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Perversity and Exponential Sums

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Respectfully dedicated to Kenkichi Iwasawa on his seventieth birthday

Introduction

This paper is devoted to the study of the mean absolute value of exponential sums. The situation we have in mind is the following. Fix an integer $r \geq 1$, and a closed subscheme X of $\boldsymbol{A}^r_{\boldsymbol{Z}} = \mathrm{Spec}(\boldsymbol{Z}[x_1, \cdots, x_r])$, the r-dimensional affine space over $\boldsymbol{Z}$. Suppose that the complex variety $X_{\boldsymbol{C}}$ is reduced and irreducible, of dimension $n \geq 1$. For each prime number p, and each r-tuple $(a) = (a_1, \cdots, a_r)$ of elements of $\boldsymbol{F}_p$, we denote by $S(p; (a))$ the exponential sum

$$S(p; (a)) := \sum_{(x) \text{ in } X(\boldsymbol{F}_p)} \exp((2\pi i/p)(\Sigma_i a_i x_i)).$$

For each prime number p we denote by $M(p)$ the mean absolute value of the "normalized" sums $S(p; (a))/(\sqrt{p})^n$;

$$M(p) := p^{-r} \sum_{(a) \text{ in } (\boldsymbol{F}_p)^r} |S(p; (a))/(\sqrt{p})^n|.$$

A useful way to think about $M(p)$ is this. For fixed p, we can view $S(p; (a))/(\sqrt{p})^n$ as a complex-valued function $f_p((a))$ on the finite abelian group $(\boldsymbol{F}_p)^r$ of (a)'s. If we endow this group with its normalized Haar measure of total mass one, then $M(p)$ is precisely the L^1 norm of the function f_p. The function f_p is, by its very definition, the Fourier transform of a certain function g_p on the Pontryagin dual group $A^r(\boldsymbol{F}_p)$, namely the function $g_p := (\sqrt{p})^{-n} \times$(the characteristic function of $X(\boldsymbol{F}_p)$).

The number of points in $X(\boldsymbol{F}_p)$ is p^n, up to an error term which is $O((\sqrt{p})^{2n-1})$. So with respect to the dual Haar measure on $A^r(\boldsymbol{F}_p)$, which gives every point mass one, the L^2 norm of the function g_p is equal to $1 + O(1/\sqrt{p})$. So by Parseval, the L^2 norm of f_p is also equal to $1 + O(1/\sqrt{p})$. Since the L^1 norm is bounded by the L^2 norm for total mass one, it follows that $M(p)$ is bounded by $1 + O(1/\sqrt{p})$.

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Our main concern is the extent to which this "trivial" bound for $M(p)$ can be improved. [In the language of probability, we are trying to bound from below the *variance* of the random variable

$$|f_p|\colon (a) \longrightarrow |S(p;(a))/(\sqrt{p})^n|$$

on $(\boldsymbol{F}_p)^r$.]

It turns out that the behavior of $M(p)$ in this regard is governed by a certain nonnegative integer invariant of the complex variety $X_{\boldsymbol{C}}$, which we call its "A-number". There are three essentially different patterns of behavior, depending on whether $A=0$, or $A=1$, or $A\geq 2$. More precisely, there exists a constant C such that

(1) If $A=0$, then for all primes p,

$$M(p)\leq C/\sqrt{p}.$$

(2) If $A=1$, then for all primes p,

$$|M(p)-1|\leq C/\sqrt{p}.$$

(3) If $A\geq 2$, then for all primes p,

$$M(p)\leq 1+C/\sqrt{p},$$

and the inequality

$$(*) \qquad M(p)\leq 1-1/4(1+A^2)$$

holds for all p in a set of primes of Dirichlet density $\geq 1/2A^4$. In fact, there exists a finite galois extension $K/\boldsymbol{Q}$ such that $(*)$ holds for any sufficiently large prime p which splits completely in K.

In view of this general result, a natural question arises: given a particular variety X, how does one determine which of the three cases one is in? In principle, one can calculate A for any given X as the difference of the Euler characteristics of $X_{\boldsymbol{C}}$ and of a general hyperplane section of $X_{\boldsymbol{C}}$ with coefficients in the intersection complex IC_X of $X_{\boldsymbol{C}}$. In practice this may be cumbersome to carry out completely, or even far enough to decide whether or not $A\geq 2$.

The case when X is a hypersurface in $\boldsymbol{A}^r$ is of particular interest because of the recent work of Hooley [Hoo]. Suppose that $r\geq 2$, and that $F(x_1,\cdots,x_r)$ is a nonzero weighted homogeneous polynomial with integral coefficients whose unique critical point (when viewed as a complex polynomial) is the origin 0 in $\boldsymbol{A}^r_{\boldsymbol{C}}$. Let k be an integer, and consider the variety X of equation

$$F(x_1, \cdots, x_r)=k.$$

We prove that X has $A\geq 2$ provided that *either* of the following conditions holds:

(1) The integer k is nonzero.

(2) $r\geq 3$, and F is a sum of one or more nonsingular homogeneous forms $F_{d(i)}$ of degrees $d(i)\geq 2$ in disjoint sets of variables, and at least one of the degrees $d(i)$ is ≥ 3.

This result is fairly sharp. For example, any hypersurface whose equation has the form $x_1+x_2=P(x_3, \cdots, x_r)$ has $A=0$, and the quadric $\Sigma_i(x_i)^2=0$ has $A=0$ or $A=1$, depending on whether $r\geq 3$ is even or odd.

The two major technical tools we use are the l-adic Fourier transform as developed in [K-L] and the theory of perversity and "middle extension" as developed in [B-B-D]. We will try to explain very roughly the way in which these two theories overlap. In trying to estimate the sum defining $M(p)$, there are two obvious difficulties. The first is that while the "general" term $S(p;(a))/(\sqrt{p})^n$ is uniformly $O(1)$, by [K-L], the less general ones can be quite a bit bigger (for instance, if $(a)=(0)$, the corresponding term is equal to ${}^{\#}X(F_p)/(\sqrt{p})^n$). (The idea of using perversity together with Fourier transform to control certain non-general exponential sums first occurs in Brylinski's seminal paper [Bry, 10.3].) The first miracle of (semi) perversity is that the total contribution to $M(p)$ of the non-general terms is $O(1/\sqrt{p})$. The second difficulty is that while the general term $S(p;(a))/(\sqrt{p})^n$ is "pure of weight zero" if X is nonsingular, it is only "mixed of weight ≤ 0" for X singular. The A-number occurs as the number of "pure of weight zero" eigenvalues in the general term. The second miracle of perversity is that by taking the Fourier transform not of (a Tate twist of a shift of) the constant sheaf on X but rather of its "middle extension" K from the smooth locus of X, we obtain as general term precisely the part of $S(p;(a))/(\sqrt{p})$ which is pure of weight zero, but we do not essentially change $M(p)$. Again, the non-general terms are not significant (this is the first miracle again), but now we deal with them by replacing K by a new "smoothed" object L whose Fourier transform agrees with that of K on the set of general points (a), and vanishes outside. It turns out that this object L also lives "over $\boldsymbol{Z}$". This fact of living over $\boldsymbol{Z}$ leads to the existence of a finite galois extension $K/\boldsymbol{Q}$ and a representation of its Galois group whose character governs the behavior of $M(p)$ as a function of p. The cases $A=0$, $A=1$, and $A\geq 2$ correspond successively to the cases of the zero representation, the trivial representation, and a representation of rank at least two.

We now turn to a summary of the various sections of this paper. The first section is concerned with technical sharpenings of some of the uniformity results of [K-L], and with some "well known" uniformity results on how the middle extension varies with parameters. The second section introduces the suitable generalization of $M(p)$ for quite general complexes K of sheaves on X, and studies the question of which changes in either K itself or in the domain of summation in the dual (a)-space are inoffensive for $M(p)$. The third section gives an elementary Parseval-based proof, which I learned from Hooley, of some of the results of Section 2. The fourth section justifies the transition first from the constant sheaf to its middle extension K, and then from K to its smoothed version L. The fifth section is devoted to the axiomatic study of the object L, and of the l-adic representations which govern the higher moments of its Fourier transform. The idea of studying the higher moments I learned from Hooley. The sixth section is devoted to the calculation of A-numbers. The seventh section summarizes the main results in what at present seem the most "applicable" cases. The appendix is devoted to studying variations on Brylinski's Radon transform and their relation to Fourier transform. Although technical, this relation is the key technical point in showing that "L lives over $\boldsymbol{Z}$".

My interest in this subject grew out of very stimulating correspondence and conversations with Hooley. Many of the ideas in sections 2, 4, and 5 emerged during invaluable discussions with Ofer Gabber, without whose essential contributions this paper could not have been written.

Section 1

(1.0.1) In this first section, we give some auxiliary lemmas on stratifications, and on how "middle extension" behaves with respect to parameters. Our basic references are [K-L section 3] and [B-B-D, section 2], whose terminology and results we will employ freely. For every prime number l, we fix a complex embedding of $\overline{\boldsymbol{Q}}_l$ into $\boldsymbol{C}$, so that we can speak unambiguously of weights. Schemes are always understood to be separated and noetherian. A good scheme S is one which admits a map f of finite type to a scheme T which is regular of dimension at most one.

(1.0.2) We begin with a technical strengthening of the results 3.2.1 and 3.3.2 of [K-L]. Let X be a good scheme, l a prime number, and K an object of $D^b_c(X[1/l], \overline{\boldsymbol{Q}}_l)$. We define supnorm $(K; X)$ to be the maximum, over all geometric points x of $X[1/l]$, of $\sum_i \dim \mathscr{H}^i(K)_x$. [In the notations of [K-L, 3.0], this supnorm is the maximum value attained by the function $\|K\|$.]

Lemma 1.1. *Let S be a good scheme all of whose maximal points have characteristic zero, X a smooth S-scheme of finite type, $D=\bigcup D_i$ a union of n smooth-over-S divisors in X with normal crossings relative to*

S, U the open set $X-D$, j: $U \to X$ the inclusion, l a prime number, and K an object of $D_c^b(U[1,/l], \bar{\mathbf{Q}}_l)$ which is lisse on $U[1/l]$. Then

$$\sup \operatorname{norm}(Rj_*K, X) \leq 2^n \sup \operatorname{norm}(K, U).$$

Proof. Under the hypotheses, Rj_*K is adapted to the stratification of X given by U and the various j-fold intersections, $j=1, \cdots, n$, of the D_i's, and the formation of Rj_*K commutes with arbitrary change of base on S. Thus it suffices to check over the maximal points of S, where the assertion is obvious on physical grounds. Q.E.D.

