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Duality Theorems for Abelian Varieties over Z_p-extensions

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Dedicated to Kenkichi Iwasawa on his 70th birthday

Our concern in this paper is to define p-adic height pairings for an abelian variety A over an algebraic number field k on the niveau of a Z_p-extension k_∞ of k. We will show that there exists a map from the Λ-torsion submodule $T_\Lambda H^1(\mathcal{O}_\infty, \mathcal{A}(p))^*$ of the Pontrjagin dual of the p-Selmer group to the adjoint α of the corresponding module for the dual abelian variety A'. Here Λ denotes the completed group ring of $\mathrm{Gal}(k_\infty/k)$ over Z_p and p is a prime number where A has good reduction. $\mathcal{A}$ denotes the Néron model defined over the ring of integers $\mathcal{O}_\infty$ of k_∞. More generally, for $i \geqq 0$ there are canonical maps

$$T_\Lambda H^i(\mathcal{O}_\infty, \mathcal{A}(p))^* \longrightarrow \alpha(T_\Lambda H^{2-i}(\mathcal{O}_\infty, \mathcal{A}'(p))^*).$$

These maps are quasi-isomorphisms if A has ordinary good reduction at p. In this case they can be regarded as non-degenerate pairings between the Λ-torsion submodules of $H^i(\mathcal{O}_\infty, \mathcal{A}(p))^*$ and of $H^{2-i}(\mathcal{O}_\infty, \mathcal{A}'(p))^*$. The pairing induced on a finite layer k_n/k coincides with the pairing defined by Schneider [8] (for $i=1$ and assuming that $H^1(\mathcal{O}_\infty, \mathcal{A}(p))^*$ is Λ-torsion and fulfills a certain semi-simplicity property).

Furthermore, we define an Iwasawa L-function in terms of characteristic polynomials of $T_\Lambda H^i(\mathcal{O}_\infty, \mathcal{A}(p))^*$:

$$L_p(A, \kappa, s) = \prod_{i=0}^{2} F_i(\kappa(\phi)^{s-1} - 1)^{(-1)^{i+1}}, \qquad s \in Z_p,$$
$$F_i(t) = p^{\mu_i} \det(t - (\phi - 1);\ T_\Lambda H^i(\mathcal{O}_\infty, \mathcal{A}(p))^* \otimes Q_p),$$

where κ is the character corresponding to k_∞, ϕ is a generator of $\mathrm{Gal}(k_\infty/k)$ and μ_i is the μ-invariant of $H^i(\mathcal{O}_\infty, \mathcal{A}(p))^*$. In the ordinary case the pairing mentioned above leads to a functional equation for $L_p(A, \kappa, s)$ with respect to $s \mapsto 2-s$. This generalizes a result of Schneider [8] and

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Mazur [4], since we do not assume $H^1(\mathcal{O}_\infty, \mathcal{A}(p))^*$ to be Λ-torsion.

In the supersingular case, i.e., if the p-rank of the reduction $\mathcal{A}/\kappa_{\mathfrak{p}}$ is zero for every prime $\mathfrak{p}$ above p, the adjoint of $T_\Lambda H^1(\mathcal{O}_\infty, \mathcal{A}(p))^*$ can be identified with the dual of the kernel of the canonical map

$$H^1(\mathcal{O}_\infty, \mathcal{A}(p)) \longrightarrow \prod_{\mathfrak{p} \in \Sigma} H^1(k_{\infty\mathfrak{p}}, A(p))$$

where Σ denotes the set of primes ramified in k_∞/k. This generalizes a result for elliptic curves with complex multiplication obtained by Billot [2].

At the end of the paper we study how the pairing for an abelian variety A which is ordinary at p behaves on the two parts of the p-Selmer group given by the p-part of the Tate-Šafarevič group $Ш_\infty(A)(p)$ and the "Mordell-Weil group" $A(k_\infty)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p$. Assuming that the p-part of Ш on each layer of k_∞/k is finite we obtain a quasi-isomorphism

$$T_\delta Ш_\infty(A)(p)^* \xrightarrow{\approx} \alpha(T_\delta Ш_\infty(A)(p)^*)$$

and a quasi-exact sequence

$$\begin{aligned} 0 \longrightarrow T_\nu Ш_\infty(A')(p)^* &\longrightarrow \alpha(T_\Lambda(A(k_\infty)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^*) \\ &\longrightarrow T_\Lambda(A'(k_\infty)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^* \longrightarrow \alpha(T_\nu Ш_\infty(A)(p)^*) \longrightarrow 0 \end{aligned}$$

where $T_\nu M$ and $T_\delta M$ of a compact Λ-module M of finite type are defined by $\varinjlim_n M^{\Gamma_n}$ and $T_\delta M = T_\Lambda M/T_\nu M$, respectively. In particular, if the group of Γ_n-invariants of $Ш_\infty(A)(p)$ is infinite then A has a k_n-rational point of infinite order. As a corollary one obtains a non-degenerate pairing

$$A(k_\infty) \times A'(k_\infty) \longrightarrow \boldsymbol{Q}_p,$$

if $(A(k_\infty)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^*$ is Λ-torsion and $Ш_\infty(A)(p)^{\Gamma_n}$ is finite for all $n \geq 0$.

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§ 0. Notations

For an abelian group M let $\mathrm{Tor}\,M$ be the torsion subgroup and $M_{\mathrm{Tor}} := M/\mathrm{Tor}\,M$, let $\mathrm{Div}\,M$ be the maximal divisible subgroup and $M_{\mathrm{Div}} := M/\mathrm{Div}\,M$. For $m \in \boldsymbol{N}$ let the groups ${}_mM$ and M_m be the kernel and cokernel of the multiplication by m, respectively, and put $M(p) = \varinjlim_m {}_{p^m}M$ for a prime number p.

For a commutative group scheme G we use contrary to the convention above the usual notation G_m for the kernel of the m-multiplication.

For a Z_p-module M let $M^*=\mathrm{Hom}(M, Q_p/Z_p)$ be the Pontrjagin dual of M. For a G-module M, G a group, M^G and M_G denote the invariants and coinvariants of G, respectively.

Throughout this paper the cohomology groups $H^i(S, \ )$ are taken with respect to the big fppf-site on a scheme S.

§ 1. Λ-modules

Let Γ be a pro-p-group isomorphic to Z_p and let $\Lambda = Z_p[\![\Gamma]\!]$ be the completed group ring of Γ. We also consider Λ as the ring of power series $Z_p[\![T]\!]$ over Z_p via the homeomorphism $\gamma \mapsto 1+T$, where γ is a generator of Γ.

Let M be a finitely generated compact Λ-module, then

$$T_\Lambda M \quad \text{and} \quad T_\mu M$$

denote the Λ-torsion submodule and the Z_p-torsion submodule of M, respectively. We define

$$F_\Lambda M := M/T_\Lambda M \quad \text{and} \quad T_\lambda M := T_\Lambda M/T_\mu M.$$

Furthermore let Γ_n be the subgroup of Γ of index p^n and let

$$T_\nu M := \varinjlim_n M^{\Gamma_n} \quad \text{and} \quad T_\delta M := T_\Lambda M/T_\nu M.$$

If ξ_r denotes the irreducible polynomial of the p^r-th root of unity, then there is a quasi-isomorphism

$$T_\nu M \approx \bigoplus_i \Lambda/\xi_i \qquad \text{for some polynomials } \xi_i.$$

If

$$T_{\delta_{i+1}} M := T_\delta(T_{\delta_i} M) \qquad \text{where } T_{\delta_1} M := T_\delta M$$

then there must be an i_0 with $T_{\delta_{i_0+1}} M = T_{\delta_{i_0}} M$ and we define

$$T_{\delta_\infty} M := T_{\delta_{i_0}} M \quad \text{and} \quad T_\varepsilon M := \ker(T_\Lambda M \to T_{\delta_\infty} M).$$

Obviously the characteristic polynomial of $T_\varepsilon M$ is a product of polynomials ξ_r and $T_{\delta_\infty} M$ has no divisor ξ_r, $r \geq 0$. For a Λ-module M let $\dot{M}$ be the Λ-module given by M with a new action of Γ

$$\gamma \cdot m := \gamma^{-1} m \qquad \text{for } m \in M, \gamma \in \Gamma.$$

If

$$\alpha(M) := \mathrm{Ext}^1_\Lambda(M, \Lambda).$$

denotes the adjoint of a compact Λ-torsion module M of finite type then according to [6] I.2.2 or [2] Corollaire 1.2, Remarque 3.4

$$\alpha(M)=\varinjlim_{i} \mathrm{Hom}_{Z_p}(M/\mathfrak{q}_i M, \boldsymbol{Q}_p/\boldsymbol{Z}_p)\approx \dot{M}$$

where $\{\mathfrak{q}_i\}$ is a sequence of divisors disjoint from the annihillator of M such that $\cap \mathfrak{q}_i=1$. If $0\to M_1\to M_2\to M_3\to 0$ is a quasi-exact sequence of compact Λ-torsion modules of finite type, then applying the contravariant functor α we obtain a quasi-exact sequence

$$0\longrightarrow\alpha(M_3)\longrightarrow\alpha(M_2)\longrightarrow\alpha(M_1)\longrightarrow 0.$$

