the sphere S^3.*

He proved for a general compact Riemannian manifold the short time existence for the evolution equation (*). In the case of the three-manifold (M, g) in the theorem, he proved that for the Riemannian metrics $\{g(t)\}$ obtained from (*), the Ricci curvature remains positive.

Furthermore, by modifying (*) by rescaling he proved long time existence in this case and showed that as $t \to \infty$, $\{g(t)\}$ converges in the C^∞-topology to a Riemannian metric of constant positive sectional curvature, showing in particular that M is diffeomorphic to S^3.

For the short time existence of (*) in the general situation, Hamilton used the Nash-Moser Implicit Function Theorem. The difficulty here lies in the fact that the linearized equation as a system of P.D.E. on symmetric 2-tensors is only semi-parabolic as a consequence of the first Bianchi identity. Furthermore, he developed a maximum principle for tensors for the heat equation to prove that the positivity of Ricci curvatures is preserved under the evolution of metrics. The convergence of $g(t)$ as $t \to \infty$ is very delicate. It relies in part on some very subtle algebraic manipulations to show that the eigenvalues of the Ricci tensor tend to converge to each other at every point of the three-manifold. In case of 3 dimensions, an Einstein manifold is necessarily of constant sectional curvature. For us the relevant part is the short time existence and the maximum principle on tensors. As will be explained later on, the short time existence in the Kähler case can be reduced to a parabolic scalar equation. We explain first of all the maximum principle of Hamilton. One considers an evolution equation satisfied by some tensor T of the form

$$(**) \qquad \frac{\partial T}{\partial t} = \Delta T + F(T).$$

Here Δ is the Laplacian on tensors as explained in (2.2), and $F(T)$ is a tensor algebraic in T. In the evolution of a Riemannian g on a compact manifold M using (*) the curvature tensor R satisfies an evolution equation of type (**) with F quadratic in T. We may consider (**) under additional restrictions on T by requiring that T has certain symmetry properties. One defines some notion of positivity for tensors T satisfying the symmetry properties. We write $T > 0$ (resp. $T \geq 0$) when T is positive (resp. nonnegative) in this sense. The maximum principle of Hamilton [Ham1, 2] is roughly

The maximum principle on tensors for the heat equation (**) on tensors

We say that F satisfies the null-vector condition if whenever $T \leq 0$ and T has zeros, $F(T)$ is nonnegative at the zeros of T. Furthermore, (for an appropriate notion of positivity) when T is evolved according to (**) and $T \leq 0$ at time $t = 0$, then $T(t) \leq 0$ for any $t \leq 0$ (whenever $T(t)$ is defined) if F verifies the null vector condition.

In [Ham1] for compact three-manifolds M it was proved that the null-vector condition was satisfied for the evolution equation of the Ricci tensor (In case of three dimensions the Ricci curvature Ric determines the sectional curvature and vice versa), when $\mathrm{Ric} > 0$ simply means that Ric is positive as a symmetric bilinear form. This was a crucial step for the proof of Theorem (2.5.1). Hamilton also proved (unpublished) that in Theorem (2.5.1) when (M, g) is only assumed to be of nonnegative Ricci curvature, then either M is diffeomorphic to S^3 or (M, g) splits isometrically locally. Roughly speaking, he proved that either for $t > 0$ the evolved metric $g(t)$ is of positive Ricci curvature, or the zeros of the Ricci tensor of $(M, g(t))$ for $t > 0$ constitute a parallel distribution on M.

Bando [Ban, 1984] initiated the application of the heat equation method of Hamilton to the study of compact Kähler manifolds of nonnegative bisectional curvature. He proved

Theorem (2.5.2) (Bando [Ban]). *Let (X, g) be be a three-dimensional compact Kähler manifold of nonnegative holomorphic bisectional curvature. Then, X is biholomorphic to $\mathbf{P}^3, \mathbf{Q}^3, \mathbf{P}^1 \times \mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$.*

The proof of this theorem is based on the curvature characterization of $\mathbf{P}^n$ and $\mathbf{Q}^n$ of [SY3] and [Siu4] and the maximum principle on tensors for the evolution equation (**). First of all, Bando proved the null-vector condition for the evolution of the bisectional curvature in case of three-dimensional compact Kähler manifold. Bando then proved that either $(X, g(t))$ satisfies condition (C) for $t > 0$ or there is a parallel complex one-dimensional distribution on T_X, which gives rise to an isometric splitting of $(X, g(t))$ and hence of (X, g).

We explain here the simple proof of short-time existence of (*) for Kähler manifolds and outline a proof of the maximum principle for evolution of bisectional curvatures in case of Theorem (2.5.2), as given in [Ban]. In the case of Kähler manifolds, the Kähler form ω is locally given by potential functions ϕ such that $\omega = \sqrt{-1}\partial\overline{\partial}\phi$. On a compact Kähler manifold, given some closed (1, 1) form ν, we can write globally $\omega = \nu + \sqrt{-1}\partial\overline{\partial}u$ for any closed (1, 1) form ω cohomologous to ν, by Hodge theory. In the evolution equation (*) write for the evolved metrics $g(t)$

$$g_{i\bar{j}}(t) = g_{i\bar{j}}(0) - tR_{i\bar{j}}(0) + \partial_i\partial_{\bar{j}}u$$

in terms of holomorphic coordinates $\{z_i\}$. Recall that the Ricci form is given by

$$(\#) \qquad R_{i\bar{j}} = -\partial_i\partial_{\bar{j}}(\log\det(g_{\alpha\bar{\beta}})).$$

The evolution equation (*) can then be written as

$$\frac{\partial}{\partial t}\left[g_{i\bar{j}}(t) = g_{i\bar{j}}(0) - tR_{i\bar{j}}(0) + \partial_i\partial_{\bar{j}}u\right] = \partial_i\partial_{\bar{j}}\left[\log\det(g_{\alpha\bar{\beta}})\right].$$

Using (#) at $t = 0$ we can rewrite (*) as

$$\partial_i\partial_{\bar{j}}(\frac{\partial u}{\partial t}) = \partial_i\partial_{\bar{j}}\log\left[\frac{\det(g_{\alpha\bar{\beta}})}{\det(g_{\alpha\bar{\beta}}(0))}\right].$$

To solve (*) it suffices to solve the scalar equation on u given by

$$\frac{\partial u}{\partial t} = \log\left[\frac{\det(g_{i\bar{j}}(0) - tR_{i\bar{j}}(0) + \partial_i\partial_{\bar{j}}u)}{\det(g_{\alpha\beta}(0))}\right].$$

The short time existence of the scalar equation is standard from the study of the Monge-Ampére operator. In fact, the linearized equation is a heat equation on a scalar variable.

The null-vector condition for the evolution of bisectional curvatures in dimension 3

The bisectional curvatures satisfy an evolution equation of type (**). In case of pairs of vectors (α, ζ) such that $R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$ we have in terms of orthonormal bases at one point the expression

$$F(R)_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = \sum_{\mu,\nu} R_{\alpha\bar{\alpha}\mu\bar{\nu}}R_{\nu\bar{\mu}\zeta\bar{\zeta}} - \sum_{\mu,\nu}|R_{\alpha\bar{\mu}\zeta\bar{\nu}}|^2 + \sum_{\mu,\nu}|R_{\alpha\bar{\zeta}\mu\bar{\nu}}|^2.$$

To prove the null vector condition for bisectional curvatures for F in case of dimension 3 it suffices to consider a unit vector μ orthogonal to the span of $\{\alpha, \zeta\}$ and prove the inequality

$$(\div) \qquad R_{\alpha\bar{\alpha}\mu\bar{\mu}}R_{\mu\bar{\mu}\zeta\bar{\zeta}} \leq |R_{\alpha\bar{\mu}\zeta\bar{\mu}}|^2 + |R_{\alpha\bar{\zeta}\mu\bar{\mu}}|^2.$$

This inequality as proved from a second variation inequality arising from the fact that (α, ζ) is a zero (and hence minimum) of the bisectional curvatures. More precisely $(\div)$ results from the inequality

$$\frac{\partial^2}{\partial\epsilon^2}R(\alpha + \epsilon\mu, \overline{\alpha + \epsilon\mu}; \zeta + \epsilon xe^{i\theta}\mu, \overline{\zeta + \epsilon xe^{i\theta}\mu}) \leq 0$$

for the real variables ϵ, x and for real numbers $\theta, 0 \leq \theta \leq 2P$. Define the right hand side as $G_\theta(x) \leq 0$. The inequality $(\div)$ follows from computing the discriminant of the quadratic polynomial G_θ and averaging over θ. From the null-vector condition one can apply the maximum principle for tensors to prove that the nonnegativity of bisectional curvatures is preserved during the evolution of (X, g) by (*). We outline in this special case the proof of the maximum principle. We consider an evolution of the form

$$(**)' \qquad \frac{\partial}{\partial t} T' \leq \Delta T' + F'(T', g),$$

where we write $A \leq B$ for two tensors of type (2, 2) (having the same universal symmetries as bisectional curvatures) to mean that $A_{\alpha\bar{\alpha}\beta\bar{\beta}} \leq B_{\alpha\bar{\alpha}\beta\bar{\beta}}$ for all choices of (α, β), etc. Here F' is an algebraic expression in terms of T' and the Kähler metric $g(t)$. Suppose F' has the property that whenever $T'_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$ and $\{\alpha, \zeta\}$ are non-zero vectors, $F'_{\alpha\bar{\alpha}\zeta\bar{\zeta}} > 0$. Suppose at time $t = 0$ $T' > 0$. Let $t = t_o$ be the first time that T' attains some non-trivial zero $T'_{\alpha\bar{\alpha}\zeta\bar{\zeta}}$ at some $x \in X$. Then we have at $t = t_o$

$$\frac{\partial}{\partial t} T'_{\alpha\bar{\alpha}\zeta\bar{\zeta}} \leq 0; \Delta T'_{\alpha\bar{\alpha}\zeta\bar{\zeta}} \geq 0 \text{ and } F'_{\alpha\bar{\alpha}\zeta\bar{\zeta}} > 0,$$

which is a plain contradiction to $(**)'$. In other words, positivity is preserved under the evolution equation $(**)'$. To relate (**) for the curvature tensor R to $(**)'$ one considers the evolution of of $T' = R + \epsilon f G$ where $\epsilon > 0, G(t)$ is the parallel (2, 2) tensor on $(X, g(t))$ defined by $G_{\alpha\bar{\beta}\gamma\bar{\delta}}(t) = (g_{\alpha\bar{\beta}} g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}} g_{\gamma\bar{\beta}})(t)$, and f is a positive function to be prescribed. From the fact that $G(t)$ is parallel on $(X, g(t))$ one verifies immediately for any $\epsilon > 0$ the inequality $(**)'$ for a smooth function f satisfying the heat equation

$$\begin{cases} \dfrac{\partial f}{\partial t} = \delta f + Cf; \\ f(o; x) = 1. \end{cases}$$

for some sufficiently large constant C independent of $\epsilon > 0$. Finally, from the positivity of $T' = T'_\epsilon$ for any $\epsilon > 0$ it follows readily that $R \geq 0$ in the process of the evolution (*).

The argument for the maximum principle on tensors also yields the fact that if bisectional curvatures are positive at some point for the initial metric, then for an evolved metric $g(t), t > 0$ the bisectional curvatures are positive everywhere. It suffices to consider the evolution equation of the tensor $T" = R - \epsilon h G$ for some function h chosen such that at time $t = 0, h \geq 0, h > 0$ at some point, $T" \geq 0$ and such that for some large enough constant C, h satisfies

$$\frac{\partial h}{\partial t} = \delta h - Ch.$$

For some choice of C and $\epsilon > 0$ one verifies that the evolution equation of T" satisfies the null vector condition. The function $h(t)$ is necessarily positive for $t > 0$ from the heat kernel method for heat equations (cf. Protter-Weinberger [PW]). From the maximum principle on tensors it follows then that $T" \geq 0$ and that $R > 0$ for $t > 0$.

Using the splitting theorem of Howard-Smyth-Wu [HSW] (Theorem (2.2.2)) one may further assume that (X, g) is of positive Ricci curvature at some point. The line of argument above also allows Bando to prove that $(X, g(t))$ is of positive Ricci curvature for $t > 0$. Furthermore, if the condition (C) is satisfied at some point, then for the evolved metrics satisfies (C) everywhere.

If on the other hand (C) is not satisfied at any point for an evolved metric, Bando proved the existence of a parallel 1-dimensional distribution in T_X. To see this we make the observation that as a consequence of the expression of F and the proof of the null-vector condition using $(\div)$ we have the fact that for an evolved metric $R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$ implies $F(R)_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$ (since $\frac{\partial}{\partial t} R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$) so that $R_{\alpha\bar{\zeta}*\bar{*}} = 0$. Since $(X, g(t))$ is of positive Ricci curvature for $t > 0$ it follows from this observation (applied to (α, α)) that an evolved metric is of positive holomorphic sectional curvature. Let now $T_x = V_x \oplus W_x$ be an orthogonal decomposition of T_x violating the condition (C) for some evolved metric $g(t)$ such that W_x is one-dimensional. With some oversimplification one may assume that $x \to V_x$ and $x \to W_x$ are smooth distributions. For $\Lambda = \alpha \in V_x$ and $\zeta \in \mathcal{N}_\alpha$ we have $R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$. We also have $\Delta R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$. Let $\gamma = \gamma(s)$ be a geodesic emanating from x with $\gamma(o) = x$ such that $\partial_s \gamma(o) = \eta$. If in translating α in a parallel manner along γ the resulting curve $\alpha(t)$ on T_X is not tangent to the vector bundle $V = \cup V_x$ one verifies by using the inequality $(\div)$ that the second derivative $\partial_\eta^2 R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = c R_{\zeta\bar{\zeta}\zeta\bar{\zeta}}$ for some positive constant c. This violates $\Delta R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$ and proves by contradiction that V and hence $V = \cup V_x$ is invariant under parallel transport.

Remark. We note that for such pairs (α, ζ) $F(R)_{\alpha\bar{\alpha}\zeta\bar{\zeta}}$ actually agrees with $F_o(R)_{\alpha\bar{\alpha}\zeta\bar{\zeta}}$ for the $F_o(R)$ appearing in Berger's formula in (2.1) given by

$$\Delta R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} - R_{\alpha\bar{\alpha},\zeta\bar{\zeta}} + F_o(R)_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0.$$

Berger's formula is useful in the Einstein case, where $R_{\alpha\bar{\alpha},\zeta\bar{\zeta}} = 0$. In the general case the parabolic Einstein evolution equation (*) has the effect, roughly speaking, of transforming the space covariant derivative $R_{\alpha\bar{\alpha},\zeta\bar{\zeta}}$ to the time derivative $\frac{\partial}{\partial t} R_{\alpha\bar{\alpha}\zeta\bar{\zeta}}$.

(2.6) Compact Kähler-Einstein manifolds of non-negative holomorphic bisectional curvature

There are two essential difficulties in higher dimensions in Bando's approach to solving the generalized Frankel conjecture for a general compact Kähler metric of nonnegative bisectional curvature (X, g). The first difficulty is to show that the null-vector condition for the evolution of bisectional curvatures remain valid in higher dimensions. The second difficulty, more fundamental in nature, is the lack of an algebro-geometric characterization of irreducible Hermitian symmetric spaces M of rank ≤ 2 other than the hyperquadric $\mathbf{Q}^n$. Essentially all the known characterization for such general M are differential-geometric in nature. In Mok-Zhong [MZ1] we study compact Kähler-Einstein manifolds of nonnegative bisectional curvature. Our objective at the time was to prove that (i) if (X, g) is Einstein then it is locally symmetric and that (ii) in the general case by using the evolution equation (*) of Hamilton by rescaling the Kähler metrics $\{g(t)\}$ would converge to a Kähler-Einstein metric. In [MZ1] we solved (i). More precisely, we proved

Theorem (2.6) (Mok-Zhong [MZ1]). *Let (X, g) be a compact Kähler manifold of nonnegative holomorphic bisectional curvature and constant scalar curvature. Then (X, g) is isometrically biholomorphic to a Hermitian locally symmetric manifold.*

We used the characterization that (X, g) is locally symmetric if and only if $\nabla R \equiv 0$. The starting point was Berger's formula $\Delta R + F_o(R) = 0$ for Kähler-Einstein manifolds in the proof of Theorem (2.2.3). Recall that if (X, g) is Kähler-Einstein and of positive holomorphic bisectional curvature and α is a unit vector of type (1, 0) at which the global maximum of holomorphic sectional curvatures is attained, we have

$$\begin{cases} \Delta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} \leq 0; \\ -F_o(R)_{\alpha\bar{\alpha}\alpha\bar{\alpha}} \geq \displaystyle\sum_{\beta\geq 2} R_{\alpha\bar{\alpha}\beta\bar{\beta}}(R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} - 2R_{\alpha\bar{\alpha}\beta\bar{\beta}}) \leq 0. \end{cases}$$

in terms of an orthonormal basis $\{e_\mu\}$ of T_x consisting of eigenvectors of the Hermitian form $H_\alpha(\xi, \zeta) = R_{\alpha\bar{\alpha}\zeta\bar{\zeta}}$ with $e_1 = \alpha$. The inequality $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} \leq 2R_{\alpha\bar{\alpha}\beta\bar{\beta}}$ remains valid under the weaker assumption that (X, g) is of nonnegative bisectional curvature. Rescale the metric so that $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = 1$. It follows in this case from Berger's formula $\Delta R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} + F_o(R)_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$ that both terms vanish so that there is a decomposition of T_x into the orthogonal direct sum

$$(*) \qquad T_x = C_\alpha + \mathcal{H}_\alpha + \mathcal{N}_\alpha$$

where $\mathcal{H}_\alpha$ is the eigenspace of H_α for the eigenvalue $1/2$ and $\mathcal{N}_\alpha$ is the null-space of H_α. Furthermore, we showed using Berger's formula that at each point x, there exists such an α realizing the global maximum of holomorphic sectional curvatures. Let $\mathcal{M}_x$ be the collection of such α's over $x \in X$, $\mathcal{M} = \cup_{x\in X} \mathcal{M}_x$ and V be the distribution $x \to \mathrm{Span}(\mathcal{M}_x)$. We proved essentially the partial symmetry $\nabla_\alpha R = 0$ for all $\alpha \in \mathcal{M}$. This allowed us to show that the distribution V is integrable. The proof was completed by induction on the dimension using the de Rham components. In order to prove $\nabla_\alpha R = 0$ we considered higher order elliptic operators on extremal $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = 1$ and $R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$ of holomorphic sectional and bisectional curvatures resp. Let $S^{(4)}$ be the fourth order operator on tensors obtained by averaging fourth order radial derivatives. From Berger's formula $\Delta R + F_o(R) = 0$ we obtained $S^{(4)}R + F_1(R) = 0$, where F_1 contain terms of (norm squares of) components of ∇R, etc. The manipulation is essentially algebraic in nature and depends very explicitly on the decomposition (*) and auxiliary curvature identities. Of significance to proving the generalized Frankel conjecture are the curvature identities associated to $R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$ for $\alpha \in \mathcal{M}$ and $\zeta \in \mathcal{N}_\alpha$. We proved from $\Delta R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} + F_o(R)_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$ that in fact

$$\Delta R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = F_o(R)_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0.$$

In case of dimension 3 the algebraic calculation reduces to Bando's proof [Ban] of the null-vector condition using the heat equation method. (cf. Theorem (2.4)). The difficulty here is to prove the inequality

$$(\div)' \qquad \sum_\mu R_{\alpha\bar{\alpha}\mu\bar{\mu}} R_{\mu\bar{\mu}\zeta\bar{\zeta}} \geq \sum_{\mu,\nu} |R_{\alpha\bar{\mu}\zeta\bar{\nu}}|^2 + \sum_{\mu,\nu} |R_{\alpha\bar{\zeta}\mu\bar{\nu}}|^2.$$

The summation can be taken over the basis $\{e_\mu\}$, or equivalently over a basis for $\mathcal{H}_\alpha$ (since all other terms are zero). In case of $n = 3$ the inequality $(\div)'$ reduces to the inequality $(\div)$ in the proof of Theorem (2.5.2) of Bando. Recall in the proof of the corresponding inequality $(\div)$ in 3 dimensions in (2.5.2) we used the variational inequality

$$\frac{\partial^2}{\partial \epsilon^2} R(\alpha + \epsilon\mu, \overline{\alpha + \epsilon\mu}; \zeta + \epsilon x e^{i\theta}\mu, \overline{\zeta + \epsilon x e^{i\theta}\mu}) \geq 0.$$

In the general case we consider similarly the inequality

$$\frac{\partial^2}{\partial \epsilon^2} R(\alpha + \epsilon\chi, \overline{\alpha + \epsilon\chi}; \zeta + \epsilon \sum C_\mu e^{i\theta_\mu} e\mu, \overline{\zeta + \epsilon \sum C_\mu e^{i\theta_\mu} e\mu}) \geq 0$$

for χ arbitrary. The expression on the left for fixed χ is now regarded as a quadratic polynomial in the real variables $\{C_\mu\}$. The inequality $(\div)'$

is then obtained by varying χ, computing subdeterminants of the associated symmetric bilinear forms (i.e., using homogeneous coordinates), averaging over the angles θ_μ and a combinatorial argument. A modification of $(\div)'$ leads to a proof of the null-vector condition for bisectional curvatures in arbitrary dimensions for the evolution on compact Kähler manifolds using the parabolic Einstein equation. Theorem (2.6) can be used in classifying compact Riemannian manifolds with nonnegative curvature operator, as will be explained in (2.7). Furthermore, recently Guan [Gu] made use of the curvature identities (such as given by the decomposition (*) and the $(\div)'$, replacing $\geq$ by $=$) to give canonical forms of the curvature tensor of Hermitian symmetric manifolds. Starting from the single identity $\nabla R \equiv 0$ he classified such canonical forms in terms of coordinates given by the decomposition (*) and further specifications. All such canonical forms are in fact realized as can be checked from classification theory (cf. Helgason [Hel]).

(2.7) Characterization of locally symmetric spaces of rank ≤ 2 by the holonomy group

Besides the characterization of Riemannian locally symmetric manifolds (M, h) by the identical vanishing of ∇R there is a more geometric characterization in terms of holonomy groups due to Berger [Ber1]. Later Simons [Si] gave a more direct proof. We have

Theorem (2.7) (Berger [Ber1] and Simons [Si]). *Let (M, h) be a complete Riemannian manifold. Let $x \in M$ and $H_x(M)$ be the restricted holonomy group at x. Suppose $H_x(M)$ does not act transitively on the unit sphere S_x of the (real) tangent space T_x at M. Then, either (M, h) is locally reducible as a Riemannian manifold; or (M, h) is a Riemannian locally symmetric manifold.*

The proof of Simons' relies on the fact that the holonomy group is in some sense determined by the curvature operator. More precisely, the curvature tensor $R = \{R_{i\bar{j}k\bar{l}}\}$ at $x \in M$ can be regarded as a symmetric bilinear form on $\wedge^2 T_x$. We call the symmetric bilinear form Q the curvature operator at x. Equivalently, Q can be regarded as a linear map $Q\colon \wedge^2 T_x \to \wedge^2 T_x^*$ can be identified with the Lie algebra $\mathfrak{so}(n)$ to the special orthogonal group $SO(n), n = \dim_{\mathbf{R}}(M)$, at x. Let $\{\gamma(t)\}$, $0 \leq t \leq 1$, be a contractible closed smooth loop emanating from x. We denote by Q_γ the image of Q under parallel transport of Q along γ from x to x. Let $\mathfrak{h} \subset \wedge^2 T_x^* \cong \mathfrak{so}(n)$ be the linear subspace generated by $\{Im(Q_\gamma)\}$. Then, h is the Lie algebra to the restricted holonomy group $H_x(M)$ at x.

To apply Theorem (2.7) one applies what can be called "the principle of holonomy": the existence of a geometric object invariant under holonomy implies a reduction of the holonomy group. Examples of this were already given in the splitting theorem Theorem (2.2.2) of Howard-Smyth-Wu [HSW] and Theorem (2.5.2) of Bando [Ban]. We describe here two further examples.

Hamilton ([Ham2, 1986]) considered the evolution of compact Riemannian manifolds (M, g) with positive curvature operator and prove that for $t > 0, (M, h(t))$ has positive curvature operator, or one of the following holds: (i) (M, h) is Kähler; (ii) (M, h) is quaternionic-Kähler; (iii) (M, g) locally symmetric of rank ≥ 2; or (iv) (M, h) splits isometrically locally. In case of four dimensions [Ham2] completely classified such manifolds (M, h). In the proof of Hamilton [Ham2] classifying compact Riemannian manifolds (M, h) with nonnegative curvature operator, he proved the null-vector condition for the evolution of the curvature operator. Furthermore, if the curvature operator of an evolved metrics $h(t)$ is not positive everywhere, then the images of the curvature operators at $x \in M$ defines a subbundle of $\wedge^2 T^*$ invariant under parallel transport. In this case from the more precise statements of the classification of possible holonomy groups of Berger [Ber1] and Simons [Si] either $H_x(M)$ does not act transitively on the unit sphere bundle, or $H_x(M) \cong U(n/2)$, $Sp(n/4)Sp(1)$. In the former case (M, h) is either locally symmetric of rank ≥ 2 or locally reducible. The latter case correspond precisely to the case of Kähler and quaternionic-Kähler manifolds.

More recently, Cao-Chow [CC] made use of the solution of the Frankel conjecture to show that in the Kähler case with positive scalar curvature M is biholomorphic to the projective space $\mathbf{P}^{n/2}$ or (M, h) is Hermitian symmetric with rank ≥ 2. Furthermore, Chow-Yang [ChY] completed the study of the quaternionic-Kähler case with positive scalar curvature by showing that (M, h) is either $\mathbf{HP}^n$ with a standard metric or a quaternionic symmetric space of rank ≥ 2. The proof of [ChY] is based on Mok-Zhong [MZ1]. More precisely, they considered the (twistor) space $\mathcal{Z}$ of almost complex structures on M compatible with the quaternionic structure, endowed with a canonical complex structure. The quaternionic metric h of positive scalar curvature gives rise to a Kähler-Einstein metric g of positive Ricci curvature on $\mathcal{Z}$. Furthermore, they showed that $(\mathcal{Z}, g)$ is of nonnegative holomorphic bisectional curvature, so that they could use the result of [MZ1].

Another application of Theorem (2.7) is to classify complete Riemannian manifolds (M, h) of nonnegative sectional curvature by Ballmann ([Ball]) and Burns-Spatzier ([BS]) (cf. also Ballmann-Gromov-Schroeder [BGS] and references there). Roughly speaking, there is a notion of rank on Riemannian manifolds of nonnegative sectional curvature defined in terms of Jacobi fields. In case of rank ≥ 2 they studied restrictions on the behavior of the geodesic flow imposed by the Jacobi fields and constructed integrals of the flow, i.e., functions invariant under the flow. This forces the restricted holonomy group to act intransitively on the unit sphere bundle, so that by Theorem (2.7) of Berger [Ber1] Simons [Si] (M, h) is locally symmetric.

(2.8) The space of minimal rational curves on Hermitian symmetric manifolds of compact type

As explained in (2.6) there are two difficulties to proving the generalized Frankel conjecture (Theorem (1.1)) using the methods of Hamilton [Ham1, 2] and Bando [Ban]. In order to resolve the fundamental difficulty of characterizing Hermitian symmetric manifolds of compact type we use Mori's theory of rational curves [Mo] and the deformation theory of complex submanifolds. To motivate our approach we describe here the structure of the space of minimal rational curves on an irreducible Hermitian symmetric manifolds of compact type M.