Lemma 1.2 (compare [K-L, 3.2.1]). *Let S be a good scheme, X an S-scheme of finite type, and $\mathscr{X}$ a stratification of X. Then there exists an integer $N \geq 1$, a dense open U in $S[1/N]$, an integer $C \geq 1$, and a stratification $\mathscr{X}'$ of X_U, with the following property: for any etale base change $X' \to X_U$, any prime number l, and any object K of $D_c^b(X'[1/l], \bar{\mathbf{Q}}_l)$ which is adapted to $\mathscr{X}$, the object $D_{X'/U}(K)$ is adapted to $\mathscr{X}'$, of formation compatible with arbitrary base change on U, and satisfies*

$$\sup \operatorname{norm}(D_{X'/U}(K), X') \leq C \sup \operatorname{norm}(K, X').$$

Proof. Simply add Lemma 1.1 to the proof of 3.2.1 as given in [K-L, 3.4.3]. Q.E.D.

(1.3.1) Let S be a good scheme, f: $X \to Y$ a morphism of S-schemes of finite type, $\mathscr{X}$ a stratification of X and $\mathscr{Y}$ a stratification of Y, and $C \geq 1$ an integer. We say that $(\mathscr{X}, \mathscr{Y}, C)$ is a strong $*$-stratification of f relative to S if for any etale base change $Y' \to Y$, any prime number l, any object K of $D_c^b(X_{Y'}[1/l], \bar{\mathbf{Q}}_l)$ which is adapted to $\mathscr{X}$, Rf_*K is adapted to $\mathscr{Y}$, of formation compatible with arbitrary change of base on S, and satisfies

$$\sup \operatorname{norm}(Rf_*K, Y') \leq C \sup \operatorname{norm}(K, X_{Y'}).$$

Lemma 1.3.2 (compare [K-L, 3.3.2]). *Let S be a good scheme, f: $X \to Y$ an S-morphism of S-schemes of finite type, and $\mathscr{X}$ a stratification of X. There exists an integer $N \geq 1$, an open dense U in $S[1/N]$, a stratification $\mathscr{Y}$ of Y_U, and an integer $C \geq 1$ such that $(\mathscr{X}, \mathscr{Y}, C)$ is a strong $*$-stratification of f_U relative to U.*

Proof. Repeat the proof of [K-L, 3.2.2]. Q.E.D.

(1.4.1) We say that a good scheme S is very good if it is irreducible and if there exists a connected regular scheme T of dimension at most one and a map $f: S \to T$ of finite type which is smooth and everywhere of some constant relative dimension n. Clearly an open set of a very good scheme is very good.

Lemma 1.4.2. *Suppose that S is a good scheme which is reduced and irreducible. There exists an integer $N \geq 1$ and a dense open U in $S[1/N]$ which is very good.*

Proof. Inverting a suitable N, we may assume that T has generic point of characteristic zero. If T is actually a field of characteristic zero, the result is obvious, and the general case follows by a standard "spreading out" argument. Q.E.D.

(1.4.3) Let S be a very good scheme, and X an S-scheme of finite type. A stratification $\mathscr{X} = \{X_a\}$ of X is said to be S-smooth, or to be a smooth stratification of X/S, if for each strat X_a there exists an integer d_a such that X_a is smooth and surjective over S with all of its geometric fibres equidimensional of dimension d_a.

Lemma 1.4.4. *Suppose S is good, reduced and irreducible. Given any stratification $\mathscr{X}$ of X/S, there exists an integer $N \geq 1$, and open dense U in $S[1/N]$ which is every good, and a smooth stratification $\mathscr{X}'$ of X_U/U which refines $\mathscr{X}_U$.*

Proof. By Lemma 1.4.2, we may suppose that S is already very good. By a standard "spreading out" argument, we may reduces to the case when S is the spectrun of a filed of characteristic zero, where the assertion is obvious. Q.E.D.

(1.5) Suppose that S is very good. Let l be a prime number, and K an object of $D^b_c(X[1/l], \bar{\mathbf{Q}}_l)$. Given a smooth stratification $\mathscr{X}$ of X/S, we say that K is strongly $\mathscr{X}$-perverse on $X[1/l]$ if it satisfies the following four conditions:

1.5.1. K is reflexive relative to S (cf. [K-L, 1.1.8]), i.e., the formation of both $K' := D_{X/S}(K)$ and $D_{X/S}(K')$ commute with arbitrary change of base on S.

1.5.2. Both K and $D_{X/S}(K)$ are adapted to $\mathscr{X}$.

1.5.3. For any integer i, and any strat X_a with $d_a > i$, we have

$$\mathscr{H}^{-i}(K) \,|\, X_a[1/l] = 0.$$

1.5.4. For any integer i, and any strat X_a with $d_a > i$, we have

$$\mathscr{H}^{-i}(D_{X/S}(K)) \,|\, X_a[1/l] = 0.$$

(1.6.1) We say that K is D-adapted to $\mathscr{X}$ if it satisfies conditions 1.5.1 and 1.5.2 above. We say that K is strongly $\mathscr{X}$-semiperverse if it satisfies conditions 1.5.1 and 1.5.3.

(1.6.2) Because S is assumed very good, the relative dualizing functor $D_{X/S}$ is just a shift and a Tate twist of the "absolute" one $D_{X/T}$, and consequently (cf. [De-2, 4.3]) $D_{X/S}$ satisfies biduality. Thus we may speak of $\mathscr{X}$-perversity, i.e., perversity with respect to the t-structure on $D_c^b(X[1/l], \bar{\mathbf{Q}}_l)$ defined by the stratification $\mathscr{X}$ of X and the perversity function $p(X_a) := -d_a$ on its strats, in the sense of [B-B-D, 2.1.2]. Condition 3., which we call $\mathscr{X}$-semiperversity, means precisely that K lies in ${}^pD^{\leq 0}$ for this t-structure. For an object K which is D-adapted [resp. adapted] to to $\mathscr{X}$, $\mathscr{X}$-perversity [resp. semiperversity] of K is equivalent to strong $\mathscr{X}$-perversity [resp. semiperversity] of K.

(1.6.3) Recall that on a scheme X of finite type over an algebracially closed field k of characteristic $\neq l$, an object K in $D_c^b(X, \bar{\mathbf{Q}}_l)$ is called semiperverse if it satisfies dimSupp $\mathscr{H}^{-j}(K) \leq i$ for all i, and that K is called perverse if both K and its dual $D_{X/k}(K)$ are semiperverse.

Lemma 1.6.4. *Suppose that S is very good. Given any K as above on $X[1/l]$, there exists an integer $N \geq 1$, and open dense U in $S[1/N]$, and a smooth stratification $\mathscr{X}$ of X_U/U such that K is D-adapted to $\mathscr{X}$.*

Proof. This follows immediately from [K-L, 3.2.1] and Lemma 1.4.4 above. Q.E.D.

Lemma 1.7. *Suppose that S is every good. If K is D-adapted* [resp. *adapted*] *to a smooth stratification $\mathscr{X}$ of X/S, and $X[1/l]$ is non-empty, then the following conditions are equivalent.*

1.7.1. *The restriction of K to every geometric fibre of $X[1/l]/S[1/l]$ is perverse* [*rrsp. semiperverse*].

1.7.2. *K is $\mathscr{X}$-perverse on $X[1/l]$* [*resp. $\mathscr{X}$-semiperverse*].

1.7.3. *K is strongly $\mathscr{X}$-perverse on $X[1/l]$* [*resp. strongly $\mathscr{X}$-semiperverse*].

Proof. The various lisse sheaves

$$\mathscr{H}^{-i}(K) \mid X_a[1/l], \qquad \mathscr{H}^{-i}(D_{x/S}(K)) \mid X_a[1/l]$$

vanish if and only if their restrictions to every geometric fibre of $X_{a[1/l]}/S[1/l]$ vanish. Q.E.D.

(1.8) Let S be a very good scheme, $j: X \to X'$ an open immersion of S-schemes, $\mathscr{X}$ a smooth stratification of X/S, $\mathscr{X}'$ a smooth stratification of X'/S, and $C \geq 1$ an integer. We say that $(\mathscr{X}, \mathscr{X}', C)$ is a !*-stratification of j if it satisfies the following two conditions:

1.8.1. $\mathscr{X}' | X$ coincides with $\mathscr{X}$.

1.8.2. For any etale $S' \to S$, and for any K on $X_{S'}[1/l]$ which is strongly $\mathscr{X}_{S'}$-perverse, its middle extension (cf. [B-B-D], 2.1.11]) $j_{!*}K$ to an $\mathscr{X}'_{S'}$-perverse sheaf on $X'_{S'}[1/l]$ is strongly $\mathscr{X}'_{S'}$-perverse, the formation of $j_{!*}K$ commutes with arbitrary base change on S', and

$$\text{sup norm}\,(j_{!*}K, X'_{S'}) \leq C\,\text{sup norm}\,(K, X_{S'})$$

Lemma 1.8.3. *Suppose that S is very good, that j: $X \to X'$ is an open immersion of S-schemes, and that $\mathscr{X}$ is a smooth stratification of X/S. Then there exists an integer $N \geq 1$, a dense open U in $S[1/N]$, an integer $C \geq 1$, and a smooth stratification $\mathscr{X}'$ of X'_U such that $(\mathscr{X}_U, \mathscr{X}', C)$ is a !-stratification of j.*

Proof. Let us denote by M the dimension of $(X' - X)$. We proceed by induction on M. If $M < 0$, there is nothing to prove. We may factor the given open immersion into a finite sequence of open immersions j: $X \to X'$ for which the closed subscheme $Z := (X' - X)$ is irreducible of some dimension $d \leq M$. By induction, we may assume the statement proven when $d < M$. If $d = M$, we apply Lemma 1.3.2 to our open immersion: after shrinking S, we obtain a stratification $\mathscr{X}'$ of X' and an integer C such that $(\mathscr{X}, \mathscr{X}', C)$ is a strong *-stratification of our open immersion j. Intersecting $\mathscr{X}'$ with Z, and taking the open strat of $\mathscr{X}' \cap Z$, say $Z - Z'$, we obtain a further factorization of $j : X \to X'$ as $X \to X' - Z' \to X'$, the second of which is O.K. by induction. Thus we may further assume that $(\mathscr{X}, (\mathscr{X}, Z), C)$ is a strong *-stratification of j. Now apply Lemma1.2 to $(\mathscr{X}, Z)$ on X'; we again shrink U and produce a stratification $\mathscr{X}''$ of X'_U such for any etale $U' \to U$, any K' on $X'_{U'}$ which is adapted to $(\mathscr{X}_{U'}, Z_{U'})$ is reflexive and $D_{X'/S}(K')$ is adapted to $\mathscr{X}''_U$. Replacing Z by the open strat of $\mathscr{X}'' \cap Z$, and correspondingly refactoring j we are reduced by induction to the case that

1. $(\mathscr{X}, (\mathscr{X}, Z), C)$ is a strong *-stratification of j

2. for any etale $U' \to U$, and any K' on $X'_{U'}$ adapted to $(\mathscr{X}_{U'}, Z_{U'})$ such that $K' | X_{U'}$ is D-adapted to $\mathscr{X}_{U'}$, K' is D-adapted to $(\mathscr{X}_{U'}, Z_{U'})$.