If $\mathfrak{m}$ denotes the maximal ideal of Λ we get for a compact Λ-module M of finite type a quasi-isomorphism

$$\beta(M):=\varinjlim_{i} \mathrm{Hom}(M/\mathfrak{m}^i M, \boldsymbol{Q}_p/\boldsymbol{Z}_p)\approx F_\Lambda M.$$

Lemma 1.1. *Let M be a compact Λ-module of finite type. Then there are quasi-isomorphisms*

(a) $\varprojlim_{n,m} ({}_{p^m}M^*)_{\Gamma_n}\approx\alpha(T_\lambda M)\approx\dot{T}_\lambda M$

(b) $\varprojlim_{n,m} (M^*{}_{p^m})^{\Gamma_n}\approx\alpha(T_\mu M)\approx\dot{T}_\mu M$

(c) $\varprojlim_{n,m} (M^{*\Gamma_n})_{p^m}\approx\alpha(T_\delta M)\approx\dot{T}_\delta M$

(d) $\varprojlim_{n,m} {}_{p^m}(M^*{}_{\Gamma_n})\approx\alpha(T_\nu M)\approx\dot{T}_\nu M$

(e) $\varprojlim_{n,m} {}_{p^m}M^{*\Gamma_n}\approx\beta(F_\Lambda M)\approx F_\Lambda M$

(f) $\varprojlim_{n,m} M^*{}_{p^m\Gamma_n}\approx 0,$

where the limit is taken with respect to the p-multiplication resp. canonical surjection and the norm map resp. canonical surjection. Here and in the following we use the notation $\dot{T}_-(M)=T_-(\dot{M})$.

Proof. All assertions are obtained easily from the general structure theory of compact noetherian Λ-modules. So we will only indicate the proof of (c) and (d).

Since

$$\varinjlim_{n,m} (M^{\Gamma_n})_{p^m}=\varinjlim_{m} (T_\nu M)_{p^m}$$

it follows

$$\varprojlim_{n,m} {}_{p^m}(M^*{}_{\Gamma_n})\approx\varprojlim_{m} \mathrm{Hom}(T_\nu M_{p^m}, \boldsymbol{Q}_p/\boldsymbol{Z}_p)\approx\dot{T}_\nu M.$$

In order to prove (c) we decompose M

$$M \approx \bar{M} \oplus T_\mu M \qquad \text{with } T_\mu \bar{M} = 0.$$

First we see

$$\varprojlim_{n,m} ((T_\mu M)^{*\Gamma_n})_{p^m} = \varprojlim_{n} \operatorname{Hom}(\varinjlim_{m} {}_{p^m}((T_\mu M)_{\Gamma_n}), \boldsymbol{Q}_p/\boldsymbol{Z}_p)$$
$$= \varprojlim_{n} \operatorname{Hom}(T_\mu M_{\Gamma_n}, \boldsymbol{Q}_p/\boldsymbol{Z}_p) \approx \dot{T}_\mu M$$

and secondly the exact sequence

$$0 \longrightarrow \bar{M} \xrightarrow{p^m} \bar{M} \longrightarrow \bar{M}_{p^m} \longrightarrow 0$$

leads to an exact sequence

$$0 \longrightarrow (\bar{M}^{\Gamma_n})_{p^m} \longrightarrow (\bar{M}_{p^m})^{\Gamma_n} \longrightarrow {}_{p^m}(\bar{M}_{\Gamma_n}) \longrightarrow 0.$$

Hence we obtain a quasi-exact sequence

$$0 \longrightarrow (\dot{T}_\nu \bar{M})^* \longrightarrow (\dot{T}_\lambda \bar{M})^* \longrightarrow \varinjlim_{n,m} {}_{p^m}(\bar{M}_{\Gamma_n}) \longrightarrow 0$$

(recall that the projective limit is an exact functor in the category of profinite groups). This proves (c).

§ 2. Duality theorems for abelian varieties

Let k be a number field and let A be an abelian variety defined over k. Let $\mathscr{A}$ be its Néron model over the ring of integers $\mathcal{O}$ of k and let $\mathscr{A}^0$ be the connected component of $\mathscr{A}$. By A' and $\mathscr{A}'$ we denote the dual abelian variety and its Néron model, respectively. We say A has good (ordinary) reduction at a prime number p, if A has good (ordinary) reduction at all primes of k above p. Since A and A' are k-isogenous A' has in that case good (ordinary) reduction too.

Theorem 2.1. *Let A be an abelian variety over k with good reduction at p.*

(i) (*Artin/Mazur*) *The cup product induces a perfect duality of finite groups*

$$H^i(\mathcal{O}, \mathscr{A}_{p^m}) \times H^{3-i}(\mathcal{O}, \mathscr{A}'_{p^m}) \longrightarrow H^3(\mathcal{O}, \boldsymbol{G}_m) \xrightarrow{\sim} \boldsymbol{Q}/\boldsymbol{Z} \qquad \textit{for all } i \geq 0.$$

The above pairing induces the following perfect pairings

(ii) $H^1(\mathcal{O}, \mathscr{A}^0)(p)_{\mathrm{Div}} \times H^1(\mathcal{O}, \mathscr{A}')(p)_{\mathrm{Div}} \longrightarrow \boldsymbol{Q}/\boldsymbol{Z}$,

(iii) (*Cassels/Tate*)

$$Ш(A)(p)_{\mathrm{Div}} \times Ш(A')(p)_{\mathrm{Div}} \longrightarrow \boldsymbol{Q}/\boldsymbol{Z}.$$

Remark. A proof of (i) is given in an unpublished paper of Artin and Mazur [1] and also by Milne [15] III. Corollary 3.2. The assertion (ii) is proved by Schneider [7] § 6 Lemma 3 (observe that $H^1(\mathcal{O}, \mathcal{A}(p))_{\mathrm{Div}} = H^1(\mathcal{O}, \mathcal{A})(p)_{\mathrm{Div}}$). The perfect duality for the Tate-Šafarevič groups was announced by Tate in [10]. A proof can be found in [5] I. Theorem 6.13, II. Theorem 5.6.

We will shortly indicate, how this also follows from the flat duality theorem and a duality theorem of Grothendieck. The exact sequence

$$0 \longrightarrow \mathcal{A}^0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{F} \longrightarrow 0$$

defines a skyscraper sheaf $\mathcal{F}$. The stalk

$$\mathcal{F}_x = \pi_0(\mathcal{A}_x) \qquad \text{for } x \in \mathcal{O}$$

is the group of connected components of $\mathcal{A}_x = \mathcal{A} \times_{\mathcal{O}} \kappa(x)$. According to [4], Appendix, the image of the middle map in the exact cohomology sequence

$$H^0(\mathcal{O}, \mathcal{F}) \longrightarrow H^1(\mathcal{O}, \mathcal{A}^0) \longrightarrow H^1(\mathcal{O}, \mathcal{A}) \longrightarrow H^1(\mathcal{O}, \mathcal{F})$$

is $Ш(A)$. Therefore we obtain a commutative and exact diagram

$$\begin{array}{ccccc}
H^1(\mathcal{O}, \mathcal{F}')(p) & \times & H^0(\mathcal{O}, \mathcal{F})(p) & \longrightarrow & \bigoplus_x H^1(\kappa(x), \boldsymbol{Q}/\boldsymbol{Z}) \\
\uparrow & & \downarrow \delta & & \downarrow \delta \\
H^1(\mathcal{O}, \mathcal{A}')(p)_{\mathrm{Div}} & \times & H^1(\mathcal{O}, \mathcal{A}^0)(p)_{\mathrm{Div}} & \longrightarrow & H^3(\mathcal{O}, \boldsymbol{G}_m) \\
\uparrow & & \downarrow & & \\
Ш(A')(p)_{\mathrm{Div}} & & Ш(A)(p)_{\mathrm{Div}}\,. & & \\
\uparrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

The vertical exact sequences are induced by the exact sequence above: observe that

$$H^i(\mathcal{O}, \mathcal{F})(p) = \bigoplus_x H^i(\kappa(x), \pi_0(\mathcal{A}_x))(p) = \bigoplus_x H^i(\kappa(x), \pi_0(\mathcal{A}_x)(p))$$

is a finite group. The right vertical map δ is defined by the exact divisor sequence

$$0 \longrightarrow \boldsymbol{G}_{m/\mathcal{O}} \longrightarrow g_* \boldsymbol{G}_{m/k} \longrightarrow \bigoplus_x (i_x)_* \boldsymbol{Z} \longrightarrow 0$$

(g: Spec $k \to$ Spec $\mathcal{O}$ and i_x: Spec $\kappa(x) \to$ Spec $\mathcal{O}$) under consideration of

$$H^1(\kappa(x), \mathbf{Q}/\mathbf{Z}) = H^2(\kappa(x), \mathbf{Z}) = H^2(\mathcal{O}, (i_x)_* \mathbf{Z}).$$

The pairing at the top is defined as follows: By SGA 7 IX 11.3.1 we have a perfect duality

$$\pi_0(\mathcal{A}'_x)(p) \times \pi_0(\mathcal{A}_x)(p) \longrightarrow \mathbf{Q}/\mathbf{Z}$$

(observe $p \neq$ char $\kappa(x)$). Now it is easy to check that the induced pairing

$$\bigoplus_x H^0(\kappa(x), \pi_0(\mathcal{A}'_x)(p)) \times \bigoplus_x H^1(\kappa(x), \pi_0(\mathcal{A}_x)(p)) \longrightarrow \bigoplus_x H^1(\kappa(x), \mathbf{Q}/\mathbf{Z})$$

coincides with the pairing given by (ii) via δ. Therefore we obtain a perfect duality for the Tate-Šafarevič group.