Our characterization of M of rank ≥ 2 will be somewhere between algebro-geometric and differential-geometric. Let $\mathcal{D}$ be the space of rational curves C on M representing the positive generator of $H^2(M, \mathbf{Z}) \cong \mathbf{Z}$ and $\mathcal{S} \subset \mathbf{P}T_X$ be the space consisting of all $[T_x(C)], x \in C$. All such C are non-singular. They will be referred to as minimal rational curves. $\mathcal{S} \subset \mathbf{P}T_X$ is of course an algebro-geometric object and does not depend on the choice of a Kähler metric. Since the complex structure J is invariant under parallel transport with respect to any Kähler metric, we can speak of parallel transports of elements of $\mathbf{P}T_X$. The crucial point is

Proposition (2.8.1). *$\mathcal{S}$ is holonomy-invariant for any choice of Kähler metric defining a Hermitian symmetric structure on M.*

The space of such metrics h is the orbit of a single one h under the identity component of the biholomorphism group, which is a complexification of the identity component of the isometry group. In case of the projective space $\mathbf{P}^n$, the minimal rational curves are precisely the rational lines. In case of the hyperquadric $\mathbf{Q}^n, n \geq 3$ and the Grassmannian $G(r, n)$, embedded into some $\mathbf{P}^N$ using standard embeddings (the Plüker embedding for $G(r, n)$) they are the rational lines in $\mathbf{P}^N$ that are already contained in M. By using the first canonical (isometric) embedding $\sigma: (M, h) \to (\mathbf{P}^N, v)$ as given in Nakagawa-Takagi [NaT] (cf. (2.1)), this remains true for any (M, h). Moreover, the group of biholomorphisms of

M are then realized as projective linear transformations. For any rational line C in M and $\alpha \in T_x(M)$ we have $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(C) = R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(\mathbf{P}^N)$ since C is totally geodesic in $(\mathbf{P}^N, v)$. On the other hand by the curvature decreasing property of submanifolds in Kähler manifolds we have

$$R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(C) \geq R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(M) \geq R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(\mathbf{P}^N)$$

so that the three curvatures are equal. Thus, all rational lines C are totally geodesic in (M, h). Since $\mathrm{Aut}(M) \subset \mathbf{P}GL(N+1), C$ is in fact totally geodesic with respect to any choice of Hermitian symmetric structure on M.

We normalize the Fubini-Study metric v on $\mathbf{P}^N$ so that all holomorphic sectional curvatures equal to 1. Then, $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(M) = 1$ for any unit vector $a \in T_X$ such that $[\alpha] \in \sigma$. The converse is also true, by the "Polysphere Theorem" on Hermitian symmetric manifolds (cf. Wolf [Wolf] and Mok [Mok7, Proposition (2.1.1), p.186]). It follows from $\nabla R \equiv 0$ that holomorphic sectional curvatures are invariant under parallel transport. Hence, for (M, h) of rank ≤ 2, we have a proper holonomy-invariant subset $\delta \subset \mathbf{P}^N$, proving Proposition (2.8.1).

In studying the deformation of rational curves C as described in (2.3) and (2.4) we need to know the splitting of the tangent bundle over C. For $[\alpha] \in \delta_x$ since α realizes the (global) maximum of holomorphic sectional curvatures of (M, h) we have by (2.6) the decomposition $(*)$ $\quad T_x = \mathbf{C}_\alpha + \mathcal{H}_\alpha + \mathcal{N}_\alpha$ of into eigenspaces of the Hermitian bilinear form $H_\alpha(\xi, \zeta) = R_{\alpha\bar{\alpha}\xi\bar{\zeta}}$. In case of the minimal rational curves C of (M, h), we have

Proposition (2.8.2). *$T_{M|C}$ splits holomorphically as $\mathcal{O}(2) \oplus \mathcal{O}^p(1)$ $\oplus\, \mathcal{O}^q$, where $p = \dim_{\mathbf{C}} \mathcal{H}_\alpha$ and $q = \dim_{\mathbf{C}} \mathcal{N}_\alpha$.*

Proof. By the Grothendieck splitting theorem we have $T_{M|C} \cong \sum \mathcal{O}(a_i)$. Since (M, h) is of nonnegative holomorphic bisectional curvature we have $a_i \geq 0$. For the canonical line bundle K of $K^{-1} \cdot C = \deg(K^{-1}|_C) = \frac{1}{2\pi}(1 + \frac{p}{2})\,\mathrm{Area}(C, h|_C)$. On other hand since $(C, h|_C)$ has constant holomorphic sectional curvature 1 and for the tangent bundle T_C of C we have $\deg(T_C) = 2$, we deduce $\mathrm{Area}(C, h|_C) = 4\pi$, so that $K^{-1} \cdot C = 2 + p$. From the embedding $C \to M$ and $T_C \cong \mathcal{O}(2)$ we know that one of the components of $T_{M|_C} \cong \sum \mathcal{O}(a_i)$ must be such that $a_i \geq 2$. It follows that if $T_{M|_C} \not\cong \mathcal{O}(2) \oplus \mathcal{O}^p(1) \oplus \mathcal{O}^q$ there would exist more than q trivial components. To prove Proposition (2.8.2) we argue

Lemma (2.8.3). *There are at most q trivial components in the Grothendieck decomposition $T_{M|_C} \cong \sum \mathcal{O}(a_i)$.*

Proof. Consider the Grothendieck decomposition $T^*_M|_C \cong \sum \mathcal{O}(-a_i)$ of the cotangent bundle T^*_M. By using the metric h^* on T^*_M induced by h, (T^*_M, h^*) has nonpositive curvature in the sense of Grffiths. Denote its curvature form by Θ. For any trivial summand $L \cong \mathcal{O}$ in $T^*_M \wedge =$ from the curvature-decreasing property of subbundles in Hermitian holomorphic vector bundles we know that $(L, h^*|_L)$ is of nonpositive curvature. From the triviality of L it follows that $(L, h^*|L)$ is flat. It follows that $\Theta_{\eta^* \overline{\eta^*} \alpha\bar{\alpha}} = 0$ for any $\eta^* \in L_x$. If $\eta \in T_x$ is obtained from η by lifting using the Kähler metric h, we have $R_{\alpha\bar{\alpha}\eta\bar{\eta}} = -\Theta_{\eta^* \overline{\eta^*} \alpha\bar{\alpha}} = 0$. In other words, the dimension of zero eigenvectors $\mathcal{N}_\alpha$ of the Hermitian form H_α must be at least equal to the number r of trivial factors in the decomposition $T_{M|_C} \cong \sum \mathcal{O}(a_i)$, i.e., $q \leq r$, proving the lemma.

(2.9) Holonomy-invariance of the space of tangents to minimal rational curves

In this section we give a proof of the generalized Frankel conjecture (Theorem (1.1)). Strictly speaking we will only need the work of Mori [Mo] and the methods of Hamilton [Haml] and Bando [Ban] using the parabolic Einstein equation. As explained in (2.6) the two essential difficulties to the proof of Theorem (2.1) are: (i) to prove the null-vector condition for the evolution of bisectional curvatures and (ii) the lack of an algebro-geometric characterization of irreducible compact Hermitian symmetric spaces of rank ≥ 2 other than the hyperquadric $\mathbf{Q}^n, n \geq 3$. To achieve (i) we consider a compact Kähler manifold (X, g) of nonnegative bisectional curvature and a zero of bisectional curvatures $R_{\alpha\bar{\alpha}\zeta\bar{\zeta}}$. We use an orthonormal basis $\{e_\mu\}$ consisting of eigenvectors of the Hermitian form $H_\alpha(\xi, \chi) = R_{\alpha\bar{\alpha}\xi\bar{\chi}}$. As in (2.6) (the Einstein case) we need to prove

$$(\div)' \qquad \sum_\mu R_{\alpha\bar{\alpha}\mu\bar{\mu}} R_{\mu\bar{\mu}\zeta\bar{\zeta}} \geq \sum_{\mu,\nu} |R_{\alpha\bar{\mu}\zeta\bar{\nu}}|^2 + \sum_{\mu,\nu} |R_{\alpha\bar{\zeta}\mu\bar{\nu}}|^2.$$

The proof of $(\div)'$ is achieved by a modification of the proof of the same inequality in the Einstein case for special zeros of bisectional curvatures. We remark that the null-vector condition for the evolution of sectional curvatures in the Riemannian case is not valid for real dimensions $m \geq 4$; the validity of $(\div)'$ relies to a certain extent on the particular symmetries of the curvature tensor in the Kähler case.

The null-vector condition implies that $(X, g(t))$ is of nonnegative holomorphic bisectional curvature for $t > 0$. We can use this to prove a splitting theorem without resorting to the Riemannian splitting theorem of Cheeger-Gromoll for complete manifolds of nonnegative Ricci curvature. By a modification of the argument of Bando [Ban] we can show that for an evolved metric $g(t), t > 0$, the dimension of the null space of the Ricci form is constant. To split off the flat directions we need only to prove that the distribution $x \to \mathrm{Ker}(\mathrm{Ric}_x)$ is parallel. As in Theorem (2.5.2) we know now that $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = 0$ if and only if $R_{\alpha\bar{\alpha}} = 0$. The rest of the argument is then the same as in the argument of [Ban] producing a parallel distribution.

We can now restrict ourselves to the case of a compact Kähler manifold (X, g) of nonnegative bisectional curvature, of positive Ricci curvature and of positive holomorphic sectional curvature. Mori's theory of rational curves (cf. (2.3)) now guarantees the existence of a rational curve C_o such that $C_o \cdot K_X^{-1} = r < n+1$. We choose any such rational curve C_o and consider the irreducible component $\mathcal{D}$ of the Douady space of X containing the cycle $[C]$. Let $\mathcal{S}_o \subset \mathbf{P}T_X$ be obtained by taking all tangent directions at smooth points to C for $[C] \in \mathcal{D}$ and let $\mathcal{S}$ be the closure of $\mathcal{S}_o$ in $\mathbf{P}T_X$. Denote by $\mathcal{D}_x$ the subset of $\mathcal{D}$ consisting of C passing through x. We have either

(i) for some point $x \in X$ the Grothendieck splitting of T_X over any C has no trivial summands; or

(ii) for every point $x \in X$ the Grothendieck splitting of T_X over any C has some trivial summands;

(iii) for every point $x \in X$ the Grothendieck splitting of T_X over some (but not all) C has some trivial summands.

The possibility (i) can occur only if $q = n+1$. In this case we have $T_X|_C \cong \mathcal{O}(2) + \mathcal{O}^{n-1}(1)$. Mori's proof of the Hartshorne conjecture applies to this case to show that X is biholomorphic to $\mathbf{P}^n$. The possibility (ii) occurs for example when $q < n+1$. In case (iii) we call the rational curve C special if the Grothendieck splitting of T_X over C has trivial factors. We define $\mathcal{S}'$ as in the definition of $\mathcal{S}$ except that we use only special rational curves. The crux of the proof of Theorem (1.1) is

Proposition (2.9). *$\mathcal{S} \subset \mathbf{P}T_X$ in case of* (ii) *and $\mathcal{S}' \subset \mathbf{P}T_X$ in case of* (iii) *are invariant under holonomy.*

We outline here a proof of case (ii). The proof of (iii) runs along a similar line but is more subtle, relying on more algebraic geometry.

Proof of holonomy-invariance of $\mathcal{S}$. To motivate the proof recall the situation of an irreducible Hermitian symmetric space (M, h) of compact type of rank ≥ 2 and $\mathcal{S} = \mathcal{S}_M$ obtained from the minimal rational curves C. In (2.9) we deduce from from $\nabla R \equiv 0$ the holonomy-invariance of $\mathcal{S}$ (with respect to any choice of h). We now explain how one can prove the latter from the much weaker statement that $\Delta R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$ whenever

$[\alpha] \in \mathcal{S}_M$ over $x \in X$ and $\zeta \in \mathcal{N}_\alpha$. Recall here the eigenspace decomposition $T_x \cong \mathbf{C}\alpha \oplus \mathcal{H}_\alpha \oplus \mathcal{N}_\alpha$ and that by Proposition (2.8.2) that $T_M|_C \cong \mathcal{O}(2)\oplus\mathcal{O}^p(1)\oplus\mathcal{O}^q$, where $p = \dim_{\mathbf{C}} \mathcal{H}_\alpha$ and $q = \dim_{\mathbf{C}} \mathcal{N}_\alpha$. From the deformation theory of rational curves as discussed in (2.3) and (2.4) we know now for the deformation space $\mathcal{D}_x$ of minimal rational curves passing through $x, \mathcal{D}_x$ is smooth and the tangent space to $\mathcal{D}_x$ at $[C]$ is parametrized by $H^o(C, N_{C|M} \otimes \mathcal{O}(-1)) = H^o(C, \mathcal{O}^p \oplus \mathcal{O}^q(-1)) \cong \mathbf{C}^p$, where $N_{C|M}$ is the normal bundle of C in $\mu \cong \mathcal{O}^p(1) \oplus \mathcal{O}^q$. Rewrite $T_M|_C \cong \mathcal{O}(2) \oplus \mathcal{O}^p(1) \oplus \mathcal{O}^q$ in the form $T_{M|C} \cong V \oplus W$ where V and W consist of the positive and W the trivial summands resp. We note that the summands in the Grothendieck splitting are not uniquely determined but the positive part V is. In fact $V \otimes \mathcal{O}(-1) \subset T_M|_C \otimes \mathcal{O}(-1)$ is the subbundle generated by the vector space of global sections of $T_M|_C \otimes \mathcal{O}(-1)$. From the description of $T_{[C]}(\mathcal{D}_x)$ one deduces $T_\alpha(\mathcal{S}_x) = V_x/\mathbf{C}\alpha \subset T_{[\alpha]}(\mathbf{P}T_x(M))$. We now give another description of V_x.

Lemma. *With respect to the decomposition* $T_x = \mathbf{C}\alpha \oplus \mathcal{H}_\alpha \oplus \mathcal{N}_\alpha$*, we have* $V_x = \mathbf{C}\alpha \oplus \mathcal{H}_\alpha$.

Proof. Consider the Grothendieck decomposition $T_M^*|_C \cong \mathcal{O}(-2)\oplus \mathcal{O}^p(-1) \oplus \mathcal{O}^q$ of $T_M^*|_C$. $T_M^*|_C \cong V' \oplus W'$ for V', W' corresponding to the negative and trivial summands resp. In this decomposition we can take $V' = \mathcal{O}(-2) \oplus \mathcal{O}^p(-1) = W^{\mathrm{Ann}}$ and $W' = \mathcal{O}^q = V^{\mathrm{Ann}}$, where ${W_x}^{\mathrm{Ann}} \subset T_M^*(M)$ denotes the annihilator of W_x with respect to the natural pairing between $T_x(M)$ and T_M^*. We note that $W' = V^{\mathrm{Ann}}$ is a uniquely determined subbundle of $T_M^*|_C$ while V' is not. Recall that Θ denotes the curvature of T_M^* and that if η^* and η corresponds to each other by lifting and lowering indices using h and $h^*, \Theta_{\eta^*\overline{\eta^*}\alpha\bar\alpha} = -R_{\alpha\bar\alpha\eta\bar\eta}$. Write Θ_α for the Hermitian form $\Theta_\alpha(\cdot,\bar\cdot) = \Theta_{\cdot\bar\cdot\alpha\bar\alpha}$ on $T_x^*, \alpha \in T_x$. From the proof of Lemma (2.8.3) we know that $W' \subset$ Null space of Θ_α. By taking their annihilators in $T_M|_C$ we have $V \supset \mathbf{C}\alpha \oplus \mathcal{H}_\alpha$. By counting dimensions we had in Lemma (2.8.3) $\dim_{\mathbf{C}} W' = \dim_{\mathbf{C}} \mathcal{N}_\alpha = q$ so that in fact $V = \mathbf{C}\alpha \oplus \mathcal{H}_\alpha$.

We continue with our proof of the holonomy-invariance of $\mathcal{S}_M$. Let $\gamma = \{\gamma(t) : -\delta < t < \delta\}$ be a curve on μ such that $\gamma(o) = x$. Let $[\alpha(t)]$ be obtained from $[\alpha]$ by parallel transport along γ. Suppose $[\alpha_t]$ is a smooth section of $\mathcal{S}_M$ over γ obtained by projecting $\alpha(t)$ to the nearest point on $\mathcal{S}_{\gamma(t)}$. We do not require α_t to be of unit length but choose α_t such that $\alpha(t) = \alpha_t + t\zeta_t$ is an orthogonal decomposition. Let C_t be the (unique) minimal rational curve passing through $\gamma(t)$ such that α_t is tangent to C_t. Let V_t be the positive part of the Grothendieck decomposition of T_M^* over C_t. Since $T_{[\alpha_t]}\mathcal{S}_{\gamma(t)} = V_t \bmod (\mathbf{C}\alpha_t)$ by our choice of the orthogonal decomposition $\alpha(t) = \alpha_t + \zeta_t, \zeta_t$ is orthogonal to V_t. To prove that $\mathcal{S}$ is invariant under holonomy it suffices to show that it is true infinitesimally, i.e., for all possible choices of x, α and $\gamma, \zeta_o = 0$. To see this write $\zeta = \zeta_o$ and let $\zeta(t)$ be the parallel translate of ζ along γ. We will show that $\zeta = \zeta_o = 0$ by using $\Delta R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$ at $x = \gamma(o)$. Using the decompositions $\alpha(t) = \alpha_t + t\zeta_t$ and $\zeta(t) = \zeta_t + t\xi_t$; $\xi_o = \xi$; we have

$$\Delta R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = R_{\zeta\bar{\zeta}\zeta\bar{\zeta}} + R_{\alpha\bar{\alpha}\xi\bar{\xi}} \geq 0$$

Since (M, h) has positive holomorphic sectional and nonnegative bisectional curva tures this can only happen if $\zeta = 0$. This completes the proof of the holonomy invariance of $\mathcal{S}_M$ based on the fact $\Delta R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$.

We return now to the proof of Proposition (2.9) (and hence Theorem (2.1)). We are going to show that for the compact Kähler manifold $(X, g(t))$ with $t > 0$ of positive holomorphic bisectional curvature and positive Ricci curvature obtained by the parabolic Einstein evolution equation, $\mathcal{S}$ as defined in Proposition (2.9) is invariant under holonomy. We consider the Grothendieck decomposition $T_X|_C \cong \sum \mathcal{O}(a_i) = V \oplus W$, with V and W corresponding to the positive and trivial summands resp., as in the case of (M, h) by using a rational curve C such that $C \cdot K_X \leq n + 1$. Since C is possibly singular we work with the normalization $\nu: \mathbf{P}^1 \to C$ and consider deformations of $C' = \mathrm{Graph}(\nu)$ in $X' = \mathbf{P}^1 \times X$. Let $T_x^* = \mathcal{P}_\alpha \oplus \mathcal{N}_\alpha$ be the decomposition of T_x^* in terms of the Hermitian form $H_a(\xi, \eta) = R_{\alpha\bar{\alpha}\xi\bar{\eta}}$ of $(X, g(t))$, where $\mathcal{P}_\alpha$ is the direct sum of eigenspaces of H_α with positive eigenvalues and $\mathcal{N}_\alpha$ is the null-space of H_α. Imitating Lemma (2.8.3) we have on $(X, g(t)), V_x \supset \mathcal{P}_\alpha$ (instead of the stronger statement $V_x = \mathcal{P}_\alpha$). On examining the proof of the holonomy invariance of $\mathcal{S}_M$ on the Hermitian symmetric manifold (M, h) of compact type, we see that the inclusion $V_x \supset \mathcal{P}_\alpha$ is actually sufficient. As was proved using the null-vector condition, $(X, g(t)), t > 0$, is also of positive holomorphic sectional curvature and $\Delta R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$ for any pair (α, ζ) such that $R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0$. The proof of Proposition (2.9) is completed. To apply Proposition (2.1) to the proof of Theorem (2.1) we note that in case $\mathcal{S} \neq \mathbf{P}T_X$ we have by Theorem (2.7) of Berger-Simons that $(X, g(t))$ is Hermitian symmetric, which implies that the same is true at $t = 0$ for $(X, g(o)) = (X, g)$.

One approach to proving the generalized Frankel conjecture using only methods of differential geometry is to generalize the argument of Bando [Ban] of using condition (C) to show that for an evolved metric $\gamma(t), t > 0$, if condition (C) is violated at some point then $(X, g(t))$ is locally reducible as a Riemannian manifold. In higher dimensions it appears that one would need a deformation theory for stable harmonic maps to accomplish this. If this can be accomplished then one can use the method of stable harmonic maps to produce rational curves C representing a generator of $H^2(X, \mathbf{Z})$ modulo torsions in case $b_2(X) = 1$. Given this, the possibility (iii) (of having special rational curves) appearing before Proposition (2.9) can be ruled out since $K_X^{-1} = n + 1$ would imply that X is biholomorphic to $\mathbf{P}^n$ by the criterion of Kobayashi-Ochiai (Theorem (2.4.2)).

Alternatively, one may try to complete the program as envisaged in Mok-Zhong [MZ1] by showing that one can use the parabolic Einstein equation to evolve (X, g) to a Kähler-Einstein manifold (necessarily of nonnegative holomorphic bisectional curvature by [Mok9]). Even in the case of the projective space $\mathbf{P}^n$ it is not known whether a Kähler metric of positive holomorphic bisectional curvature evolves into a Kähler-Einstein metric using the parabolic Einstein equation. In the special case of $n = 1$ this was proved (unpublished) by Hamilton.

Lecture III. Compactification of Complete Kähler Manifolds of Positive Curvature

The Frankel Conjecture for open manifolds

In analogy with the Frankel conjecture for compact Kähler manifolds, Greene-Wu [GW1] and Siu [Siu1] formulated the following conjecture on complete Kähler manifolds of positive curvature.

Conjecture. *Let X be an n-dimensional complete non-compact Kähler manifold of positive Riemannian sectional curvature. Then, X is biholomorphic to the complex Euclidean space $\mathbf{C}^n$.*

In case of $n = 1$ this is a theorem of Blanc-Fiala [BF]. It is a consequence of the uniformization theorem in one complex variable and the fact X is simply-connected and supports no non-trivial positive harmonic function. In the general case Cheeger-Gromoll-Meyer [GM] [CG2] proved that a complete non-compact m-dimensional Riemannian manifold of positive sectional curvature is diffeomorphic to R^m. On the other hand, Greene-Wu [GW1] proved by using Busemann functions that X is necessarily Stein. These are the only known supporting evidences for the general conjecture. It should be remarked that the conjecture is formulated for the more restrictive condition of positive Riemannian sectional curvature instead of positive holomorphic bisectional curvature. On the one hand it is not known whether the theorem of Cheeger-Gromoll-Meyer [GM] [CG2] can be extended to include this case. On the other hand it is not even known whether such a complex manifold is Stein.

Both are due to the lack of a Toponogov Theorem for the case of positive holomorphic bisectional curvature. In Mok [Mok3] we attempted to study the non-compact version of the Frankel Conjecture by imposing additional quantitative geometric conditions. We proved an embedding theorem of such manifolds into $\mathbf{C}^n$ as affine-algebraic varieties. More precisely, we have

Theorem (3.1.1) (Mok [Mok3]). *Let (X, g) be a complex n-dimensional complete Kähler manifold of positive holomorphic bisectional curvature. Suppose X satisfies*

(i) $0 <$ *Bisectional curvatures* $<$ Const. $/d(x_o; x)^2$;

(ii) Volume$(B(x_o; R)) \geq cR^{2n}$ *for some* $c > 0$.

Then, X is biholomorphic to an affine-algebraic variety.

Here x_o is any fixed base point and $d(\cdot\ ;\cdot)$,$B(\cdot\ ;\cdot)$ denote geodesic distances and geodesic balls resp. We note that for (ii) by standard comparison theorems we always have the opposite inequality

$$\text{Volume}(B(x_o; R)) \leq CR^{2n} \text{ for some } C > 0$$

for a complete Riemannian manifold of nonnegative Ricci curvature. Henceforth we will say in general that a Riemannian manifold (M, h) is of quadratic curvature decay if the absolute values of Riemannian sectional curvatures decay quadratically (or faster). We will say that (M, h) is of Euclidean volume growth if there exists positive constants c and C such that

$$cR^m \leq \text{Volume}(B(x_o; R)) \leq CR^m.$$

Regarding Theorem (3.1) we note that the proof of the theorem extends to the case when X is of nonnegative holomorphic bisectional curvature and of positive Ricci curvature. On the other hand, in case of $n = 2$, a theorem in algebraic geometry due to Ramanujam [Ra] asserted that a quasi-projective surface diffeomorphic to $\mathbf{R}^4$ is necessarily biregular to $\mathbf{C}^2$. As a consequence we have

Theorem (3.1.2) (Mok [Mok1,3]). *Let (X, g) be a complete Kähler surface of positive Riemannian sectional curvature. Suppose X is of quadratic curvature decay and of Euclidean volume growth. Then, X is biholomorphic to $\mathbf{C}^2$.*

We will also consider in this lecture the problem of embedding complete Kähler manifolds of positive Ricci curvature. We prove

Theorem (3.1.3) (Mok [Mok14]). *Let (X, g) be an n-dimensional complete Kähler manifold of positive Ricci curvature. Suppose X is of quadratic curvature decay and of Euclidean volume growth. Suppose furthermore that $\int_X \mathrm{Ric}^n < \infty$. Then, X is biholomorphic to a non-singular quasi-projective variety.*

The embedding theorems presented in this lecture will be regarded as special cases of embedding theorems of non-compact complete Kähler manifolds. From this perspective we will try to explain a general scheme of compactifying such manifolds. This scheme will be relevant in other contexts as well. Later we will explain a result of Demailly on characterizing affine-algebraic varieties based on a similar scheme developed independently in [De1]. The approach developed in this lecture will also be relevant in the next lecture, where we will study the question of compactifying complete Kähler-Einstein manifolds of negative Ricci curvature.

(3.2) Techniques of L^2-estimate of $\bar{\partial}$ for the embedding problem

The most classical embedding theorem for complex manifolds is the Kodaira Embedding Theorem

Theorem (3.2.1) (Kodaira, cf. Kodaira-Morrow [KM]). *Let X be a compact complex manifold admitting a Hermitian holomorphic line bundle (L, h) of positive curvature. Then, X is biholomorphic to a non-singular projective-algebraic variety.*

The original formulation of the proof of the Kodaira Embedding Theorem is based on the Kodaira Vanishing Theorem for the positive line bundle L, which asserts that for a large enough positive integer $p_o, H^i(X, L^p) = 0$ for all $p \geq p_o$ and $i > 0$. The proof of this is based on the Bochner-Kodaira formula. The Kodaira Vanishing Theorem, combined with the Theorem of Riemann-Roch, allows one to construct enough holomorphic sections on L^p for a large enough p to construct an embedding into some projective space.

A somewhat different formulation of the proof is to construct directly holomorphic sections in $\Gamma(X, L^p)$ for p large enough. This formulation is based on the same Bochner-Kodaira formula but avoids using the Theorem of Riemann-Roch. To formulate it consider the problem of finding a single non-trivial holomorphic section s of some positive power of L. A smooth section s is holomorphic if and only if $\bar{\partial} s = 0$. Let $x_o \in X$ be any point and U be a Euclidean coordinate neighborhood of x_o with coordinates $(z_1, \ldots, z_n)$ such that x_o corresponds to the origin and such that $L|_U$ is holomorphically trivial with a holomorphic basis e. We look for some positive integer p and $s \in \Gamma(X, L^p)$ such that $s(x_o) \neq 0$. Let ρ be a cut-off function on U such that $\rho \equiv 1$ on a neighborhood of x_o and $\mathrm{Supp}(\rho) \subset\subset U$. Consider the $\bar{\partial}$-equation on $\bar{\partial} u = \bar{\partial}(\rho e^p)$ on L^p. The section $s = \rho e^p - u$ then satisfies $\bar{\partial} s = 0$ and defines a holomorphic section of L^p. The requirement $s(x_o) \neq 0$ is guaranteed if $u(x_o) = 0$. To solve the equation $\bar{\partial} u = \bar{\partial}(\rho e^p)$ with the additional condition $u(x_o) = 0$ we use the following existence theorem with L^2-estimates for the $\bar{\partial}$ -equation which in the compact case is a consequence of the Bochner-Kodaira formula, smoothing arguments (Friedrich's lemma) and the Riesz representation theorem in functional analysis:

Theorem (3.2.2) (Andreotti-Vesentini [AV] and Hörmander [Hör1]). *Let (X, ω) be a complete Kähler manifold or a Stein manifold equipped with a Kähler metric ω. Let (Λ, θ) be a Hermitian holomorphic line bundle with curvature form $c(\Lambda, \theta)$ and* Ric *be the Ricci form of (X, ω). Let φ be a smooth function on X. Suppose c is a positive function on X such that $c(\Lambda, \theta) + \mathrm{Ric} + \sqrt{-1}\partial\bar{\partial}\varphi \geq c\omega$ everywhere on X. Let f be a $\bar{\partial}$-closed L^2 Λ-valued $(0,1)$-form on X such that $\int_X \|f\|^2/c < \infty$. Then, there exists an L^2 Λ-valued section u satisfying $\bar{\partial} u = f$ and the estimate*

$$\int_X \|u\|^2 e^{-\varphi} < \int_X \frac{\|f\|^2 e^{-\varphi}}{c} < \infty.$$

Furthermore, if f is smooth u is also smooth.