Again shrinking S and then Z, we may further assume that $(\mathscr{X}, Z)$ is a smooth stratification of X'/S. Let us now show that we may take $(\mathscr{X}, Z)$ together with the integer C as the required $\mathscr{X}'$. Denoting by d the relative dimension of Z over S, we see (by [B-B-D], 2.2.11] that for K $\mathscr{X}_{U'}$-perverse on $X_{U'}$, $j_{!*}K$ is just $\tau^{Z}_{<-d}(Rj_*K)$, which in virtue of 1 and 2. above is reflexive, adapted and hence D-adapted to $(\mathscr{X}_{U'}, Z_{U'})$, and of formation compatible with arbitrary change of base on S. By Lemma 1.7, we see that $j_{!*}K$ is indeed $(\mathscr{X}_{U'}, Z_{U'})$-perverse, since on each geometric fibre X'_s of $X'_{U'}/U'$ it is perverse, being the middle extension of $K|X_s$. The estimate for supnorms is obvious, since the operation $\tau^{Z}_{\leq -d}$ decreases the supnorm. Q.E.D.

Lemma 1.9. *In the situation* 1.8 *above, suppose that* S *is of finite type over* $\boldsymbol{Z}$, *and that* $(\mathscr{X}, \mathscr{X}', C)$ *is a* !*-*stratification of* j. *If* K *on* $X[1/l]$ *is strongly* $\mathscr{X}$-*perverse and is pure of some weight* w, *then the strongly* $\mathscr{X}'$-*perverse sheaf* $j_{!*}K$ *on* $X'[1/l]$ *is pure of the same weight* w.

Proof. That $j_!K$ and $j_{!*}D_{X/S}(K)$ on $X'_U[1/l]$ are mixed follows from [De-1, 6.2.3]. The punctual weights of their cohomology sheaves may be checked fibre by fibre over S, whence the assertion results from the purity theorem [B-B-D, 5.4.3] applied to the fibres. Q.E.D.

Lemma 1.10. *Suppose that* S *is a very good scheme of finite type over* $\boldsymbol{Z}$, *and that* V *is a smooth* S-*scheme with geometrically connected fibres all of dimension* $n\geq 0$. *Then there exists an integer* $N\geq 1$ *and a dense open* U *in* $S[1/N]$ *with the following property*: *For any prime number* l, *and any lisse* $\bar{\boldsymbol{Q}}_l$-*sheaf* $\mathscr{F}$ *on* $V_U[1/l]$ *which is pure of some weight* w, *the set of geometric points* u *in* $U[1/l]$ *such that* $\mathscr{F}\,|\,X_u$ *is irreducible as a* $\bar{\boldsymbol{Q}}_l$-*representation of* $\pi_1(V_u)$ *is either empty or it contains all the geometric points of* $U[1/l]$.

Proof. Simply apply [K-L, 3.1.2] to the morphism $\pi\colon V\to S$ to produce a U of the asserted type with the property that for any lisse $\bar{\boldsymbol{Q}}_l$-sheaf $\mathscr{F}$ on $V_U[1/l]$, all the $R^i\pi_!\mathscr{F}$ are lisse on $U[1/l]$. For $\mathscr{F}$ which is pure, each $\mathscr{F}\,|\,V_u$ is semi-simple (by [De-1, 3.4.12]), so the set of points in question is precisely the locus where the lisse sheaf $R^{2n}\pi_!(\mathrm{End}\,(\mathscr{F}))$ on $U[1/l]$ has fibres of dimension one. Q.E.D.

Lemma 1.11. *Let* S *be a scheme of finite type over* $\boldsymbol{Z}$, $d\geq 0$ *an integer, and* $\pi\colon W\to S$ *an* S-*scheme of finite type all whose geometric fibres have dimension* $\leq d$. *There exists a constant* K *such that for any pair* $(\boldsymbol{F}, s)$ *consisting of a finite field* $\boldsymbol{F}$ *and an* $\boldsymbol{F}$-*valued point* s *of* S, *we have*

$$\# W_s(F)\leq Kq^d,$$

with q *the cardinality of* $\boldsymbol{F}$.

Proof. Successively inverting two distinct primes on S, we may reduce to the case in which a prime number l is invertible on S. In this case we may take $K=\sup\mathrm{norm}\,(R\pi_!\bar{\boldsymbol{Q}}_l, S)$. Q.E.D.

Section 2

(2.0.1) In this section, we will begin our investigation of L^1 estimates for Fourier transforms.

(2.0.2) We fix a subring R of $\boldsymbol{C}$ which is finitely generated as a $\boldsymbol{Z}$-algebra, and which is smooth over $\boldsymbol{Z}$ everywhere of some relative dimension $d \geq 0$. We denote Spec (R) by S, which is thus a very good scheme over $\boldsymbol{Z}$. We fix an integer $r \geq 1$, and a rank r vector bundle $\boldsymbol{E}$ over S, with dual bundle $\boldsymbol{E}^{\vee}$ and canonical pairing $\langle\ ,\ \rangle$: $\boldsymbol{E}x\boldsymbol{E}^{\vee} \to \boldsymbol{A}^1_S$. Let l be a prime number, and K an object of $D^b_c(\boldsymbol{E}[1/l], \bar{\boldsymbol{Q}}_l)$.

(2.0.3) For each quadruple $(\boldsymbol{F}, s, \psi, e^{\vee})$ consisting of a finite field $\boldsymbol{F}$ of characteristic $\neq l$, an $\boldsymbol{F}$-valued point s of S, a nontrivial $\bar{\boldsymbol{Q}}_l$-valuep additive character ψ of $\boldsymbol{F}$, and an $\boldsymbol{F}$-valued point $e^{\vee}$ of $(\boldsymbol{E}^{\vee})_s$, we may form the exponential sum $S(K, \boldsymbol{F}, s, \psi, e^{\vee})$ defined

$$S(K, \boldsymbol{F}, s, \psi, e^{\vee}) := \sum_{e \text{ in } \boldsymbol{E}_s(\boldsymbol{F})} \psi(\langle e, e^{\vee}\rangle) \operatorname{trace}(\operatorname{Frob}_{e,\boldsymbol{F}} | K_e),$$

where we have written

$$\operatorname{trace}(\operatorname{Frob}_{e,\boldsymbol{F}} | K_e) := \textstyle\sum_i (-1)^i \operatorname{trace}(\operatorname{Frob}_{e,\boldsymbol{F}} | \mathscr{H}^i(K)_e).$$

(2.1) The L^1 norms we are interested in are the sums $|(K, \boldsymbol{F}, s, \psi)|$ defined as

$$|(K, \boldsymbol{F}, s, \psi)| := q^{-r} \sum_{e^{\vee} \text{ in } \boldsymbol{E}^{\vee}_s(\boldsymbol{F})} |S(K, \boldsymbol{F}, s, \psi, e^{\vee})|,$$

where we have written q for the cardinality of $\boldsymbol{F}$. Let us record the basic elementary properties of this norm.

2.1.1. The norm $|(K, \boldsymbol{F}, s, \psi)|$ is independent of the choice of the nontrivial additive character ψ of $\boldsymbol{F}$: indeed any other one is of the form $\psi_\alpha(x) = \psi(\alpha x)$ for some α in $\boldsymbol{F}^{\times}$, and the individual sums $S(K, \boldsymbol{F}, s, \psi, e^{\vee})$ obviously satisfy $S(K, F, s, \psi_\alpha, e^{\vee}) = S(K, \boldsymbol{F}, s, \psi, \alpha^{-1}e^{\vee})$.

2.1.2. For any shift $K[n]$ of K, $|(K[n], \boldsymbol{F}, s, \psi)| = |(K, \boldsymbol{F}, s, \psi)|$; indeed we have $(-1)^n S(K[n], \boldsymbol{F}, s, \psi, e^{\vee}) = S(K, \boldsymbol{F}, s, \psi, e^{\vee})$ for each $e^{\vee}$.

2.1.3. If K is a successive extension of finitely many K_i's, we have

$$|(K, \boldsymbol{F}, s, \psi)| \leq \textstyle\sum_i |(K_i, \boldsymbol{F}, s, \psi)|;$$

indeed we have $S(K, \boldsymbol{F}, s, \psi, e^{\vee}) = \sum_i S(K_i, \boldsymbol{F}, s, \psi, e^{\vee})$ for each $e^{\vee}$.

(2.2.1) In terms of the Fourier Transform $\mathrm{FT}_{s,\psi}$(denoted $\mathscr{F}_{s,\psi}$ in [K-L, 4.0]) of $K|\boldsymbol{E}_s$, we have

$$\begin{aligned} S(K, \boldsymbol{F}, s, \psi, e^{\vee}) &= (-1)^r \operatorname{trace}(\operatorname{Frob}_{e^{\vee},\boldsymbol{F}} | \mathrm{FT}_{s,\psi}(K|\boldsymbol{E}_s)) \\ &:= \textstyle\sum_i (-1)^{i+r} \operatorname{trace}(\operatorname{Frob}_{e^{\vee},\boldsymbol{F}} | \mathscr{H}^i(\mathrm{FT}_{s,\psi}(K|\boldsymbol{E}_s))). \end{aligned}$$

(2.2.2) This expression makes it natural to consider the auxiliary sums, one for each integer i, denoted $S_i(K, \boldsymbol{F}, s, \psi, e^\vee)$ and defined as

$$S_i(K, \boldsymbol{F}, s, \psi, e^\vee) := \mathrm{trace}\,(\mathrm{Frob}_{e^\vee, \boldsymbol{F}} \,|\, \mathscr{H}^i(\mathrm{FT}_{s,\psi}(K\,|\,\boldsymbol{E}_s))).$$

(2.2.3) The corresponding L^1 norms, one for each integer i, and denoted $|(K, \boldsymbol{F}, s, \psi)|_i$, are defined as

$$|(K, \boldsymbol{F}, s, \psi)|_i := q^{-r} \sum_{e^\vee \text{ in } \boldsymbol{E}_s^\vee(\boldsymbol{F})} |\mathrm{trace}\,(\mathrm{Frob}_{e^\vee, \boldsymbol{F}} \,|\, \mathscr{H}^i(\mathrm{FT}_{s,\psi}(K\,|\,\boldsymbol{E}_s)))|$$

$$:= q^{-r} \sum_{e^\vee \text{ in } \boldsymbol{E}_s^\vee(\boldsymbol{F})} |S_i(K, \boldsymbol{F}, s, \psi, e^\vee)|.$$

(2.2.4) Because the Fourier transform defined using ψ is carried into the one using ψ_α by the homothety "α" of $\boldsymbol{E}_s^\vee$, the norms $|(K, F, s, \psi)|_i$ are indepandent of the choice of the nontrivial additive character ψ.