Now let k_∞ be a $\mathbf{Z}_p$-extension of k and let k_n be the n-th layer of k_∞/k. Let $\mathcal{O}_n$ and $\mathcal{O}_\infty$ be the ring of integers of k_n and k_∞, respectively. We denote by Σ the finite set of primes of k which are ramified in k_∞ (and which therefore lie above p).

Theorem 2.2. *Let A be an abelian variety over k with good reduction at p. Then the flat duality induces quasi-isomorphisms*

(i) $$\alpha(H^0(\mathcal{O}_\infty, \mathcal{A}(p))^*) \approx T_\Lambda H^2(\mathcal{O}_\infty, \mathcal{A}'(p))^*$$

(ii) $$\beta(F_\Lambda(H^1(\mathcal{O}_\infty, \mathcal{A}(p))^*) \approx F_\Lambda H^2(\mathcal{O}_\infty, \mathcal{A}'(p))^* \oplus \Lambda^s$$

and a quasi-exact sequence

$$0 \longrightarrow F_\Lambda H^2(\mathcal{O}_\infty, \mathcal{A}'(p))^* \longrightarrow \beta(F_\Lambda H^1(\mathcal{O}_\infty, \mathcal{A}(p))^*) \searrow \bigoplus_{\mathfrak{p} \in \Sigma} (A'(k_{\infty\mathfrak{p}})/N'_\mathfrak{p} \otimes \mathbf{Q}_p/\mathbf{Z}_p)^*$$

$$\swarrow\; 0 \leftarrow \beta(F_\Lambda H^2(\mathcal{O}_\infty, \mathcal{A}(p))^*) \oplus \alpha(T_\Lambda H^1(\mathcal{O}_\infty, \mathcal{A}^0(p))^*) \leftarrow H^1(\mathcal{O}_\infty, \mathcal{A}(p))^*$$

where the third term is quasi-isomorphic to Λ^{2s},

$$s = \sum_{\mathfrak{p} \in \Sigma} (\dim A - r_\mathfrak{p})[k_\mathfrak{p} : \mathbf{Q}_p]$$

$$r_\mathfrak{p} = p\text{-rank of the reduction } \mathcal{A}/\kappa(\mathfrak{p}),$$

and where $N_\mathfrak{p}$ denotes the group of "universal norms in $A(k_{\infty\mathfrak{p}})$"

$$N_\mathfrak{p} = \bigcup_n \bigcap_{m \geq n} N_{k_{m\mathfrak{p}}/k_{n\mathfrak{p}}}(A(k_{m\mathfrak{p}})).$$

In particular, the above sequence induces a quasi-exact sequence

$$0 \longrightarrow \Lambda^s \longrightarrow \Lambda^{2s} \longrightarrow \Lambda^s \oplus T_\Lambda H^1(\mathcal{O}_\infty, \mathcal{A}'(p))^* \longrightarrow \dot{T}_\Lambda H^1(\mathcal{O}, \mathcal{A}^0(p))^* \longrightarrow 0.$$

Remark 2.3. (i) If k_∞ is the cyclotomic Z_p-extension it is conjectured that $F_\Lambda H^2(\mathcal{O}_\infty, \mathcal{A}(p))^* \approx 0$. This is proved for elliptic curves with complex multiplication by an order in an imaginary quadratic field K defined over an abelian extension of K with good ordinary reduction at p, see [3] Proposition 15, and in the case that the reduction of the abelian variety A/k is supersingular for every $\mathfrak{p}/p$ and the Iwasawa-μ-invariant of $k(A_p)$ is zero, [9] Theorem 5, Remark 1.

(ii) The canonical map

$$H^i(\mathcal{O}_\infty, \mathcal{A}(p))^* \longrightarrow H^i(\mathcal{O}_\infty, \mathcal{A}^0(p))^*$$

is a quasi-isomorphism except for the μ-part if $i=1$. Indeed, we have

$$\begin{aligned} H^r(\mathcal{O}_\infty, \mathcal{F}(p))^* &= \varprojlim_n \bigoplus_x H^r(\kappa(x), \pi_0(\mathcal{A}_x)(p))^* \\ &\cong \bigoplus_{x \in B} \bigoplus_j Z/p^{n_j(x)}[\![\Gamma]\!] \end{aligned}$$

where B is the set of all bad primes $x \in \mathcal{O}$ splitting completely in k_∞/k and the integers $n_j(x)$ for $x \in B$ are given by

$$H^r(\kappa(x), \pi_0(\mathcal{A}_x)(p))^* \cong \bigoplus_j Z/p^{n_j(x)}.$$

In order to prove Theorem 2.2 we need

Lemma 2.4. *Let N be a discrete Γ-module.*

(i) *There are isomorphisms*

$$H^1(\Gamma, N_{\mathrm{Tor}}) \cong (N \otimes Q_p/Z_p)^\Gamma / ((N_{\mathrm{Tor}})^\Gamma \otimes Q_p/Z_p).$$

$$H^2(\Gamma, N) \cong (N \otimes Q_p/Z_p)_\Gamma.$$

(ii) *Let $(N \otimes Q_p/Z_p)^*$ be a Λ-module of finite type. Then there is a quasi-exact sequence*

$$0 \longrightarrow \varprojlim_{n,m} (N_{\mathrm{Tor}})^{\Gamma_n} \otimes Z_p \longrightarrow \beta(F_\Lambda(N \otimes Q_p/Z_p)^*) \longrightarrow \varprojlim_{n,m} {}_{p^m}H^1(\Gamma_n, N_{\mathrm{Tor}}) \longrightarrow 0$$

Proof. Taking cohomology of the exact sequence

$$0 \longrightarrow N_{\mathrm{Tor}} \longrightarrow N_{\mathrm{Tor}} \otimes Z\left[\frac{1}{p}\right] \longrightarrow N \otimes Q_p/Z_p \longrightarrow 0$$

leads to an exact sequence

$$0 \longrightarrow N^{\Gamma}_{\mathrm{Tor}} \longrightarrow N^{\Gamma}_{\mathrm{Tor}} \otimes \boldsymbol{Z}\left[\frac{1}{p}\right] \longrightarrow (N \otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^{\Gamma} \longrightarrow H^1(\Gamma, N_{\mathrm{Tor}}) \longrightarrow 0$$

and an isomorphism

$$H^2(\Gamma, N) = H^2(\Gamma, N_{\mathrm{Tor}}) \cong H^1(\Gamma, N \otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p) \cong (N \otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)_{\Gamma}.$$

This proves (i). Taking Γ_n instead of Γ and applying the projective limit to the exact sequence

$$0 \longrightarrow {}_{pm}(N^{\Gamma_n}_{\mathrm{Tor}} \otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p) \longrightarrow {}_{pm}(N \otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^{\Gamma_n} \longrightarrow {}_{pm}H^1(\Gamma_n, N_{\mathrm{Tor}}) \longrightarrow 0$$

implies the result (ii).

Proof of Theorem 2.2. From the global flat duality theorem we obtain a perfect pairing

$$\varprojlim_{n,m} H^i(\mathcal{O}_n, \mathcal{A}_{pm}) \times H^{3-i}(\mathcal{O}_\infty, \mathcal{A}'(p)) \xrightarrow{\cup} \varprojlim_{n,m} H^2(\mathcal{O}_n, \boldsymbol{G}_m) \rightleftarrows \boldsymbol{Q}/\boldsymbol{Z}$$

where the projective limits are taken with respect to the norm map and the multiplication by p. In order to compute $\varprojlim_{n,m} H^i(\mathcal{O}_n, \mathcal{A}_{pm})$ we consider the descent diagram [8] p. 332, [7] Lemmas 6.1, 6.3:

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & & H^1(\mathcal{O}_n, \mathcal{A}_{pm}) & & & & \\
 & & \downarrow & & & & \\
0 \longrightarrow H^1(\Gamma_n, \mathcal{A}_{pm}(k_\infty)) & \longrightarrow & H^1(\mathcal{O}_\infty/\mathcal{O}_n, \mathcal{A}_{pm}) & \longrightarrow & H^1(\mathcal{O}_\infty, \mathcal{A}_{pm})^{\Gamma_n} \longrightarrow 0 \\
 & & \downarrow & & & & \\
 & & \bigoplus_{\mathfrak{p} \in S} {}_{pm}H^1(\Gamma_{n\mathfrak{p}}, A(k_{\infty\mathfrak{p}})) & & & & \\
 & & \downarrow & & & & \\
 & & H^2(\mathcal{O}_n, \mathcal{A}_{pm}) & & & & \\
 & & \downarrow \psi_{n,m} & & & & \\
0 \longrightarrow H^1(\mathcal{O}_\infty, \mathcal{A}_{pm})_{\Gamma_n} & \longrightarrow & H^2(\mathcal{O}_\infty/\mathcal{O}_n, \mathcal{A}_{pm}) & \longrightarrow & H^2(\mathcal{O}_\infty, \mathcal{A}_{pm})^{\Gamma_n} \longrightarrow 0.
\end{array}$$