The statement can be reduced to the case when the weight function is simply 1 when we replace the Hermitian line bundle (Λ, θ) by $(\Lambda, \theta e^{-\varphi})$ since we have $c(\Lambda, \theta e^{-\varphi}) = c(\Lambda, \theta) + \sqrt{-1}\partial\bar{\partial}\varphi$. In this case the Bochner-Kodaira formula implies the following: Let $\bar{\partial}^*$ denote the adjoint operator of $\bar{\partial}$ with respect to the Kähler metric ω on X and the Hermitian metric θ on Λ. Then, for any square-integrable Λ-valued section v belonging to the domains of definition of $\bar{\partial}$ and $\bar{\partial}^*$, we have the inequality

$$\|\bar{\partial} v\|^2 + \|\bar{\partial}^* v\|^2 \geq c\|v\|^2 + \|\bar{\nabla} v\|^2$$

everywhere on X. In the compact case, Theorem (3.2.2) (including the regularity statement) is a consequence of this inequality (the fundamental estimate), a density lemma of smooth forms in $\mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$ and the Riesz representation theorem in functional analysis.

Returning to the problem of finding a non-trivial holomorphic section $s \in \Gamma(X, L^p)$ such that $s(x_o) \neq 0$ in Theorem (3.2.1) we consider for the time being the Hermitian holomorphic line bundles (L^p, h^p).We want to solve for $\bar\partial u = \bar\partial(\rho e^p)$ on (L^p, h^p) with a weight φ with L^2-estimates while requiring $u(x_o) = 0$. Since $\bar\partial^2 = 0, f = \bar\partial(\rho e^p)$ is $\bar\partial$ -closed. Consider the singular weight function $\varphi = n\rho \log \sum |z_i|^2$ on X (φ being defined as zero outside U). Since $\log \sum |z_i|^2$ is plurisubharmonic for p large enough one verifies that $c(L^p, h^p e^{-\varphi}) \geq c\omega$ for some positive function c on X. Suppose Theorem (3.2.2) remains valid for the singular weight φ. Then, we have from the theorem a smooth solution of $\bar\partial u = \bar\partial(\rho e^p)$ with the estimate

$$\int_X \|u\|^2 e^{-\varphi} < \int_X \frac{\|\bar\partial(\rho e^p)\|^2}{c} e^{-\varphi} < \infty.$$

The second integral is finite because $\bar\partial(\rho e^p)$ is zero on a neighborhood of x_o. It then follows from $\int_X \|u\|^2 e^{-\varphi} < \infty$ that in fact $u(x_o) = 0$. To justify applying Theorem (3.2.2) to the singular weight φ one can simply approximate φ by the smooth weights $\varphi_\epsilon = n\rho \log \sum(|z_i|^2 + \epsilon)$.

Similar arguments allow us to construct holomorphic sections $s_o, \ldots, s_N$ in $\Gamma(X, L^p)$ for p large enough so that $\sigma = [s_o, \ldots, s_N]: X \to \mathbf{P}^N$ defines a holomorphic embedding of X into some projective space $\mathbf{P}^N$. Here if $s_i = t_i e^p$ in terms of the holomorphic basis e of $L|_U, [s_o, \ldots, s_N]$ is defined to be $[t_o, \ldots, t_N]$, which is independent of the choice of e. More precisely, to ensure that σ separates points we construct holomorphic sections with prescribed values at any two distinct points by using a weight function with two singularities. To ensure that σ gives a local embedding at a given point x_o we construct holomorphic sections vanishing at x_o with prescribed first derivatives (in local coordinates). This is done by using the singular weight $(n+1)\rho \log \sum |z_i|^2$. Finally, we obtain σ by using a covering argument.

Applying L^2-estimates of $\bar\partial$ to non-compact manifolds

Let (X, ω) be a complete Kähler manifold admitting a positive holomorphic line bundle (L, h) such that for some integer r,$rc(L, h) + \mathrm{Ric}$ is positive. Theorem (3.2.2) applies to yield holomorphic sections of high powers of L. In case the line bundle L is holomorphically trivial we can even construct holomorphic functions. In order to embed X as an affine-algebraic variety or a quasi-projective variety the first difficulty is that there may be too many holomorphic sections. The vector spaces $\Gamma(X, L^p)$ may be infinite-dimensional. Even if they are finite-dimensional, the procedure of constructing holomorphic sections with prescribed values and first derivatives at a point x_o may only work with an exponent $p(x_o)$ that blows up as x_o diverges to infinity. The use of compactness in the Kodaira Embedding Theorem has to be replaced by finiteness theorems at each stage of the scheme of embedding. The first theorem of embedding complete Kähler manifolds was the following affirmative answer to a conjecture of Greene-Wu [GW1] due to Siu-Yau [SY1, 2] :

Theorem (3.2.3) (Siu-Yau [SY1,2]). *Let (X,ω) be a simply-connected complete Kähler manifold of nonpositive Riemannian sectional curvature satisfying*

$$-C/(d(x_o;x)^{2+\epsilon} \leq \textit{Sectional curvatures } \leq 0$$

for some base point x_o, some $\epsilon > 0$ and some constant C. Then, X is biholomorphic to the complex Euclidean space $\mathbf{C}^n$.

In the proof they used comparison theorems in Riemannian geometry to construct plurisubharmonic functions on X of logarithmic growth such that the complex Hessian dominates $-$Ric. This allows them to work with the trivial line bundle and construct using L^2-estimates of n holomorphic sections of minimal growth which are candidates for Euclidean coordinates on $\mathbf{C}^n$. The difficulty of the proof is in showing that these n functions $(f_1, \ldots, f_n)$ do give a proper holomorphic map F into $\mathbf{C}^n$. The proof involves two steps: the map F is shown to be a locally a biholomorphism by estimating the volume of the zero-divisor of $df_1 \wedge \cdots \wedge df_n$ using the Poincaré-Lelong equation. The technique of showing that F is proper involves using L^2-estimates of $\bar{\partial}$ and standard elliptic estimates.

It was shown later on that the manifold (X,ω) in Theorem (3.2.3) is actually flat. The technique of proof yields nonetheless the following embedding theorem for (X,ω) with mixed curvature.

Theorem (3.2.4) (Mok-Siu-Yau [MSY]). *For every $\epsilon > 0$ there is a positive constant A_ϵ for which the following is true: Let (X,ω) be a simply-connected complete Kähler manifold satisfying*

$$|\textit{Sectional curvatures}| \leq A_\epsilon/(d(x_o;x)^{2+\epsilon}$$

for some base point x_o. Then, X is biholomorphic to the complex Euclidean space $\mathbf{C}^n$.

Both Theorems (3.2.3) and (3.2.4) are rather special since we can construct plurisubharmonic weight functions by using comparison theorems. In case of Theorem (2.1) we constructed in Mok-Siu-Yau [MSY] such functions by solving the Poincaré-Lelong equation. Since the Ricci form is positive a real solution φ to the Poincaré-Lelong equation $\sqrt{-1}\partial\bar{\partial}\varphi = \mathrm{Ric}$ is a plurisubharmonic function. In [Le1] Lelong solved such equations $\sqrt{-1}\partial\bar{\partial}\psi = T$ on $\mathbf{C}^n$ for a closed positive $(1,1)$ current T on $\mathbf{C}^n$ of minimal growth by a reduction of the equation to $\Delta\psi = 2\,\mathrm{Trace}(T)$. In case of manifolds of nonnegative holomorphic bisectional curvature of quadratic curvature decay and Euclidean volume growth we solved in [MSY] $\sqrt{-1}\partial\bar{\partial}\varphi = \rho$ for a closed positive $(1,1)$ form ρ of quadratic decay by finding a solution of logarithmic growth to the equation $\Delta\varphi = 2\,\mathrm{Trace}(\rho)$. The nonnegativity of bisectional curvatures is used in showing that $\Delta\|\sqrt{-1}\partial\bar{\partial}\varphi - \rho\|^2 \geq 0$, as in Theorem (2.2.1). We remark that in the argument and notations there we do not need ν to be a harmonic form, but only a closed $(1,1)$ form with constant trace. Here $\nu = \sqrt{-1}\partial\bar{\partial}\varphi - \rho$ has constant trace 0. The growth condition on ρ is then used to show that the subharmonic function $u = \|\sqrt{-1}\partial\bar{\partial}\varphi - \rho\|^2$ would violate the mean-value inequality unless it is identically zero. In the proof we used estimates of the Green Kernel and its gradient based on the isoperimetric inequality of Croke [Cro] and gradient estimates of Yau [Y1]. Both the Green function $G(-,y)$ and its gradient decay exactly as in the Euclidean case.

Using the fact that Ric is of quadratic decay one can deduce that φ is in fact an exhaustion function, i.e., $\{\varphi < c\} \Subset X$ for any c. As a consequence X is a Stein manifold by Grauert's solution to the Levi problem [Gr1].

Equipped with a plurisubharmonic function φ of logarithmic growth we can then use L^2-estimates of $\bar{\partial}$ to construct holomorphic functions of polynomial growth. If X can be embedded as an affine-algebraic variety one expects the algebra $P(X)$ of functions of polynomial growth to be finitely generated. (We will henceforth call such functions polynomials on X.) However, it is in general very difficult even in the compact case to show that a graded algebra of holomorphic sections of line bundles is finitely generated. Instead we pass to the "birational geometry" of the open Kähler manifold X by considering the quotient field $R(X)$ of "rational functions" of $P(X)$.

(3.3) Siegel's Theorem for the field of rational functions

Our first finiteness theorem is a Siegel's Theorem on the field $R(X)$. To explain this we consider first the classical Siegel's Theorem on compact complex spaces:

Theorem (3.3.1). *Let W be a connected compact complex space* *of dimension n. Denote by $M(W)$ the field of meromorphic functions on Z. Then $M(W)$ is of transcendence degree at most n. When the transcendence degree is equal to n we have $M(W) = C(f_1, \ldots, f_n, g)$ where $f_1, \ldots, f_n$ are algebraically independent and g is algebraic over $C(f_1, \ldots, f_n)$.*

We give here an explanation of the classical Siegel's Theorem in the case of compact Kähler manifolds (W, ν). The explanation will be relevant to our study of the complete Kähler manifold (X, ω). To prove Siegel's Theorem for W it suffices to consider a single holomorphic line bundle L and the field $M(W, L)$ of meromorphic sections of the form s/t, where s and t are holomorphic sections of L^p for some positive p. The proof of the Siegel's Theorem for W is then reduced to

Dimension estimate.

There exists a positive constant C such that for all positive integers p

$$\dim_{\mathbf{C}} \Gamma(W, L^p) < Cp^n.$$

In fact, if $\mathbf{C}(f_1, \ldots, f_n)$ is a purely transcendental extension and g is transcendental over $\mathbf{C}(f_1, \ldots, f_n)$ then by taking $f_i = s_i/u, g = t/u$ with the common denominator $u \in \Gamma(W, L^q)$ for some q fixed we can construct by forming monomials with $s_1, \ldots, s_n, t$ sections in $\Gamma(W, L^{qp})$ of dimension $\sim p^{n+1}$, contradicting the dimension estimate.

A second reduction is to reduce the dimension estimate to the following

Multiplicity estimate.

Fix a base point $x_o \in W$. Then, there exists a positive constant C such that for any nontrivial holomorphic section $s \in \Gamma(W, L^p)$ with zero-divisor $[Zs]$, we have

$$\mathrm{mult}([Zs]; x_o) \le Cp.$$

We argue by contradiction. If the dimension estimate is invalid, then by considering the Taylor expansion of sections $s \in \Gamma(W, L^p)$ at x_o and counting the number of coefficients of total degree $\le$ Const.p we can obtain holomorphic sections s whose order of vanishing violates the multiplicity estimate.

Finally, the multiplicity estimate can further be reduced to the following

Volume estimate.

There exists a positive constant C such that for any nontrivial holomorphic section $s \in \Gamma(W, L^p)$ with zero-divisor $[Zs]$, we have

$$\text{Volume }([Zs],\, X_0) \leq Cp.$$

The multiplicity estimate follows immediately from the volume estimate and the following lemma of Bishop-Lelong (cf. Lelong [Le2])

Lemma. *Let $[Z]$ be a subvariety of pure dimension m on the Euclidean unit ball B^n. Let C_m be the Euclidean volume of the unit ball B^m in $\mathbf{C}^m$. Then, in terms of the Euclidean Kähler form β we have*

$$\text{Volume}([Z], b) \geq C_m \, \text{mult}([Z]; o).$$

For compact Kähler manifolds W the volume estimate and hence Siegel's Theorem can now be proved by using the Poincaré-Lelong equation. Equip the holomorphic line bundle with a Hermitian metric h and denote by $c(L, h)$ its curvature form. Let $s \in \Gamma(W, L^p)$ be a non-trivial section. Then, measuring sections by h the function $\log \|s\|^2$ satisfies the Poincaré-Lelong equation

$$\sqrt{-1}\partial\bar{\partial} \log \|s\|^2 = 2\pi[Zs] - pc(L, h).$$

Integrating against the closed $(n-1, n-1)$ form ν^{n-1} and applying Stokes' Theorem we obtain immediate the volume estimate.

Siegel's Theorem for the complete Kähler manifold X

Returning to the n-dimensional complete Kähler manifold (X, ω) we are going to show

Proposition (3.3.2). *The field of rational functions $R(X)$ is a finite extension of a purely transcendental extension of transcendence degree n.*

By using the weight function φ and L^2-estimates of $\bar{\partial}$ we construct polynomials $f_1, \ldots, f_n$ such that $df_1 \wedge \cdots \wedge df_n$ does not vanish identically. In particular $f_1, \ldots, f_n$ are algebraically independent. To prove the proposition we are going to establish the multiplicity estimate for polynomials f of degree p. (The reduction of Siegel's Theorem to the multiplicity estimate does not make use of compactness.) As in the compact case we relate $\text{mult}([Zf]; x_o)$ with volumes of $[Zf]$ and hence with the degree of f (defined in the obvious way in terms of geodesic distances). From the Poincaré-Lelong equation

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|f|^2 = [Zf].$$

By the Bishop-Lelong inequality applied to a small Euclidean neighborhood of x_o we obtain for $R \geq R_o$ and for some constant c

$$\text{mult}([Zf];x_o) \leq c\int_{B(R)}\sqrt{-1}\partial\bar{\partial}\log|f|^2 \wedge \text{Ric}^{n-1}$$

From Stokes' Theorem and integrating with respect to R we obtain for $R \geq 2R_o$ and for some constant c

$$\text{mult}([Zf];x_o) \leq \frac{c}{R^{2n-1}}\int_{B(R)-B(\frac{R}{2})}\|\bar{\partial}\log|f|^2\|.$$

The estimates of the Green functions and their gradients yields immediately the integral representation of $\log|f|^2$

$$\frac{1}{2\pi}\log|f(x)|^2 = \lim_{R\to\infty}\left(\int_{B(R)\cap[Zf]} G_R(x;y)dy + c_R\right)$$

for some choice of normalizing constants c_R, where $G_R(-;-)$ denotes a the Green Kernel on the geodesic ball $B(R) = B(x_o;R)$. This allows us to transform the multiplicity estimate to one involving the volumes of the zero-divisor $[Zf]$ over geodesic balls by using the gradient estimate on the Green Kernel. At the same time the volumes of $[Zf]$ over geodesic balls is related to the degree of the polynomial f using the integral representation. Combining the two estimates this allows us to obtain the multiplicity estimate

$$\text{mult}([Zf];x_o) \leq C\deg(f)$$

which yields Siegel's Theorem for $R(X)$ as in the compact case.

Siegel's Theorem for $R(X)$ allows us to embed X "birationally" into an affine algebraic hypersurface Z_o in $\mathbf{C}^{n+2}$ by $F_o = (f_1,\ldots,f_n,g,h)$, where $R(X) = C(f_1,\ldots,f_n,g/h)$. From the fact that $P(X)$ separates points it follows that $F_o|_{X-V}$ is an embedding into Z_o. We need to show that F_o can be completed to a proper holomorphic embedding $F\colon X \to \mathbf{C}^N$ onto an affine algebraic variety Z by adjoining a finite number of polynomials to F_o. This involves a number of finiteness theorems.

We will say that a holomorphic map $F_o: X \to Z_o$ is quasi-surjective if there exists an algebraic subvariety T of Z such that $F_o(X)$ contains $Z - T$. To complete F_o to a proper holomorphic embedding we will show that (i) F_o is quasi-surjective, (ii) F_o can be desingularized by adjoining a finite number of polynomials and (iii) the resulting holomorphic map $F_1: X \to Z_1$ can be made into a proper map by adjoining a finite number of polynomials. The proofs of (i) – (iii) are a combination of analytic and elementary algebraic techniques. The analytic techniques involve essentially the L^2-estimate of $\bar\partial$ and comparison theorems.

(3.4) L^2-estimates for the ideal problem and quasi-surjectivity

We proceed to explain the proof of

Proposition (3.4.1). *$F_o: X \to Z_o$ is quasi-surjective.*

To present the idea of the proof we will consider the simple situation where $n = 2$ and we have in fact $F_o: X \to \mathbf{C}^2$, i.e., $R(X)$ is a purely transcendental extension of $\mathbf{C}$. For the purpose of illustration consider a holomorphic map $G: \mathbf{C}^2 \to \mathbf{C}^2, G = (g_1, g_2)$ defined by $G(z_1, z_2) = (z_1, z_1 z_2) = (w_1, w_2)$. G is a birational map. The image of G misses all of the w_2-axis except the origin. G is quasi-surjective in the sense of (3.3). We explain how the w_2-axis can be singled out by solving an ideal problem. The point $(0, 1)$ lies outside the image of G. In other words, the two polynomials $g_1, g_2 - 1$ have no common zero. By the Hilbert Nullstellensatz the constant function 1 lies in the ideal generated by $g_1, g_2 - 1$ i.e., the equation

$$g_1 h_1 + (g_2 - 1) h_2 = 1$$

admits a polynomial solution (h_1, h_2). Indeed an explicit solution is given by

$$g_1 z_2 + (g_2 - 1)(-1) = z_1 z_2 + (z_1 z_2 - 1)(-1) = 1.$$

Since G is birational we can think about the polynomials $z_2 - 1$ as pull-backs of rational functions on the (w_1, w_2)-plane $\mathbf{C}^2$. As such $z_2 = z_1 z_2 / z_1 = G^*(w_2 / w_1)$. The rational function w_2/w_1 on the (w_1, w_2)-plane $\mathbf{C}^2$ has the property that it has a pole at the point (0,1) missed by G.

In general if we have a birational map $G: \mathbf{C}^2 \to \mathbf{C}^2$ then for any point (a, b) missed by G we can solve the equation

$$(g_1 - a) h_1 + (g_2 - b) h_2 = 1$$

We argue that writing $(h_1, h_2) = G^*(r_1, r_2)$ either r_1 or r_2 must have a pole at (a, b). Otherwise from the equation

$$(w_1 - a)r_1 + (w_2 - b)r_2 = 1$$

on the (w_1, w_2)-plane $\mathbf{C}^2$ the R.H.S. would vanish at the point (a, b). We explain a procedure to exhaust all the algebraic curves on the (w_1, w_2)-plane $\mathbf{C}^2$ that are essentially missed by G. Starting with a point (a_1, b_1) we obtain $(h_{1,1}, h_{2,1}) = G^*(r_{1,1}, r_{2,1})$ such that either $r_{1,1}$ or $r_{2,1}$ has a pole at (a_1, b_1). Since $h_{1,1}, h_{2,1}$ are holomorphic and $(h_{1,1}, h_{2,1}) = G^*(r_{1,1}, r_{2,1})$. G must miss a dense open set of the union P_1 of poles of $r_{1,1}$ and $r_{2,1}$. Suppose the image of G does not contain $\mathbf{C}^2 - P_1$. We then pick a point (a_2, b_2) on $(\mathbf{C}^2 - P_1) - G(\mathbf{C}^2)$ and repeat the procedure to obtain $(h_{1,2}, h_{2,2}) = G^*(r_{1,2}, r_{2,2})$. We obtain thus the union P_2 of the pole sets of $(r_{1,2}, r_{2,2})$. This procedure can be repeated a finite number of times to realize the closure of $\mathbf{C}^2 - G(\mathbf{C}^2)$ as the union of pole sets $P_1, \ldots, P_k$ of rational functions.

Denote the rational function $r_{1,i}$ or $r_{2,i}$ having a pole at (a_i, b_i) by R_i. Important to us is the observation that $R_1, \ldots, R_k$ is a set of linearly independent rational functions. In fact, if $\sum c_i R_i = 0$ then c_k must vanish since R_k has a pole at (a_k, b_k) while R_j are holomorphic at (a_k, b_k) for $1 \leq j \leq k - 1$.

It is precisely this procedure that we adopted to show that the holomorphic map $F_o: X \to Z_o$ is quasi-surjective. Essentially, assuming as above $Z_o = \mathbf{C}^2, F_o = (f_1, f_2)$, and proceeding as in the case of $G: \mathbf{C}^2 \to \mathbf{C}^2$ to solve

$$(f_1 - a)h_1 + (f_2 - b)h_2 = 1$$

we are going to get estimates on the solutions (h_1, h_2) showing that their degrees are uniformly bounded by some K independent of the choice of (a, b). It then follows from the linear independence of $R_1, \ldots, R_k$ that we obtain at least k linearly polynomials of degree K. This then contradict with the dimension estimate used in the proof of Siegel's Theorem for $R(X)$.

In order to solve the ideal problem, we use the L^2-estimates of Skoda [Sko1] on the ideal problem, modified to include the case of complete Kähler manifolds. Our formulation of the estimates is less precise than that given in [Sko1].

Theorem (3.4.2) (Skoda [Sko1]). *Let (X, ω) be an n-dimensional complete Kähler manifold or Stein manifold equipped with a Kähler metric. Denote the Ricci form by* Ric *and let φ be a smooth function such* *that* $\sqrt{-1}\partial\bar{\partial}\varphi + \mathrm{Ric}$ *is a nonnegative* $(1,1)$ *form,. Let* $f_1, \ldots, f_p, h$ *be holomorphic functions on* X, k *be a positive constant,* $\alpha > 1$ *arbitrary such that:*

$$\int_X \frac{|h|^2}{(\sum |f_i|^2)^{\alpha n+1}} e^{-k\varphi} < \infty.$$

Then, there exists a solution $(g_1, \ldots, g_p)$ *of the equation*

$$f_1 g_1 + \cdots + f_p g_p = h$$

satisfying the estimate

$$\int_X \frac{|g_j|^2}{(\sum |f_i|^2)^{\alpha n}} e^{-k\varphi} \leq C_\alpha \int_X \frac{|h|^2}{(\sum |f_i|^2)^{\alpha n+1}} e^{-k\varphi}$$

for each $j, 1 \leq j \leq p$ *and for some constants* C_α *independent of* $f_1, \ldots, f_p$ *and* h.

The proof of Skoda's L^2-estimate of the ideal problem is based on an approach similar to the L^2-estimate of $\bar{\partial}$ in Theorem (3.2.2). In order to apply Skoda's estimate to prove that $F_o: X \to Z_o$ is quasi-surjective, we use the weight function φ arising from the Ricci form. We prove the elementary estimate.

Lower estimate of distance to the boundary

On Z_o there exists an algebraic subvariety S such that for all $b \in Z - S - F(X)$

$$\mathrm{dist}(F(x); b) > \frac{C(b)}{d^k(x; x_o)},$$

where dist denotes the Euclidean distance in $\mathbf{C}^{n+2}$, $C(b)$ is a constant depending on b and k is a constant depending only on the sum of degrees of $f_1, \ldots, f_n, g$ and h.

The estimate is obtained by finding a maximal Euclidean ball (under some coordinate projection) on which one can invert the mapping F near $F(x)$. The proof involves the sub-mean-value inequality and solving an ODE. The estimate allows us to show that the solutions to the ideal problem are indeed of degrees bounded independent of the point $b \in Z - S - F(X)$, yielding the proposition that $F_o: X \to Z_o$ is quasi-surjective.

(3.5) Desingularizing the quasi-surjective embedding F_o

We explain in this section how to adjoin holomorphic functions to F_o in order to get a holomorphic embedding $F_1: X \to Z_1$ onto an open subset of an affine algebraic variety Z_1. From the proof of quasi-surjectivity it follows that the image of F_1 is a Zariski-open subset of Z_1.

Again to simplify the notation we consider the case of $n = 2$ and $Z_o = \mathbf{C}^2$. Although in case of 2 dimensions there is a simpler argument we will present here a method that generalizes easily to arbitrary dimensions. Let V be the branching locus of F_o. Let $\{V_i\}$ be an enumeration of irreducible components of V. The problem is to prove that there are at most a finite number of such irreducible components. The map $F|_{X-V}$ maps $X - V$ biholomorphically onto $\mathbf{C}^2 - T$ for some algebraic curve T on $\mathbf{C}^2$. We are going to show

Proposition (3.5.1). *For each irreducible component V_i (of dimension 1) there exists a complex-analytic curve C_i on X intersecting V_i at isolated points such that the closure K_i of $F_o(C_i)$ is an algebraic curve of degree bounded independent of i.*

Assuming the proposition for the time being we explain the proof of the finiteness of V_i (still under the assumption that $F_o\colon X \to Z_o = \mathbf{C}^2$). Suppose all the K_i are of degree bounded by m. Parametrize the space $\{K_\zeta\}$ of algebraic curves of degree $\leq m$ by the coefficients of the defining equations. Thus $\{K_\zeta\}$ is parametrized by some Euclidean space (minus the origin), say $\mathbf{C}^N$. Let $E \subset \mathbf{C}^2 \times \mathbf{C}^N$ be the subspace such that $E_\zeta = E \cap \mathbf{C}^2 \times \{\zeta\}$ corresponds to the curve K_ζ. By the proposition for each 1-dimensional irreducible component V_i of V there exists some $K_i \in \{K_\zeta\}$ and a corresponding complex analytic curve C_i (with at most a finite number of irreducible components) such that C_i intersects V_i at (finitely many) isolated points. Our idea is to invert the holomorphic mapping $F_o\colon X \to \mathbf{C}^2$ along the affine-algebraic curves $\{K_\zeta\}$ in order to recover all the irreducible components $\{V_i\}$. Recall that there is an algebraic curve T on $Z_o = \mathbf{C}^2$ such that $F_o|_{X-V}\colon X - V \to \mathbf{C}^2 - T$ is a biholomorphism. The holomorphic mapping $F_o\colon X \to Z_o$ gives rise to a holomorphic mapping $E - (T \times \mathbf{C}^N) \to X$. Consider the special case when $E - (T \times \mathbf{C}^N) \to X$ extends holomorphically to $E \to X$. This means in particular that F_o can be inverted along each K_ζ.K_ζ intersects the affine-algebraic curve T at finitely many points. The number of intersection points is bounded independent of K_ζ. Since F_o can be inverted holomorphically on K_ζ each point in $K_\zeta \cap T$ corresponds to some point P lying on some V_i. As ζ varies the points $K_\zeta \cap T$ and hence P vary holomorphically in some sense so that the points P can only move on a finite number of irreducible components $\{V_i\}$. From the proposition it follows then the number of 1-dimensional irreducible components $\{V_i\}$ is finite.

In general, the holomorphic mapping $E - (T \times \mathbf{C}^N) \to X$ does not extend holomorphically to E. It may happen for example that a point on some K_ζ corresponds to infinity in X. We argue that $E - (T \times \mathbf{C}^N) \to X$ extends in some sense meromorphically to E. To make sense of this consider a partial quasi-surjective embedding $F_p: X \to Z_p$ of the Kähler manifold X into some affine algebraic variety Z_p which dominates the initial quasi-surjective embedding $F_o: X \to Z_o$. Then, the induced holomorphic mapping $H_p: E - (T \times \mathbf{C}^N) \to Z_p$ extends meromorphically to E. If $F_q: X \to Z_q$ dominates $F_p: X \to Z_p$ then the pole set of H_q contains the pole set of H_p. Thus, we can take some $F_p: X \to Z_p$ such that the pole set of H_p is the union of all possible pole sets of $\{H_q\}$.