(2.3) Shrinking S, we can, in virtue of Lemma 2, find a smooth stratification $\mathscr{E}$ of $\boldsymbol{E}$ to which K is adapted. Further shrinking S, we can (again in virtue of Lemma 2) find a smooth stratification $\mathscr{E}^\vee$ of $\boldsymbol{E}^\vee$ and an integer $C \geq 1$ which "works" in Theorem 4.1 of [K-L] with respect to $\mathscr{E}$, and with "$N=1$". For simplicity, we will refer to such an $(\mathscr{E}, \mathscr{E}^\vee, C)$ as a smooth FT stratification to which K is adapted.

Theorem 2.4. *Suppose that $(\mathscr{E}, \mathscr{E}^\vee, C)$ is a smooth* FT *stratification (cf.* 2.3*). There exists an integer $M_1 \geq 1$ depending only on $\mathscr{E}^\vee$, such that we have:*

Let l be a prime number, and K an object K of $D_c^b(\boldsymbol{E}[1/l], \overline{\boldsymbol{Q}}_l)$ which is adapted to $\mathscr{E}$, mixed of weight $\leq w$, and strongly $\mathscr{E}$-semiperverse. Then for any triple $(\boldsymbol{F}, s, \psi)$ consisting of a finite field $\boldsymbol{F}$ of characteristic $\neq l$, an $\boldsymbol{F}$-valued point s of S, and a nontrivial $\overline{\boldsymbol{Q}}_l$-valued additive character ψ of $\boldsymbol{F}$, we have the following estimates for the L^1 norms $|(K, \boldsymbol{F}, s, \psi)|_i$:

2.4.1. $|(K, \boldsymbol{F}, s, \psi)|_i = 0$ *if* $i > 0$.

2.4.2. $|(K, \boldsymbol{F}, s, \psi)|_i \leq M_1 C \sup \mathrm{norm}\,(K, \boldsymbol{E})(\sqrt{q})^{w-r-i}$ *if* $i + r \geq 0$.

2.4.3. $|(K, \boldsymbol{F}, s, \psi)|_i \leq C \sup \mathrm{norm}\,(K, \boldsymbol{E})(\sqrt{q})^{w+r+i}$ *if* $i + r < 0$.

Proof. We first explain where the constant M_1 comes from. Each strat $E_\alpha^\vee$ of $\mathscr{E}^\vee$ is of finite type over S with fibres of dimension d_α, so Lemma 1.11 shows that there exists a constant M_α such that for any pair (F, s) consisting of a finite field $\boldsymbol{F}$ and an $\boldsymbol{F}$-valued point s of S, we have the estimate $\#(E_\alpha^\vee)_s(\boldsymbol{F}) \leq M_\alpha(q)^{d_\alpha}$,. We then take for M_1 the sum of the M_α's over the finitely many strats of $\mathscr{E}^\vee$.

By hypothesis, the Fouier transform $\mathrm{FT}_{s,\psi}(K\,|\,\boldsymbol{E}_s)$ on $(E^\vee)_s$ is adapted to $(\mathscr{E}^\vee)_s$, and its supnorm is bounded by C sup norm $(K, \boldsymbol{E})$. Because

$K|E_s$ is semiperverse and mixed of weight $\leq w$, it follows (from the basic miracle [K-L, 2.1.3] of Fourier Transform, Weil II, and Artin's result [SGA4, XIV 3.1] as reformulated in [B-B-D, 4.1.1]) that $\mathrm{FT}_{s,\psi}(K|E_s)$ on $(E^\vee)_s$ is semiperverse and mixed of weight $\leq w+r$.

Now fix an integer i, and consider the i'th cohomology sheaf $\mathscr{H}^i := \mathscr{H}^i(\mathrm{FT}_{s,\psi}(K|E_s))$. By semiperversity, $\mathscr{H}^i=0$ if $i>0$, whence 2.4.1. For $0\geq i\geq -r$, $\mathscr{H}^i$ is mixed of weight $\leq w+r+i$, its fibres have rank at most $C\,\mathrm{sup\,norm}(K, E)$, it is adapted to $(\mathscr{E}^\vee)_s$, and by semiperversity it is supported in the union of those strats $(E_a^\vee)_s$ whose d_a is $\leq -i$. Therefore the number of points of $(E^\vee)_s(F)$ at which $\mathscr{H}^i$ is nonzero is bounded by $M_1 q^{-i}$, and at each of these points the absolute value of the trace of Frobenius is bounded by $C\,\mathrm{sup\,norm}(K, E)(\sqrt{q})^{w+r+i}$, whence 2.4.2. If $i<-r$, then $\mathscr{H}^i$ is mixed of weight $\leq w+r+i$, and its fibres have rank at most $C\,\mathrm{sup\,norm}(K, E)$. Since the number of points of $(E^\vee)_s(F)$ at which $\mathscr{H}^i$ is nonzero is trivially bounded by q^r, we get 2.4.3. Q.E.D.

Corollary 2.5. *Hypotheses and notations as in Theorem* 2.4 *above, we have the following estimates*:

(2.5.1) $$||(K, F, s, \psi)|-|(K, F, s, \psi)|_{-r}| \leq (4+4M_1)C\,\mathrm{sup\,norm}(K, E)(\sqrt{q})^{w-1}.$$

(2.5.2) $$|(K, F, s, \psi)|_{-r}\leq M_1 C\,\mathrm{sup\,norm}(K, E)(\sqrt{q})^{w}.$$

(2.5.3) $$|(K, F, s, \psi)|\leq(4+5M_1)C\,\mathrm{sup\,norm}(K, E)(\sqrt{q})^{w}.$$

Proof. We trivially have the inequality

$$||(K, F, s, \psi)|-|(K, F, s, \psi)|_{-r}|\leq \textstyle\sum_{j=-r}|(K, F, s, \psi)|_j.$$

We separately sum the estimates for $j=0, -1, \cdots, -(r-1)$ and the estimates for $j=-(r+1), -(r+2), \cdots$ and then bound the geometric series in $1/\sqrt{q}$. The "4" comes as the bound for $1/(1-1/\sqrt{q})$ in the worst case $q=2$, Q.E.D.

(2.6) We now introduce another variant on our L^1 norms. Let $U^\vee$ be an open set of $E^\vee$ which maps surjectively to S. We define the L^1 norms $|(K, F, s, \psi, U^\vee)|$ and $|(K, F, s, \psi, U^\vee)|_i$ by

$$|(K, F, s, \psi, U^\vee)| := q^{-r} \sum_{e^\vee \text{ in } (U^\vee)_s(F)} |S(K, F, s, \psi, e^\vee)|,$$

$$|(K, F, s, \psi, U^\vee)|_i := q^{-r} \sum_{e^\vee \text{ in } (U^\vee)_s(F)} |\mathrm{trace}(\mathrm{Frob}_{e^\vee, F}|\mathscr{H}^i(\mathrm{FT}_{s,\psi}(K|E_s)))|$$

$$:= q^{-r} \sum_{e^\vee \text{ in } (U^\vee)_s(F)} |S_i(K, F, s, \psi, e^\vee)|.$$

(2.6.1) If the open set $U^\vee$ is homogeneous, i.e., stable under the homotheties of $E^\vee$, then just as in 2.2.4 the norms $|(K, F, s, \psi, U^\vee)|_i$ are independent of the choice of the nontrivial additive character ψ. In the applications to follow, $U^\vee$ will in fact be homogeneous, but for the time being we do not need to suppose this.

Lemma 2.7. *Let $U^\vee$ be an open set of $E^\vee$ which maps surjectively to S. There exists a constant M_2 such that for any pair (F, s) consisting of a finite field F and F-valued point s of S, we have the estimate*

$$|{}^{\#}(U^\vee)_s(F) - q^r| \leq M_2 q^{r-1},$$

with q the cardinality of F.

Proof. Let us denote by $W := E^\vee - U^\vee$ the closed complement, and by $\pi: W \to S$ the projection. Then all the geometric fibres of π have dimension at most $r-1$, and we apply Lemma 1.11. Q.E.D.

Corollary 2.8. *Suppose that $(\mathscr{E}, \mathscr{E}^\vee, C)$ is a smooth FT stratification* (cf. 2.3), *and $U^\vee$ is an open set of $E^\vee$ which maps surjectively to S. Let M denote the integer $M := \sup(M_1, M_2)$, which depends only on $(\mathscr{E}^\vee, U^\vee)$. Let l be a prime number, and K an object K of $D^b_c(E[1/l], \bar{Q}_l)$ which is adapted to $\mathscr{E}$, mixed of weight $\leq w$, and strongly $\mathscr{E}$-semiperverse. Then for any triple (F, s, ψ) consisting of a finite field of F characteristic $\neq l$, an F-valued point s of S, and a nontrivial $\bar{Q}_l$-valued additive character ψ of F, we have the following estimates*:

$$||(K, F, s, \psi)| - |(K, F, s, \psi, U^\vee)|_{-r}| \leq |(5+4M)C \sup \operatorname{norm}(K, E)(\sqrt{q})^{w-1}.$$

$$||(K, F, s, \psi, U^\vee)| - |(K, F, s, \psi, U^\vee)|_{-r}|$$
$$\leq (4+4M)MC \sup \operatorname{norm}(K, E)(\sqrt{q})^{w-1}.$$

$$|(K, F, s, \psi, U^\vee)| \leq |(K, F, s, \psi)| \leq (4+5M)C \sup \operatorname{norm}(K, E)(\sqrt{q})^{w}.$$

$$||(K, F, s, \psi, U^\vee)| - |(K, F, s, \psi)|| \leq (9+8M)C \sup \operatorname{norm}(K, E)(\sqrt{q})^{w-1}.$$

Proof. It is immediate from the fact that $\mathscr{H}^{-r}$ is mixed of weight $\leq w$, with fibres of dimension at most $C \sup \operatorname{norm}(K, E)$, that we have the inequality

$$||(K, F, s, \psi)|_{-r} - |(K, F, s, \psi, U^\vee)|_{-r}| \leq MC \sup \operatorname{norm}(K, E)(\sqrt{q})^{w-2}.$$

which, in view of Corollary 2.5, proves the first assertion. The second and third assertions follows from and third assertions follows from Theorem 2.4 and the trivial inequalities

$$|(K, F, s, \psi, U^{\vee})|_i \leq |(K, F, s, \psi)|_i,$$
$$|(K, F, s, \psi, U^{\vee})| \leq |(K, F, s, \psi)|,$$

exactly as in the proof of Corollary 2.5. The fourth is immediate from the first two. Q.E.D.

Section 3

(3.0) We now explain, following Hooley, how an inequality of the form

$$|(K, F, s, \psi)| \leq O(\sqrt{q}^{w})$$

is an elementary consequence of Parseval's identity for the classical Fourier Transform on finite abelian groups. We do *not* know an elementary proof of the inequality

$$||(K, F, s, \psi, U^{\vee})| - |(K, F, s, \psi)|| \leq O(\sqrt{q}^{w-1}),$$

which seems to require the full mechanism of perversity.