Here $H^i(\mathcal{O}_\infty/\mathcal{O}_n, -)$ denotes the equivariant cohomology, [7] Appendix, and $\Gamma_{n\mathfrak{p}}$ is the decomposition group of Γ_n with respect to $\mathfrak{p}$. We calculate the projective limit of the finite groups in the diagram:

$$\varprojlim_{n,m} H^i(\mathcal{O}_n, \mathcal{A}_{pm}) \cong H^{3-i}(\mathcal{O}_\infty, \mathcal{A}'(p))^*,$$

$$\varprojlim_{n,m} H^i(\Gamma_n, H^j(\mathcal{O}_\infty, \mathcal{A}_{p^m})) \cong \varprojlim_{n,m} H^i(\Gamma_n, H^j(\mathcal{O}_\infty, \mathcal{A}^0_{p^m})).$$

The exact Kummer sequence, SGA 7 IX 2.2.1

$$0 \longrightarrow \mathcal{A}^0_{p^m} \longrightarrow \mathcal{A}^0(p) \xrightarrow{p^m} \mathcal{A}^0(p) \longrightarrow 0$$

implies an exact sequence

$$0 \longrightarrow H^{i-1}(\mathcal{O}_\infty, \mathcal{A}^0(p))_{p^m} \longrightarrow H^i(\mathcal{O}_\infty, \mathcal{A}^0_{p^m}) \longrightarrow {}_{p^m}H^i(\mathcal{O}_\infty, \mathcal{A}^0(p)) \longrightarrow 0$$

and therefore we obtain an exact sequence

$$\begin{aligned} 0 &\longrightarrow (H^{i-1}(\mathcal{O}_\infty, \mathcal{A}^0(p))_{p^m})^{\Gamma_n} \longrightarrow H^i(\mathcal{O}_\infty, \mathcal{A}^0_{p^m})^{\Gamma_n} \\ &\longrightarrow {}_{p^m}H^i(\mathcal{O}_\infty, \mathcal{A}^0(p))^{\Gamma_n} \longrightarrow H^{i-1}(\mathcal{O}_\infty, \mathcal{A}^0(p))_{p^m\Gamma_n} \\ &\longrightarrow H^i(\mathcal{O}_\infty, \mathcal{A}^0_{p^m})_{\Gamma_n} \longrightarrow ({}_{p^m}H^i(\mathcal{O}_\infty, \mathcal{A}^0(p)))_{\Gamma_n} \longrightarrow 0. \end{aligned}$$

By Lemma 1.1 we obtain quasi-isomorphisms

$$\varprojlim_{n,m} H^i(\mathcal{O}_\infty, \mathcal{A}^0_{p^m})^{\Gamma_n} \approx \alpha(T_\mu H^{i-1}(\mathcal{O}_\infty, \mathcal{A}^0(p))^*) \oplus \beta(F_\Lambda H^i(\mathcal{O}_\infty, \mathcal{A}^0(p))^*),$$

$$\varprojlim_{n,m} H^i(\mathcal{O}_\infty, \mathcal{A}^0_{p^m})_{\Gamma_n} \approx \alpha(T_\lambda H^i(\mathcal{O}_\infty, \mathcal{A}^0(p))^*)$$

inducing quasi-isomorphisms

$$\varprojlim_{n,m} H^i(\mathcal{O}_\infty/\mathcal{O}_n, \mathcal{A}^0_{p^m}) \approx \alpha(T_\Lambda H^{i-1}(\mathcal{O}_\infty, \mathcal{A}^0(p))^*) \oplus \beta(F_\Lambda H^i(\mathcal{O}_\infty, \mathcal{A}(p))^*).$$

Next, for $\mathfrak{p} \in \Sigma$ we want to show

Claim 1. $\varprojlim_{n,m} {}_{p^m}H^1(\Gamma_n, A(k_{\infty\mathfrak{p}})) \cong (A'(k_{\infty\mathfrak{p}})/N'_\mathfrak{p} \otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^* \approx \boldsymbol{Z}_p[\![\Gamma_\mathfrak{p}]\!]^{2(\dim A - r_\mathfrak{p})[k_\mathfrak{p}:\boldsymbol{Q}_p]}$

Proof. According to [12], Theorem 2.2 the group

$$H^2((\Gamma_\mathfrak{p})_n, A(k_{\infty\mathfrak{p}})) \cong (A(k_{\infty\mathfrak{p}}) \otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)_{(\Gamma_\mathfrak{p})_n}$$

(Lemma 2.4.i) is of finite order independent of n, n big enough. Therefore we obtain a quasi-exact sequence:

$$\begin{aligned} 0 &\longrightarrow \varprojlim_{n,m} {}_{p^m}H^1((\Gamma_\mathfrak{p})_n, A(k_{\infty\mathfrak{p}})) \longrightarrow \varprojlim_{n,m} {}_{p^m}H^1(k_{\mathfrak{p}n}, A) \\ &\longrightarrow \varprojlim_{n,m} {}_{p^m}H^1(k_{\infty\mathfrak{p}}, A)^{(\Gamma_\mathfrak{p})_n} \longrightarrow \varprojlim_{n,m} H^1((\Gamma_\mathfrak{p})_n, A(k_{\infty\mathfrak{p}}))_{p^m}. \end{aligned}$$

Again by [12] Theorem 2.2 the modules

$$\varprojlim_{n,m} {}_{p^m}H^1(k_{\mathfrak{p}n}, A) \cong (\varinjlim_{n,m} A'(k_{\mathfrak{p}n})_{p^m})^* = (A'(k_{\infty\mathfrak{p}}) \otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^*,$$
$$\varprojlim_{n,m} {}_{p^m}H^1(k_{\infty\mathfrak{p}}, A)^{(\Gamma_\mathfrak{p})^n} \approx \beta(H^1(k_{\infty\mathfrak{p}}, A)^*) \approx F_\Lambda H^1(k_{\infty\mathfrak{p}}, A)^*$$

are quasi-free of rank $(2\dim A - r_\mathfrak{p})[k_\mathfrak{p}: \boldsymbol{Q}_p]$ and $r_\mathfrak{p}[k_\mathfrak{p}: \boldsymbol{Q}_p]$, respectively. Since the fourth module in the sequence above is $\boldsymbol{Z}_p[\![\Gamma_\mathfrak{p}]\!]$-torsion we prove Claim 1.

Now the proof of the theorem will be accomplished once we have shown the quasi-surjectivity of the map

$$\psi = \varprojlim_{n,m} \psi_{n,m} : \varprojlim_{n,m} H^2(\mathcal{O}_n, \mathscr{A}_{p^m}) \longrightarrow \varprojlim_{n,m} H^2(\mathcal{O}_\infty/\mathcal{O}_n, \mathscr{A}_{p^m}).$$

(Observe that $F_\Lambda H^2(\mathcal{O}_\infty, \mathscr{A}(p))^*$ can be divided out of the first exact sequence in 2.2 (iii) in order to obtain the second, since a quasi-exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of compact Λ-modules of finite type induces a quasi-exact sequence

$$0 \longrightarrow M_1 \longrightarrow \ker(M_2 \twoheadrightarrow F_\Lambda M_3) \longrightarrow T_\Lambda M_3 \longrightarrow 0.)$$

Now, according to [8] Lemma 3 we have a commutative and exact diagram

$$\begin{array}{ccccccc}
H^{i-1}(Y_n, \mathscr{A}_{p^m}) & \longrightarrow & H^i_{\Sigma_n}(\mathcal{O}_n, \mathscr{A}_{p^m}) & \longrightarrow & H^i(\mathcal{O}_n, \mathscr{A}_{p^m}) & \longrightarrow & H^i(Y_n, \mathscr{A}_{p^m}) \\
\| & & \downarrow{\scriptstyle \varphi_{n,m}} & & \downarrow{\scriptstyle \psi_{n,m}} & & \| \\
H^{i-1}(Y_n, \mathscr{A}_{p^m}) & \longrightarrow & H^i_{\Sigma_n}(\mathcal{O}_\infty/\mathcal{O}_n, \mathscr{A}_{p^m}) & \longrightarrow & H^i(\mathcal{O}_\infty/\mathcal{O}_n, \mathscr{A}_{p^m}) & \longrightarrow & H^i(Y_n, \mathscr{A}_{p^m}),
\end{array}$$

where $Y_n = \mathcal{O}_n \backslash \Sigma_n$. If $A^i_{n,m}$ and $B^i_{n,m}$ denote the kernel and cokernel of the map $\varphi_{n,m}$ and $C^i_{n,m}$ and $D^i_{n,m}$ the kernel and cokernel of $\psi_{n,m}$, respectively, then we obtain exact sequences

$$0 \longrightarrow B^i_{n,m} \longrightarrow D^i_{n,m} \longrightarrow A^{i+1}_{n,m} \longrightarrow C^{i+1}_{n,m} \longrightarrow 0.$$