If H_p has poles along $(T \times \mathbf{C}^N)$ we can discard those points on $E \cap (T \times \mathbf{C}^N)$ which belongs to the pole set of H_p and which are not points of indeterminacy of H_p. In fact such points corresponds to points on $K_\zeta \cap T$ which are mapped to infinity under the inverse of F_o. This means that we can now consider a quasi-affine subvariety W of $E \cap (T \times \mathbf{C}^N)$ which could possibly correspond to points on some $\{V_i\}$ when we invert F_o along the algebraic curves K_ζ. With some oversimplification this procedure allows us to inductively cut down $E \cap (T \times \mathbf{C}^N)$ to obtain a finite number of irreducible quasi-affine algebraic variety $\{W_j\}$ such that the points on W_j correspond to points lying on some V_i and such that as we move along a fixed W_j the corresponding points on V must move along at most a finite number of irreducible component V_i, proving that the set $\{V_i\}$ is finite. (One has to take normalizations of the curves $K_\zeta \cap T$ so that a point on $K_\zeta \cap T$ may correspond to a finite number of points on the normalization and hence on V.)

From the finiteness of $\{V_i\}$ we can adjoin a finite number of polynomials to F_o to obtain $F': X \to Z'$ whose branching locus is reduced to at most an isolated set of points. By passing to the normalization these points are removed by the Riemann Extension Theorem for bounded holomorphic functions.

We now proceed to explain the proof of the proposition. The key point is to obtain a uniform multiplicity estimate as follows:

Uniform multiplicity estimate

Let f be a given polynomial in $P(X)$. Then, except for an isolated set of points $\{x_i\}$ we have for all $x \in [Zf]$ the estimate

$$\mathrm{mult}([Zf]; x) \le C \deg(f)$$

where the constant C is independent of f.

In (3.3) we explained a multiplicity estimate which depends on the choice of base point x_o. That estimate was enough to establish Siegel's Theorem for $R(X)$. On the other hand the uniform multiplicity estimate is valid with the possible exception of a discrete set of points so that neither estimate implies the other.

We explain the ideas involved in the uniform multiplicity estimate. We recall first of all a proof of the analogous estimate in $\mathbf{C}^n$ (with no exceptional set $\{x_i\}$). Let f be a polynomial on $\mathbf{C}^n$, $[Zf]$ be its zero-divisor, and x be an arbitrary point on $\mathbf{C}^n$. An integral formula of Lelong [Le2] then shows that the ratio $\frac{\mathrm{Volume}(V(x;R)\cap[Zf])}{R^{2n-2}}$ is a nondecreasing function of R. The limiting ratio as $R \to 0$ yields essentially the multiplicity $\mathrm{mult}([Zf];x)$. (For more details, cf. (4.7)). On the other hand, the limiting ratio as $R \to \infty$ yields essentially the degree of f independent of the choice of x. Lelong's proof makes use of the fact that the Euclidean volume form is given by $\sqrt{-1}\partial\bar{\partial}\|z\|^2$ and that $\sqrt{-1}\partial\bar{\partial}\log\|z\|^2$, which defines the Fubini-Study metric, is positive semi-definite.

In our situation to imitate the proof we have to look for a potential φ for a metric comparable to the Kähler metric ω with the property that $\log\varphi$ is also plurisubharmonic. We can do this on geodesic balls by using comparison theorems. Recall that the volume of $[Zf]$ on geodesic balls is controlled by the degree of f. To establish the uniform multiplicity estimate we want to show that the volume of $[Zf]$ on geodesic balls would grow too fast if $\mathrm{mult}([Zf];x)$ is too large. For large geodesic balls we have to use a covering argument. Fix a polynomial $f \in P(X)$ and a point $x \in [Zf]$. The set $\{y : \mathrm{mult}([Zf];y) \geq \mathrm{mult}([Zf];x)\}$ is complex-analytic. Since X is Stein, any compact complex-analytic subvariety of X is zero-dimensional. This means that except for an isolated set of points $\{x_i\}$ we can assume that the multiplicity $\mathrm{mult}([Zf];x)$ is propagated. Because X is of quadratic curvature decay, the conjugate radius expands linearly. We show furthermore that there exists $c > 0$ such that for $d(x;x_o) = R$ the exponential map on the Euclidean ball $B_o(x;R)$ in $T_x(X)$ is at most λ-sheeted for a λ independent of R. For purposes of estimating volumes this is essentially the same as proving that the injectivity radius expands like R. To get a lower estimate of $\mathrm{Volume}(B(x,R)\cap[Zf])$ in terms of $\mathrm{mult}([Zf];x)$ we cover a portion of $(B(x,R)\cap[Zf])$ with geodesic balls $B(y,c'r), r = d(y;x_o)$, $c' > 0$ and fixed such that $\mathrm{mult}([Zf];y) \geq \mathrm{mult}([Zf];x)$. The union of the geodesic balls $B(y,cr')$ cover in some sense a positive fraction of X. On $B(y,c'r)$ we can then estimate $\mathrm{Volume}(B(y,c'r)\cap[Zf])$ from below in terms of $\mathrm{mult}([Zf];y) \geq \mathrm{mult}([Zf];x)$ by using Lelong's integral formula and comparison theorems. By rescaling and the assumption of quadratic curvature decay it suffices to work on geodesic balls B of fixed radius $c > 0$ and a Kähler manifold of sectional curvature pinched between 0 and +1. We can therefore compare B to a portion of the complex projective space equipped with a Fubini-Study metric of sectional curvature $\geq +1$.

To explain the deduction of Proposition (3.5) from the uniform multiplicity estimate we assume again that $n = 2$ and that $F_o: X \to Z_o = \mathbf{C}^2$. Write $F_o = (f_1, f_2)$ and let (w_1, w_2) be Euclidean coordinates on $\mathbf{C}^2$. Let $\{V_i\}$ be an enumeration of all the 1-dimensional irreducible components of the branching locus V. The functions f_1, f_2 are necessarily constant on V_i.

Suppose $f_1 = a_i$ on V_i. From the uniform multiplicity estimate the vanishing order of $f_1 - a_i$ on V_i is uniformly bounded independent of i. To simplify notations we make the technical assumption here that $f_1 - a_i$ has simple poles on V_i. (Otherwise we will need to work with local k-th roots of $f_1 - a_i$, where k is the order of vanishing.) Fix a V_i and pick a smooth point x_i on $V_i - \cup_{j \neq i} V_j$. Choose a neighborhood U_i of x_i in X parametrized by the unit bidisc with Euclidean coordinates (z_1, z_2) such that $z_1 = f_1 - a_i$. We expand f_2 in Taylor series in the form

$$f_2 = \sum_{\nu=0}^{\infty} c_\nu(z_2) z_1^\nu.$$

Suppose $c_\mu(z_2)$ is the first non-constant coefficient. Since $dz_1 = df_1$ the holomorphic 2-form $df_1 \wedge df_2$ vanishes at least to the order μ at x_i. On the other hand the holomorphic 2-form $df_1 \wedge df_2$ is also of polynomial growth and the argument of the uniform multiplicity estimate also shows that the vanishing order of $df_1 \wedge df_2$ at x_i is uniformly bounded independent of i. This shows that μ is uniformly bounded. Write now

$$f_2 = \sum_{\nu=0}^{\mu-1} c_\nu z_1^\nu + h_\mu(z_1, z_2) z_1^\mu.$$

Since the holomorphic function $h_\mu(z_1, z_2)$ is non-constant on the curve V_i the meromorphic function on X

$$g_\mu = \frac{f_2 - \sum_{\nu=0}^{\mu-1} c_\nu (f_1 - a_i)^\nu}{(f_1 - a_i)^\mu}$$

is holomorphic in a neighborhood of x_i and non-constant on V_i. Hence, for some choice of c_μ the curve C_i^o defined by

$$f_2 = \sum_{\nu=0}^{\mu-1} c_\nu (f_1 - a_i)^\nu + c_\mu (f_1 - a_i)^\mu$$

contains an irreducible branch which intersects V_i at isolated points. Define C_i as $\overline{C_i^o - V}$. Define an algebraic curve K_i on $\mathbf{C}^2$ by the equation

$$w_2 = \sum_{\nu=0}^{\mu-1} c_\nu (f_1 - a_i)^\nu + c_\mu (f_1 - a_i)^\mu.$$

Then, C_i and K_i, which correspond to each other under F_o, satisfy the requirement of the proposition. Combining (3.4) and (3.5) we have now obtained a holomorphic embedding $F_1 : X \to Z_1$ of X onto a Zariski-open subset of an affine algebraic variety Z_1.

(3.6) Completion to a proper holomorphic embedding

Suppose for the time being that the affine-algebraic variety Z_1 is non-singular. Since X is Stein, $Z_1 - F_1(X) = S$ is an algebraic hypersurface of Z_1. By using the line bundle $[S]$ on Z_1 and a standard procedure for constructing meromorphic functions (using vanishing theorems for algebraic coherent sheaves on the affine variety Z_1) one sees that the quasi-affine algebraic variety $Z_1 - S$ is necessarily biregular to an affine variety.

When Z_1 is singular the problem is that S may not define a line bundle. The algebraic proof breaks down. In fact there does exist an affine-algebraic variety V and an algebraic subvariety Q such that $V - Q$ is not biregular to an affine algebraic variety (cf. Goodman [Good]). To complete $F_1 : X \to Z_1$ to a proper holomorphic embedding we have to do analysis on the Kähler manifold X. To formulate the problem we define first of all the notion of rational convexity. Let W be a quasi-projective variety. We say that W is rationally convex if for any compact subset K of W there is a compact set $K' \supset K$ such that for any $y \in W - K'$ there exists a rational function f which is regular (holomorphic) on W and satisfies $|f(y)| > \sup_{x \in K} |f(x)|$. We prove using vanishing theorems in Algebraic Geometry

Theorem (3.6.1) (Mok [Mok3], Theorem (9.1), p. 252ff.). *Let V be an affine-algebraic variety, possibly singular, and let Q be an algebraic subvariety of V. Suppose $V - Q$ is rationally convex. Then, $V - Q$ is biregular to an affine-algebraic variety.*

On the complete Kähler manifold X we can define the corresponding notion of convexity with respect to $P(X)$. Since $R(X)$ corresponds to the field of rational functions on Z_1 under F_1, $Z_1 - S$ is rationally convex if X is convex with respect to $P(X)$. (The converse is also true.) We prove

Proposition (3.6.2). *X is convex with respect to the algebra $P(X)$ of polynomials.*

The proposition is proved by using the technique of Runge approximation in Several Complex Variables. By using algebraic techniques (Theorem of Cartan-Oka) or by using L^2-estimates of $\bar{\partial}$ we have the standard

Theorem (Runge approximation on Stein manifolds, cf. Hörmander [Hör2, 4.3]) . *Let M be a Stein manifold with a plurisubharmonic exhaustion function ψ. Let c be an arbitrary real number. Then, the open set $M_c = \{\psi < c\} \Subset M$ is Runge in M in the sense that holomorphic functions on M_c can be approximated uniformly on compact subsets by global holomorphic functions on M.*

In our case take K to be a compact set and y to be a point far away so that $\varphi(y) > \sup_{x \in K} \varphi(x)$. Let χ be a plurisubharmonic weight function with the only singularity at y. Then, by considering $\psi = \varphi + \epsilon\chi$ for a sufficiently small ϵ, we can choose c such that $K \subset M_c = \{\psi < c\}$ contains at least two disjoint components, one of which contains y, and such that $K \subset M_c$. The technique of Runge approximation then allows us to approximate the holomorphic function which is identically 1 on the component containing y and 0 on all other components. We can choose χ to be of logarithmic growth. Then, the L^2-estimate of $\bar{\partial}$ allows us to approximate the function f by using polynomials in $P(X)$. This proves the proposition and completes the construction of a proper holomorphic embedding $F: X \to Z$ onto an affine-algebraic variety using polynomials, proving Theorem (3.1.1) and (3.1.2).

We mention in passing that the holomorphic embedding $F: X \to Z$ is "biregular" in the sense that both F and F^{-1} are of polynomial growth, when distances on X and on Z are measured in terms of the Kähler metric and the Euclidean metric resp. This was proved in Mok [Mok2] by using an extension theorem of closed positive $(1,1)$ currents and L^2-estimates of $\bar{\partial}$.

(3.7) Embedding complete Kähler manifolds of positive Ricci curvature

A good portion of the estimates for embedding complete Kähler manifolds of positive bisectional curvature also works for the case of positive Ricci curvature. There are however two essential differences. First of all, the technique of producing plurisubharmonic weight functions φ by solving the Poincaré-Lelong equation only works under the assumption of nonnegative holomorphic bisectional curvature. Secondly, because of the lack of plurisubharmonic exhaustion functions there may be positive-dimensional compact subvarieties.

Let now X be a complete Kähler manifold of positive Ricci curvature, of quadratic curvature decay and Euclidean volume growth. In the absence of plurisubharmonic functions we work with powers of the positive line bundle K^{-1}. Let $R(X, K^{-1})$ be the quotient field of the graded algebra $\oplus_{q>0} P(X, K^{-q})$ of pluri-anticanonical sections (i.e. sections in $\Gamma(X, K^{-q})$ for some $q > 0$) of polynomial growth. As the estimates for the Green Kernel and its gradient only requires nonnegative Ricci curvature and Euclidean volume growth, the proof of Siegel's Theorem works essentially for $R(X, K^{-1})$. For $f \in \cup_{q>0} \cdot P(X, K^{-q})$, the function $\log \|s\|^2$ satisfies the Poincaré-Lelong equation

$$\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \|s\|^2 = [Zs] - \frac{p}{2\pi} \operatorname{Ric}.$$

Hence, in adapting the proof of Siegel's Theorem for compact Kähler manifolds to our $R(X, K^{-1})$ there is also an interior term. The proof of Siegel's Theorem then goes through under the additional assumption $\int_X \operatorname{Ric}^n < \infty$. Under this additional assumption we can embed X "birationally" onto a quasi-projective variety Z. To follow the scheme of compactification in the case of positive bisectional curvature, the quasi-surjectivity presents difficulty since solving the ideal problem using Skoda's L^2-estimates requires some plurisubharmonic exhaustion function. On the other hand, the uniform multiplicity estimate is valid under a weaker form. Because of the possible presence of positive-dimensional compact subvarieties, the multiplicity may fail to propagate on disjoint unions of compact subvarieties. Here we will explain only the first difficulty. For the second difficulty we need to prove a finiteness theorem for such positive-dimensional compact subvarieties and we will need the technique of Bézout estimates on complete Kähler manifolds. This technique will be developed in relation to the problem of compactifying complete Kähler-Einstein manifolds of finite volume. For this reason we will postpone the explanation to the next lecture.

To obtain plurisubharmonic weight functions we use the theory of pseudoconvex Riemann domains of Oka. We give first of all a brief description of domains of holomorphy in $\mathbf{C}^n$. A domain of holomorphy is essentially a maximal domain of existence of holomorphic functions. Consider the Hartogs' domain $H = \Delta \times (\Delta - \Delta(a)) \cup \Delta(b) \times \Delta$ in $\mathbf{C}^2$, where $0 < a, b < 1$. Every holomorphic function f on H extends to the bidisc (by using Laurent series expansions), which is the hull of holomorphy of H, i.e., a minimal domain of holomorphy containing H. Essential to Oka's theory of pseudoconvex domains is the notion of the Kontinuittäzssatz:

Definition. We say that a domain Ω verifies the Kontinuittäzssatz if and only if for any smooth one-parameter family of closed holomorphic discs Δ_t with boundary indexed by the smooth parameter $t \in [0,1]$ such that Δ_t belongs to Ω for $0 \leq t < 1$ and such that $\partial\Delta_1$ is also in Ω, then the entire closed holomorphic disc Δ_1 lies on the domain Ω.

For Riemann domains $\pi: \Omega \to \mathbf{C}^n$ spread over $\mathbf{C}^n$ we can also define similarly the notion of domains of holomorphy, hulls of holomorphy and the Kontinuitätssatz.

Returning to the situation of Riemann domains $\pi: \Omega \to \mathbf{C}^n$ we can define the Euclidean distance $\delta(x) = \delta(x; \partial\Omega)$ to the boundary as follows. There is a maximal connected neighborhood U of x in Ω such that $\pi|U$ maps U biholomorphically to a Euclidean ball B centered at $\pi(x)$. We define $\delta(x)$ to be the Euclidean radius of B. Central to Oka's theory is the

Theorem (Oka [Oka1, 2]). *A Riemann domain $\pi: \Omega \to \mathbf{C}^n$ is a Riemann domain of holomorphy if and only if the (continuous) function $-\log\delta$ is plurisubharmonic. Furthermore, this is the case if and only if Ω satisfies Oka's Kontinuitätssatz.*

By a theorem of Griffiths [Gri2] a Riemann domain $\pi: \Omega \to \mathbf{C}^n$ is a Riemann domain of holomorphy if it carries a complete Hermitian metric of nonnegative holomorphic sectional curvature. In case of Kähler manifolds Mok-Yau [MY] proved the following

Theorem (3.7.1) (Mok-Yau [MY]). *A Riemann domain $\pi: \Omega \to \mathbf{C}^n$ is a Riemann domain of holomorphy if it carries a complete Kähler metric ω with asymptotically nonpositive holomorphic bisectional curvature, i.e., if $K(x)$ denotes the maximum of all holomorphic bisectional curvatures at $x \in \Omega, d(-;-)$ denotes geodesic distances on (Ω, ω) and $x_o \in \Omega$ is a base point*

$$\limsup_{d(x;x_o)\mapsto\infty} K(x) \leq 0.$$

We return now to our problem of embedding the complete Kähler manifold of positive Ricci curvature (X, ω). Let $F_o: X \to Z_o \subset P^N$ be a "birational" embedding of X into a projective-algebraic variety Z_o, which we may take to be non-singular by Hironaka desingularization. There exists a complex-analytic hypersurface V in X such that

$F_o|_{X-V}: X - V \to Z_o$ is an open holomorphic embedding. We may choose V so that $F_o(X-V)$ lies in some Euclidean space $\mathbf{C}^N \subset P^N$ and and that some Euclidean coordinate projection π realizes $\Omega = F_o(X-V)$ as a Riemann domain over $\mathbf{C}^n$. Recall that (X,ω) is of quadratic curvature decay. While (Ω,ω) is no longer complete, the proof of the theorem above can be adapted to show that the Riemann domain $\pi:\Omega \to \mathbf{C}^n$ is a Riemann domain of holomorphy. By Oka's Theorem we have at our disposal a plurisubharmonic weight function $-\log\delta$ defined on Ω. Skoda's L^2-estimate for the ideal problem (cf. (3.5)) applies in particular to Ω. (A Riemann domain $\pi:\Omega \to \mathbf{C}^n$ is a Riemann domain of holomorphy if and only if it is Stein.) Suppose $\Omega \subset Z_o - T$, where $Z_o - T \subset \mathbf{C}^N$ is quasi-affine and the coordinate projection π also realizes $Z_o - T$ as a Riemann domain. We can adapt the proof of quasi-surjectivity in (3.4) to show that $\Omega = Z_o - T - S$, where S is a projective-algebraic subvariety of Z_o. In solving the ideal problem we use the weight functions $-\log\delta$ and $\log\sum|z_i|^2$, where $(z_1,\ldots,z_n)$ are the Euclidean coordinates on $\mathbf{C}^n$. As these functions arise from Euclidean distances we use the Euclidean metric instead of the Kähler metric ω in solving the ideal problem (in the space of holomorphic functions). We obtain this way linearly independent rational functions $R_1,\ldots,R_k,\ldots$ in the same way as in (3.4). In order to prove quasi-surjectivity the key point is to relate these functions to holomorphic sections in $\cup_{q>0}P(X,K^{-q})$, for which we have dimension estimates (in the proof of Siegel's Theorem). We have the estimate

Lower estimate of distance to the boundary

There exists some $s \in \cup_{q>0}P(X,K^{-q})$ and a positive integer k such that on the domain Ω we have the estimate

$$\delta(x) \geq \frac{1}{\|s\| d(x;x_o)^k}.$$

This estimate allows us to show that for some $t \in \cup_{q>0}P(X,K^{-q})$ the meromorphic pluri-anticanonical sections $\{R_k t\}$ on $X - V$ (identifying Ω with $X - V$) extend holomorphically to X to give holomorphic sections $s_k \in P(X,K^{-q})$ for the same exponent q such that the degree of $\{s_k\}$ is uniformly bounded. Here the degree of s is defined by

$$\deg(s) = \limsup_{d(x;x_o)\to\infty} \frac{\log\|s\|}{\log d(x;x_o)}.$$

This allows us to prove quasi-surjectivity as in (3.4). The uniform multiplicity estimate in (3.5) can now be formulated in to the case of positive Ricci curvature in such a way that the estimate may fail for a disjoint union of compact analytic subvarieties $\{E_k\}$. In the next lecture we will develop Bézout estimates to show that there are at most a finite number of positive-dimensional components in $\{E_k\}$. Given this, the scheme of desingularization in (3.5) will allow us to show that X is biholomorphic to a quasi-projective variety.

(3.8) Characterization of affine-algebraic varieties

In [De1] Demailly considered the question of characterizing affine-algebraic varieties by complex-analytic and geometric conditions. Consider a non-singular affine-algebraic variety $X \subset \mathbf{C}^N \subset P^N$. The Euclidean metric restricts to X to yield a complete Kähler metric with non-positive holomorphic bisectional curvature, hence of nonpositive Ricci curvature. On the other hand the Fubini-Study metric restricts to X to give a Kähler metric of finite volume. Complex-analytically X is a Stein manifold equipped with an exhaustion function $\varphi = \log(1 + \sum |z_i|^2)$, where $(z_1, \ldots z_n)$ are the Euclidean coordinates. The function φ is related to both the Euclidean metric and the Fubini-Study metric. In fact, φ is a potential for the Fubini-Study metric and e^φ is a potential for the Euclidean metric. In the non-singular case [De1] Demailly proved

Theorem (3.8) (Demailly [De1], 9, p.70ff.). *Let X be a connected n-dimensional complex manifold. Then, X is biholomorphic to a non-singular affine-algebraic variety if and only if the even Betti numbers of X are finite and there is a smooth strictly plurisubharmonic exhaustion function φ such that for the Kähler forms $\alpha = \sqrt{-1}\partial\bar\partial\varphi$ and $\beta = \sqrt{-1}\partial\bar\partial(e^\varphi)$ we have*

(i) Volume$(X, \alpha) < \infty$,

(ii) *there exists a locally integrable function ψ verifying $\psi \le A\varphi + B$ for some constants A and B such that*

$$\mathrm{Ric}(X, \beta) \ge -\sqrt{-1}\partial\bar\partial\psi.$$

The necessary condition is obtained by using $\varphi = \log(1 + \sum |z_i|^2)$ as above. Let $(P_1, ..P_m)$ be generators of the ideal $I(X)$ of polynomials vanishing on X and let $\{J_1, \ldots, J_M\}$ be an enumeration of all the partial Jacobians of n-tuples in $\{p_1, \ldots, P_m\}$. Define $\psi = 2\log\sum |J_k|^2$. It was shown in [De1] that ψ satisfies the assumption (ii). We have actually the stronger property

$$\text{(ii)}' \qquad 0 \ge \mathrm{Ric}(X, \beta) \ge -\sqrt{-1}\partial\bar\partial\psi.$$

We give a brief outline of the scheme of proof of the sufficient part of Theorem (3.8) adopted in [De1]. From the existence of the exhaustion function φ and Grauert's solution to the Levi problem (Grauert [Gr1]) we know that X is a Stein manifold. Applying the L^2-estimate of $\bar{\partial}$ one produces holomorphic functions f satisfying the L^2-condition

$$\int_X |f|^2 e^{-C\varphi}\beta^n < \infty.$$

In solving $\bar{\partial}$ because of the Ricci term appearing in the Bochner-Kodaira formula we use the weight functions $C\varphi+\psi$ to get $\sqrt{-1}\partial\bar{\partial}(C\varphi+\psi) + \mathrm{Ric}(X,\beta) > 0$. Since $\psi \le A\varphi + B$ the holomorphic functions still satisfy the L^2-condition stated. To prove Siegel's theorem we need to work with a space of holomorphic functions closed under multiplication. Define for $0 < p < \infty$ the vector space $\Gamma^p(X,\varphi)$ of holomorphic functions satisfying the condition

$$\int_X |f|^2 e^{-C\varphi}\beta^n < \infty$$

for some constant $C \ge 0$. Define $\Gamma^o(X,\varphi)$ as $\cup_{0<p<\infty}\Gamma^p(X,\varphi)$. Then, $\Gamma^o(X,\varphi)$ is closed under multiplication. The first step of the proof of Theorem (3.8) is to prove Siegel's Theorem for the quotient field $R(X,\varphi)$ of the algebra $\Gamma^o(X,\varphi)$.

Consider the "pseudoballs" $B(r) = \{\varphi < r\}$ and the "pseudospheres" $S(r) = \{\varphi = r\}$. $S(r)$ is smooth for all but a discrete set of $\{r\}$. Associated to $S(r)$ is the measures μ_r defined by

$$\mu_r(h) = \int_X h(\sqrt{-1}\partial\bar{\partial}\varphi)^{n-1} \wedge \sqrt{-1}\bar{\partial}\varphi.$$

They can also be defined on singular pseudospheres. μ_r are called Monge-Ampère measures in [De1]. In case of the Euclidean space and $\varphi = \log(1+\sum|z_i|^2)$, μ_r is up to a fixed constant the normalized spherical measure on the Euclidean sphere $\partial B(o;(e^r-1)^{1/2})$. These Monge-Ampère measures appear in the following basic

Jensen-Lelong formula (Demailly [De1], Theorem 0.3)

Every plurisubharmonic function V is μ_r-integrable for all r and verifies the integral formula

$$\int_\infty^r dt \int_{B(t)} \sqrt{-1}\partial\bar{\partial}V \wedge (\sqrt{-1}\partial\bar{\partial}\varphi)^{n-1} = \mu_r(V) - \int_{B(r)} V(\sqrt{-1}\partial\bar{\partial}\varphi)^n.$$

For a holomorphic function f on X define the degree $\delta(f)$ of f by the formula

$$\delta(f) = \limsup_{r\to\infty} \frac{1}{r}\mu_r \log^+ |f|,$$

where $\log^+(\alpha) = \max(\log\alpha, 0)$ for a positive number α. In the Euclidean case this gives

$$\delta(f) = \limsup_{R\to\infty} \frac{1}{2\log R}(\text{Average of } \log^+ |f| \text{ over } B(o;R)).$$

Hence, $\delta(f) < \infty$ if and only if f is a polynomial. In the general context of Theorem (3.8), Demailly considered the space of holomorphic functions f of finite degree $\delta(f)$ and proves a volume estimate for $[Zf]$ in the metric $\alpha = \sqrt{-1}\partial\bar\partial\varphi$. Furthermore he proved that any $f \in \Gamma^o(X, \varphi)$ is of finite degree. This yields the Siegel Theorem for $R(X,\varphi)$. The volume estimate

$$2\pi \int_X [Zf] \wedge \alpha^{n-1} \leq \delta(f)$$

is obtained by using the Jensen-Lelong formula associated to the Monge-Ampère measures μ_r.

Siegel's Theorem for $R(X,\varphi)$ gives rise to a "birational" holomorphic mapping $F_o: X \to Z_o$ into an affine-algebraic variety Z_o. To simplify the presentation we consider the case when $Z_o = \mathbf{C}^n$ and F_o is already a holomorphic embedding. The proof of the quasi-surjectivity of F_o makes use of a current extension theorem. Identify X with its image $F_o(X) \subset \mathbf{C}^n \subset P^n$. In [Del] it was proved that α can be extended trivially to a closed positive current T on P^n. The proof of this is motivated by the proof of Skoda [Sko2] and El Mir [El] of Bishop's Extension Theorem for positive closed (k,k)-currents across closed pluripolar sets. The current extension theorem allows one to solve the Poincaré-Lelong equation $\sqrt{-1}\partial\bar\partial u = T$ on $\mathbf{C}^n$ with a solution u of logarithmic growth. On the other hand $\sqrt{-1}\partial\bar\partial\varphi = \alpha$ on X so that on X we have two potentials for the same current T. The difference $\tau = u - \varphi$ is then a pluriharmonic function on X. As u is locally bounded from above on $\mathbf{C}^n$ and φ is an exhaustion function the function τ tends to $-\infty$ on $\partial X \subset \mathbf{C}^n$, so that ∂X is pluripolar. In particular, ∂X is locally connected.