Proposition 3.1 (Hooley). *Suppose that $\mathscr{E}$ is a smooth stratification of E/S. There exists an constant $M_3 > 0$ depending only on $\mathscr{E}$, such that we have*:

Let l be a prime number, and K an object K of $D^b_c(E[1/l], \bar{Q}_l)$ which is adapted to $\mathscr{E}$, mixed of weight $\leq w$, and strongly $\mathscr{E}$-semiperverse. Then for any triple (F, s, ψ) consisting of a finite field F of characteristic $\neq l$, an F-valued point s of S, and a nontrivial $\bar{Q}_l$-valued additive character ψ of F, we have the following estimate:

$$|(K, F, s, \psi)| \leq M_3 \sup \operatorname{norm}(K, E) \sqrt{q}^{w}.$$

Proof. The M_a are selected exactly as in the proof of Theorem 2.4, but this time with respect to the stratification $\mathscr{E}$ itself, so that that for any pair (F, s) consisting of a finite field and an F-valued point s of S, we have the estimate $^{\#}(E_a)_s(F) \leq M_a(q)^{d_a}$, with q the cardinality of F. The constant M_3 is then taken to be the sum of the $\sqrt{M_a}$'s. Let us fix a triple (F, s, ψ) consisting of a finite field F of characteristic $\neq l$, an F-valued point s of S, and a nontrivial $\bar{Q}_l$-valued additive character ψ of F. The Parseval identity for the usual Fourier transform from $E_s(F)$ to $E_s^{\vee}(F)$

$$\sum_{e \text{ in } E_s(F)} |\operatorname{trace}(\operatorname{Frob}_{e,F} | K_e)|^2 = q^{-r} \sum_{e^{\vee} \text{ in } E_s^{\vee}(F)} |S(K, F, s, \psi, e^{\vee})|^2,$$

and the Cauchy-Schwarz inequality on $E_s^{\vee}(F)$

$$q^{-r}\sum_{e^\vee \text{ in } \boldsymbol{E}_s^\vee(\boldsymbol{F})} |S(K, \boldsymbol{F}, s, \psi, e^\vee)| \leq (q^{-r}\sum_{e^\vee \text{ in } \boldsymbol{E}_s^\vee(\boldsymbol{F})} |S(K, \boldsymbol{F}, s, \psi, e^\vee)|^2)^{1/2}$$

give

$$\begin{aligned} |(K, \boldsymbol{F}, s, \psi)| &:= q^{-r}\sum_{e^\vee \text{ in } \boldsymbol{E}_s^\vee(\boldsymbol{F})} |S(K, \boldsymbol{F}, s, \psi, e^\vee)| \\ &\leq (\sum_{e \text{ in } \boldsymbol{E}_s(\boldsymbol{F})} |\mathrm{trace}(\mathrm{Frob}_{e,\boldsymbol{F}} | K_e)|^2)^{1/2} \\ &\leq \sum_{\text{strats } E_a} (\sum_{e \text{ in } (E_a)_s(\boldsymbol{F})} |\mathrm{trace}(\mathrm{Frob}_{e,\boldsymbol{F}} | K_e)|^2)^{1/2} \\ &\leq \sum_{\text{strats } E_a} (M_a q^{d_a} \sup_{e \text{ in } (E_a)_s(\boldsymbol{F})} |\mathrm{trace}(\mathrm{Frob}_{e,\boldsymbol{F}} | K_e)|^2)^{1/2} \\ &\leq \sum_{\text{strats } E_a} (M_a q^{d_a})^{1/2} \sup_{e \text{ in } (E_a)_s(\boldsymbol{F})} |\mathrm{trace}(\mathrm{Frob}_{e,\boldsymbol{F}} | K_e)|. \end{aligned}$$

We now easily estimate the individual Sup's. Since K is strongly $\mathscr{E}$-semiperverse, we have $\mathscr{H}^i K | E_a = 0$ unless $i + d_a \leq 0$. At each point e in $(E_a)_s(F)$, $\mathscr{H}^i K_e$ has weight $\leq w+i$, so we have

$$|\mathrm{trace}(\mathrm{Frob}_{e,\boldsymbol{F}} | K_e)| \leq \sup \mathrm{norm}(K, \boldsymbol{E})(\sqrt{q})^{w-d_a} \qquad \text{for } e \text{ in } (E_a)_s(\boldsymbol{F}).$$

Thus we obtain

$$|(K, \boldsymbol{F}, s, \psi)| \leq (\sum_{\text{strats } E_a} \sqrt{M_a}) \,\mathrm{supnorm}(K, \boldsymbol{E}) \sqrt{q}^{\,w},$$

as required. Q.E.D.

Section 4

(4.0) We continue to work in the setting of the previous two sections. We fix a subring R of $\boldsymbol{C}$ which is finitely generated as a $\boldsymbol{Z}$-algebra, and which is smooth over $\boldsymbol{Z}$ everywhere of some relative dimension $d \geq 0$. We denote $\mathrm{Spec}(R)$ by S, which is thus a very good scheme over $\boldsymbol{Z}$. We fix an integer $r \geq 1$, and a rank r vector bundle $\boldsymbol{E}$ over S, with dual bundle $\boldsymbol{E}^v$ and canonical pairing $\langle\ ,\ \rangle \colon \boldsymbol{E} \times \boldsymbol{E}^v \to \boldsymbol{A}^1_S$. We fix a closed subscheme X of $\boldsymbol{E}$, $i\colon X \to \boldsymbol{E}$ its inclusion, an open set V in X, and an integer $n \geq 0$, about which we make the following hypotheses:

4.0.1. The geometric fibres of X/S are all irreducible of dimension n.

4.0.2. V/S is smooth.

4.0.3. The geometric fibres of V/S are all irreducible of dimension n.

(4.1) We view V as being endowed with the S-lisse stratification $\mathscr{V} = \{V\}$ whose only strat is V alone. We also suppose given a smooth stratification $\mathscr{Z}$ of the complement $Z := X - V$. At the expense of shrinking S, we can find a smooth stratification $\mathscr{X}$ of X, and an integer $C_1 \geq 1$ such that $(\mathscr{V}=\{V\}, \mathscr{X}, C_1)$ is a !*-stratification for the inclusion $j: V \to X$. Shrinking S and refining $\mathscr{X} \cap Z$, we may assume that $\mathscr{X} \cap Z$ refines $\mathscr{Z}$. Another shrinking of S allows us to pick an integer $C_2 > 0$ such that $(\mathscr{V}=\{V\}, \mathscr{S}=\{S\}, C_2)$ is a universal !-stratification (cf. [K-L, 3.1.2] of the tautological morphism $\pi: V \to S$. We now view $\boldsymbol{E}$ as being endowed with the S-smooth stratification $\mathscr{E}:=(\mathscr{X}, E-X)$. Yet another shrinking of S allows us to find a smooth stratification $\mathscr{E}^\vee$ and an integer $C_3 \geq 1$ such that $(\mathscr{E}, \mathscr{E}^\vee, C_3)$ is a smooth FT stratification (2.3). A final shrinking of S allows us to pick a *homogeneous* open set $\boldsymbol{U}^\vee$ of $\mathscr{E}^\vee$ which maps surjectively to S and which "works" in Corollary 4.2 of [K-L] (cf. [K-L, 4.2.3]). We denote by M_1, M_2, and M_3 the constants in Theorem 2.4, Lemma 2.7, and Proposition 3.1 respectively; these depend on $\mathscr{E}^\vee$, $\boldsymbol{U}^\vee$, and $\mathscr{E}$ respectively. We denote by M_4 the constant

$$M_4 := 5 + 4 \sup(M_1, M_2) + M_3(1+C_1).$$

(4.2) Suppose now that l is a prime number, and $\mathscr{F}$ is a constructible $\bar{\boldsymbol{Q}}_l$-sheaf on $X[1/l]$ which is mixed of weight ≤ 0, adapted to $\mathscr{X}$, and whose restriction $j^*\mathscr{F}$ to $V[1/l]$ is lisse and pure of weight zero. By placing $j^*\mathscr{F}$ in degree $-n$ on $V[1/l]$, we obtain an object $j^*\mathscr{F}[n]$ on $V[1/l]$ which is strongly $\mathscr{V}$-perverse and pure of weight n. By Lemma 1.9, $j_{!*}(j^*\mathscr{F}[n])$ is strongly $\mathscr{X}$-perverse on $X[1/l]$ and pure of weight n. Let us denote it by K;

$$K := j_{!*}(j^*\mathscr{F}[n]).$$

Thus i_*K is strongly $\mathscr{E}$-perverse on $\boldsymbol{E}[1/l]$ and pure of weight n.

Lemma 4.3. *Notations and hypotheses as in* (4.0), (4.1), *and* (4.2) *above, for any triple* $(\boldsymbol{F}, s, \psi)$ *consisting of a finite field* $\boldsymbol{F}$ *of characteristic* $\neq l$, *an* $\boldsymbol{F}$*-valued point* s *of* S, *and a nontrivial* $\bar{\boldsymbol{Q}}_l$*-valued additive character* ψ *of* $\boldsymbol{F}$, *we have the estimate*

$$\|(i_*K, \boldsymbol{F}, s, \psi)| - |(i_*\mathscr{F}, \boldsymbol{F}, s, \psi)\| \leq M_3(1+C_1) \sup \operatorname{norm}(\mathscr{F}, X) \sqrt{q}^{\,n-1}.$$

Proof. The L^1 norms do not change if we replace K by $K[-1]$ and $\mathscr{F}$ by $\mathscr{F}[n-1]$. Let us denote by Z the closed set $Z := X - V$, and by $k := Z - X$ the inclusion. We first compare $\mathscr{F}[n-1]$ to $j_! j^*\mathscr{F}[n-1]$, and then we compare $j_! j^*\mathscr{F}[n-1]$, to $K[-1] = j_{!*}(j^*\mathscr{F}[n-1])$. In the first case, the difference is $k_*k^*\mathscr{F}[n-1]$, which is strongly $\mathscr{X}$-semiperverse and mixed of weight $\leq n-1$, with supnorm bounded by that of $\mathscr{F}$. So Proposition 3.1 gives

$$||(i_*\mathscr{F}[n-1], \boldsymbol{F}, s, \psi)|-|(i_*j_!j^*\mathscr{F}[n-1], \boldsymbol{F}, s, \psi)|| \\ \leq M_3 \sup \operatorname{norm}(\mathscr{F}, X)\sqrt{q}^{\,n-1}.$$

In the second case, the difference is $k_*k^*K[-1]$, whose supnorm is bounded by that of K, which is in turn bounded by C_1 supnorm $(\mathscr{F}, X)$. It remains to see that $k_*k^*K[-1]$ is strongly $\mathscr{X}$-semiperverse and mixed of weight $\leq n-1$. For once we know this, Proposition 3.1 will give

$$||(i_*K[-1], \boldsymbol{F}, s, \psi)|-|(i_*j_!j^*\mathscr{F}[n-1], \boldsymbol{F}, s, \psi)|| \\ \leq C_1 M_3 \sup \operatorname{norm}(\mathscr{F}, X)\sqrt{q}^{\,n-1}.$$

That $k_*k^*K[-1]$ is mixed of weight $\leq n-1$ is a formal consequence of the fact (cf. Lemma 1.9) that K is itself mixed of weight $\leq n$. Since $k_*k^*K[-1]$ is adapted to $\mathscr{X}$, its strongly $\mathscr{X}$-semiperversity may be checked (cf. Lemma 1.7) on the geometric fibres of X/S. Recall (cf. Lemma 1.8) that K is fibre by fibre over $S[1/l]$ the middle extension of a local system on $V[1/l]$ placed in degree $-n$. Therefore [B-B-D, 2.1.11] in each fibre we have $\dim \operatorname{Supp} \mathscr{H}^{-i}K \leq i-1$ for $i<n$. Since $\mathscr{H}^{-i}K=0$ for $i>n$, we see that for *every* i, $\dim \operatorname{Supp} \mathscr{H}^{-i}(k_*k^*K) \leq i-1$. This means exactly that $k_*k^*K[-1]$ is fibre by fibre semiperverse, as required. Q.E.D.