Claim 2. $\varprojlim_{n,m} B^2_{n,m} \approx 0.$

Proof. Because

$$H^2_{\Sigma_n}(\mathcal{O}_n, \mathscr{A}_{p^m}) = \bigoplus_{\mathfrak{p} \in \Sigma_n} {}_{p^m}H^1(k_{n\mathfrak{p}}, A),$$
$$H^2_{\Sigma_n}(\mathcal{O}_\infty/\mathcal{O}_n, \mathscr{A}_{p^m}) = \bigoplus_{\mathfrak{p} \in \Sigma_n} {}_{p^m}H^1(k_{\infty\mathfrak{p}}, A)^{\Gamma_{n\mathfrak{p}}},$$

[4] 5.1, 5.2 and [8] Lemma 7, we have

$$B^2_{n,m} = \bigoplus_{\mathfrak{p} \in \Sigma_n} \operatorname{coker}({}_{p^m}H^1(k_{n\mathfrak{p}}, A) \longrightarrow {}_{p^m}H^1(k_{\infty\mathfrak{p}}, A)^{\Gamma_{n\mathfrak{p}}}).$$

Hence by the exact sequence in the proof of Claim 1:

$$\varprojlim_{n,m} B^2_{n,m} \subseteq \varprojlim_{n,m} H^1(\Gamma_{n\mathfrak{p}}, A(k_{\infty\mathfrak{p}}))_{pm}.$$

From Lemma 2.4 (i) we obtain a surjection

$$((A(k_{\infty\mathfrak{p}})\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^{\Gamma_{n\mathfrak{p}}})_{pm} \twoheadrightarrow H^1(\Gamma_{n\mathfrak{p}}, A(k_{\infty\mathfrak{p}})_{\mathrm{Tor}})_{pm}.$$

Because

$$\varprojlim_{n,m} H^1(\Gamma_{n\mathfrak{p}}, \mathrm{Tor}(A(k_{\infty\mathfrak{p}})))_{pm} \approx 0,$$

$$\varprojlim_{n,m} ((A(k_{\infty\mathfrak{p}})\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^{\Gamma_{n\mathfrak{p}}})_{pm} \approx \dot{T}_\delta(A(k_{\infty\mathfrak{p}})\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^* \approx 0$$

by Lemma 1.1 and [12] Theorem 2.2 we obtain

$$\varprojlim_{n,m} H^1(\Gamma_{n\mathfrak{p}}, A(k_{\infty\mathfrak{p}}))_{pm} \approx 0$$

proving Claim 2.

Claim 3. $\varprojlim_{n,m} A^3_{n,m}$ and $\varprojlim_{n,m} C^3_{n,m}$ are finitely generated $\boldsymbol{Z}_p$-modules of the same rank.

Proof. We have the (quasi-) isomorphisms

$$\varprojlim_{n,m} H^3_{\Sigma_n}(\mathcal{O}_n, \mathscr{A}_{pm}) \cong (\varinjlim_{n,m} \bigoplus_{\mathfrak{p}\in\Sigma_n} H^0(\mathcal{O}_{n\mathfrak{p}}, \mathscr{A}'_{pm}))^* = \bigoplus_{\mathfrak{p}\in\Sigma} A'(k_{\infty\mathfrak{p}})(p)^*,$$

$$\varprojlim_{n,m} H^3_{\Sigma_n}(\mathcal{O}_\infty/\mathcal{O}_n, \mathscr{A}_{pm}) \cong \varprojlim_{n,m} \bigoplus_{\mathfrak{p}\in\Sigma_n} ({}_{pm}H^1(k_\infty A))_{\Gamma_{n\mathfrak{p}}} \approx \bigoplus_{\mathfrak{p}\in\Sigma} \dot{T}_\lambda H^1(k_{\infty\mathfrak{p}}, A)^*$$
$$\approx \bigoplus_{\mathfrak{p}\in\Sigma} T_\lambda A'(k_{\infty\mathfrak{p}})(p)^* \approx \bigoplus_{\mathfrak{p}\in\Sigma} A'(k_{\infty\mathfrak{p}})(p)^*$$

(by local flat duality, Lemma 1.1, [12] Theorem 2.2 and Theorem 3.4),

$$\varprojlim_{n,m} H^3(\mathcal{O}_n, \mathscr{A}_{pm}) \cong A'(k_\infty)(p)^*,$$

$$\varprojlim_{n,m} H^3(\mathcal{O}_\infty/\mathcal{O}_n, \mathscr{A}_{pm}) \cong \varprojlim_{n,m} H^2(\mathcal{O}_\infty, \mathscr{A}_{pm})_{\Gamma_n} \approx \varprojlim_{n,m} ({}_{pm}H^2(\mathcal{O}_\infty, \mathscr{A}^0(p)))_{\Gamma_n}$$
$$\approx \dot{T}_\lambda H^2(\mathcal{O}_\infty, \mathscr{A}^0(p))^* \approx A'(k_\infty)(p)^*$$

(by Lemma 1.1 and the assertion (i) of this theorem proven above). Because $A^4_{n,m}=0$ there is an isomorphism

$$\varprojlim_{n,m} B^3_{n,m} \cong \varprojlim_{n,m} D^3_{n,m}.$$

Together with the quasi-isomorphisms above this proves Claim 3.

Now, from Claims 2 and 3 it follows

$$\operatorname{coker}\psi=\varprojlim_{n,m}\operatorname{coker}\psi_{n,m}=\varprojlim_{n,m}D^2_{n,m}\approx 0.$$

This completes the proof of Theorem 2.2.

Corollary 2.5. *Let A be an abelian variety over k with good supersingular reduction at every prime above p. Assume $H^2(\mathcal{O}_\infty, \mathcal{A}(p))^*$ to be a Λ-torsion module. Then the Pontrjagin dual of the kernel of the canonical map*

$$H^1(\mathcal{O}_\infty, \mathcal{A}(p))\longrightarrow\prod_{\mathfrak{p}\in\Sigma}H^1(k_{\infty\mathfrak{p}}, A(p))$$

is quasi-isomorphic to $\dot{T}_\Lambda H^1(\mathcal{O}_\infty, \mathcal{A}'(p))^*$.

This result is also obtained by Billot [2] for elliptic curves which have complex multiplication. The Corollary 2.5 follows easily from the theorem, because the above map is dual to the projective limit of the maps

$$\bigoplus_{\mathfrak{p}\in\Sigma_n}H^1(k_{n\mathfrak{p}}, A'_{p^m})\longrightarrow\bigoplus_{\mathfrak{p}\in\Sigma_n}{}_{p^m}H^1(k_{n\mathfrak{p}}, A')$$
$$\hookleftarrow\bigoplus_{\mathfrak{p}\in\Sigma_n}{}_{p^m}H^1(\Gamma_{n\mathfrak{p}}, A(k_{\infty\mathfrak{p}}))\longrightarrow H^2(\mathcal{O}_n, \mathcal{A}_{p^m}).$$

Observe that the middle map is an isomorphism, i.e., the universal norms $NA(k_{n\mathfrak{p}})$ are zero in the supersingular case, [9] Theorem 1. We conclude this section with some easy consequences of the assumption that $Ш_n = Ш(A(k_n))(p)$ is finite for all n.

Proposition 2.6. *Let $Ш_n$ be finite for all n. Then*

i) $\operatorname{rank}_\Lambda H^2(\mathcal{O}_\infty, \mathcal{A}(p))^*\leq\operatorname{rank}_\Lambda(A(k_\infty)\otimes\boldsymbol{Q}_p/\boldsymbol{Z}_p)^*$,

ii) $\operatorname{rank}_\Lambda Ш^*_\infty\leq\sum_{\mathfrak{p}\in\Sigma}(\dim A-r_\mathfrak{p})[k_\mathfrak{p}:\boldsymbol{Q}_p]=s$.

Corollary 2.7. *Let $Ш_n$ be finite for all n. If $(A(k_\infty)\otimes\boldsymbol{Q}_p/\boldsymbol{Z}_p)^*$ is a Λ-torsion module, then* $\operatorname{rank}_\Lambda H^2(\mathcal{O}_\infty, \mathcal{A}(p))^*=0$ *and* $\operatorname{rank}_\Lambda Ш^*_\infty=s$.