To show that $\mathbf{C}^n - X$ is algebraic Demailly [Del] considered that holomorphic 1-form $h = \partial\tau$. It was proved (in the case under consideration) that h extends to a rational 1-form on P^n. To explain the approach one can define $\delta(V)$ for any plurisubharmonic function V in such a way that for a holomorphic function (by abuse of notation)

$\delta(f) = \delta(\log^+ |f|)$. If $\delta(V) < \infty$ we say that V is of finite degree. Analogous to the volume estimate for holomorphic functions of finite degree we have the corresponding mass estimate that

$$\int_X \sqrt{-1}\partial\bar{\partial}V \wedge \alpha^{n-1} < \delta(V).$$

From the logarithmic growth of u on $\mathbf{C}^n$ one obtains upper estimates of $\tau = u - \varphi$, which can be used to show that the function $\theta = \log(1 + e^\tau)$ is of finite degree. The mass estimate for $V = \sqrt{-1}\partial\bar{\partial}\theta$ then yields an estimate on the gradient of τ. One can then deduce that each component h_j of $h = \sum h_j dz^j = \partial\tau$ is of finite degree, hence rational in view of the construction of F_o using Siegel's Theorem. Let S be the union of the pole sets of h_j. We have $d\tau = h + \overline{h}$. We argue that X is precisely $\mathbf{C}^n - S$. For every point $b \in \mathbf{C}^n - S$ one can integrate the closed real 1-form $h + \overline{h}$ in some neighborhood U to obtain a smooth solution of $d\sigma = h + \overline{h}$. Since X is locally connected at every boundary point we may assume that $U \cap X$ is connected and so $\sigma - \tau$ is constant. As τ tends to $-\infty$ on ∂X and σ is smooth, it is impossible that $b \notin X$, proving that $X = \mathbf{C}^n - S$, in particular, X is Zariski-open in $\mathbf{C}^n$.

Additional difficulties arise in the general situation $F_o: X \to Z_o$. The principle difficulty is that in general one cannot solve the Poincaré-Lelong equation $\sqrt{-1}\partial\bar{\partial}u = T$ on Z_o. In general we can take a smooth projective compactification M of Z_o and extend T to a closed positive $(1,1)$-current on M. Then from Hodge theory there exists a smooth closed $(1,1)$-form ν on M and a smooth function u on X bounded from above such that $T = \nu + \sqrt{-1}\partial\bar{\partial}u$ on M. For $\tau = u - \varphi$, the 1-form $h = \partial\tau$ is not holomorphic but $\bar{\partial}h = \sqrt{-1}\nu$ is $\bar{\partial}$-closed. One can therefore solve with L^2-estimates on the affine part Z_o the equation $\bar{\partial}\mu = \sqrt{-1}\nu$ for some smooth $(1,0)$ form μ to get a holomorphic 1-form $h - \mu$.

The step of desingularizing the quasi-surjective holomorphic mapping $F_o: X \to Z_o$ to an open embedding is attained by using the topological condition that even Betti numbers are finite. This was achieved by the following topological lemma due to Demailly.

Lemma (Demailly [De1]). *Let X be a complex manifold of complex dimension n. Let Y be a subvariety of dimension $\leq p$ in X and write $d = n - p = \operatorname{codim}_{\mathbf{C}} Y$. Then, the relative cohomology groups $H^q(X, X - Y; \mathbf{R})$ are zero if $q \leq 2d$ and $H^{2d}(X, X - Y; \mathbf{R}) = \mathbf{R}^{\mathrm{Card}(J)}$, where $(Y_j)_{j \in J}$ is the set of irreducible components of dimension p in Y. As a consequence, if $H^{2d}(X; \mathbf{R})$ and $H^{2d-1}(X - Y; \mathbf{R})$ are finite-*

dimensional, then the number of irreducible components of complex dimension p is finite.

From the lemma we can inductively desingularize F_o to obtain a holomorphic embedding $F_1: X \to Z_1$ onto a Zariski open subset of an affine-algebraic variety. To complete F_1 to a proper holomorphic embedding into some $\mathbf{C}^N$ one uses Theorem (3.6.1) characterizing affine-algebraic varieties among quasi-projective varieties in terms of rational convexity. The rational convexity of $F_1(X)$ is proved by using Runge approximation in terms of the exhaustion function φ.

Remarks. Here and in Lecture IV extension of closed positive currents play an important role in proving quasi-surjectivity of certain "birational" embeddings. For a general reference on techniques based on integration by parts used in proving extension theorems of closed positive currents we refer the reader to Skoda [Sko3] and Sibony [Sib].

Lecture IV. Compactification of Complete Kähler-Einstein Manifolds of Finite Volume

(4.1) Compactification of arithmetic quotients of bounded symmetric domains and generalizations

In this lecture we will discuss the problem of compactifying complete Kähler manifolds of negative Ricci curvature, including in particular Kähler-Einstein manifolds. If X is a quasi-projective variety admitting a complete Kähler metric ω with bounded and negative Ricci curvature bounded from above by a negative constant it follows from the Ahlfors-Schwarz lemma for volume forms that X is necessarily of finite volume.

The first examples of non-compact complete Kähler-Einstein manifolds of finite volume are hyperbolic non-compact finite Riemann surfaces S (i.e. compact Riemann surfaces with a finite number of points removed) equipped with the Poincaré metric. The Poincaré metrics are obtained from the Poincaré metric on the unit disc, which is invariant under Möbius transformations. The first higher dimensional examples are obtained from quotients X of bounded symmetric domains Ω by torsion-free arithmetic lattices $\Gamma \subset \mathrm{Aut}(\Omega)$ (cf. (1.1)). The bounded symmetric domains Ω are equipped with Bergman metrics ds^2_B invariant under $\mathrm{Aut}(\Omega)$. (Ω, ds^2_B) are automatically Kähler-Einstein, inducing complete Kähler-Einstein metrics $\omega = \omega_{KE}$ on X. (X, ω) are automatically of finite volume (cf. Raghunathan [Rag]). The theory of compactification of such X was first considered by Satake [Sat1, 1956] who gave certain quotients of Siegel upper half spaces topological compactifications. This was later extended to other bounded symmetric domains and arithmetic groups in [Sat2, 1960]. Baily [Bai, 1958], and Baily-Borel [BB, 1966], based in part on the works of Satake, endowed such topological compactifications with complex structures making X into a Zariski-open subset of a (highly singular) projective-algebraic variety $X_{\min}$, which we call Satake-Baily-Borel compactifications. They did this for all arithmetic quotients of bounded symmetric domains. One can always find non-singular compactifications of X by using Hironaka desingularization. For the purpose of explicit computations it was desirable however to give natural compactifications $\bar{X}$ such that $\bar{X} - X$ is a union of smooth divisors intersecting at normal crossings. There is a long history of efforts in this direction, culminating in the work of Ash-Mumford-Rapapport-Tai [AMRT] on toroidal compactifications of arithmetic quotients on bounded symmetric domains.

There were also classical results on compactifying X using complex analysis. In Andreotti-Grauert [AG2] and Andreotti [A] the theory of pseudoconcave manifolds was developed. In particular, [A] proved Siegel's Theorem for pseudoconcave manifolds. In [AG1, 1960] it was verified that certain arithmetic quotients X of the Siegel upper half planes are pseudoconcave. At that time the arithmetic theory of compactification of such manifolds had been developed (by Satake [Sat1] and Baily [Bai]). The primary interest of [AG1] was to give an elementary analytic proof of the extendability of meromorphic functions to Satake-Baily compactifications, but their method can also be used to embed X as open subsets in certain projective varieties Z independent of the work of Satake-Baily. (cf. Pjatetskii-Shapiro [Pja]). It was later proved that all irreducible arithmetic quotients of dimension ≥ 2 are pseudoconcave (Spiker [Spi] , Borel [Bo3]). In the case of strongly pseudoconcave manifolds X Andreotti-Tomassini [AT] proved a general theorem for embedding X as open subsets of projective varieties Z. Very recently, the approach of compactifying complete Kähler manifolds by using pseudoconcavity was taken up again by Nadel-Tsuji [NT]. Their result completed in particular the efforts started by Andreotti-Grauert [AG1] to give a complex-analytic proof that arithmetic quotients of bounded symmetric domains are biholomorphic to quasi-projective varieties.

On the other hand, we can regard arithmetic quotients of bounded symmetric domains (by torsion-free lattices) as particular examples of Kähler- Einstein manifolds of finite volume. In this direction, very recently Mok [Mok12] , Mok-Zhong [MZ2] pursued a program of compactifying complete Kähler-Einstein manifolds of finite volume and bounded curvature. The results imply in particular the fact that quotients of bounded symmetric domains of finite volume are necessarily biholomorphic to quasi-projective varieties. In this lecture we will discuss both the results of Nadel-Tsuji using pseudoconcavity and the results of Mok-Zhong using curvature conditions.

(4.2) Siegel's Theorem on pseudoconcave manifolds

Pseudoconcave manifolds are roughly speaking complex manifolds which behave like compact manifolds complex-analytically, e.g., the maximum principle for plurisubharmonic functions is valid so that they support no non-constant plurisubharmonic functions. Before giving a precise formulation we start with a proof of Siegel's Theorem for general compact complex spaces. This proof uses a covering argument and will be generalized to pseudoconcave manifolds.

Proof of Siegel's Theorem for compact complex spaces. By desingularization we need to consider only compact complex manifolds M. Let L be a holomorphic line bundle on M. We need to get dimension estimates for $\Gamma(M, L^p), p > 0$. Instead of reducing to multiplicity estimates at a single point as in (3.3) we use a covering argument. Let U''_α be a covering of X by Euclidean open sets U''_α with the following additional properties: (i) $L|U''_\alpha$ is holomorphically trivial; (ii) there exists open sets $U_\alpha \Subset U'_\alpha \Subset U''_\alpha$ such that U''_α and U''_α are open subcoverings of X and such that there exist points $x_\alpha \in U_\alpha$ and biholomorphisms $\Phi_\alpha: U'_\alpha \to B(2)$ with $\Phi_\alpha(x_\alpha) = o$ and $\Phi_\alpha(U_\alpha) = B(1)$. Here $B(r)$ denotes the Euclidean ball $B(o; r)$. Equip L with a Hermitian line bundle h. The proof of the dimension estimate for the system $\{\Gamma(M, L^p), p > 0\}$ as in (3.3) is reduced to the

Multiplicity estimate.

There exists a positive constant C such that for any integer $p > 0$ and for any $s \in \Gamma(M, L^p)$ it is not possible that $\text{mult}([Zs]; x_\alpha) > Cp$ for all x_α.

To prove this we use the Schwarz lemma for bounded holomorphic functions:

Schwarz lemma for holomorphic functions.

Let $f: B(1) \to \Delta$ be a holomorphic function such that the vanishing order of f at o is at least k. Then, we have

$$|f(z)| \leq |z|^k \text{ for any } z \in B(1).$$

Let e^α be a holomorphic basis of $L|U''_\alpha$. Let $\|.\|$ denote Hermitian norms on L^p in terms of the Hermitian metrics h^p. Let $s \in \Gamma(M, L^p)$ with $\sup_M \|s\| = 1$ be such that the vanishing order of s at each x_α is at least k, i.e., $\text{mult}([Zs]; x_\alpha) \geq k$. Over each U''_α write $s = s_\alpha(e^\alpha)^p$. There exists a positive constant $C \geq 1$ such that

$$\frac{|s_\alpha|}{C^p} < \|s\| < C^p |s_\alpha|$$

for all $s \in \Gamma(M, L^p)$ and for all α. Since $\sup_M \|s\| = 1$ we have

$$\sup_\alpha \sup_{U_\alpha} |s_\alpha| \geq \frac{1}{C^p};$$
$$\sup_\alpha \sup_{U'_\alpha} \leq C^p.$$

On the other hand from the Schwarz lemma we have

$$\sup_{U_\alpha} |s_\alpha| \leq (\sup_{U'_\alpha} |s_\alpha|) \left(\frac{1}{2}\right)^k.$$

It follows immediately that $(1/2)^k \leq C^{2p}$, showing that $k \leq C'p$ for some constant C'. This establishes the multiplicity estimate and proves Siegel's Theorem for M.

Pseudoconcave spaces.

Let U be a complex space and V be a subset of U. We define the hull $\widehat{V}_U$ of V in U by

$$\widehat{V}_U = \{x \in U : |f(x)| \leq \sup_V |f|\}.$$

Definition. Let X be a complex space and $Y \subset X$ be open in X. Let y_o be a point on the boundary ∂Y. We say that Y is pseudoconcave at the point y_o if and only if there exists a fundamental system of open neighborhoods $\{U_\nu\}_{1 \leq \nu < \infty}$ of y_o in X such that y_o is an interior point of $(\widehat{U_\nu \cap Y})_{U_\nu}$ for all ν. The complex space X is said to be pseudoconcave if there exist an open subset $Y \Subset X$ such that Y is pseudoconcave at every boundary point $y_o \in \partial Y$.

Let φ be a C^2 function on a complex manifold. We say that φ is strictly q-plurisubharmonic if and only if the Levi form $\sqrt{-1}\partial\bar{\partial}\varphi$ has at least $n - q$ strictly positive eigenvalues. In practice complex manifolds X are shown to be pseudoconcave by constructing exhaustion functions ψ (i.e.$\{\psi < c\} \Subset X$ for any c) such that $-\psi$ is at $(n-2)$-plurisubharmonic outside a compact set. In case $-\psi$ is strictly plurisubharmonic outside a compact set we say that M is a strongly pseudoconcave manifold.

Let now X be a psuedoconcave manifold. We assert

Theorem (4.2). *Siegel's Theorem is true for pseudoconcave manifolds.*

Proof. We get the multiplicity estimate as in the compact case. Let $Y \Subset X$ be an open subset such that Y is pseudoconcave in X at every boundary point $b \in \partial Y$. By using the definition of pseudoconcavity we can get three finite collections of open subsets $U_\alpha, U'_\alpha, U''_\alpha$ of X such that $U_\alpha \Subset U'_\alpha \Subset U''_\alpha, Y \subset \cup U_\alpha, U''_\alpha$ are coordinate Euclidean balls, $L|U''_\alpha$ are holomorphically trivial and such that

$$U_\alpha \subset (\widehat{U_\alpha \cap Y})_{U''_\alpha};$$
$$U'_\alpha \subset (\widehat{U'_\alpha \cap Y})_{U''_\alpha}.$$

We can now apply the proof of Siegel's Theorem for compact complex spaces to the case of pseudoconcave manifolds X by working with the two finite open covers $U_\alpha \cap Y$ and $U'_\alpha \cap Y$ of Y. In applying the argument using the Schwarz lemma we work with $U_\alpha \Subset U'_\alpha \Subset U''_\alpha$ and observe that from definition for any holomorphic function f on U''_α we have

$$\sup_{U_\alpha \cap Y} |f| = \sup_{U_\alpha} |f|;$$
$$\sup_{U'_\alpha \cap Y} |f| = \sup_{U'_\alpha} |f|.$$

We also need the following weak form of the Schwarz lemma for domains.

Schwarz lemma for domains.

Let $\Omega \Subset \Omega'$ be domains in $\mathbf{C}^n$ with $o \in \Omega$. Then, there exist positive constants C and c such that for any bounded holomorphic function f on Ω we have

$$\sup_{\Omega} |f| \leq C \sup_{\Omega'} |f| (\frac{1}{2})^{c \cdot \mathbf{mult}([Zf];o)}.$$

This Schwarz lemma can be easily deduced from the Poisson-Jensen formula on domains. Without loss of generality we may assume that Ω' is a bounded domain with smooth boundary. Let $G(-;-)$ be the (negative) Green Kernel on Ω' for the Dirichlet boundary condition.

Then, for a bounded holomorphic function f on Ω' we have from the Poisson-Jensen formula

$$\log|f(x)| \leq \sup_{\Omega'} \log|f| + \frac{1}{2\pi}\int_{[Zf]} G(x;y)\,dy.$$

Here the integration is performed over the smooth part of $[Zf]$ and dy denotes the Euclidean volume there. The Schwarz lemma follows from this inequality and the fact that the Euclidean volume of $[Zf] \cap \Omega$ dominates $c.\,\mathrm{mult}([Zf];o)$ for some positive constant c, by the Bishop-Lelong inequality (cf. (3.2)).

(4.3) Embedding certain pseudoconcave manifolds

Andreotti-Grauert [AG1] proved that the quotient of the Siegel upper half-plane $H_n = \{n \times n$ symmetric matrices $Z : \mathrm{Im} Z > 0\}$ by the Siegel modular group $\Gamma_o = Sp(n, \mathbf{Z})$ is pseudoconcave for $n \geq 2$. By taking torsion-free subgroups $\Gamma \subset \Gamma_o$ of finite index we obtain examples of pseudoconcave manifolds $X = H_n/\Gamma$ which are pseudoconcave and admit positive line bundles K_X. The result of Andreotti-Grauert [AG2] is obtained by explicit construction of exhaustion functions ψ and showed that X is strongly q-pseudoconcave for some $q \leq n-2$. These functions ψ has the additional property that $-\psi$ is weakly plurisubharmonic. By using Siegel's Theorem for X and using the fact that X admits no rational curves it can be proved that X is biholomorphic to an open subset of a quasi-projective variety. To explain the use of pseudoconcavity in proving embedding theorems we start with the case of strongly pseudoconcave manifolds with the following theorem of Andreotti-Tomassini [AT] :

Theorem (Andreotti-Tomassini [AT]). *Let X be a strongly pseudoconcave manifold admitting a complete Kähler metric ω and a Hermitian holomorphic line bundle (L,h) such that both L and $L \otimes K_X^{-1}$ are positive, i.e., $c(L,h)$ and $c(L,h) + \mathrm{Ric}$ are positive. Then, X is biholomorphic to an open subset (in the Euclidean topology) of some projective-algebraic variety M.*

Proof. The curvature assumptions on (X,ω) and (L,h) implies that one can solve $\bar\partial$ with L^2-estimates to obtain holomorphic sections in $\Gamma(X, L^p), p > 0$. Let $R(X,L)$ be the space of meromorphic functions arising from quotients of $\Gamma(X, L^p)$. From Siegel's Theorem we can "birationally" embed X into some non-singular projective-algebraic variety Z by $F = [s_o, \ldots, s_N]$, say, with $s_o, \ldots, s_N$ belonging to some

$\Gamma(X, L^p), p > 0$. The base point set $B(F)$ of $[s_o, \ldots, s_N]$ is a complex-analytic subvariety of X. Since X is strongly pseudoconcave there exists a $\mathcal{C}^2$ exhaustion function ψ such that $-\psi$ is strictly plurisubharmonic outside some compact subset K of X. By solving $\bar{\partial}$ we can choose F such that $B(F) \cap K = \emptyset$ and F is a holomorphic embedding on some neighborhood of K. Since ψ is an exhaustion function $-\psi$ restricted to any subvariety of X must achieve its maximum at some point. Since $B(F) \cap K = \emptyset$ it follows from the maximum principle for plurisubharmonic functions that $B(F)$ consists at most of isolated points. The same argument shows that there exists at most an isolated set of points $\{x_\nu\}$ such that $F|X - \{x_\nu\}$ is a biholomorphism onto its image in Z. Since Z is non-singular what could happen is that each x_ν is blown up to a divisor P_ν of Z such that the $\{P_\nu\}$ are mutually disjoint. Thus, $\{P_\nu\}$ are disjoint exceptional divisors on Z. By Grauert's criterion for exceptional sets it follows easily that $\{P_\nu\}$ represent $\mathbf{R}$-linearly independent homology classes in $H_{2n-2}(X, \mathbf{R})$. In particular, the set $\{P_\nu\}$ is finite. By adjoining new holomorphic sections we can modify F to obtain an open embedding, proving the theorem.

Remark. In case of surfaces Grauert's criterion asserts that the intersection matrix $\{P_\nu . P_\mu\}$ is negative definite, implying that $\{P_\nu\}$ represent $\mathbf{R}$-linearly independent classes in $H_2(X, \mathbf{R})$. For the general case one can reduce to the 2-dimensional situation by taking hyperplane sections.

Returning to the result deduced from Andreotti-Grauert [AG1] on embedding quotients of the Siegel upper half plane the function $-\psi$ is pseudoconvex outside a compact set K such that the Levi form possesses at least 2 positive eigenvalues outside K. The method stated above using the maximum principle shows that all irreducible components of the base point set $B(F)$ of F (with $B(F) \cap K = \emptyset$) are compact and of codimension ≥ 2 because the restriction of $-\psi$ to the non-singular part of a hypersurface has at least one positive eigenvalue. The same applies to the branching locus of F. Thus we have a discrete set of disjoint compact subvarieties $\{C_\nu\}$ of X of codimension ≤ 2 and disjoint subvarieties P_ν of the non-singular projective variety Z such that $F|X - \cup\{C_\nu\}$ is a biholomorphism onto its image Ω_o and C_ν is transformed to P_ν under the meromorphic map F. One can adjoin $\{P_\nu\}$ to Ω_o to get a domain Ω on X. By using the fact that X admits no rational curves (X being uniformized by a bounded domain) one deduces that F^{-1} extends holomorphically to $F^{-1}: \Omega \to X$ (cf. Fujiki [Fu2, Proposition 1, p.113]). Let $\omega \in P_\nu, x = F(w)$, and $(z_1, \ldots, z_n)$ be holomorphic local coordinates at x. Then, P_ν must identify locally with the zero-set of $(F^{-1})^*(dz_1 \wedge \cdots \wedge dz_n)$, hence of codimension one. On the other hand using pseudoconcavity one can show that there are at most a finite number of P_ν's of codimension one, proving that $\{P_\nu\}$ must be a finite set. One can then modify F to obtain a holomorphic embedding.

Still denoting by $F: X \to Z$ the modified holomorphic embedding from the existence of the plurisubharmonic function $-\psi$ outside some compact subset K of X it follows that the $Z - X$ is pluripolar and hence of zero measure with respect to any Hermitian metric on Z. In particular X is dense in Z. We have the additional property that the field of all meromorphic functions $M(X)$ of X identifies with the field of rational functions of $R(Z)$. By construction we only showed that $R(X, K)$ identifies with $R(Z)$, where $R(X, K)$ is the field of meromorphic functions arising from taking quotients of pluricanonical sections. From Siegel's Theorem for pseudoconcave manifolds $M(X)$ is at most a finite extension field of $R(X, K)$. From this we can get a meromorphic map $F': X \to Z'$ with $(F')^* R(Z') = M(X)$ and a finite meromorphic map $\nu: Z' \to Z$ such that $F = \nu \circ F'$. Since both F' and F are generically injective, and X is dense in Z this is impossible unless ν is birational, i.e., $R(X) = R(X, K)$.

Remark. The preceding arguments are due to Nadel-Tsuji [NT].

(4.4) Scheme for compactifying certain pseudoconcave manifolds of negative Ricci curvature

The principle difficulty in compactifying pseudoconcave manifolds using Siegel's Theorem and positive line bundles is to prove quasi-surjectivity. Following Nadel-Tsuji [NT] we will call a complex manifold very strongly q-pseudoconcave if there exists a $\mathcal{C}^2$ exhaustion function ψ such that $-\psi$ is weakly plurisubharmonic and the Levi form of $-\psi$ has at least q positive eigenvalues. They proved

Theorem (4.4) (Nadel-Tsuji [NT]). *Let (X, ω) be a complete Kähler manifold of dimension n of negative Ricci curvature. Assume that X is uniformized by a Stein manifold and that X is very strongly $(n-2)$-pseudoconcave. Then, X is biholomorphic to a quasi-projective variety.*

As explained in (4.3) we have a holomorphic embedding $F: X \to Z$ of X onto a non-singular projective-algebraic variety Z. We will henceforth identify X with its image under F. To show that $Z - X$ is a subvariety of Z [NT] developed an L^2-Riemann-Roch inequality based on the proof of complex Morse inequalities of Demailly's [De1]. They also use existence theorems for complete Kähler-Einstein metrics on bounded domains of holomorphy and certain quasi-projective varieties. With some oversimplification the proof goes as follows. From the pseudoconcavity of X it follows that $Z-X$ contains at most a finite number of irreducible hypersurfaces D_i of Z. Denoting the union of D_i by D we are going to show that $X = Z-D$. By removing some hypersurface V in Z and using existence theorems for complete Kähler-Einstein metrics one can show that both $W = X-V$ and $W' = (Z-D)-V$ admit Kähler-Einstein metrics of Ricci curvatures $\equiv -1$. Denote the Kähler forms by μ and μ' resp. From the Ahlfors-Schwarz lemma for volume forms we have $\mu^n \geq {\mu'}^n$. To show that $W = W'$ and hence $X = Z - D$ it suffices to show that $\mu^n \equiv {\mu'}^n$. In fact, since Kähler-Einstein metrics are determined by their volume forms this would imply $\mu \equiv \mu'$ and hence $W = W'$. Denote by K the canonical line bundle and by [Del] the divisor line bundle. We relate the volumes of (W,μ) and (W,μ') to asymptotic growth of the dimensions of square-integrable holomorphic sections $\Gamma^2(W,(K+[D]+[V])^p)$ and $\Gamma^2(W',(K+[D]+[V])^p)$. As pseudoconcavity implies extension theorems for holomorphic sections of line bundles from W to W' the asymptotic growths of $\Gamma^2(W,(K+[D]+[V])^p)$ and $\Gamma^2(W',(K+[D]+[V])^p)$ are the same. On the one hand for the quasi-projective variety W' the asymptotic growth of $\Gamma^2(W',(K+[D]+[V])^p)$ is determined by the volume of (W',μ'). On the other hand, the L^2-Riemann-Roch inequality of Nadel-Tsuji gives a lower bound for the asymptotic growth of $\Gamma^2(W,(K+[D]+[V])^p)$ by the volume of (W,μ). Equating the two asymptotic rates yields the inequality $\text{Volume}(X,\mu) \leq \text{Volume}(X',\mu')$. Combined with $\mu^n \geq {\mu'}^n$ this yields the desired identity $\mu^n \equiv {\mu'}^n$ and hence $X = Z - D$.

Remark. Here we used the notation $[D]$ for the divisor line bundle defined by D. We use interchangeably the additive notation $+$ and the multiplicative notation $\otimes$ for tensor products of holomorphic line bundles.

(4.5) Existence theorems on complete Kähler-Einstein metrics on non-compact manifolds

On any complex manifold M there is at most one complete Kähler-Einstein metric of Ricci curvature -1. This follows from the Ahlfors-Schwarz lemma on volume forms (Yau [Yau2]) which asserts in particular that any holomorphic mapping between complete Kähler-Einstein manifolds of Ricci curvature -1 is volume-decreasing. As a consequence the volume forms of such complete Kähler-Einstein metrics on X are identical. Since a Kähler-Einstein metric is uniquely determined by its volume form, this proves the uniqueness assertion. In particular, such metrics are invariant under (biholomorphic) automorphisms and descend to any quotient of M by a torsion-free discrete group of automorphisms.

We explain in this section existence of the complete Kähler-Einstein metrics of negative Ricci curvature for two classes of non-compact manifolds: bounded domains of holomorphy and certain quasi-projective varieties. Based in Yau's solution of the Calabi conjectures [Yau3] on compact Kähler manifolds Cheng-Yau [CY2] studied the problem of constructing complete Kähler-Einstein metrics (of negative Ricci curvature) on bounded domains of holomorphy which can be thought of as generalizations of the Poincaré metrics in one dimension. They introduced the notion of complete Kähler manifolds of bounded geometry and developed the continuity method for the Monge-Ampère equation on such manifolds. They proved in particular the existence of such metrics on bounded weakly pseudoconvex domains with C^2 boundary. In Mok-Yau [MY] the results are completed to yield:

Theorem (4.5) (Cheng-Yau [CY2] and Mok-Yau [MY]). *Let $\pi: \Omega \to \mathbf{C}^n$ be a Riemann domain with $\pi(\Omega)$ bounded. Then, there exists a unique complete Kähler-Einstein metric of constant negative Ricci curvature -1.*

The converse to this statement is also true ([MY]). We prove that if a bounded Riemann domain supports a complete Kähler metric of bounded negative Ricci curvature, it must satisfy the Kontinuitätssatz and hence be a Riemann domain of holomorphy.