Lemma 4.4 (Hooley). *Notations and hypotheses as in* (4.0), (4.1), *and* (4.2) *above, suppose the restriction of $\mathscr{F}$ to each geometric fibre V_t of $V[1/l]/S[1/l]$ is an irreducible $\bar{\boldsymbol{Q}}_l$-representation of $\pi_1(V_t)$. Then for any triple $(\boldsymbol{F}, s, \psi)$ consisting of a finite field $\boldsymbol{F}$ of characteristic $\neq l$, an $\boldsymbol{F}$-valued point s of S, and a nontrivial $\bar{\boldsymbol{Q}}_l$-valued additive chaarcter ψ of $\boldsymbol{F}$, we have the estimate*

$$|(i_*\mathscr{F}, \boldsymbol{F}, s, \psi)| \\ \leq \sqrt{q}^{\,n}+[(1/2)C_2 \operatorname{rank}(j^*\mathscr{F})^2+M_3 \sup \operatorname{norm}(\mathscr{F}, X)]\sqrt{q}^{\,n-1}.$$

Proof. We have already noted in 4.3 above that comparing $\mathscr{F}$ to $j_!j^*\mathscr{F}$ yields, via Proposition 3.1,

$$||(i_*\mathscr{F}, \boldsymbol{F}, s, \psi)|-|(i_*j_!j^*\mathscr{F}, \boldsymbol{F}, s, \psi)|| \leq M_3 \sup \operatorname{norm}(\mathscr{F}, X)\sqrt{q}^{\,n-1}.$$

In order to estimate $|(i_*j_!j^*\mathscr{F}, \boldsymbol{F}, s, \psi)|$, we repeat the first part of the proof of Proposition 3.1 to find an inequality

$$|(i_*j_!j^*\mathscr{F}, \boldsymbol{F}, s, \psi)| \leq \Big(\sum_{e \text{ in } V_s(\boldsymbol{F})} |\operatorname{trace}(\mathrm{Frob}_{e,\boldsymbol{F}} | \mathscr{F}_e)|^2\Big)^{1/2}.$$

Because $j^*\mathscr{F}$ is pure of weight zero and lisse, we have

$$|\mathrm{trace}(\mathrm{Frob}_{e,\boldsymbol{F}}\,|\,\mathscr{F}_e)|^2 = \mathrm{trace}(\mathrm{Frob}_{e,\boldsymbol{F}}\,|\,\mathscr{E}nd(j^*\mathscr{F})_e).$$

By the Lefschetz trace formula, we find

$$\sum_{e \text{ in } V_s(\boldsymbol{F})} |\mathrm{trace}(\mathrm{Frob}_{e,\boldsymbol{F}}\,|\,\mathscr{F}_e)|^2$$
$$= \textstyle\sum_i (-1)^{i+1}\,\mathrm{trace}(\mathrm{Frob}_{s,\boldsymbol{F}}\,|\,H_c^i(V_s \otimes \bar{\boldsymbol{F}}, \mathscr{E}nd(j^*\mathscr{F}))).$$

The H_c^i with $i=2n$ is canonically $\bar{\boldsymbol{Q}}_l(-n)$ via the trace (thanks to the hypothesis of fibre by fibre geometric irreducibility of $j^*\mathscr{F}$), the lower ones have weight $\leq 2n-1$, and the higher ones vanish. Thus we have the estimate

$$\sum_{e \text{ in } V_s(\boldsymbol{F})} |\mathrm{trace}(\mathrm{Frob}_{e,\boldsymbol{F}}\,|\,\mathscr{F}_e)|^2 \leq q^n + C_2\,\mathrm{rank}(j^*\mathscr{F})^2 \sqrt{q}^{\,2n-1}.$$

Now divide both sides by q^n, take the square root, and estimate the right hand side by the inequality $\sqrt{1+x} \leq 1+x/2$ for $x \geq 0$. Q.E.D.

Lemma 4.5. *Notations and hypotheses as in* (4.0), (4.1) *and* (4.2) *above, for any triple* $(\boldsymbol{F}, s, \psi)$ *consisting of a finite field* $\boldsymbol{F}$ *of characteristic* $\neq l$, *an* $\boldsymbol{F}$*-valued point* s *of* S, *and a nontrivial* $\bar{\boldsymbol{Q}}_l$*-valued additive character* ψ *of* $\boldsymbol{F}$, *we have the estimate*

$$\|(i_*K, \boldsymbol{F}, s, \psi, \boldsymbol{U}^\vee)|_{-r} - |(i_*\mathscr{F}, \boldsymbol{F}, s, \psi)\| \leq M_4 \sup\mathrm{norm}(\mathscr{F}, X)\sqrt{q}^{\,n-1}.$$

Proof. Simply combine Lemma 4.3 and Corollary 2.8, part 1. Q.E.D.

Lemma 4.6. *Notations and hypotheses as in* (4.0), (4.1), *and* (4.2) *above, there exists an integer* $A \geq 0$ (*depending on* l *and* $\mathscr{F}$) *such that for any triple* $(\boldsymbol{F}, s, \psi)$ *consisting of a finite field* $\boldsymbol{F}$ *of characteristic* $\neq l$, *an* $\boldsymbol{F}$*-valued point* s *of* S, *and a nontrivial* $\bar{\boldsymbol{Q}}_l$*-valued additive character* ψ *of* $\boldsymbol{F}$, *the restriction to* $(\boldsymbol{U}^\vee)_s$ *of the Fourier Transform* $\mathrm{FT}_{s,\psi}(i_*K\,|\,\boldsymbol{E}_s)$ *is a single lisse sheaf* $\mathscr{G}(s, \psi)$ *of rank* A *on* $(\boldsymbol{U}^\vee)_s$, *pure of weight* n, *placed in degree* $-r$. *If the restriction of* $\mathscr{F}$ *to* V_s *is geometrically irreducible, then the lisse sheaf* $\mathscr{G}(s, \psi)$ *on* $(\boldsymbol{U}^\vee)_s$ *is geometrically irreducible. The integer* A *is given by the follownig formula*: *Over the generic point of* S (*i.e., over the fraction field* L *of* R), *pick an isomorphism of* $\boldsymbol{E}$ *with standard* $\boldsymbol{A}^r$, *via coordinates* $x_1, \cdots, x_r$. *Then pick* $r+1$ *complex constants* $\alpha_1, \cdots, \alpha_r, \beta$ *which are algebraically independent over* L. *Denote by* H *the intersection with* $X_{\boldsymbol{C}}$ *of the affine hyperplane of* $\boldsymbol{E}_{\boldsymbol{C}}$ *of equation* $\sum_i \alpha_i x_i = \beta$. *Then* A *is given by the formula*

$$A = \chi(X_{\boldsymbol{C}}; K) - \chi(H; K).$$

Proof. This is [K-L, 4.2 and 4.2.1], except for the irreducibility, which is [Bry, 9.11], and the sign, which is *incorrect* in [K-L] because the shift $[r]$ in the definition of Fourier Transform was temporarily forgotten in 4.1.1, 4.2.1, and 4.3.1 of [K-L]. Q.E.D.

Lemma 4.7. *Notations and hypotheses as in* (4.0), (4.1) *and* (4.2) *above, there exists an object L in $D^b_c(\boldsymbol{E}[1/l], \bar{\boldsymbol{Q}}_l)$ such that for any triple $(\boldsymbol{F}, s, \psi)$ consisting of a finite field $\boldsymbol{F}$ of characteristic $\neq l$, an $\boldsymbol{F}$-valued point s of S, and a nontrivial $\bar{\boldsymbol{Q}}_l$-valued additive character ψ of $\boldsymbol{F}$, the Fourier Transform* $\mathrm{FT}_{s,\psi}(L \mid \boldsymbol{E}_s)$ *is supported in $(\boldsymbol{U}^\vee)_s$, and on $(\boldsymbol{U}^\vee)_s$ it is isomorphic to $\mathscr{G}(s, \psi)$, the restriction to $(\boldsymbol{U}^\vee)_s$ of the Fourier Transform* $\mathrm{FT}_{s,\psi}(i_* K \mid \boldsymbol{E}_s)$.

Proof. Under Fourier Transform, additive convolution on $\boldsymbol{E}$ goes over to tensor product on $\boldsymbol{E}^\vee$ with a shift of $[-r]$,

$$\mathrm{FT}(K * L)[r] = \mathrm{FT}(K) \otimes \mathrm{FT}(L)$$

so it suffices to produce an L_0 such that for any triple $(\boldsymbol{F}, s, \psi)$ consisting of a finite field $\boldsymbol{F}$ of characteristic $\neq l$, an $\boldsymbol{F}$-valued point s of S, and a nontrivial $\bar{\boldsymbol{Q}}_l$-valued additive character ψ of $\boldsymbol{F}$, the Fourier Transform $\mathrm{FT}_{s,\psi}(L_0 \mid \boldsymbol{E}_s)$ is $[(\boldsymbol{U}^\vee)_s][r]$, the constant sheaf on $(\boldsymbol{U}^\vee)_s$ concentrated in degree $-r$, extended by zero. For once we have such an L_0, its convolution with $i_* K$ will serve as the required L. The key is that $\boldsymbol{U}^\vee$ is a *homogeneous* open set in $\boldsymbol{E}^\vee$, and consequently its characteristic sheaf $([\boldsymbol{U}^\vee])$ is the extension by zero to $\boldsymbol{E}^\vee$ of a sheaf on $\boldsymbol{E}^\vee - \{0\}$ which is itself the pullback of a sheaf from $\boldsymbol{P}\boldsymbol{E}^\vee$. For such sheaves $\mathscr{F}$, one knows (see the Appendix, Proposition A3) a universal Brylinski-Radon-transform type description over S of a complex $\mathscr{G}$ whose fibres over S Fourier transform to those of $\mathscr{F}$. Q.E.D.