Proof. Because of

$$\operatorname{rank}_\Lambda(A(k_\infty)\otimes\boldsymbol{Q}_p/\boldsymbol{Z}_p)^*+\operatorname{rank}_\Lambda Ш^*_\infty=s+\operatorname{rank}_\Lambda H^2(\mathcal{O}_\infty, \mathcal{A}(p))^*,$$

[8] Lemma 2.2, the second assertion follows from the first. Now, since $Ш(A(k_n))(p)$ is finite for all n, we obtain by the flat duality theorem and [8] Lemma 1.4 an isomorphism

$$\varprojlim_n \mathscr{A}^0(\mathcal{O}_n)\otimes \boldsymbol{Z}_p \rightrightarrows \varprojlim_{n,m} H^1(\mathcal{O}_n, \mathscr{A}_{p^m}) \cong H^2(\mathcal{O}_\infty, \mathscr{A}'(p))^*.$$

Hence

$$\begin{aligned}\operatorname{rank}_\Lambda H^2(\mathcal{O}_\infty, \mathscr{A}(p))^* &= \operatorname{rank}_\Lambda \varprojlim_n \mathscr{A}^0(\mathcal{O}_n)\otimes \boldsymbol{Z}_p \\ &= \operatorname{rank}_\Lambda \varprojlim_n A(k_n)\otimes \boldsymbol{Z}_p \qquad \text{(see Remark 2.3 ii)} \\ &\leq \operatorname{rank}_\Lambda F_\Lambda(A(k_\infty)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^*\end{aligned}$$

by Lemma 2.4 (ii).

§ 3. Ordinary reduction

In this section we will consider abelian varieties which have ordinary good reduction at p. As a direct consequence of Theorem 2.2 we obtain the following result

Theorem 3.1. *Let A be an abelian variety with ordinary good reduction at p. Then for $i \geq 0$ there are quasi-isomorphisms induced by the global flat duality*

$$T_\Lambda H^i(\mathcal{O}_\infty, \mathscr{A}'(p))^* \approx \alpha(T_\Lambda H^{2-i}(\mathcal{O}_\infty, \mathscr{A}^0(p))^*),$$
$$F_\Lambda H^i(\mathcal{O}_\infty, \mathscr{A}'(p))^* \approx \beta(F_\Lambda H^{3-i}(\mathcal{O}_\infty, \mathscr{A}(p))^*).$$

Remark 3.2. The quasi-isomorphism

$$T_\lambda H^i(\mathcal{O}_\infty, \mathscr{A}'(p))^* \approx \alpha(T_\lambda H^{2-i}(\mathcal{O}_\infty, \mathscr{A}(p))^*)$$

can be understood as pairing

$$T_\lambda H^i(\mathcal{O}_\infty, \mathscr{A}'(p))^* \times T_\lambda H^{2-i}(\mathcal{O}_\infty, \mathscr{A}(p))^* \longrightarrow \boldsymbol{Z}_p$$

with finite kernels. Indeed, the quasi-isomorphism is obtained from the discrete-compact pairing

$$\begin{array}{ccc}
H^i(\mathcal{O}_\infty, \mathscr{A}'(p)) & \times \varprojlim_{n,m} H^{3-i}(\mathcal{O}_n, \mathscr{A}_{p^m}) \longrightarrow \boldsymbol{Q}_p/\boldsymbol{Z}_p \\
\downarrow & \downarrow \approx \\
(T_\Lambda H^i(\mathcal{O}_\infty, \mathscr{A}'(p))^*)^* & \varprojlim_{n,m} H^{3-i}(\mathcal{O}_\infty/\mathcal{O}_n, \mathscr{A}_{p^m}) \\
\uparrow & \uparrow \text{quasi-injective} \\
 & \varprojlim_{n,m} ({}_{p^m}H^{2-i}(\mathcal{O}_\infty, \mathscr{A}(p)))_{\Gamma_n} \\
 & \uparrow \approx \\
(T_\lambda H^i(\mathcal{O}_\infty, \mathscr{A}'(p))^*)^* & \alpha(T_\lambda H^{2-i}(\mathcal{O}_\infty, \mathscr{A}(p))^*)
\end{array}$$

According to [11] Lemma 7.6 we obtain a pairing of compact Λ-modules. Now let

$$\kappa\colon G_k \longrightarrow Z_p^*$$

be the continuous character of the absolute Galois group of k corresponding to the Z_p-extension k_∞/k and let ϕ be a generator of $\Gamma = G(k_\infty/k)$. We define an Iwasawa L-function of A with respect to κ by

$$L_p(A, \kappa, s) = \prod_{i\geq 0} F_i(\kappa(\phi)^{s-1} - 1)^{(-1)^{i+1}}, \quad s \in Z_p,$$

where

$$F_i(t) = p^{\mu_i} \det(t - (\phi - 1);\, T_\Lambda H^i(\mathcal{O}_\infty, \mathscr{A}(p))^* \otimes Q_p)$$

is the characteristic polynomial of the Λ-torsion module $T_\Lambda H^i$ and μ_i denotes the μ-invariant of $H^i(\mathcal{O}_\infty, \mathscr{A}(p))^*$ (see also [8] § 2, where L_p is defined assuming that $H^i(\mathcal{O}_\infty, \mathscr{A}(p))$ is a Λ-torsion module). Using a polarization we obtain from Theorem 3.1 and [4] Lemma 7.1 a quasi-isomorphism

$$T_\lambda H^i(\mathcal{O}_\infty, \mathscr{A}(p))^* \approx \dot{T}_\lambda H^{2-i}(\mathcal{O}_\infty, \mathscr{A}(p))^*, \qquad i \geq 0,$$

which implies the following result

Corollary 3.3. *Let A be an abelian variety with good ordinary reduction at p. Then the Iwasawa L-function satisfies a functional equation with respect to $s \mapsto 2-s$:*

$$L_p(A, \kappa, s) = \varepsilon \cdot \kappa(\phi)^{(s-1)(2\lambda_0 - \lambda_1)} L_p(A, \kappa, 2-s),$$

where

$$\lambda_i = \operatorname{rank}_{Z_p} T_\lambda H^i(\mathcal{O}_\infty, \mathscr{A}(p))^*$$

$$\varepsilon = (-1)^r, \qquad r = \operatorname{ord}_{t=0} F_1(t).$$

Remark. The above corollary generalizes a result of Mazur and Schneider [4] Corollary 7.8, [8] p. 342, where k_∞/k is an admissible Z_p-extension, i.e., the bad primes split only finitely in k_∞/k and $H^i(\mathcal{O}_\infty, \mathscr{A}(p))^*$ are assumed to be Λ-torsion modules.

Proposition 3.4. *If A has good ordinary reduction at p and Ш$_n$ is finite for all n, then Ш$_\infty^*$ is a Λ-torsion module.*

This is a direct consequence of Proposition 2.6 (ii).

Proposition 3.5. *Let A be ordinary at p and let $Ш_n$ be finite for all n. Then*

$$T_\delta(A(k_\infty)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^* \approx 0,$$

i.e.,

$$(A(k_\infty)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^* \approx \Lambda^{\rho_2}\oplus \bigoplus_i \Lambda/\xi_i$$

for some polynomials ξ_i and $\rho_2 = \operatorname{rank}_\Lambda H^2(\mathcal{O}_\infty, \mathcal{A}(p))^*$.

Remark 3.6. This result should hold true without any conditions. For trivial reasons one also obtains this assertion if $(A(k_\infty)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^*$ is a Λ-torsion module: Since $A(k_\infty)$ is a discrete Γ-module, it is easy to see that $(A(k_\infty)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^*$ is fixed under the action of Γ_n, n big enough. Indeed, for $m \geq n$ there are injections

$$\begin{array}{ccc} (A(k_\infty)_{\mathrm{Tor}})^{\Gamma_n}\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p & \hookrightarrow & (A(k_\infty)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^{\Gamma_n} \\ \downarrow & & \downarrow \\ (A(k_\infty)_{\mathrm{Tor}})^{\Gamma_m}\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p & \hookrightarrow & (A(k_\infty)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^{\Gamma_m}, \end{array}$$

Lemma 2.4 (i). Since $A(k_\infty)_{\mathrm{Tor}}$ is discrete, i.e., $A(k_\infty)_{\mathrm{Tor}} = \bigcup_n (A(k_\infty)_{\mathrm{Tor}})^{\Gamma_n}$, we obtain

$$(A(k_\infty)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^* = \varprojlim_n ((A(k_\infty)_{\mathrm{Tor}})^{\Gamma_n}\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^*$$

where the limit is taken over the surjective norm maps. Since $(A(k_\infty)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^*$ is by assumption of finite $\boldsymbol{Z}_p$-rank, the projective system will become stationary. Hence

$$(A(k_\infty)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^* = (A(k_\infty)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^{*\Gamma_n}$$

for some $n \geq 0$. Now the general structure theory of compact Λ-modules of finite type proves the assertion above.

Proof of Proposition 3.5. We have to show that

$$\dot{T}_\delta(A(k_\infty)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^* \approx \varprojlim_{n,m} ((A(k_\infty)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^{\Gamma_n})_{p^m} \approx \varprojlim_{n,m} H^1(\Gamma_n, A(k_\infty)_{\mathrm{Tor}})_{p^m}$$

is finite. (Here we used Lemma 1.1 and 2.4i).