In Cheng-Yau [CY2], by exhausting Ω by strongly pseudoconvex domains with smooth boundary Ω_k and taking limits of complete Kähler-Einstein metrics it was already shown that the limiting metric exists and is Kähler-Einstein. The missing point was the proof of completeness. In Mok-Yau [MY] the completeness was deduced from a lower estimate for the Kähler-Einstein volume forms on Ω_k and using the gradient estimate of Yau [Yau1] on complete Riemannian manifolds. As we will be using the lower estimate of volume forms later on, we sketch here a proof based on the maximum principle for the Monge-Ampère operator.

Proposition. *Let D be a bounded strictly pseudoconvex domain with smooth boundary in $\mathbf{C}^n$ and $g = (g_{i\bar{j}})$ be the complete Kähler-Einstein metric of constant negative Ricci curvature -1. Let $\delta(x) = \delta(x, \partial D)$ denote the Euclidean distance to the boundary. Assume that the diameter of D is less than $1/e$ (so that $-\log\delta \geq 1$). Then there* *exists a constant C depending only on n such that*

$$\det(g_{i\bar{j}}) \geq \frac{C}{\delta^2(-\log\delta)^2}.$$

Proof. To motivate the proof consider the case of $n = 1$. For the case of D being a punctured disc the Poincaré metric is exactly a constant multiple of $\frac{|dz|^2}{\delta^2(-\log\delta)^2}$ near the puncture. By putting D into punctured discs centered at each boundary point b on ∂D and applying the Ahlfors-Schwarz lemma this yields the desired estimate. In case of higher dimensions this breaks down because a punctured ball $B^n - \{o\}, n \geq 2$ cannot support a complete Kähler-Einstein metric. However, one can prove the proposition in case of dimension 1 directly by using the maximum principle for the Laplace operator. This proof can be generalized to higher dimensions by using the maximum principle for the Monge-Ampère operator and the Theorem of Oka which asserts that $-\log\delta$ is plurisubharmonic for a domain of holomorphy. To be precise the function $u = \log\det(g_{i\bar{j}})$ satisfies the Monge-Ampère equation

$$\det(u_{i\bar{j}}) = e^u.$$

On the other hand for the function $\nu = \log\left(\frac{1}{\delta^2(-\log\delta)^2}\right) + |z|^2$ we have

$$\begin{aligned}&\sqrt{-1}\partial\bar{\partial}\left(-2\log\delta - \log(-\log\delta) + |z|^2\right)\\ =&2\sqrt{-1}\partial\bar{\partial}(-\log\delta) - \frac{2\sqrt{-1}\partial\bar{\partial}(-\log\delta)}{-\log\delta} + \frac{2\sqrt{-1}\partial\delta\wedge\bar{\partial}\delta}{\delta^2(-\log\delta)^2} + \sqrt{-1}\partial\bar{\partial}|z|^2.\end{aligned}$$

From Oka's Theorem $\sqrt{-1}\partial\bar{\partial}(-\log d) \geq 0$. Since $-\log\delta \geq 1$ the sum of the first two terms is nonnegative. From $\|d\delta\| = 1$ almost everywhere, taking the n-th exterior power and replacing v by $v+c$ for some constant c we have the Monge-Ampère inequality

$$\det(v_{i\bar{j}}) \geq e^v$$

almost everywhere. By the construction (on strictly pseudoconvex domains) of [CY2] $u = \infty$ on ∂D. Comparing u and v and applying the maximum principle for the Monge-Ampère operator (modulo smoothing arguments) we obtain $u \geq v$, proving the proposition. The function u is bounded from below and satisfies the differential equation $\Delta u = -n$. One can then apply the gradient estimate of Yau [Yau1] to get

$$\|\nabla(\log u)\| \leq C_n$$

for some constant C_n depending only on the dimension n. From the proposition $\log u$ blows up on ∂D. The gradient estimate then implies that the Kähler-Einstein metric on D is complete with a lower estimate for geodesic distances independent of D. This allows one to pass to limit to prove the theorem of Mok-Yau [MY] that the Kähler-Einstein metrics constructed in Cheng-Yau [CY2] on bounded domains of holomorphy Ω are indeed complete.

For the purpose of generalizing to manifolds the use of the maximum principle can be reformulated in term of the following general Ahlfors-Schwarz lemma for volume forms:

Ahlfors-Schwarz lemma for volume forms (Mok-Yau [MY, (1.1), p. 43ff.]). *Let M be an n-dimensional complete Hermitian manifold with scalar curvature bounded from below by $-K$ and let N be a complex manifold of the same dimension with a volume form (i.e., a positive (n,n)-form) V_N such that the Ricci is negative definite and such that, writing $V_N = V_N^o(\sqrt{-1}/2)dz^1 \wedge \cdots \wedge dz^n$, we have*

$$\left(\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log V_N^o\right)^n \geq K_2\, V_N.$$

Suppose $f\colon M \to N$ is a holomorphic map and the Jacobian is nonvanishing at some point. Then, $K_1 > 0$ and we have

$$\sup_M \frac{f^*V_N}{V_M} \leq \frac{K_1^n}{n^n K_2}.$$

The continuity method for the Monge-Ampère equation can also be used to prove the existence of complete Kähler-Einstein metrics on certain quasi-projective varieties. Such results were first announced in Yau [Yau3]. In particular, we have

Theorem (Kobayashi [Ko2], Cheng-Yau [CY3] and Tian-Yau [TY]). *Let M be a non-singular projective-algebraic variety and D be a union of smooth divisors with at worst normal crossings such that for the canonical line bundle K_M of M the holomorphic line bundle $K_M + [D]$ admits a Hermitian metric of positive curvature. Then, there exists on $Y = M - D$ a unique complete Kähler-Einstein metric μ of Ricci curvature -1.*

From the method of construction of the Kähler-Einstein metric on Y is uniformly equivalent to the Poincaré metric on punctured polydiscs in neighborhoods of points of D. Moreover, the Ricci form of (W', μ')

represents $c_1(K_M + [D])$ in the sense that the volume form $(\mu')^n$ defines a Hermitian metric on $K_M + [D]$ with logarithmic singularities along D and V. (To be more precise $(\mu')^n$ defines a good metric for $K_M + [D]$ in the sense of Mumford [Mum2], so that as far as Chern integrals are concerned, $(\mu')^n$ is as good as a smooth metric on $K_M + [D]$.) In particular, if h is a smooth Hermitian metric for $K_M + [D]$ over M and c_1 denotes first Chern classes, then

$$\int_M c_1^n(M, h) = \int_{M-D} c_1^n(M - D, \mu) = \int_{M-D} \left(\frac{\mu}{2\pi}\right)^n .$$

We will write for short $K_M + [D] > 0$ for the sufficient condition in the theorem, which is in general not necessary. In Kobayashi [Ko1] and Cheng-Yau [CY3] they also investigated in case of dimension 2 weaker sufficient conditions in terms of algebro-geometric or differential-geometric notions of semipositivity of the line bundle $K_M + [D]$. Such conditions for arbitrary dimensions were investigated by Tsuji [Ts] in certain special cases and Tian-Yau [TY] in broader generality. However, there is still a large gap between necessary and sufficient conditions for the existence of complete Kähler-Einstein metrics on $Y = M - D$.

We return now to the theorem of Nadel-Tsuji [NT]. Recall in the notations of (4.4) that the pseudoconcave manifold X is already embedded as an open subset of a non-singular projective-algebraic variety Z. Let D be the union of all hypersurfaces in $Z - X$. Without loss of generality we may assume that D is a union of smooth divisors with at worst normal crossings. Since $K_Z + [D]$ may not be positive to construct complete Kähler-Einstein metrics we have to work with $W' = (Z - D) - V$ obtained by removing some positive hyperplane sections so that $D \cup V$ is still a union of smooth divisors with at worst normal crossings. Writing $W = X - V \subset W'$ we need for the scheme of proof of [NT] to guarantee the existence of a complete Kähler-Einstein metric μ on W.

The Kontinuitätssatz for complex manifolds

To proceed further we introduce the Kontinuitätssatz for complex manifolds. We have

Definition. A connected complex manifold M is said to verify the Kontinuitätssatz if and only if for any smooth one-parameter family of closed holomorphic discs Δ_t with boundary indexed by the smooth parameter $t \in [0,1)$ such that $\cup_{0\leq t<1}\partial\Delta_t \Subset M$ we have $\cup_{0\leq t<1}\Delta_t \Subset M$.

One can also define a discrete version of the Kontinuitätssatz by using a discrete sequence of holomorphic discs with boundary in place of a one-parameter family. In case of domains Ω in $\mathbf{C}^n$ this definition appears to be stronger than Oka's. However, by Oka's Theorem (3.7.1) the two notions are equivalent. In fact, it follows from Oka's Theorem that a domain Ω in $\mathbf{C}^n$ is a domain of holomorphy if if and only if it satisfies the discrete version of the Kontinuitätssatz. Here and henceforth by the Kontinuitätssatz we will always mean the smooth one-parameter version.

As a consequence of Oka's Theorem Docquier-Grauert [DG] proved that a domain Ω in a Stein manifold N is Stein if and only if Ω satisfies the Kontinuitätssatz. Furthermore, Stein [St] proved using Oka's Theorem that if $\nu: M' \to M$ is a holomorphic covering map between complex manifolds such that M is Stein, then M' is also Stein. By using plurisubharmonic exhaustion functions and the maximum principle it follows readily that a Stein manifold verifies the Kontinuitätssatz. As an immediate consequence of the definition, if $\nu: M' \to M$ is a holomorphic covering map between complex manifolds, then M verifies the Kontinuitätssatz if M' does. Furthermore, if M verifies the Kontinuitätssatz and V is a complex-analytic hypersurface in M, then $M - V$ also verifies the Kontinuitätssatz. However, the converse is not true.

Existence of complete Kähler-Einstein metrics on some $X - V = W$

To make use of the theorem of Mok-Yau [MY] we use a theorem of Griffiths [Gri3] on finding quasi-projective varieties uniformized by a bounded domain of holomorphy by the method of simultaneous uniformization (Ahlfors [Ahl] and Bers [Be]). Griffiths' result implies that we can choose V with the additional property that $W' = (Z - D) - V$ is uniformized by a bounded domain of holomorphy Ω'. Write $\rho: \Omega' \to W' \supset W$ for the universal covering map and let Ω be a connected component of $\rho^{-1}(W)$. Since X is uniformized by a Stein manifold it verifies the Kontinuitätssatz, so does $W = X - V$ obtained by removing a hypersurface. Since $W \subset W'$ is an open set of the affine-algebraic, hence Stein manifold W', W is Stein by Docquier-Grauert [DG]. As $\rho|_\Omega: \Omega \to W$ is a holomorphic covering map, by Stein [St] $\Omega \subset \Omega' \subset \mathbf{C}^n$ is a bounded domain of holomorphy, so that by [MY] it admits a complete Kähler-Einstein metric of Ricci curvature -1. Replacing Ω by its universal covering $\widetilde{\Omega}$ and using the invariance of such metrics by automorphisms this induces a complete Kähler-Einstein metric on $W = X - V$.

(4.6) L^2-Riemann-Roch inequality of Nadel-Tsuji

To formulate the L^2-Riemann-Roch inequality of Nadel-Tsuji [NT] we start with Demailly's generalization [De2] of Weyl's formula for the asymptotic spectrum for sections of powers of a Hermitian holomorphic line bundle (L, h) on a Hermitian complex manifold (M, θ). Let Ω be an open subset of X. Denote the formal adjoint of $\bar{\partial}$ on L^ν-valued $(0, q)$ forms (with respect to the Hermitian metrics ω and h^ν) by $\bar{\partial}^*$. Let $Q_\Omega^{(0,q)}$ denote the quadratic form on smooth compactly supported L^ν-valued $(0, q)$-forms on Ω defined by $Q_\Omega^{(0,q)}(f) = \int_\Omega \|\bar{\partial} f\|^2 + \|\bar{\partial}^* f\|^2$. Let $\mathcal{H}_\Omega^{(0,q)}$ denote the Hilbert space of square-integrable L^ν-valued $(0, q)$-forms. Extend the domains of $\bar{\partial}$ by taking closure in the graph norm $f \to (\|f\|^2 + \|\bar{\partial} f\|^2)^{\frac{1}{2}}$ and extend the domain of $\bar{\partial}^*$ by taking the Hilbert space adjoint. Thus, $Q_\Omega^{(0,q)}$ is extended to $\mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$. Let $\Box$ denote the box operator $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$. When Ω is a bounded domain with smooth boundary, the spectrum is discrete. We denote by $N_\Omega^{(0,q)}(\mu; L^\nu)$ the number of eigenvalues of $\Box$ less than μ, counted with multiplicities. In this case Demailly [De2] obtained asymptotic inequalities in terms of ν for counting $N_\Omega^{(0,q)}(\nu\lambda; L^\nu)$ when $\lambda > 0$. We will only need the case of $q = 0$. We write $N_\Omega(\mu; L^\nu)$ for $N_\Omega^{(0,0)}(\mu; L^\nu)$. In this case we have

Asymptotic lower estimate for $N_\Omega(\nu\lambda; K_M^\nu)$ (consequence of Demailly [De2])

For $\lambda > 0, c_1(K_M)$ denoting the first Chern form of the canonical line bundle K_M on M induced by the Hermitian metric θ and Ω denoting a bounded domain in M with smooth boundary , we have

$$\liminf_{\nu \to \infty} \frac{N_\Omega(\nu\lambda; K_M^\nu)}{\nu^n} \geq \frac{1}{n!} \int_\Omega c_1^n(K_M).$$

We give a very brief explanation of the methods of the proof. For details we refer the reader to Demailly [De2] and the Bourbaki lecture of Siu [Siu8]. First, by the Bochner-Kodaira formula for Hermitian manifolds, writing $D\nu = \nabla\nu + \overline{\nabla}\nu$ for the full gradient, i.e., the sum of the $(1,0)$-gradient and $(0,1)$-gradient for L^ν-valued $(0, q)$ forms f with respect to the Hermitian metrics h^ν and θ, we have for $f \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$

$$\int_\Omega \|\bar{\partial} f\|^2 + \|\bar{\partial}^* f\|^2 = \int_\Omega \|D_\nu f + Sf\|^2 + \nu K(f, f) + R(f, f) + T(f, f)$$

where K, R, T are Hermitian forms arising from the curvatures of (L, h) and (M, θ) and the torsion of (M, θ) resp., while S is a zero-order operator arising from the torsion of (M, θ). In case of $q = 0$, in terms of local coordinates, if Γ denotes the connection 1-form of (L,h) and D denotes the the full gradient for the flat connection, then $D_\nu f = Df + \nu\Gamma \otimes f$.

To count $N_\Omega^{(0,q)}(\nu\lambda; L^\nu)$ asymptotically one uses a localization procedure and the minimax principle. The localization procedure is analogous to the method initiated by Weyl [Weyl] on the asymptotic distribution of eigenvalues of the Laplace operator. One uses coverings of Ω with sets of diameter $r(n) \to 0$ as $\nu \to \infty$ in a specific way. The problem can be formulated in terms of real coordinates $(x_1, \ldots, x_{2n})$ using normal geodesic coordinates. Rescaling one is led to study the case of the trivial line bundle on cubes $P(R)$ of side R on $\mathbf{R}^{2n}$ equipped with a Hermitian unitary connection of constant curvature B such that the (purely imaginary) connection form $\sqrt{-1}A$ is chosen to have linear coefficients. (We have $dA = -\sqrt{-1}B$.) If the (skew-symmetric) curvature form $B = \sum_{1\le j\le s} B_j dx^j \wedge dx^{n+j}$ for constants B_j we can choose the connection form $A = \sum_{1\le j\le s} \sqrt{-1}B_j dx^{n+j}$. By rescaling we are led to study the quadratic form on functions

$$
\begin{aligned}
&Q_{P(R)}(u) \\
&= \int_{P(R)} \sum_{j=1}^{s} \left(\left|\frac{\partial u}{\partial x_j}\right|^2 + \left|\frac{\partial u}{\partial x_{j+s}} + \sqrt{-1}B_j x_j u\right|^2 \right) + \sum_{j>2s} \left|\frac{\partial u}{\partial x_j}\right|^2 + V|u|^2.
\end{aligned}
$$

Here V is a constant potential and $R \to \infty$ as $\nu \to \infty$. The potentials V arise from the quadratic form K above in the Bochner-Kodaira formula and has the effect of translating the spectrum by V. In the rescaling the effects of the torsion and the curvature of (M,θ) (i.e., S, T and R) become negligible since they are constant independent of ν, while both the connection form and the curvature forms are proportional to ν. Performing Fourier transforms on the variables x_{j+s}, $1 \le j \le s$, and making linear changes of variables, one is led in the limiting case as $\nu \to \infty$ (i.e.,$R \to \infty$) to comparisons (using the minimax principle) with the quadratic form

$$P(u) = \int_{-\infty}^{\infty} \left|\frac{du}{dx}\right| + x^2|u|^2$$

and the corresponding second order ordinary differential operator

$$H(u) = \frac{d^2u}{dx^2} + x^2 u,$$

which is the Schroedinger operator for the harmonic oscillator. The eigenvalues of H, which are discrete, and the corresponding eigenfunctions are both explicitly known. This information allows us to count asymptotically the eigenvalue distribution of $Q_{P(R)}(u)$ as $R \to \infty$. Denoting by $N_{Q_{P(R)}}(\lambda)$ the number of eigenvalues $\leq \lambda$ we have in fact an exact formula for $\lim\limits_{R\to\infty} R^{-2n} N_{Q_{P(R)}}(\lambda)$. The localization procedure and the minimax principle then allows us to get the asymptotic Morse inequalities of Demailly, in particular the estimate stated for the line bundle $L = K_M$.

Deduction of the L^2-Riemann-Roch inequalities from Demailly's estimate

We state the L^2-Riemann-Roch inequality of Nadel-Tsuji [NT].

Proposition. *In the notations of the preceding paragraphs we have*

$$\liminf_{\nu\to\infty} \frac{\dim \Gamma^2(M, K_M^\nu)}{\nu^{-n}} \geq \frac{1}{n!} \int_M c_1^n(K_M).$$

In order to get the L^2-Riemann Roch inequality from Demailly's estimates we exhaust the complete Kähler manifold M by a sequence of smooth bounded domains with smooth boundary. From the minimax principle it follows that for $\lambda > 0$, $N_\Omega(\nu\lambda; L^\nu) \leq N_M(\nu\lambda; L^\nu)$. Applying Demailly's estimate to the exhausting sequence we obtain the inequality that for every $\lambda > 0$, we have

$$\liminf_{\nu\to\infty} \frac{\dim N_M(M, K_M^\nu)}{\nu^{-n}} \geq \frac{1}{n!} \int_M c_1^n(K_M).$$

To prove the proposition it suffices to show that there exists some $\lambda > 0$ such that

$$N_M(\nu\lambda; L^\nu) = \dim \Gamma^2(M, K_M^\nu)$$

for ν sufficiently large. This is in fact true for $0 < \lambda < 1/2$. First recall the Bochner-Kodaira formula for (0, 1)-forms

$$(\Box f, f) = \|\bar\nabla f\|^2 + (\nu - 1)\operatorname{Ric}(f, f).$$

From this it follows readily that $Q_M^{(0,1)}(f) \geq (\nu-1)\|f\|^2$, so that for $\nu \geq 2, Q_M^{(0,1)}(f) \geq \frac{\nu}{2}\|f\|^2$. Consider now the case of sections u. We have

$$(\Box u, \Box u) = (\bar\partial^*\bar\partial u, \bar\partial^*\bar\partial u)$$

since $\bar\partial^*$ is zero for functions. On the other hand, since $\bar\partial(\bar\partial u) = 0$, $(\bar\partial^*\bar\partial u, \bar\partial^*\bar\partial u)$ is the same as $Q_M^{(0,1)}(\bar\partial u)$ for the $(0,1)$ form $\bar\partial u$. Thus, for

$\nu \geq 2$

$$(\Box u, \Box u) = Q_M^{(0,1)}(\bar{\partial} u) \geq \frac{\nu}{2}\|\bar{\partial} u\|^2 = \frac{\nu}{2}(\Box u, u).$$

In particular, if u is an eigensection of K_M^ν with $\Box u = \mu u$ we have $\mu \geq \frac{\nu}{2}$. This proves our assertion and establishes the L^2-Riemann-Roch inequality.

We have now the necessary ingredients for completing the proof of Theorem (4.4) of Nadel-Tsuji. Recall that we have $W \subset W' \subset Z - D$ with $W = X - V$ affine-algebraic and $W' = (Z - D) - V$ and complete Kähler-Einstein metrics μ and μ' on W and W' resp. Let σ be an L^2-holomorphic section of K_W on (W, μ). We claim that σ extends to a meromorphic section of K_Z on Z. First, by the Ahlfors-Schwarz lemma on volume forms the Kähler-Einstein metric μ is dominated the Poincaré metric on punctured polydiscs near the divisors D and V, so that σ extends meromorphically across D and V with poles of order at most $\nu - 1$ along D and V. Let now τ be any meromorphic section of K_Z on Z. Then σ/τ is a meromorphic function on W which extends meromorphically across V to X. By our choice of embedding we know on the other hand from (4.3) that every meromorphic function on X extends to the compactification Z. This implies that σ can be meromorphically extended to Z. More precisely, we have the inclusion

$$\Gamma^2(W, K_W^\nu) \subset \Gamma(Z, K_Z \otimes [D]^{\nu-1} \otimes [V]^{\nu-1}) \subset \Gamma(Z, (K_Z + [D] + [V]^\nu).$$

Since the metric μ is Kähler-Einstein with constant Ricci curvature -1, by the L^2-Riemann-Roch inequality, we have

$$\frac{1}{n!}\int_W \left(\frac{1}{2\pi}\mu\right)^n \leq \liminf_{\nu\to\infty} \frac{\dim \Gamma^2(W, K_W^\nu)}{\nu^n}.$$

On the other hand, on the compact manifold $K_Z + [D] + [V]$ is a positive line bundle so that from Riemann-Roch and the Kodaira Vanishing Theorem we have

$$\lim_{\nu\to\infty} \frac{\dim \Gamma(Z, (K_Z + [D] + [V])^\nu}{\nu^n} = \frac{1}{n!}\int_Z c_1^n(K_Z + [D] + [V]),$$

where the first Chern form on the right is computed with respect to any choice of Hermitian metric. From the discussion in (4.5) we have

$$\int_w \left(\frac{1}{2\pi}\mu\right)^n \leq \int_Z c_1^n(K_Z + [D] + [V]) = \int_{w'} \left(\frac{1}{2\pi}\mu'\right)^n.$$

Since $W \subset W'$ by the Ahlfors-Schwarz lemma on volume forms $\mu^n \geq \mu'^n$. By assumption there exists a plurisubharmonic function $-\psi$ on $X - K$

for some compact subset K such that $-\psi$ converges to $-\infty$ along $\partial X = Z - X$, so that ∂X is pluripolar, in particular of measure zero with respect to any Riemannian metric on Z. Since $W' - W \subset Z - X$ it follows from $\mu \geq \mu'$ and the inequality on volumes that in fact $\mu^n \equiv \mu'^n$. As Kähler-Einstein metrics are determined by their volume forms we have in fact $\mu \equiv \mu'$. Since (W, μ) is complete this can only happen if $W = W'$, i.e., $X = Z - D$. The proof of the Theorem of Nadel-Tsuji [NT] is completed.

Remarks. 1) In the Theorem of Nadel-Tsuji [NT] the volume of the Kähler metric is not assumed to be finite. If one assumes in addition that the Ricci form is bounded from above by a negative constant it follows from the theorem and the Ahlfors-Schwarz lemma for volume forms that the volume is finite.
2) The negativity of the Ricci curvature was only used in solving $\bar{\partial}$ with L^2-estimates to construct holomorphic sections.

(4.7) A local compactification theorem on bounded domains

In this section we discuss a characterization of "local Zariski-density" on bounded domains of holomorphy in terms of the complete Kähler-Einstein metric. Suppose D is a domain of holomorphy and S is a complex-analytic hypersurface. Then, $\Omega = D - S$ is a bounded domain of holomorphy. Let $b \in \partial\Omega$ and $U \Subset D$ be a neighborhood of b in D. Let ω denotes the Kähler-Einstein form on Ω. By desingularizing S and using the Ahlfors-Schwarz lemma for volume forms it follows readily that Volume$(\Omega \cap U, \omega) < \infty$. We prove the converse

Theorem (4.7) (Mok [Mok11]). *Let (Ω, ω) be a bounded domain of holomorphy in $\mathbf{C}^n$ equipped with the canonical complete Kähler-Einstein metric on Ω of Ricci curvature -1. Suppose b is a point of $\partial\Omega$ such that for some open neighborhood U of b in $\mathbf{C}^n, \Omega \cap U$ is of finite volume with respect to ω. Then, $S = U - \Omega$ is a complex-analytic subvariety of U of pure codimension* 1.

The proof of Theorem (4.7) is based on the estimate of the Kähler-Einstein volume form in (4.5), an extension theorem for closed positive (1, 1)-currents and a criterion for the complex-analyticity of a closed set using plurisubharmonic functions due to Bombieri [Bom1]. One reason for discussing this elementary local compactification theorem is to give a motivation for techniques of proving quasi-surjectivity of embedded Kähler-Einstein manifolds. Moreover, as will be explained later, one can also use Theorem (4.7) to give a more immediate deduction of the Theorem of Nadel-Tsuji [NT] from the L^2-Riemann-Roch inequality. Concerning the criterion of Bombieri [Bom1] we have

Characterization of complex-analyticity in terms of plurisubharmonic functions

Proposition. *Let Ω be a bounded domain of holomorphy and φ be a plurisubharmonic function on Ω. Then, there exists a non-trivial holomorphic function f on Ω square-integrable with respect to the weight function $e^{-\varphi}$. As a consequence, the set E of points $x \in \Omega$ at which $e^{-\varphi}$ is not locally summable lies on the complex-analytic subvariety Z consisting of zeros of the holomorphic function f.*

Before explaining Bombieri's result it is convenient to introduce at this point the notion of the Lelong number. Let T be a closed positive (p,p)-current on the Euclidean ball $B(o;r_o)$. We defined the mass of T on a subset E of $B(o;r_o)$ with respect to a Kähler metric ν by integrating $\nu^p/p! \wedge T$ over E and denote this by $\mathrm{Mass}(T|E;\nu)$. When $T = [V]$ is the integration current over the non-singular part of a complex-analytic subvariety V of pure codimension p, the mass $\mathrm{Mass}(T|E;\nu)$ is the same as $\mathrm{Volume}([V] \cap E;\nu)$. Let now β denote the Kähler form of the Euclidean metric. For the mass $\mathrm{Mass}(T|B(o;r);\beta)$ we have the monotonicity of

$$\frac{\mathrm{Mass}(T|_{B(o;r)};\beta)}{C_p r^{2p}} = \Theta(T;r;o),$$

where the constant C_p is chosen to be the volume of the unit ball in $\mathbf{C}^p$. More precisely, we have, for $r_2 > r_1 > 0$ the Fubini-Study metric σ on $\mathbf{P}^{n-1}$ with potential $(1/2\pi)\log\|z\|^2$ when lifted to $\mathbf{C}^n - \{0\}$, we have

$$\Theta(T;r_2;o) - \Theta(T;r_1;o) = \mathrm{Mass}(T|_{\overline{B(o;r_2)}-B(o;r_1)};\sigma).$$

The nonincreasing limit of $\Theta(T;r;o)$ as $r \to 0$ is defined to be the Lelong number $n(T;o)$ of T at o, which in case of the subvariety $[V]$ reduces to the multiplicity (Thie [Th]). The Bishop-Lelong inequality of (3.3) generalizes to

$$\mathrm{Mass}(T|_{B(o;r)};\beta) \geq C_p r^{2p} \cdot n(T;o).$$

In case of closed positive $(1,1)$-currents T we can write locally $T = \sqrt{-1}\partial\bar{\partial}\varphi$. The concentration of mass of T near a point is then reflected by the local summability or non-summability of the functions $e^{-c\varphi}$, for c a positive constant.