Corollary 4.7.1. *Notations and hypotheses as in Lemma* 4.7 *above, we have*

$$|(i_* K, \boldsymbol{F}, s, \psi, \boldsymbol{U}^\vee)|_{-r} = |(L, \boldsymbol{F}, s, \psi)|,$$

and

$$\big||(L, \boldsymbol{F}, s, \psi)| - (i_* \mathscr{F}, \boldsymbol{F}, s, \psi)\big| \leq M_4 \sup \mathrm{norm}(\mathscr{F}, X) \sqrt{q}^{\,n-1}.$$

Proof. The first equality is a tautology in view of the defining property of L, and the second results from the first by Lemma 4.5. Q.E.D.

Let us begin by explicating the two extreme cases of low A, namely $A = 0$ and $A = 1$.

Corollary 4.8. *Notations and hypotheses as in* (4.0), (4.1), (4.2) *above, suppose that $A=0$. Then for any triple $(\boldsymbol{F}, s, \psi)$ consisting of a finite field $\boldsymbol{F}$ of characteristic $\neq l$, an $\boldsymbol{F}$-valued point s of S, and a nontrivial $\bar{\boldsymbol{Q}}_l$-valued additive character ψ of $\boldsymbol{F}$,*

$$|(i_*\mathscr{F}, \boldsymbol{F}, s, \psi)| \leq M_4 \sup \operatorname{norm}(\mathscr{F}, X)\sqrt{q}^{\,n-1}.$$

Proof. Indeed, if $A=0$ then $|(L, \boldsymbol{F}, s, \psi)|=0$, so the result is immediate from Corollary 4.7.1. Q.E.D.

Corollary 4.9. *Notations and hypotheses as in* (4.0), (4.1), (4.2) *above, suppose that $A=1$. Then for any triple $(\boldsymbol{F}, s, \psi)$ consisting of a finite field $\boldsymbol{F}$ of characteristic $\neq l$, an $\boldsymbol{F}$-valued point s of S, and a nontrivial $\bar{\boldsymbol{Q}}$-valued additive character ψ of $\boldsymbol{F}$,*

$$\big||(i_*\mathscr{F}, \boldsymbol{F}, s, \psi)| - \sqrt{q}^{\,n}\big| \leq 2M_4 \sup \operatorname{norm}(\mathscr{F}, X)\sqrt{q}^{\,n-1}.$$

Proof. Indeed, if $A=1$ then $|(L, \boldsymbol{F}, s, \psi)| = q^{-r}\,{}^{\#}(\boldsymbol{U}^{\vee})_s(\boldsymbol{F})\sqrt{q}^{\,n}$, so the result is immediate from Lemma 2.7 and Corollary 4.7.1. Q.E.D.

With these trivial cases taken care of, we can now state the main result.

Theorem 4.10. *Notations and hypotheses as in* (4.0) *and* (4.1) *above, there exists an open dense set U in S such that: for any constant M'', and for any l, $\mathscr{F}$ satisfying* (4.2) *whose restriction to every geometric fibre of $V[1/l]/S[1/l]$ is geometrically irreducible and whose A is ≥ 2, the set of closed points s of $U[1/l]$ for which, denoting by $\boldsymbol{F}(s)$ the residue field at s and by $q(s)$ the cardinality of $\boldsymbol{F}(s)$, the inequality*

$$(\sqrt{q})^{-n}\,|(i_*\mathscr{F}, \boldsymbol{F}(s), s, \psi)| \leq 1 - 1/4(1+A^2) - M''/\sqrt{q}(s)$$

holds for any (*or equivalently for every*) *choice of nontrivial ψ, contains a subset having Dirichlet density $\geq 1/2A^4$.*

Proof. In view of Lemma 4.7, it suffices to prove this with $i_*\mathscr{F}$ replaced by L, in which case it will be proven in 5.18.2. Q.E.D.

Remark (4.11.0). In passing from K to L in 4.7, we "lose" the explicit stratification $\mathscr{E}$ to which i_*K was adapted. The general results of [K-L] show that if we invert some integer $N\geq 1$, then we can find a stratification $\mathscr{E}'$ of $\boldsymbol{E}$, depending only on $\boldsymbol{U}^{\vee}$, to which L_0 is adapted. Applying [K-L] to the "sum" map on $\boldsymbol{E}$, we can, if we invert some integer $N'\geq 1$, produce a stratification $\mathscr{E}''$, depending only on $\mathscr{E}$ and $\mathscr{E}'$, to which

L is adapted. In fact, we will, in the following section 5, need also to consider the two-fold and four-fold convolutions $L_{1,1}$ and $L_{2,2}$ of L with itself and with $[x\to -x]^*DL$, then to pull these back to S by the zero-section, and then find a dense open U of S on which these pullbacks are both lisse. This is the open set U which figures in the statement of Theorem 4.10 above. In view of the above description of its provenance, U can be specified in terms only of the original $\mathscr{E}$, $U^\vee$ with which we began, but we will not make this explicit.

Section 5

(5.0.1) This section is devoted to the proof of Theorem 4.10. In view of Lemmas 4.6 and 4.7, the real object of study is the complex L whose Fourier Transform is so remarkable. In particular, the actual provenance of L can be completely forgotten.

(5.0.2) We will give estimates for the L^1 norm of L by studying the higher moments of L and the l-adic representations which govern these moments.

We begin by explaining the "real analysis" basis of such estimates.

Lemma 5.1. *Let (X, μ) be a measure space, with μ a positive measure of total mass one. Let f be a measurable complex-valued function on X, $\varepsilon\geq 0$ a real number. Suppose that*

1. *$|f|$ is bounded by a real constant M.*
2. $\displaystyle\int_X |f|^2 d\mu \leq 1+\varepsilon.$
3. $\displaystyle\int_X |f|^4 d\mu \geq 1+C-\varepsilon,$ *with $C\geq 0$.*

Then

$$\int_X |f| d\mu \leq 1+2\varepsilon - C/2(1+M)^2.$$

Proof. Replacing f by $|f|$, we may suppose that f is nonnegative. The ideal is to exploit the identity $(f^2-1)^2=(f+1)^2(f-1)^2$, in the following way.

$$\int (f^2-1)^2 d\mu = \int (f^4+1-2f^2) d\mu \geq 1+C-\varepsilon+1-2-2\varepsilon$$
$$\geq C-3\varepsilon.$$

$$\int (f+1)^2(f-1)^2 d\mu \leq (1+M)^2 \int (f-1)^2 d\mu$$
$$\leq (1+M^2) \int (f^2+1-2f) d\mu$$

$$\leq (1+M)^2\left(1+\varepsilon+1-2\int fd\mu\right)$$
$$\leq (1+M)^2\left(\varepsilon+2\left(1-\int fd\mu\right)\right).$$

Comparing, we find

$$C-3\varepsilon \leq (1+M)^2\left(\varepsilon+2\left(1-\int fd\mu\right)\right),$$

and the assertion follows. Q.E.D.

(5.2) We now turn to the l-adic side of the story. For simplicity (cf. 4.7.0), we fix the prime number l, and we suppose that S is a very good scheme of finite type over $\boldsymbol{Z}[1/l]$. Recall that $\boldsymbol{E}$ is a vector bundle of rank $r \geq 1$ over S, $\boldsymbol{E}^\vee$ is the dual vector bundle, and $\boldsymbol{U}^\vee$ is a homogeneous open set in which maps surjectively to S. We are given an object L in $D_c^b(E, \bar{\boldsymbol{Q}}_l)$, and integers n and A, about which we assume the following:

(5.2.1) for any triple $(\boldsymbol{F}, s, \psi)$ consisting of a finite field $\boldsymbol{F}$ of characteristic $\neq l$, an $\boldsymbol{F}$-valued point s of S, and a nontrivial $\bar{\boldsymbol{Q}}_l$-valued additive character ψ of $\boldsymbol{F}$, the Fourier Transform $\mathrm{FT}_{s,\varphi}(L|\boldsymbol{E}_s)$ is supported in $(\boldsymbol{U}^\vee)_s$, and on $(\boldsymbol{U}^\vee)_s$ it is a single lisse geometrically irreducible sheaf $\mathcal{G}(s, \psi)$ of rank $A \geq 1$, pure of weight n, placed in degree $-r$.

We will refer to the situation (5.2) above as the "axiomatic situation".

(5.3.1) We denote by

$$\mathcal{G}(s, \psi)^\vee$$

the lisse sheaf on $(\boldsymbol{U}^\vee)_s$ which is dual to $\mathcal{G}(s, \psi)$. Because $\mathcal{G}(s, \psi)$ is pure of weight n, the traces of $\mathcal{G}(s, \psi)^\vee(-n)$ are the complex conjugates of those of $\mathcal{G}(s, \psi)$.

(5.3.2) We denote by

$$\bar{\mathcal{G}}(s, \psi) := \mathcal{G}(s, \psi)^\vee(-n)$$

the lisse sheaf on $(\boldsymbol{U}^\vee)_s$ which is the "complex conjugate" of $\mathcal{G}(s, \psi)$.

(5.3.3) For each pair (a, b) of nonnegative integers, we define a lisse sheaf $\mathcal{G}_{a,b}(s, \psi)$ of weight $n(a-b)$ on $(\boldsymbol{U}^\vee)_s$ by

$$\mathcal{G}_{a,b}(s, \psi) := ((\mathcal{G}(s, \psi))^{\otimes a}) \otimes ((\bar{\mathcal{G}}(s, \psi))^{\otimes b})(nb),$$
$$:= ((\mathcal{G}(s, \psi))^{\otimes a}) \otimes ((\mathcal{G}(s, \psi)^\vee)^{\otimes b}),$$

so that

$$\mathscr{G}_{1,0}(s, \psi)=\mathscr{G}(s, \psi), \qquad \mathscr{G}_{0,1}(s, \psi)=\mathscr{G}(s, \psi)^{\vee}.$$

Lemma 5.4. *Suppose we are in the axiomatic situation* 5.2. *Then for each pair* (a, b) *of nonnegative integers, there exists an object* $L_{a,b}$ *in* $D_c^b(\boldsymbol{E}, \bar{\boldsymbol{Q}}_l)$ *with the following property: for any triple* $(\boldsymbol{F}, s, \psi)$ *consisting of a finite field* $\boldsymbol{F}$ *of characteristic* $\neq l$, *an* $\boldsymbol{F}$*-valued point* s *of* S, *and a nontrivial* $\bar{\boldsymbol{Q}}_l$*-valued additive character* ψ *of* $\boldsymbol{F}$, *the Fourier Transform* $\mathrm{FT}_{s,\psi}(L_{a,b}|\boldsymbol{E}_s)$ *is supported in* $(\boldsymbol{U}^{\vee})_s$, *and on* $(\boldsymbol{U}^{\vee})_s$ *it is the lisse sheaf* $\mathscr{G}_{a,b}(s, \psi)$, *placed in degree* $-r$.