From the spectral sequence

$$H^i(\Gamma_n, H^j(\mathcal{O}_\infty, \mathcal{A})) \Longrightarrow H^{i+j}(\mathcal{O}_\infty/\mathcal{O}_n, \mathcal{A}),$$

[7] Appendix, we obtain exact sequences

$$
\begin{array}{ccccccc}
0 \longrightarrow H^1(\Gamma_n, A(k_\infty)) \longrightarrow H^1(\mathcal{O}_\infty/\mathcal{O}_n, \mathscr{A})(p) \xrightarrow[\varphi_n]{} H^1(\mathcal{O}_\infty, \mathscr{A})(p)^{\Gamma_n} \\
\downarrow \psi_n \\
H^2(\Gamma_n, A(k_\infty)) \\
\downarrow \\
(*) \quad 0 \longleftarrow H^2(\mathcal{O}_\infty, \mathscr{A})(p)^{\Gamma_n} \longleftarrow H^2(\mathcal{O}_\infty/\mathcal{O}_n, \mathscr{A})(p) \longleftarrow F_1(n) \longleftarrow 0 \\
\downarrow \\
H^1(\Gamma_n, H^1(\mathcal{O}_\infty, \mathscr{A})) \\
\downarrow \\
0
\end{array}
$$

where $F_1(n)$ denotes the first filtration step of $H^2(\mathcal{O}_\infty/\mathcal{O}_n, \mathscr{A})$. Since $H^1(\mathcal{O}_\infty, \mathscr{A})$ is a torsion group, we have

$$
\begin{aligned}
&H^2(\Gamma_n, H^1(\mathcal{O}_\infty, \mathscr{A}))=0,\\
&H^2(\mathcal{O}_\infty/\mathcal{O}_n, \mathscr{A}(p))\cong H^2(\mathcal{O}_\infty/\mathcal{O}_n, \mathscr{A})(p),\\
&H^2(\mathcal{O}_n, \mathscr{A}(p))\cong H^2(\mathcal{O}_n, \mathscr{A})(p),
\end{aligned}
$$

hence

$$F_1(n)\cong H^1(\mathcal{O}_\infty, \mathscr{A}(p))_{\Gamma_n}.$$

From the second spectral sequence

$$H^i(\mathcal{O}_n, R^j\pi_{\Gamma_n *}\mathscr{A})\Longrightarrow H^{i+j}(\mathcal{O}_\infty/\mathcal{O}_n, \mathscr{A}),$$

see [7] Appendix, we obtain the exact sequence

$$
\begin{aligned}
0\longrightarrow H^1(\mathcal{O}_n, \mathscr{A})(p)\longrightarrow H^1(\mathcal{O}_\infty/\mathcal{O}_n, \mathscr{A})(p)\longrightarrow \bigoplus_{\mathfrak{p}\in\Sigma} H^1(\Gamma_{n\mathfrak{p}}, A(k_{\infty\mathfrak{p}}))\\
\longrightarrow H^2(\mathcal{O}_n, \mathscr{A})(p)\longrightarrow H^2(\mathcal{O}_\infty/\mathcal{O}_n, \mathscr{A})(p)\longrightarrow 0
\end{aligned}
$$

using [8] Proposition 1.2. Therefore

$$H^i(\mathcal{O}_n, \mathscr{A})(p)\approx H^i(\mathcal{O}_\infty/\mathcal{O}_n, \mathscr{A})(p), \qquad i\geq 0,$$

where the defect is independent of n, n big enough, [8] Proposition 1.1 (iii). By the perfect duality 2.1 (iii) and the finiteness of $H^1(\mathcal{O}_n, \mathscr{A})(p)$ we obtain a quasi-exact sequence

$$0\longrightarrow \varprojlim_n H^1(\Gamma_n, A(k_\infty))\longrightarrow H^1(\mathcal{O}_\infty, \mathscr{A}'^0)(p)^*\longrightarrow \varprojlim_n H^1(\mathcal{O}_\infty, \mathscr{A})(p)^{\Gamma_n}$$

where the next term

$$\varprojlim_n H^2(\Gamma_n, A(k_\infty))\cong(T_\nu(A(k_\infty)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^*)^*$$

(Lemma 2.4i) is Z_p-torsion. Now $H^1(\mathcal{O}_\infty, \mathscr{A})(p)^*$ is a Λ-torsion module (3.4), hence

$$H^1(\mathcal{O}_\infty, \mathscr{A})(p)^* \approx T_{\delta_\infty} H^1(\mathcal{O}_\infty, \mathscr{A})(p)^* \oplus T_\varepsilon H^1(\mathcal{O}_\infty, \mathscr{A})(p)^*$$

and therefore

$$\varprojlim_n H^1(\mathcal{O}_\infty, \mathscr{A})(p)^{\Gamma_n} \approx \dot{T}_{\delta_\infty} H^1(\mathcal{O}_\infty, \mathscr{A})(p)^* \oplus (T_\varepsilon H^1(\mathcal{O}_\infty, \mathscr{A})(p)^* \otimes_{Z_p} Q_p).$$

Hence the quasi-exact sequence above shows

$$\varprojlim_n H^1(\Gamma_n, A(k_\infty)) \otimes Q_p = 0,$$

i.e., $\varprojlim_n H^1(\Gamma_n, A(k_\infty))$ is Z_p-torsion, and therefore

$$\varprojlim_{n,m} H^1(\Gamma_n, A(k_\infty))_{p^m}$$

can only have a μ-part. But this is impossible, since $T_\delta(A(k_\infty) \otimes Q_p/Z_p)^*$ has zero μ-invariant.

Theorem 3.6. *Let A be an abelian variety over k with ordinary good reduction at p and let $Ш(A(k_n))(p)$ be finite for all n. Then the global flat duality induces quasi-isomorphisms*

(i) $$T_\delta H^1(\mathcal{O}_\infty, \mathscr{A}')(p)^* \approx \alpha(T_\delta H^1(\mathcal{O}_\infty, \mathscr{A}^0)(p)^*),$$
$$T_\delta Ш_\infty(A')(p)^* \approx \alpha(T_\delta Ш_\infty(A)(p)^*)$$

and a quasi-exact sequence

(ii) $$0 \longrightarrow T_\nu Ш_\infty(A')(p)^* \longrightarrow \alpha(T_\Lambda(A(k_\infty) \otimes Q_p/Z_p)^*)$$
$$\longrightarrow T_\Lambda(A'(k_\infty) \otimes Q_p/Z_p)^* \longrightarrow \alpha(T_\nu Ш_\infty(A)(p)^*) \longrightarrow 0$$

Corollary 3.7. *Let the assumptions of* 3.4 *be fulfilled, then the following is true.*

(i) *Any divisor of the form ξ_r of $Ш_\infty(A)(p)^*$ is also a divisor of $(A(k_\infty) \otimes Q_p/Z_p)^*$. In particular, if $Ш_\infty(A)(p)^{\Gamma_n}$ is infinite then A has a k_n-rational point of infinite order.*

(ii) *The following assertions are equivalent*:

(a) *The Λ-torsion submodule* of $H^1(\mathcal{O}_\infty, \mathscr{A}(p))^*$ *is semi-simple by $\zeta_{p^n} - 1$ for all $n \geq 0$, i.e.,*

$$T_\varepsilon H^1(\mathcal{O}_\infty, \mathscr{A}(p))^* \approx \bigoplus_r \Lambda/\xi_r.$$

(b) *The Γ_n-invariants $Ш_\infty(A)(p)^{\Gamma_n}$ of $Ш_n(A)(p)$ are finite groups for all $n \geq 0$.*

(c) *There is a quasi-isomorphism*

$$\alpha(T_\Lambda(A(k_\infty)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^*) \approx T_\Lambda(A'(k_\infty)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^*$$

induced by the global flat duality.

Corollary 3.8. *Let A be an abelian variety with ordinary good reduction at p. Let $Ш_\infty(A)(p)^{\Gamma_n}$ be finite for all n and assume that $(A(k_\infty)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^*$ is a Λ-torsion module. Then there is a non-degenerate pairing*

$$A(k_\infty)\times A'(k_\infty) \longrightarrow \boldsymbol{Q}_p.$$

In particular, there are non degenerate pairings

$$A(k_n)\times A'(k_n) \longrightarrow \boldsymbol{Q}_p$$

for all n.