We return now to the criterion of Bombieri [Bom1]. In case φ is at least smooth at one point, the proof of this proposition is an immediate consequence of the technique of constructing holomorphic functions using singular plurisubharmonic weight functions as discussed in (3.2), which is a consequence of the L^2-estimate of $\bar{\partial}$. What we need is that $e^{-\varphi}$ is at least locally summable at one point x_o so that we can solve $\bar{\partial}$ with singular weights at x_o. In Bombieri [Bom1] it was proved that *a priori* the set of points at which $e^{-\varphi}$ is not locally summable is at most a closed subset with Hausdorff $(2n-1)$-dimensional measure zero. This was done by relating the local summability of $e^{-\varphi}$ to the concentration of mass of $T = \sqrt{-1}\partial\bar{\partial}\varphi$ to Euclidean balls. We have in fact (Bom [Bom1, 2])

Proposition. *There exist two positive numbers $\gamma_1 < \gamma_2$ depending only on n such that the following is valid: Suppose φ is a plurisubharmonic function on an open subset Ω of $\mathbf{C}^n, T = \sqrt{-1}\partial\bar{\partial}\varphi$ and $x \in \Omega$. We have*

(i) *If $n(T;x) < \gamma_1$, then $e^{-\varphi}$ is locally summable at x.*

(ii) *If $n(T;x) > \gamma_2$, then $e^{-\varphi}$ is not locally summable at x.*

To find the open subset $\Omega_o \subset \Omega$ on which $e^{-\varphi}$ is locally summable it suffices to show that the set $E_{\gamma_1}(T) \subset \Omega$ consisting of points of Lelong number $\geq \gamma_1$ is of zero $(2n-1)$-dimensional Hausdorff measure. This follows readily from the monotonicity of $\Theta(T;r;x)$ in r and a covering argument.

We return now to the proof of the local compactification theorem. Let ω denote the Kähler form of the complete Kähler-Einstein metric of Ricci curvature -1 on the bounded domain Ω. Let $b \in \partial\Omega$ and U be as in the theorem. We prove that $\omega|_{U\cap\Omega}$ extends trivially as a closed positive (1, 1)-current T to U. Choosing U to be a Euclidean ball we can solve the Poincaré-Lelong equation $T = \sqrt{-1}\partial\bar{\partial}\varphi$ to obtain a potential φ for T on U. On the other hand, writing $(g_{i\bar{j}})$ for the Kähler-Einstein metric, we have $\sqrt{-1}\partial\bar{\partial}\log\det(g_{i\bar{j}}) = \omega$ on $U \cap \Omega$. The difference $h = \varphi - \log\det(g_{i\bar{j}})$ is a pluriharmonic function on $U \cap \Omega$. φ is bounded from above on U while by the volume estimate of the Kähler-Einstein metric given in (4.5) we know that the function $\log\det(g_{i\bar{j}})$ blows up along $U \cap \partial\Omega$. It follows that h can be extended to a plurisubharmonic function on U by defining h to be $-\infty$ on $U \cap \partial\Omega$. Moreover from the estimate

$$\det(g_{i\bar{j}}) \geq \frac{C}{\delta^2(-\log\delta)^2}.$$

of (4.5) we deduce readily that $e^{-(n+1)h}$ is not locally summable at every point of $U \cap \partial\Omega$. It follows therefore by Bombieri's criterion that $U \cap \partial\Omega$ is contained in a complex-analytic subvariety S of U. Since Ω is a domain of holomorphy it forces $U \cap \partial\Omega$ itself to be a complex-analytic hypersurface.

To prove the extension of the closed positive (1, 1) form $\omega|_{U\cap\Omega}$ as a current to U we need the following

Ahlfors-Schwarz lemma for metrics (Yau [Yau2] , Chen- Cheng-Lu [CCL] , Royden [Roy]). *Let (M, ω_M) be a complete Kähler manifold of Ricci curvature bounded from below by a negative constant $-K_1$. Let (X, ω_X) be a Kähler manifold with holomorphic sectional curvature bounded from above by a negative constant $-K_2$. Then, there exists a constant C depending only on K_1, K_2 and the dimension of M such that for any holomorphic mapping $f: M \to X$, we have*

$$f^*\omega_X \le C\omega_M.$$

The significance of this Ahlfors-Schwarz lemma is that we can control the metric (M, ω_M) without any condition on the full curvature. Applying to the situation where $(M, \omega_M) = (\Omega, \omega)$ and $(X, \omega_X) = (B, \omega_B)$ is a larger Euclidean ball equipped with the Poincaré metric, we have the estimate that $(g_{i\bar{j}}) \ge cd_{i\bar{j}}$ for some positive constant c. It follows from this and Volume$(U \cap \Omega; \omega) < \infty$ that in fact the trace $\sum g_{i\bar{i}}$ and hence the coefficients $g_{i\bar{j}}$ of ω are integrable on $U \cap \Omega$. We can therefore define the trivial extension T of $w|_{U\cap\Omega}$ to U. To prove the closedness of T we use the same Ahlfors-Schwarz lemma. On the complete Kähler manifold (Ω, ω) one can construct smooth cut-off functions ρ_R such that $\|\nabla\rho_R\| \le 1/R$ and such that $\rho_R \equiv 1$ on increasing large concentric geodesic balls of radius R. It suffices to show that $d(\rho_R\omega)$ converges weakly to zero on U as a current. Let now ψ be a smooth $(2n-3)$-form with compact support on U. We have to show

$$\lim_{R\to\infty} \int_{U\cap\Omega} d(\rho_R) \wedge \psi = 0.$$

To do this we measure everything in the integrand in terms of the Kähler-Einstein metric ω. From $(g_{i\bar{j}}) \le (1/c)\delta_{i\bar{j}}$ it follows that $\|\psi\|$ is bounded. From $\|\nabla\rho_R\| \le 1/R$ it follows immediately that the integrand is bounded by C/R for some positive constant C. As Volume$(U \cap \Omega, \omega) < \infty$ this proves (*) and hence the closedness of T. The proof of Theorem (4.7) is completed.

We explain how one can deduce the embedding theorem of Nadel-Tsuji [NT] from the L^2-Riemann-Roch inequality and the local compactification theorem. Embed as in (4.5) X as an open subset of some non-singular projective-algebraic variety Z. Take an arbitrary point $b \in$

$\partial X \subset Z$. It suffices to show that for some open neighborhood U of b in $Z, U - X$ is complex-analytic. Take a finite union of hyperplane sections V avoiding b such that $Z - V$ is uniformized by a bounded domain Ω_o. Write $\rho: \Omega_o \to Z - V$ for the universal covering map. Let Ω be a connected component of $\rho^{-1}(X - V)$ and $\tilde{b} \in \partial\Omega$ be such that $\rho(\tilde{b}) = b$. To show that $Z - V$ is complex-analytic at b it suffices to show that $\Omega_o - \Omega$ is complex analytic at $\tilde{b}$. $X - V$ supports a complete Kähler-Einstein metric (with Kähler form) μ of Ricci curvature -1 (cf. (4.5)). Furthermore, it has finite volume because of the L^2-Riemann-Roch inequality and pseudoconcavity. Lifting to Ω via ρ we conclude that for some open neighborhood U of b in Ω_o, Volume$(\Omega, \omega) < \infty$ for the complete Kähler-Einstein metric ω on Ω. $U - \Omega$ is then a complex analytic variety by the local compactification theorem (Theorem (4.7)).

(4.8) Compactifying complete Kähler manifolds of finite volume with pinched strictly negative sectional curvature

As a differential-geometric generalization to the compactification of arithmetic quotient of bounded symmetric domains of rank 1 Siu-Yau proved earlier on

Theorem (4.8) (Siu-Yau [SY4]). *Let (X, ω) be a complete Kähler metric of finite volume such that for some constant C*

$$-C \leq \textit{Sectional curvatures } \leq -1.$$

Then, X is biholomorphic to a Zariski open subset of a non-singular projective-algebraic variety Z such that the complement $Z - X$ is an exceptional set of Z that can be blown down to a finite number of isolated singularities.

The proof of [SY4] uses a combination of differential-geometric techniques and pseudoconcavity. Using Margulis lemma on complete Riemannian manifolds of pinched strictly negative curvature they construct a decomposition of X into a compact piece and a finite union of ends X_i as in the arithmetic rank-1 case. They construct a function φ on X strictly plurisubharmonic outside a compact subset K and diverging to $-\infty$ by using Busemann functions B_γ associated to a geodesic ray γ. Here for a geodesic ray $\{\gamma(r), 0 \leq r < \infty\}$ parametrized by arc length on the universal covering X (which is a Cartan-Hadamard manifold) the Busemann function B_γ is defined by

$$B_\gamma(x) = \lim_{r \to \infty} [d(x; \gamma(r)) - r].$$

They showed that for c sufficiently large, when restricted to $B_\gamma < -c$, the Busemann function descends to Busemann functions B_γ on X. The function φ on X is constructed from the minimum of a finite number of such B_γ. By applying the embedding theorem of Andreotti-Tomassini one can then embed X as an open subset of a non-singular projective-algebraic variety Z. To show that $Z - X$ is a subvariety (necessarily a divisor) of Z that can be blown down to isolated singularities they used the Ahlfors-Schwarz lemma in (4.7). We give here an alternative argument using the local compactification theorem and the Ahlfors-Schwarz lemma. For a generic $c > 0$ sufficiently large the open set $G = \{\varphi < -c\} \cup (Z - X) \subset Z$ is a manifold with smooth strictly pseudoconvex boundary. By Grauert [Gr2] there exists a maximal compact subvariety E of Ω and a proper holomorphic map $\pi: G \to G'$ onto a complex manifold G' which is a biholomorphism on $G - E$ and which blows down E to a finite number of points. We claim that $X \cap G = G - E$. Since X is of curvature ≤ 0 the universal covering manifold $\tilde{X}$ is Stein ([GW2]). As a consequence X verifies the Kontinuitätssatz. To prove $X \cap G = G - E$ it suffices therefore to show:

Proposition. *Let B be a Euclidean open ball centered at the origin o and W be a closed subvariety with at most an isolated singularity at o. Suppose Ω is a Stein open subset of $W - \{o\}$ with $W - \Omega \Subset W$ such that there exists a Kähler metric ω on Ω with Ricci curvature ≥ -1 satisfying* (i) *for a base point x_o the geodesic distances $d(x_o; x) \to \infty$ for $x \to W - \Omega$ and* (ii) Volume$(\Omega, \omega) < \infty$. *Then, $\Omega = W - \{o\}$.*

Remark. The assumption that Ω is a Stein open subset is the same as the assumption that Ω verifies the Kontinuitätssatz, by Oka's Theorem.

To prove the proposition observe that for a fixed compact subset K of W there exists $\delta > 0$ such that for any point $x \in K \cap \Omega$ the geodesic ball $B(x; \delta) = \{y \in \Omega : d(x; y) < \delta\}$ is relatively compact in Ω. Applying the Ahlfors-Schwarz lemma in (4.7) together with the localization argument of Cheng-Yau [CY1] we conclude that there exists a positive number ϵ such that $\omega \geq \epsilon\beta$ for β the Euclidean Kähler form on B. For any point $x \in \Omega$ there exists an open neighborhood U of x in W such that $U \Subset W - \{o\}$ is biholomorphic to a Euclidean ball and hence $U \cap \Omega$ is biholomorphic to a domain of holomorphy. Although $(U \cap \Omega, \omega)$ is not complete the localization argument of Cheng-Yau [CY1] still allows one to get the lower estimate for the volume form of of $(U \cap \Omega, \omega)$ near points of $U - \Omega$ using the Ahlfors-Schwarz lemma for volume forms of (4.5). As in the proof Theorem (4.7) one can then prove the current extension theorem and the fact that $U-\Omega$ is a complex-analytic hypersurface of U.

Remarks. 1) In the proof the assumption that Ω is Stein is in fact unnecessary. One can use the existence of ω to prove that Ω satisfies the Kontinuitätssatz, as in Mok-Yau [MY, Main Theorem, p.42ff.]

2) Knowing that X is strongly pseudoconcave, Theorem (4.8) follows immediately from the proof of the Theorem of Nadel-Tsuji [NT]. In fact, in the proof the [NT] and in the notations of the proof of Theorem (4.8) the complement $D = Z - X$ was shown to be the maximal compact hypersurface in G. From the strict pseudoconcavity of G it follows that D is the maximal compact subvariety of Ω, i.e., the exceptional set E of Ω. The original proof using the Ahlfors-Schwarz lemma given in [SY4] and the one given here are however much more elementary.

(4.9) Siegel's Theorem and Bézout estimates on Kähler manifolds of finite volume

As a differential-geometric generalization of the compactification of arithmetic quotients of bounded symmetric domains Mok [Mok11] and Mok-Zhong [MZ2] considered the question of compactifying complete Kähler manifolds of finite volume under curvature assumptions. We proved in particular

Theorem (4.9.1) (Mok-Zhong [MZ2]). *Let X be a complex manifold with finite even Betti numbers. Let ω be a complete Kähler metric on X of finite volume and negative Ricci curvature. Suppose furthermore that the sectional curvatures are bounded. Then, X is biholomorphic to a quasi-projective variety.*

The topological assumption on X was used in the same way as in the Theorem of Demailly [De1] on characterizing affine-algebraic varieties in the process of desingularizing "birational" embeddings. Without this additional assumption, the result was that there exists a complex-analytic subvariety V of X such that $X-V$ is biholomorphic to a quasi-projective variety. We mention that Theorem (4.9) can already be regarded as a differential-geometric generalization of the compactification of quotients of bounded symmetric domains of finite volume. In fact, we have

Theorem (Gromov et al., cf. Ballmann-Gromov-Schroeder [BGS]). *Let (M,g) be a real-analytic Riemannian manifold such that*

$$-1 \leq \textit{Sectional curvatures } \leq 0.$$

Suppose the injectivity radius $Inj(x)$ converges to zero as x diverges to infinity on M. Then, M is diffeomorphic to compact manifold with smooth boundary.

Let now (X,ω) be a quotient of a bounded symmetric domain of finite volume in the Kähler-Einstein metric ω, which is real-analytic. From the boundedness of the sectional curvature and the finiteness of the volume (X,ω) verifies easily the condition on the injectivity radius. While Theorem (4.9) can be regarded as a generalization of Theorem (4.8) of Siu-Yau in the case when the sectional curvature is pinched and strictly negative, the proof of our result is very different since we do not have at our disposal the complex-analytic property of pseudoconcavity. Under additional assumptions of partial strict negativity of (bi-)sectional curvatures it is not known whether X can be compactified by adding varieties of dimensions $< n-1, n = \dim_{\mathbb{C}} X$. It is not even known whether Theorem (4.8) remains valid under the weaker assumption

$$-C \leq \text{Bisectional curvatures} \leq -1.$$

As in the embedding theorems of Lecture III, we prove Siegel's theorem for a field of meromorphic functions arising from a special class of sections of some line bundles. This part of the estimate is valid without any boundedness assumption on the full curvature tensor. On a complete Kähler manifold (X,ω) we say that $s \in \Gamma(X, K^p)$ is of class N^2 if $\log^+ \|s\|$ is square-integrable. We write $s \in \Gamma_{N^2}(X, K^p)$. We have

Siegel's Theorem for complete Kähler manifolds of inite volume and bounded negative Ricci curvature

Let (X,ω) be a complete n-dimensional Kähler manifold of finite volume such that

$$-1 \leq \text{Ricci curvature } < 0.$$

Let $R_{N^2}(X,K)$ denote the field of meromorphic functions on X obtained by taking quotients of (holomorphic) pluricanonical sections of class N^2. Then, $R_{N^2}(X,K)$ is a finite extension of a purely transcendental extension of $\mathbf{C}$ of transcendental degree n.

Remark. Siegel's Theorem remains valid for any field of meromorphic functions R such that $R \subset R_{N^2}(X,K)$ and such that R contains quotients of square-integrable pluricanonical sections.

Since $\text{Ric}(X,\omega) < 0$ to prove Siegel's Theorem it suffices to prove volume estimates for $s \in \Gamma_{N^2}(X,K^p)$ as in (3.2). We mention that such volume estimates for $\sigma \in \Gamma^2(X,K^p)$ (Γ^α meaning holomorphic and of class L^α) are straight-forward. As in Lecture III to prove Siegel's Theorem we have however to work with a graded algebra of holomorphic sections of powers of some line bundle. $\Gamma_{N^2}(X, K^p); p > 0$ is obviously closed under multiplication.

We proceed to obtain

Volume estimate.

There exists a constant $C > 0$ such that for any positive integer p and for any $s \in \Gamma_{N^2}(X, K^p)$ we have

$$\mathrm{Volume}([Zs]) \leq Cp.$$

Proof of the volume estimate. Let $s \in \Gamma_{N^2}(X, K^p)$ and for $\epsilon > 0$ define u_ϵ by

$$u_\epsilon = \log\left(\frac{\|s\|^2 + \epsilon^2}{\epsilon^2}\right) \geq 0.$$

Write Ric for $\mathrm{Ric}(X, \omega)$. From direct calculation we have $\sqrt{-1}\partial\bar{\partial}u_\epsilon > -p.\mathrm{Ric}(X, \omega)$. Define

$$\zeta_\epsilon = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_\epsilon + p\,\mathrm{Ric} \geq 0.$$

As $\epsilon \to 0$, $\zeta_\epsilon \to [Zs]$ as closed positive currents. Let $\{\rho_R\}$ be the family of cut-off functions on X as in (4.8) with $\|\nabla\rho_R\| \leq C/R$. From Stokes' Theorem we have

$$\int_X d(\rho_R\sqrt{-1}\bar{\partial}u_\epsilon) \wedge \omega^{n-1} = 0.$$

If $\bar{\partial}u_\epsilon$ is integrable we have by letting $R \to \infty$ readily $\int_X \sqrt{-1}\partial\bar{\partial}u_\epsilon = 0$ and thus $\mathrm{Mass}(\zeta_\epsilon; \omega) = Cp$ for some constant C coming from $\int_X \mathrm{Ric} \wedge \omega^{n-1}$. From $\zeta_\epsilon \to [Zs]$ as currents and the Fatou's lemma we have readily the Volume Estimate

$$\mathrm{Volume}([Zs]; \omega) \leq Cp,$$

as desired. We are going in fact to show that $\bar{\partial}u_\epsilon$ is square-integrable (with the integral diverging to ∞ as $\epsilon \to 0$ if s has zeros). To achieve this we perform integration by parts with the formula

$$\int_X d(\rho_R^2 \cdot \sqrt{-1}u_\epsilon\bar{\partial}u_\epsilon \wedge \omega^{n-1}) = 0.$$

Obviously in such an estimate the Levi form $\sqrt{-1}\partial\bar{\partial}u_\epsilon$ reappears. The observation is that it appears with the opposite sign in

$$\begin{aligned}&\rho^2\sqrt{-1}\partial u_\epsilon\wedge\bar{\partial}u_\epsilon\\ =&d(\rho_R^2\sqrt{-1}u_\epsilon\bar{\partial}u_\epsilon)-2\rho_R\sqrt{-1}\partial\rho_R\wedge u_\epsilon\bar{\partial}u_\epsilon-\rho_R^2\sqrt{-1}u_\epsilon\partial\bar{\partial}u_\epsilon.\end{aligned}$$

because by definition $u_\epsilon\geq 0$. It follows therefore from the assumption $s\in\Gamma_{N^2}(X,K^p)$ that u_ϵ is square-integrable. Writing the formula as

$$\begin{aligned}&\rho^2\sqrt{-1}\partial u_\epsilon\wedge\bar{\partial}u_\epsilon\\ =&d(\rho_R^2\sqrt{-1}u_\epsilon\bar{\partial}u_\epsilon)-2\rho_R\sqrt{-1}\partial\rho_R\wedge u_\epsilon\bar{\partial}u_\epsilon-\rho_R^2\sqrt{-1}u_\epsilon\zeta_\epsilon+pu_\epsilon\,\mathrm{Ric},\end{aligned}$$

taking wedge product with ω^{n-1} and integrating over X, we obtain readily from the Cauchy-Schwarz inequality and taking limits as $R\to\infty$ an upper estimate of $\int_X\|\bar{\partial}u_\epsilon\|^2$ showing that it is finite. As explained above, this estimate implies the volume estimate and hence Siegel's Theorem.

As before from Siegel's Theorem we obtain a "birational" embedding $F\colon X\to Z$ of X into a non-singular quasi-projective variety Z. In order to simplify the presentation we consider the situation when F is already a holomorphic embedding. Identifying X with its image $F(X)$, we will show that $Z-X$ is a complex analytic variety. Recall $F\colon X\to Z\subset\mathbf{P}^N$. In addition to $F=F_1$ we can consider $F_k\colon X\to Z\subset\mathbf{P}^{N(k)}$ obtained by composing F with iterations of the Veronese map. If $[z_o,\ldots,z_n]$ are homogeneous coordinates on $\mathbf{P}^n$ then the Veronese map

$$\tau\colon\mathbf{P}^n\to\mathbf{P}^{\frac{n(n+1)}{2}}$$

is defined by $\tau([z_0,\ldots,z_n])=[z_iz_j]_{0\leq i\leq j\leq n}$. The Veronese mapping is a holomorphic embedding.

Let S be a smooth curve on Z obtained by slicing Z by $(n-1)$ hyperplane sections with respect to some embedding F_k in general positions. If X is Zariski-open in Z, then for a generic choice of S, $C=S\cap X$ is a finite Riemann surface obtained by removing form S a finite number of points whose cardinality is bounded independent of the choice of hyperplane sections. We will prove this as a first step in proving that $Z-X$ is complex-analytic. To study C we first prove that the Area of C is finite and bounded independent of the choice of S. By a strengthening of the volume estimate leading to Siegel's Theorem we are able in fact to obtain the following

Bézout estimate.

Let X be as in Theorem (4.8). Let $s_1, \ldots, s_k$ be k sections in $\Gamma^o(X, K^p)$ for some integer $p > 0$ such that they intersect at some point $x \in X$ at normal crossings. Let $[V]$ be irreducible component of $[Zs_1] \cap \cdots \cap [Zs_k]$. Then, there exists a constant C independent of the choice of $s_1, \ldots, s_k$ such that

$$\text{Volume}([V]; \omega) \leq Cp^k.$$

Here $\Gamma^0(X, K^p) = \cup_{\alpha > 0} \Gamma^\alpha(X, K^p)$ and the intersection $[Zs_1] \cap \cdots \cap [Zs_k]$ is counted with multiplicities. In particular, if the multiplicity of $[Zs_1], \ldots, [Zs_k]$ are resp. $m_1, \ldots, m_k$ at x then $[V]$ is counted with multiplicity $m_1 \ldots m_k$.

Naively the idea of proof of the Bézout estimate is to restrict the holomorphic sections s_i inductively to hyperplane sections. If it were possible to show for example that $s_2 | Zs_1$ is of class Γ^o, then $Zs_1 \cap Zs_2 = Z(s_2 | Zs_1)$ can be estimated in the same way as in the volume estimate for Zs_1. While this does not appear possible, one can instead "restrict" the sections s_i to the closed positive currents (h, h) currents $\zeta_{1,\epsilon} \wedge \cdots \wedge \zeta_{h,\epsilon}$, where $\zeta_{1,\epsilon} = \sqrt{-1}\partial\bar{\partial} \log(\|s_i\|^2 + \epsilon^2) + p \operatorname{Ric} \geq 0$ is a closed nonnegative (1, 1)-form. We remark first of all that the stronger assumption that $s_i \in \Gamma^o(X, K^p)$ implies the stronger gradient estimate that for some $\alpha_i > 0$, $\partial e^{\alpha_i u_{i,\epsilon}}$ is square-integrable (by a similar argument). To give an idea consider the case of $n = 2$ and $k = 2$. As in the volume estimate for $[Zs_i]$ we obtain estimates of the integral of $\zeta_{1,\epsilon} \wedge \zeta_{2,\epsilon}$ over X and uniform in ϵ and apply the Fatou's lemma to get the volume estimate for $[Zs_1] \cap [Zs_2]$. The missing step is then to show that there exists some $\alpha > 0$ such that $\|s_2\|^\alpha$ is integrable with respect to the measure $\zeta_{1,\epsilon} \wedge \omega$. Writing $\|s_i\|^2 + \epsilon^2 = e^{u_{2,\epsilon}}$ it suffices to show that for some $\alpha > 0$

$$\int_X e^{\alpha u_{2,\epsilon}} \zeta_{1,\epsilon} \wedge \omega < \infty.$$

Expanding $\zeta_{1,\epsilon} = \sqrt{-1}\partial\bar{\partial} u_{1,\epsilon} + p \operatorname{Ric}$ this reduces readily by integrating by parts to estimating the square norm of $\bar{\partial} e^{\alpha u_{2,\epsilon}}$ over X. As remarked, this gradient estimate could be obtained and it now suffices to set $\alpha = \alpha_2$. More generally, for some $\alpha = \alpha_j > 0$ we have inductively the gradient estimate

$$\int_{B(R)} \sqrt{-1}\partial e^{\alpha u_\epsilon} \wedge \bar{\partial} e^{\alpha u_\epsilon} \wedge \zeta_{1,\epsilon} \wedge \cdots \wedge \zeta_{j-1,\epsilon} \wedge \omega^{n-j} < \infty.$$

We conclude this section by giving an application of Siegel's Theorem study the automorphism of complete Kähler manifolds of finite volume and negative Ricci curvature.

Theorem (4.9.2). *Let (M, ξ) be a complete Kähler manifold of finite volume such that*

$$-1 < \textit{Ricci curvatures} < 0.$$

Then, the group Bimer(X) *of bimeromorphic self-mappings on X is finite. In particular, the group* Aut(X) *of biholomorphic automorphisms of X is finite.*

Proof. By Sakai [Sak] for any integer $p > 0$ the space $L^{2/p}(X, K^p)$ of pluricanonical sections of class $L^{2/p}$ is equipped with a norm $\|.\|$ invariant under biholomoprhic automorphisms. Since $N^2(X, K^p) \subset L^{2/p}(X, K^p) \subset L^2(X, K^p)$ from the volume estimate accompanying Siegel's Theorem we deduce that $L^{2/p}(X, K^p)$ is finite dimensional for each X. Let $M_o(X)$ be the field of meromorphic functions arising from quotients of pluricanonical sections belonging to some $L^{2/p}(X, K^p), p$ a positive integer. We have Siegel's Theorem for $M_o(X)$. Let now $F': X \to Z'$ be a birational map constructed from $M_o(X)$ using $s_o, \ldots, s_N \in L^{2/q}(X, K^q)$. The argument of Sakai [Sak] then shows that Bimer(X) is represented as a closed subgroup H of the group Γ of isometries of $(L^{2/q}(X, K^q), \|.\|)$. Since $L^{2/q}(X, K^q)$ is finite-dimensional, it follows from the maximum principle that H is finite.

(4.10) Estimates of Gauss-Bonnet integrals and a criterion of Zariski-openness using the Kontinuitätssatz

To make use of the Bézout estimates we will modify the "birational" embedding of (4.9) by using instead the field $R_{L^o}(X, K)$ of meromorphic functions arising from quotients of pluricanonical sections of class L^o. We retain the notation $F: X \to Z$ for the new "birational" embedding using $R_{L^o}(X, K)$ and assume for simplicity of presentation that F is actually a holomorphic embedding. Let $S_o \subset Z$ be non-singular and sliced by hyperplane sections in general positions. Let $S \subset S_o$ be an irreducible component and $C = S \cap X$ be non-empty. We want to prove

Proposition (4.10.1). *The cardinality of $S - C$ is finite and bounded independent of the choice of S.*

Recall that by assumption (X, ω) is of bounded sectional curvature. Since bisectional curvatures (in particular holomorphic sectional curvatures) of Kähler submanifolds are smaller or equal to those of the ambient manifold, $(C, \omega|_C)$ has curvature bounded from above by a constant. By the Bézout estimate of (4.9) the volume of $(C, \omega|_C)$ is finite and bounded independent of S, so that the total Gauss-Bonnet integral of $(C, \omega|_C)$ is well defined as an element of $[-\infty, \infty)$. We will prove that $\mathrm{Card}(S - C)$ is finite and bounded independent of S by showing that the total Gauss-Bonnet integral of $(C, \omega|_C)$ is bounded from below independent of S. While the implication can be deduced from the Cohn-Vossen inequality [Cohn] , we give here a self-contained argument more in the spirit of the proof of Siegel's Theorem. We consider the space $\Gamma'(C, K_C^2)$ of integrable quadratic differentials on $S - C$, which is independent of the choice of Hermitian metrics. To bound $\mathrm{Card}(S - C)$ we show that

(i) $\dim_{\mathbf{C}} \Gamma^1(C, K_C^2) \geq 3\mathrm{genus}\ (S) + \mathrm{Card}(S - C) - 3$;

(ii) for some choice of Hermitian metric θ on C one has the estimate

$$\dim_{\mathbf{C}} \Gamma^1(C, K_C, \theta) \leq C$$

for some constant depending only on an upper bound K of sectional curvatures on (X, ω).

Since a meromorphic quadratic differential on S with at most simple poles at a finite number of points on $S - C$ is integrable, (i) follows immediately from the Riemann-Roch Theorem. (ii) follows from a modification of the proof of volume estimate (hence the dimension estimate) of (4.9). We consider first of all the metric $\omega|_C$. If we use $\omega|_C$ the problem is that the curvature of the induced metric s on K_C may be too positive. The assumption that the total Gauss curvature of $(C, \omega|_C)$ is finite implies only that the curvature of $(K_C^2, (\omega|_C)^{-2})$ is integrable. An examination of the proof of the volume estimate shows that this is not enough to show $\dim_{\mathbf{C}} \Gamma^1(C, K_C^2, \omega) \leq C$. (One needs the curvature to be of class $L^{1+\epsilon}$ for some $\epsilon > 0$.) Instead we modify the Hermitian metric to remove the negative part of the curvature of C. Decompose the curvature form Ric of $(C, \omega|_C)$ into the positive and negative parts. Thus $\mathrm{Ric} = \mathrm{Ric}^+ + \mathrm{Ric}^-$, where both Ric^+ and Ric^- are integrable (1, 1) forms on C. On S we can solve the equation $\sqrt{-1}\partial\bar{\partial}\varphi = -\mathrm{Ric}^- + \mu$ for some integrable smooth (1, 1) form μ on S supported on a compact subset of C. The (1, 1)-form $\theta = e^{-\varphi}\omega|_C$ then defines a $\mathcal{C}^{(1,1)}$ Hermitian metric on C of nonnegative curvature outside a compact subset. By construction φ is subharmonic and hence bounded from above outside a compact subset of C. It follows that the Hermitian metric θ is complete but the volume is no longer finite in general. The curvature of (K_C^2, θ^{-2}) is then nonpositive outside a compact subset of C. For $s \in \Gamma^1(C, K_C^2, \theta)$ the volume estimate of (4.9) goes through, yielding the estimate that

$\mathrm{Card}([Zs]) \leq C$ independent of s and hence the dimension estimate in (ii).

Remark. In the volume estimate of (4.9) we did not actually need the finiteness of the volume but only the fact that u_ϵ is both integrable and square-integrable. It follows readily from $s \in \Gamma^1(C, K_C^2, \theta^{-2})$ that u_ϵ is L^α for any $\alpha, 0 < \alpha < 0$, without assuming that $\mathrm{Volume}(C, \theta) < \infty$.

We now proceed to explain the estimate of the Gauss-Bonnet integral of $(C, \omega|_C)$. This is equivalent to estimating the integral of $\|\sigma\|^2$, where σ is the second fundamental form of $(C, \omega|_C)$ in (X, ω). We do not have a general method for estimating $\int_S \|s\|^2$ on a submanifold S. However, in case of Riemann surfaces, this can be done by transforming the problem to a problem of Bézout estimates on the projectivized tangent bundle $\mathbf{P}T_X$. The idea comes from the fact that for a minimal surface Σ in $\mathbf{R}^3$, the Gauss-Bonnet integral is up to a constant the area of the image of the Gauss map. The Gauss map $\gamma: \Sigma \to S^2$ (unit 2-sphere) associates to every point $x \in \Sigma$ a unit normal vector. Equivalently, we associate to the (real) tangent plane $T_x(\Sigma) \in G_{\mathbf{R}}(2,3)$, the Grassmannian of real 2-planes in $\mathbf{R}^3$.

In our situation one has the analogue of lifting C to a curve $\widehat{C}$ in $\mathbf{P}T_X$. We first equip $\mathbf{P}T_X$ with a Kähler form. By Hopf blow-up we can associate to the trivial vector bundle $\mathbf{C}^n \to \{o\}$ the tautological line bundle $L_o \to P^{n-1}$. L_o is isomorphic to the dual of the hyperplane section line bundle H. The Euclidean metric on $\mathbf{C}^n$ then corresponds on $L \to P^{n-1}$ of negative curvature. For the Hermitian holomorphic vector bundle $T_X \to X$ one can do the (fiber-by-fiber) Hopf blow-up of the zero section $\cong X$ to get the tautological line bundle $L \to \mathbf{P}T_X$. The Hermitian metric h on T_X (coming from ω) induces a Hermitian metric h on $L \to \mathbf{P}^{n-1}$. The first Chern form $c_1(L,h)$ is related to the curvature form of (T_X, h). In fact, $c_1(L,h)$ is negative definite if and only if (T_X, h) is negative in the sense of Griffiths [Gri1] , i.e., (X, ω) is of negative holomorphic bisectional curvature. Since the restriction L_x of L to $\mathbf{P}T_x(X) \cong \mathbf{P}^{n-1}$ is isomorphic to the tautological line bundle $L_o \to \mathbf{P}^{n-1}$, the restriction of $-c_1(L,h)$ to the fibers of $\pi: \mathbf{P}T_X \to X$ is always positive definite. From the boundedness of bisectional curvatures of (X, ω) one can construct a Kähler form ν on $\mathbf{P}T_X$ by setting $\nu = [-c_1(L,h)] + k\pi^*\omega$ for a sufficiently positive constant k. For such positive k, $(\mathbf{P}T_X, \nu)$ is a complete Kähler manifold of finite volume. Let now $\Theta: C \to \mathbf{P}T_X$ be the holomorphic map which associates to each $x \in C$ the point $(x, [T_x(C)])$, where $[T_x(C)]$ denotes the element in $\mathbf{P}T_x(X)$ defined by the holomorphic plane $T_x(C)$. A direct computation using complex geodesic coordinates then yields

The Gauss-Bonnet integral as a difference of surface areas

Writing $\nu = [-c_1(L,h)] + k\pi^*\omega$ as in the above and denoting by $K(x)$ the holomorphic sectional curvature for $(C, \omega|_C)$ at $x \in C$ we have

$$\int_{\widehat{C}} \nu = k\text{Area } (C, \omega|_C) - \frac{1}{2\pi} \int_C K(x).$$

We now return to the proof of Proposition (4.10.1). From the preceding discussion to bound the cardinality $\text{Card}(S - C)$ it suffices to bound $\text{Area}(\widehat{C}, \nu)$. We do this by proving Bézout estimates on $(\mathbf{P}T_X, \nu)$. Recall that $\text{Volume}(\mathbf{P}T_X, \nu) < 0$. From the definition $\nu = [-c_1(L,h)] + k\pi^*\omega$ it follows that (L,h) is of bounded curvature in $(\mathbf{P}T_X, \nu)$. Denote by s the Hermitian metric on K_X induced by the Kähler metric ω. The proof of the Bézout estimate sketched in (4.9) applies equally well to $(\mathbf{P}T_X, \nu)$ and line bundles (Λ, θ) coming from tensor products of (L,h) and (π^*K_X, π^*s) and their inverses. To prove Proposition (4.10.1) it suffices therefore to show that $\widehat{C} \subset \mathbf{P}T_X$ can be realized as an irreducible component of the intersection of $(2n-2)$ holomorphic sections of class L^o of such line bundles (Λ, θ). Let S be the common zero set of $s_1, \ldots s_{n-1} \in \Gamma^o(X, L^p)$. For $x \in C$ let $s_o \in \Gamma^o(X, L^p)$ be such that $s_o(x) \neq 0$. For p sufficiently large such an s_o always exists. (One can compose $F: X \to Z \subset P^N$ with iterated powers of the Veronese map.) The meromorphic 1-forms $d(s_i/s_o), 1 \le i \le n$, annihilate $T_x(X)$. To get holomorphic sections write

$$d\left(\frac{s_i}{s_o}\right) = \frac{1}{s_o^2}\left(s_o \nabla s_i - s_i \nabla s_o\right),$$

where ∇ denotes covariant differentiation on (X, ω). (The formula for $d(s_i/s_o)$ is valid for any choice of Hermitian connections on K^p.) The section $s_o^2 d(s_i/s_o) = s_o \nabla s_i - s_i \nabla s_o$ is then a K^{2p}-valued holomorphic 1-form on X. $s_o \nabla s_i - s_i \nabla s_o$ corresponds to a holomorphic section λ_i of $(\pi^* K^{2p} \otimes L)^{-1}$. Recall that for s of class Γ^o we have gradient estimates for ∇s. This allows us to show that $\mu_i = \pi^*(s_o s_i)\lambda_i$ is of class Γ^o. As $\widehat{C}$ is a connected component of the zero set of the $(2n-2)$ holomorphic sections $\pi^* s_1, \ldots, \pi^* s_n, \mu_1, \ldots \mu_{n-1}$, Bézout estimates on $(PT_X \nu)$ implies the necessary estimates for Gauss-Bonnet integrals to conclude the proof of Proposition (4.10.1).

We give in this section a method of proving Zariski-openness assuming that X verifies the Kontinuitätssatz. Assuming as we did that

$F: X \to Z$ is a holomorphic embedding and Z is non-singular, this would be the case if (X, ω) is of nonpositive holomorphic bisectional curvature (cf. (3.7)). The additional difficulties coming from the existence of positive bisectional curvatures will be dealt with in the next section by using Bombieri's criterion for complex analyticity. As will be seen, the two approaches are complementary to each other and can be used together to yield a proof of Theorem (4.8).

By Proposition (4.10.1) and assuming the Kontinuitätssatz we need

Proposition (4.10.2) (Narasimhan). *Let $D \subset \Delta^n$ be a domain of holomorphy contained in the unit polydisc such that for every $a \in \Delta^{n-1}$ and for $D_a = w \in \Delta : (a, w) \in D, \Delta - D_a$ is a finite set with cardinality $\leq m$ for an integer m independent of a. Then, $D = \Delta - V$ for some subvariety V of Δ^n.*

Remarks. i) For every $b \in \partial X$ there is always a neighborhood U biholomorphic to Δ^n such that $X \cap U$ corresponds to the domain D in the proposition. This is done by taking an affine part of Z and projecting onto some Euclidean space. We will call such coordinates affine Euclidean coordinates at b and call U a proper affine Euclidean polydisc on Z.
ii) In the case when $m = 1$ the proposition is due to Hartogs. In this case $\Delta^n - D$, if non-empty, is the graph of a continuous function f (as a consequence of Hartogs'extension theorem). For $z \in D$ let $R_i(z)$ be the Hartogs radius in the z_i-direction defined as the radius of the largest disc on D_a centered at $z = (a, w)$. Then, $-\log R_i$ is plurisubharmonic. In our case we have $R_n(z) = |z_n - f(z)|$ and a direct computation shows that $\sqrt{-1}\partial\bar{\partial}(-\log R_n) \geq 0$ in the sense of distribution if and only if f is holomorphic. The case of general m was deduced from the case $m = 1$ using Rado's Theorem in [R. Narasimhan, *Several Complex Variables*, Chicago Lectures in Math. Series, University of Chicago Press, Chicago and London, Chapter 4, p.50ff.].

(4.11) Zariski-openness using plurisubharmonic potentials

When some bisectional curvatures of the manifold (X, ω) are positive, X may no longer satisfy the Kontinuitätssatz. We will say that $X \subset Z$ is locally Stein if for every point $x \in \partial X$ there exists an open coordinate neighborhood $U \cap X$ is a Stein domain (i.e., domain of holomorphy) in U. (This notion depends on the choice of Z and the embedding $F: X \to Z$. We do not have an example of an $(X, \omega), X \subset Z$, satisfying the hypothesis of Theorem (4.8) which is not locally Stein. In fact, in case of $n = 2$ as a consequence of Mok-Yau [MY, Main Theorem, p.42ff.]

and the localization argument of Cheng-Yau [CY1]) one can deduce that X is necessarily locally Stein in Z (so that $Z - X$ is a curve). However, Theorem (4.8) can be regarded as a special case of a Kodaira Embedding Theorem for complete Kähler manifolds of bounded curvature and finite volume (Mok-Zhong [MZ2, Theorem 1]). In this formulation there is a simple example on $\mathbf{P}^2 - \{o\}$ (cf. Mok-Yau [MY, 4, p.53-54]). To motivate our approach we examine this example. The example in [MY] was constructed on $B^2 - \{o\}$ using a potential. We define $\sigma = \sqrt{-1}\partial\bar{\partial}f(|z|^2)$, where $f(x) = \log \frac{x}{-\log x}$. Roughly speaking, ω is obtained by piecing together the Fubini-Study and the Poincaré metrics. By direct computation $(B^2 - \{o\}, \sigma)$ is complete, of finite volume and bounded sectional curvature. By using a cut-off function to extend the potential to $\mathbf{P}^2$ and adding on a multiple of some Kähler metric on $\mathbf{P}^2$ one can modify σ to get an example ω on the quasi-projective variety $\mathbf{P}^2 - \{o\}$. By construction on $B^2 - \{o\}$, $\omega | B^2 - \{o\} = \sqrt{-1}\partial\bar{\partial}\varphi$ for some plurisubharmonic function φ. The function φ, like $f(|z|^2)$, is of logarithmic decay near the origin. In particular, there exists a positive number c such that $e^{-c\varphi}$ is not integrable on $B^2 - \{o\}$. Taking $X = \mathbf{P}^2 - \{o\}$ and $Z = \mathbf{P}^2, X$ is not locally Stein at o and the complement $Z - X = \{o\}$ can be recovered as the set where $e^{-c\varphi}$ is not locally summable.

The example above suggests an approach to proving Zariski-openness of $X \subset Z$ in the proof of Theorem (4.8): to look for an extension of the closed positive (1, 1)-form ω trivially to a closed positive current T on Z and recover the complement $Z - X$ locally as a set of local non-summability of some local potentials φ on Z. This approach alone is however not possible in general, as suggested by the simple example of the punctured disc equipped with the Poincaré metric ω. In this case the metric can be defined as $\sqrt{-1}\partial\bar{\partial}\varphi$, where $\varphi = \log(-\log|z|^2)$ and $e^{-c\varphi}$ is locally summable at o for any positive integer c. Comparing to Proposition (4.7.2) this simply comes from the fact that the trivial extension T has no mass at o (as a consequence of the definition). We have the following general fact: If Ω is an open set, T is a complex-analytic hypersurface in Ω and T_o is a closed positive (1, 1)-current on $\Omega - V$ admitting a trivial extension T to Ω, then $n(T, x) = 0$ for almost all $x \in V$. (cf. Siu [Siu2, §6]). In other words, if $(Z - X) \cap \Omega = V$ is in fact a complex-analytic hypersurface for some open set Ω, it is impossible to recover V using the latter method. In this case however we had the method of (4.10) based on the Kontinuitätssatz and estimates of Gauss-Bonnet integrals. A method must therefore be devised to incorporate both situations.

The example of $(\mathbf{P}^2 - \{o\}, \omega)$ suggested that the approach using

Bombieri's criterion for complex-analyticity may work for points $b \in Z - X$ at which the (local) Kontinuitätssatz fails. Let U be an open coordinate neighborhood of b in Z biholomorphic to a polydisc. We consider the hull of holomorphy $H(U \cap X)$. In general the hull of holomorphy of a domain may only be a Riemann domain. This is analogous to the situation of analytic continuation of a single holomorphic function f on a plane domain D, e.g. the logarithm, in which case the natural domain of existence becomes a Riemann surface. We have obviously

Lemma. *Let $\Omega \subset \Delta$ be a connected open subset. Suppose for any Ω' such that $\Omega \subset \Omega' \subset \Delta^n, \Omega'$ is locally connected at any point $b \in \Delta^n - \Omega'$. Then the hull of holomorphy $H(\Omega)$ is a domain.*

In our situation $X \subset Z$ consider $b \in \partial X$ and a proper affine Euclidean polydisc U on Z containing b. The condition in the lemma is satisfied for $\Omega = U \cap X$ by a slicing argument. We have $\Omega \subset H(\Omega) \subset \Delta^n$. To prove $\Delta^n - \Omega$ is a subvariety, we proceed to prove: (i) $\Delta^n - H(\Omega)$ is a subvariety of Δ^n; (ii) $H(\Omega) - \Omega$ is a subvariety of $H(\Omega)$ and (iii) the union $(\Delta^n - H(\Omega)) \cup (H(\Omega) - \Omega) = \Delta^n - \Omega$ is a subvariety of Δ^n. Step (i) follows from the argument of (4.9) using the Kontinuitätssatz. Steps (ii) and (iii) will both involve producing plurisubharmonic functions φ with large singularities.

$H(\Omega) - \Omega$ is complex-analytic

To explain the idea of (ii) recall the proof of the local compactification theorem in (4.7). There we used the Ricci curvature and the volume form. We used the lower estimate of the volume form coming from the maximum principle for the Monge-Ampère equation. In the present case the lower estimate of the volume form is no longer valid (as can be seen by blowing up points on the boundary). Instead we consider the curvature form $c_1(L, h)$ of the tautological line bundle. Recall that $\nu = -c_1(L, h) + k\pi^*\omega$ is a Kähler form on $\mathbf{P}T_X$ for $\pi : \mathbf{P}T_X \to X$ the canonical projection. We prove

Proposition (4.11.1). *ω (resp. ν) can be extended trivially as a closed positive current from X to Z (resp. from $\mathbf{P}T_X$ to $\mathbf{P}T_Z$.)*

The proof of the proposition follows from the simple argument of the extension theorem in (4.7) (the local compactification theorem) and Bézout's estimate. Fixing a Kähler form β on Z to prove that the trivial extension of ω exists one estimates the mass $\mathrm{Mass}(\omega; \beta)$. This can be done by restricting to algebraic curves S sliced by hyperplane sections and applying Fubini's Theorem. The problem is then reduced to area estimates of $(C, \omega|_C)$ for $C = S \cap X$, which was accomplished by the Bézout estimate of (4.9). The closedness of the trivial extension can be reduced to Bézout estimates on algebraic surfaces sliced by hyperplane sections. In fact as in (4.7) one estimates integrals of $d(\rho_R \omega) \wedge \xi$ for μ a smooth $(2n-3)$-form using cut-off functions $\{\rho_R\}$ with $\|\nabla \rho_R\| \leq C/R$ for constant C. Consider the case $\xi = \psi \wedge \beta^{n-1}$, where ψ is of type $(1,0)$. Applying the Cauchy-Schwarz inequality and writing $\sqrt{-1}\partial\rho_R \wedge \bar{\partial}\rho_R \leq \frac{C'}{R^2}\omega$, $\sqrt{-1}\psi \wedge \bar{\psi} \leq C''\beta$

$$\begin{aligned} &2\int_Z \|d(\rho_R\omega) \wedge \psi \wedge \beta^{n-2}\| \\ \leq &R\frac{C'}{R^2}\int_Z \omega^2 \wedge \beta^{n-2} + \frac{1}{R}\int_Z \omega \wedge C''\beta^{n-1}. \end{aligned}$$

The two integrals are finite by using Bézout estimates and slicing by algebraic surfaces and curves resp. The case of a general ξ being similar, we have proved by letting $R \to \infty$ that the trivial extension $\tilde{\omega}$ of ω to Z is d-closed. The argument for the extension of ν from $\mathbf{P}T_X$ to $\mathbf{P}T_Z$ is similar. Denote the closed trivial extension of ν by $\tilde{\nu}$.

Let $b \in H(\Omega) - \Omega$ and $p \in \mathbf{P}T_Z$ lying above b. Let φ and ζ resp. be local potentials for $\pi^*\omega$ (φ depending only on the base) and ν in a small neighborhood U of p. $\zeta - k\varphi$ is then a potential for $-c_1(L, h)$. On the other hand there is a natural potential for $-c_1(L,h)$ given by the metric, $(1/2\pi)\sqrt{-1}\partial\bar{\partial}(\log\gamma)$, where the function γ represents the Hermitian metric h in local coordinates. Comparing the two potentials we have over U

$$(*) \qquad \zeta - k\varphi = (1/2\pi)\log\gamma + h$$

for some pluriharmonic function h. We now use the assumption $b \in H(\Omega) - \Omega$ to assert that h extends to a pluriharmonic function on $H(\Omega)$. This follows from extending the holomorphic 1-form ∂h and integrating $dh = \partial h + \bar{\partial} h$. The fact that there are no periods comes from the fact that every smooth curve on $H(\Omega)$ is homotopic to a smooth curve on Ω (on which there are certainly no periods of dh), as can be deduced from Proposition (4.10.1). We want to show that $e^{-c\varphi}$ is not locally summable at b. Since h extends to a pluriharmonic function on $H(\Omega)$ it can be ignored in (*). Shrinking U we may assume that the plurisubharmonic function ζ is bounded from above on U. On the other hand, since (X, ω) is complete $\gamma(w)$ must blow up as $w \to p$ in some integral sense. One can in fact show that there exists some constant c' such that $e^{c'\log\gamma} = \gamma^{c'}$ is not locally summable at p. From (*) this is only possible if $k > 0$ and $\varphi(x)$

approaches $-\infty$ in some integral sense as $x \to b$. In fact, from Hölder inequality one deduces readily that there exists a constant c independent of the choice of b such that $e^{-c\varphi}$ is not locally summable at b. As the function φ is smooth on $\Omega, H(\Omega) - \Omega$ is precisely the common zero set of all holomorphic functions f on $H(\Omega)$ square-integrable with respect to $e^{-c\varphi}$. This proves that $H(\Omega) - \Omega$ is a complex-analytic variety.

$\Delta^n - \Omega$ is complex-analytic

To show that $\Delta^n - \Omega = (\Delta^n - H(\Omega)) \cup (H(\Omega) - \Omega)$ is complex-analytic we produce a plurisubharmonic function with large singularities on both $\Delta^n - H(\Omega)$ and $(H(\Omega) - \Omega)$. We may assume that φ is defined on Δ^n. By Step (i) $\Delta^n - H(\Omega)$ is a complex-analytic hypersurface W of Δ^n. By solving the Poincaré-Lelong equation $\sqrt{-1}\partial\bar{\partial}\tau = [W]$ we obtain a holomorphic function f such that $\tau = 2\pi(\log|f|^2)$ and $Zf = W$. Since $n([W], x) \geq 1$ for any $x \in W$ it follows that there exists some positive constant c such that $e^{-c(\varphi+\tau)}$ is not locally summable at $W \cup (\Delta^n - H(\Omega))$.

Under the assumption that $F: X \to Z$ is already a holomorphic embedding we have proved that X is Zariski-open in Z. In the general case there are additional technical difficulties coming from the base point set and the branching locus. In the general case there is a complex-analytic subvariety V of X such such that $F|X - V$ is a biholomorphism onto its image $X' \subset Z$. Working with X' we have the drawback that the Kähler metric ω is no longer complete on X'. To remedy for this we make use of the holomorphic sections $t_1, \ldots, t_k$ of class Γ^o defining V. In the slicing argument Step (i) modifications have to be made in estimates of the Gauss-Bonnet integrals. This involves essentially estimating the number of intersection points of the curves C on X (obtained by slicing) with V, which can be done by Bézout estimates. In Step (ii) the function γ does not have the necessary divergence property since ω is not complete on X'. As this is caused by V and the function $\log\|t\|^2 = \log\sum\|t_i\|^2$ is equal to $-\infty$ on V one hopes to add $\log\|t\|^2$ to the function φ. The function $\log\|t\|^2$ is however not plurisubharmonic. We consider instead the closed positive (1, 1)-form $\chi = \sqrt{-1}\partial\bar{\partial}\log\|t\|^2 + r\omega$ for some $r > 0$ and prove the extendibility of χ to Z as a closed positive current in order to get a potential function ζ on Δ^n playing the role of $\log\|t\|^2 + \varphi$. ζ is comparable to $\log\|t\|^2 + r\varphi$ because their difference is a pluriharmonic function on Ω which can be extended to $H(\Omega)$. This way we get a function $\varphi' = \varphi + \zeta$ playing their role of φ in Step (ii). The same function φ' can play the role of φ in Step (iii) to complete the proof in the general case that $\Delta^n - \Omega$ is complex-analytic.

Having proved that $F: X \to Z$ is in general quasi-surjective, it remains to apply the topological lemma of Demailly in (3.8) to conclude the proof of Theorem (4.8) to give a compactification of (X, ω).

(4.12) Bézout estimates on complete Kähler manifolds of positive Ricci curvature

Recall from (3.7) we considered the problem of compactifying complete Kähler manifolds (X, ω) of positive Ricci curvature, quadratic curvature decay and Euclidean volume growth such that $\int_X \mathrm{Ric}^n < \infty$. We sketched the scheme of a proof that X is biholomorphic to a quasi-projective variety. In the step of desingularizing a quasi-surjective embedding there was the difficulty that the uniform multiplicity estimate may fail for a countable union of compact subvarieties of positive dimension. In order to overcome this difficulty, we resort to the technique of Bézout estimates of (4.9). Recall that we use the algebra $R(X, K^{-1})$ arising from pluri-anticanonical sections of polynomial growth. From the Siegel's Theorem (generalization of Proposition (3.3.2)) we know that $R(X, K^{-1}) = \mathbf{C}(f_1, \ldots, f_n, h)$, where $f_i = s_i/s_o$ for some s_i, s_o of class L^2, and h is finite over $\mathbf{C}(f_1, \ldots, f_n)$. By using the algebraic relations one can reduce the problem to pluricanonical sections of class L^α, for some $\alpha > 0$. We have

Proposition (4.12). *Let $s_1, \ldots, s_k$ be pluricanonical sections of class Γ^α for some $\alpha > 0$. Then, there are at most a finite number of compact irreducible components $\{E_j\}$ of $[Zs_1] \cap \cdots \cap [Zs_k]$ of codimension k.*

We explain how Proposition (4.12) can be formulated as a problem on Bézout estimates. For simplicity assume that at each E_j the zero-sets $Zs_1, \ldots, Zs_k$ intersects at normal crossings at some point. The line bundle K^{-1} is positive over X and hence E_j. For a compact complex manifold M and a positive line bundle L the Chern number $\int_M c_1^n(L)$ is a positive integer. For M singular this remains true when integration is performed over the smooth part, as can be seen by desingularization. Thus, to show that E_j is a finite set it suffices to prove that the total integral of Ric^{n-k} over $\cup E_j$ is finite. In the notations of (4.9) the problem is reduced to showing that there exists a constant C such that

$$\int_X \zeta_{1,\epsilon} \wedge \cdots \wedge \zeta_{k,\epsilon} \wedge \mathrm{Ric}^{n-k} \leq C$$

independent of $\epsilon > 0$. Recall here that $\zeta_{i,\epsilon} \to [Zs_i]$ as closed positive currents. We can use the technique of (4.9) to get such an estimate. The assumption of the finiteness of volume of (4.9) is now replaced by the assumption that $\int_X \mathrm{Ric}^n$ is finite. When Ric^q appears with $1 \le q < n$, we can only use the information that

$$\mathrm{Ric}^q \le \left(\frac{C\omega}{1+R^2}\right)^q ;$$

which is not integrable when $q = n$. For the gradient estimate we proceed inductively (in j) with the non-closed form $\left(\frac{\omega}{1+R^2}\right)^{n-j}$, and obtain the weaker gradient estimate

$$\begin{aligned}&\int_{B(R)} \sqrt{-1}\partial e^{\alpha u_\epsilon} \wedge \bar{\partial} e^{\alpha u_\epsilon} \wedge \zeta_{1,\epsilon} \wedge \cdots \wedge \zeta_{j-1,\epsilon} \wedge \left(\frac{\omega}{1+R^2}\right)^{n-j}\\ =&\epsilon^{2\alpha} O(\log R) + C_\epsilon\end{aligned}$$

for some $\alpha > 0$ depending on j and for constants C_ϵ depending on ϵ. This weaker gradient estimate is however enough for the Bézout estimate that we need.

With the Bézout estimate of Proposition (4.12.1) we conclude the proof of the embedding theorem for positive Ricci curvature (Theorem (3.1.3)). It is interesting to know whether the finiteness of $\int_X \mathrm{Ric}^n$ is really necessary for the proof. In the uniform multiplicity estimate, the assumptions of quadratic curvature decay and Euclidean volume growth are sufficient. However, in both the proof of the Siegel's Theorem and the Bézout estimate above the finiteness of $\int_X \mathrm{Ric}^n$ is used. Even when X is assumed to be Stein, it is not known whether the uniform multiplicity estimate can be used to deduce Siegel's Theorem.

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Department of Mathematics
Columbia University
New York, New York 10027 U.S.A.