Proof of Theorem 3.6. From the descent diagram we derive the commutative and quasi-exact diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \varprojlim_{n,m} H^1(\mathcal{O}_n, \mathscr{A}^0)_{p^m} & \longrightarrow & \varprojlim_{n,m} H^1(\mathcal{O}_\infty/\mathcal{O}_n, \mathscr{A}^0)_{p^m} & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varprojlim_{n,m} H^2(\mathcal{O}_n, \mathscr{A}_{p^m}) & \longrightarrow & \varprojlim_{n,m} H^2(\mathcal{O}_\infty/\mathcal{O}_n, \mathscr{A}_{p^m}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \varprojlim_{n,m} {}_{p^m}H^2(\mathcal{O}_n, \mathscr{A}^0) & \longrightarrow & \varprojlim_{n,m} {}_{p^m}H^2(\mathcal{O}_\infty/\mathcal{O}_n, \mathscr{A}^0). & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

We will compute the projective limits. First, from the lower exact sequence in the diagram (*) of the proof of 3.5 with $\mathscr{A}^0$ instead of $\mathscr{A}$ we obtain a quasi-exact sequence

$$0 \longrightarrow \varprojlim_{n,m} {}_{p^m}(H^1(\mathcal{O}_\infty, \mathscr{A}^0(p))_{\Gamma_n}) \longrightarrow \varprojlim_{n,m} {}_{p^m}H^2(\mathcal{O}_\infty/\mathcal{O}_n, \mathscr{A}^0)$$
$$\longrightarrow \varprojlim_{n,m} {}_{p^m}H^2(\mathcal{O}_\infty, \mathscr{A}^0)^{\Gamma_n} \longrightarrow 0$$

hence

$$\varprojlim_{n,m} {}_{p^m}H^2(\mathcal{O}_\infty/\mathcal{O}_n, \mathcal{A}^0) \approx \alpha(T_\nu H^1(\mathcal{O}_\infty, \mathcal{A}(p))^*) \oplus \beta(F_\Lambda H^2(\mathcal{O}_\infty, \mathcal{A}(p))^*).$$

Because of

$$\varprojlim_{n,m} H^2(\mathcal{O}_\infty/\mathcal{O}_n, \mathcal{A}_{p^m}) \approx \alpha(T_\Lambda H^1(\mathcal{O}_\infty, \mathcal{A}^0(p))^*) \oplus \beta(F_\Lambda H^2(\mathcal{O}_\infty, \mathcal{A}(p))^*)$$

(see proof of 2.2) we obtain

$$\varprojlim_{n,m} H^1(\mathcal{O}_\infty/\mathcal{O}_n, \mathcal{A}^0)_{p^m} \approx \alpha(T_\delta H^1(\mathcal{O}_\infty, \mathcal{A}^0(p))^*).$$

Now, since $Ш_n$ is finite for all n we have by Theorem 2.1 (ii) an isomorphism

$$\varprojlim_{n,m} H^1(\mathcal{O}_\infty, \mathcal{A}^0)_{p^m} \cong H^1(\mathcal{O}_n, \mathcal{A}')(p)^*.$$

Furthermore

$$\varprojlim_{n,m} H^2(\mathcal{O}_n, \mathcal{A}_{p^m}) \cong H^1(\mathcal{O}_\infty, \mathcal{A}'(p))^*$$

hence

$$\varprojlim_{n,m} {}_{p^m}H^2(\mathcal{O}_n, \mathcal{A}^0) \cong (A'(k_\infty) \otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^*.$$

Therefore, the diagram above induces the commutative and quasi exact diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^1(\mathcal{O}_\infty, \mathcal{A}')(p)^* & \hookrightarrow & \alpha(T_\delta H^1(\mathcal{O}_\infty, \mathcal{A}^0(p))^*) \\
\downarrow & & \downarrow \\
H^1(\mathcal{O}_\infty, \mathcal{A}'(p))^* & \xrightarrow{\sim} & \alpha(T_\Lambda H^1(\mathcal{O}_\infty, \mathcal{A}^0(p))^*) \oplus \beta(F_\Lambda H^2(\mathcal{O}_\infty, \mathcal{A}(p))^*) \\
\downarrow & & \downarrow \\
(A'(k_\infty) \otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^* & \twoheadrightarrow & \alpha(T_\nu H^1(\mathcal{O}_\infty, \mathcal{A}(p))^*) \oplus \beta(F_\Lambda H^2(\mathcal{O}_\infty, \mathcal{A}(p))^*). \\
\downarrow & & \downarrow \\
0 & & 0
\end{array} \tag{+}$$

Since $H^1(\mathcal{O}_\infty, \mathcal{A}')(p)^*$ is Λ-torsion (3.4) we have a quasi-isomorphism

$$F_\Lambda(A'(k_\infty) \otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^* \approx \beta(F_\Lambda H^2(\mathcal{O}_\infty, \mathcal{A}(p))^*)$$

and a quasi-exact and commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(\mathcal{O}_\infty, \mathscr{A}')(p)^* & \longrightarrow & T_\Lambda H^1(\mathcal{O}_\infty, \mathscr{A}')(p)^* & \longrightarrow & T_\Lambda(A'(k_\infty)\otimes \mathbf{Q}_p/\mathbf{Z}_p)^* & \longrightarrow & 0 \\
 & & & & \uparrow \approx & & \vdots\uparrow \psi & & \\
0 & \longleftarrow & \alpha(H^1(\mathcal{O}_\infty, \mathscr{A}^0)(p)^*) & \longleftarrow & \alpha(T_\Lambda H^1(\mathcal{O}_\infty, \mathscr{A}^0(p))^*) & \longleftarrow & \alpha(T_\Lambda(A(k_\infty)\otimes \mathbf{Q}_p/\mathbf{Z}_p)^*) & \longleftarrow & 0
\end{array}$$

where the map ψ is induced by the quasi-isomorphism in the middle (Theorem 3.1).

Therefore the characteristic polynomial of $T_\Lambda(A(k_\infty)\otimes \mathbf{Q}_p/\mathbf{Z}_p)^* \approx T_\nu(A(k_\infty)\otimes \mathbf{Q}_p/\mathbf{Z}_p)^*$ (3.5) devides the characteristic polynomial of $T_\nu H^1(\mathcal{O}_\infty, \mathscr{A}(p))^*$. This shows that all horizontal maps in the diagram (+) are quasi-isomorphisms:

$$(3.9)\qquad \begin{aligned} &(A'(k_\infty)\otimes \mathbf{Q}_p/\mathbf{Z}_p)^* \approx \alpha(T_\nu H^1(\mathcal{O}_\infty, \mathscr{A}(p))^*)\oplus\beta(F_\Lambda H^2(\mathcal{O}_\infty, \mathscr{A}(p))^*), \\ &H^1(\mathcal{O}_\infty, \mathscr{A}')(p)^* \approx \alpha(T_\delta H^1(\mathcal{O}_\infty, \mathscr{A}(p))^*). \end{aligned}$$

Thus we obtain from the diagram above the quasi-exact sequence

$$0 \longrightarrow T_\nu\, \text{Ш}_\infty (A')(p)^* \longrightarrow \alpha(T_\Lambda(A(k_\infty)\otimes \mathbf{Q}_p/\mathbf{Z}_p)^*) \xrightarrow{\ \psi\ } T_\Lambda(A'(k_\infty)\otimes \mathbf{Q}_p/\mathbf{Z}_p)^*).$$

Obviously the cokernel of ψ is quasi-isomorphic to $\alpha(T_\nu\, \text{Ш}_\infty (A)(p)^*)$. Furthermore, the diagram above implies a quasi-injection

$$T_\delta H^1(\mathcal{O}_\infty, \mathscr{A}')(p)^* \hookrightarrow T_\delta H^1(\mathcal{O}_\infty, \mathscr{A}')(p)^*.$$

Hence, taking the adjoint and combining it with the quasi-isomorphism (3.9) we obtain a quasi-surjection

$$H^1(\mathcal{O}_\infty, \mathscr{A})(p)^* \approx \alpha(T_\delta H^1(\mathcal{O}_\infty, \mathscr{A}'(p))^*) \twoheadrightarrow \alpha(T_\delta H^1(\mathcal{O}_\infty, \mathscr{A}')(p)^*).$$

This proves Theorem 3.6.

Proof of Corollary 3.7. Since

$$T_\nu(A(k_\infty)\otimes \mathbf{Q}_p/\mathbf{Z}_p)^* \quad \text{and} \quad T_\nu H^1(\mathcal{O}_\infty, \mathscr{A}(p))^*$$

have the same characteristic polynomials the following assertions are equivalent:

$$\begin{aligned} &H^1(\mathcal{O}_\infty, \mathscr{A}(p))^* \text{ is semi-simple by } \zeta_{p^n}-1 \text{ for all } n\geq 0, \\ &T_\nu H^1(\mathcal{O}_\infty, \mathscr{A}(p))^* \approx T_\varepsilon H^1(\mathcal{O}_\infty, \mathscr{A}(p))^*, \\ &T_\nu H^1(\mathcal{O}_\infty, \mathscr{A})(p)^* \approx 0, \\ &\text{Ш}_\infty (A)(p)^{*\Gamma_n} \text{ is finite for all } n\geq 0. \end{aligned}$$

Because $\text{Ш}_\infty (A)(p)^*$ is Λ-torsion the last assertion is equivalent to

$$Ш_\infty (A)(p)^{\Gamma_n} \text{ is finite for all } n \geq 0.$$

The equivalence to (c) and the first assertion follow immediately from 3.6 (ii).

Proof of Corollary 3.8. This is a consequence of 3.7 (ii), observing that the maps

$$(A(k_\infty)\otimes \boldsymbol{Q}_p)^{\Gamma_n} \longrightarrow (A(k_\infty)\otimes \boldsymbol{Q}_p)_{\Gamma_n}, \qquad n \geq 0,$$

induced by the identity are isomorphisms because

$$(A(k_\infty)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^* \approx T_v(A(k_\infty)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p)^*.$$

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