Proof. The existence of $L_{0,0}$ is explained in the proof of Lemma 4.7, where $L_{0,0}$ is denoted simply L_0. The object L itself is $L_{1,0}$. Because Fourier transform carries convolution to tensor product with a shift of $[-r]$, it suffices to construct $L_{0,1}$, for once we have this, then $L_{a,b}$ is obtained by convolving together L a times and $L_{0,1}$ b times. Because FT commutes with duality relative to the base in the form

$$\mathrm{FT}_{s,\psi}([x\to -x]^*(D(K)))=D(\mathrm{FT}_{s,\psi}(K)),$$

the object $[x\to -x]^*(D(L))(r)$ has the desired Fourier transform on $(\boldsymbol{U}^{\vee})_s$, but its Fourier transform may not be supported in $(\boldsymbol{U}^{\vee})_s$. But since we have already constructed $L_{0,0}$, this support problem is no obstacle: we construct $L_{0,1}$ as the convolution of $L_{0,0}$ with $[x\to -x]^*(D(L))(r)$. Q.E.D.

Lemma 5.5. *Suppose we are in the axiomatic situation* 5.2. *Let us denote by* $[0]_S\colon S\to \boldsymbol{E}$ *the zero-section. For any triple* $(\boldsymbol{F}, s, \psi)$ *consisting of a finite field* $\boldsymbol{F}$ *of characteristic* $\neq l$, *an* $\boldsymbol{F}$*-valued point* s *of* S, *and a nontrivial* $\bar{\boldsymbol{Q}}_l$*-valued additive character* ψ *of* $\boldsymbol{F}$, *we have for every integer* i *an isomorphism of* $\bar{\boldsymbol{Q}}_l[\mathrm{Frob}_{s,\boldsymbol{F}}]$*-modules*

$$s^*(([0]_S)^*\mathscr{H}^{i-2r}(L_{a,b}))=H_c^i((\boldsymbol{U}^{\vee})_s\otimes\bar{\boldsymbol{F}}, \mathscr{G}_{a,b}(s, \psi))(r).$$

Proof. This is just Fourier inversion, applied along the zero-section fibre by fibre. Q.E.D.

(5.6) Suppose we are in the axiomatic situation 5.2. For each a, b, we pick a dense open set $U_{a,b}$ of S over which $([0]_S)^*\mathscr{H}^0(L_{a,b})$ is lisse. We dentoe by $\rho_{a,b}$ the corresponding l-adic representation of $\pi_1(U_{a,b})$. Notice that, in virtue of Lemma 5.4 and the fact that $\mathscr{G}_{a,b}(s, \psi)$ is pure of weight $n(a-b)$, $\rho_{a,b}$ is pure of weight $n(a-b)$. We denote by $M_{a,b}$ the supnorm$(L_{a,b}, \boldsymbol{E})$. For any triple $(\boldsymbol{F}, s, \psi)$ consisting of a finite field $\boldsymbol{F}$ of characteristic $\neq l$, an $\boldsymbol{F}$-valued point s of $U_{a,b}$, and a nontrivial $\bar{\boldsymbol{Q}}_l$-valued additive character ψ of $\boldsymbol{F}$, we denote by $\mathrm{Moment}_{a,b}(L, \boldsymbol{F}, s, \psi)$ the sum

$$\mathrm{Moment}_{a,b}(L, F, s, \psi) = q^{-r} \sum_{e^\vee \text{ in } (U^\vee)_s(F)} S(L_{a,b}, F, s, \psi, e^\vee).$$
$$= q^{-nb-r} \sum_{e^\vee \text{ in } (U^\vee)_s(F)} S(L, F, s, \psi, e^\vee)^a \overline{S(L, F, s, \psi, e^\vee)^b}.$$

With these notation, we have

Theorem 5.7. *Notations as in* 5.6 *above, suppose we are in the axiomatic situation* 5.2. *For each pair* (a, b) *of nonnegative integers, and for any triple* (F, s, ψ) *consisting of a finite field* F *of characteristic* $\neq l$, *an* F*-valued point* s *of* $U_{a,b}$, *and a nontrivial* $\bar{Q}_l$*-valued additive character* ψ *of* F, *we have the estimate*

$$|\mathrm{trace}(\rho_{a,b}(\mathrm{Frob}_{s,F})) - \mathrm{Moment}_{a,b}(L, F, s, \psi)| \leq M_{a,b}(\sqrt{q})^{n(a-b)-1}.$$

Proof. The Lefschetz trace formula expresses $\mathrm{Moment}_{a,b}(L, F, s, \psi)$ as the alternating sum

$$\textstyle\sum_i (-1)^i \mathrm{trace}(\mathrm{Frob}_{s,F} \,|\, H_c^i((U^\vee)_s \otimes \bar{F}, \mathcal{G}_{a,b}(s, \psi))(r)).$$

Because $\mathcal{G}_{a,b}(s, \psi)$ is pure of weight $n(a-b)$, the top term, $i=2r$, is pure of weight $n(a-b)$, and for s in $U_{a,b}$ this top term is equal to $\mathrm{trace}(\rho_{a,b}(\mathrm{Frob}_{s,F}))$. The lower terms are mixed of weight $\leq n(a-b)-1$. As the sum of the dimensions of all the H_c^i is bounded by $M_{a,b}$ in virtue of Lemma 5.5, the assertion is obivous. Q.E.D.

Theorem 5.8. *Suppose we are in the axiomatic situation* 5.2. *Then the representations* $\rho_{a,b}$ *are all projectively finite, i.e., their associated projective representations all have finite images. Moreover, if* $a=b$ *then the representations* $\rho_{a,a}$ *themselves have finite images.*

Proof. Fix (a, b). Let us begin by showing for any triple (F, s, ψ) consisting of a finite field F of characteristic $\neq l$, an F-valued point s of $U_{a,b}$, and a nontrivial $\bar{Q}_l$-valued additive character ψ of F, *some* power of each $\rho_{a,b}(\mathrm{Frob}_{s,F})$ is a scalar, and is 1 if $a=b$. This amounts to the statement that some power of $\mathrm{Frob}_{s,F}$ acts as a scalar (respectively, acts trivially if $a=b$) on

$$H_c^{2r}((U^\vee)_s \otimes \bar{F}, \mathcal{G}(s, \psi)^{\otimes a} \otimes ((\mathcal{G}(s, \psi))^\vee)^{\otimes b}).$$

This holds simply in virtue of the fact that $\mathcal{G}(s, \psi)$ is lisse and geometrically irreducible on $(U^\vee)_s$. Let us denote by $\Lambda\colon \pi_1 \to Gl(W)$ the irreducible l-adic representation of $\pi_1 := \pi_1((U^\vee)_s)$ corresponding to $\mathcal{G}(s, \psi)$, and by G the Zariski closure of $\Lambda(\pi_1^{\mathrm{geom}})$ in $Gl(W)$. Then by [De-1, 1.3.11] there exists an integer $N \geq 1$ and an element γ in π_1 of degree N such that $\Lambda(\gamma)$ lies in $\boldsymbol{G}_m \cdot G$ inside $GL(W)$. The sheaf $\mathcal{G}(s, \psi)^{\otimes a} \otimes ((\mathcal{G}(s, \psi))^{\vee})^{\otimes b}$ is the representation $W^{\otimes a} \otimes (W^{\vee})^{\otimes b}$ of π_1, and the action of $(\mathrm{Frob}_{s,\boldsymbol{F}})^N$ on $H_c^{2r}((\boldsymbol{U}^{\vee})_s \otimes \bar{\boldsymbol{F}}, \mathcal{G}(s, \psi)^{\otimes a} \otimes ((\mathcal{G}(s, \psi))^{\vee})^{\otimes b})(r)$ is the action of the element $\Lambda(\gamma)$ on the G-coinvariants $(W^{\otimes a} \otimes (W^{\vee})^{\otimes b})_G$. Because $\Lambda(\gamma)$ lies in $\boldsymbol{G}_m \cdot G$, say $\Lambda(\gamma) = \lambda g$ with g in G, its action on these twisted G-coinvariants is as the scalar λ^{a-b}.

To complete the proof, we apply the following general lemma lemma, either directly to $\rho_{a,a}$ or, if $a \neq b$, to the adjoint representation $\mathrm{End}(\rho_{a,b})$.

Lemma 5.8.1. *Let S be a connected scheme of finite type over $\boldsymbol{Z}$, l a prime number which is invertible on S, and $\rho\colon \pi_1 \to GL(V)$ a continuous finite-dimensional $\bar{\boldsymbol{Q}}_l$-representation of $\pi_1(S)$ which is definable over some finite extension E_λ of Q_l (i.e., ρ is a lisse $\bar{\boldsymbol{Q}}_l$-sheaf on S). Suppose that for every closed point s of S, some strictly positive power of* Frob_s *acts trivially on V. Then ρ has a finite image.*

Proof. In any finite-dimensional l-adic representation definable over some finite extension E_λ of $\boldsymbol{Q}_l$, all the eigenvalues of all the $\rho(\mathrm{Frob}_s)$ lie in a single finite extension E'_λ of E_λ (simply because these eigenvalues are all algebraic over E_λ of degree at most $\dim(\rho)$, and E_λ has only a finite number of extensions of any given degree). In our case, all these eigenvalues are certainly roots of unity, hence roots of unity in some finite extension E'_λ of E_λ, and hence they are all N'th roots of unity for some integer $N \geq 1$. Therefore each $(\rho(\mathrm{Frob}_s))^N$ is unipotent. But as *some* power of $(\rho(\mathrm{Frob}_s))^N$ is trivial, and as we are in characteristic zero, it follows that already $(\rho(\mathrm{Frob}_s))^N = 1$ for every closed point s in S. By Chebataroff, it follows that $\rho(\gamma)^N = 1$ for every γ in π_1. Now let G be the Zariski closure of the image of ρ in $GL(V)$. Then G is an algebraic subgroup of $GL(V)$ in which every element satisfies $g^N = 1$. Therefore $\mathrm{Lie}(G)$ is killed by N, and hence $\mathrm{Lie}(G)$, being an E_λ-vector space, vanishes. Therefore G is finite, and hence the image of ρ is finite. Q.E.D.

Corollary 5.9. *In the axiomatic situation* 5.2, *$a = b$ then $\rho_{a,a}$ has real-valued trace which is everywhere nonnegative.*

Proof. The moments $\mathrm{Moment}_{a,a}(L, \boldsymbol{F}, s, \psi)$ are real and nonnegative. For a given triple $(\boldsymbol{F}, s, \psi)$ consisting of a finite field $\boldsymbol{F}$ of characteristic $\neq l$, an $\boldsymbol{F}$-valued point s of $U_{a,a}$, and a nontrivial $\bar{\boldsymbol{Q}}_l$-valued additive character ψ of $\boldsymbol{F}$, the element $\rho_{a,a}(\mathrm{Frob}_{s,\boldsymbol{F}})$ is of some finite order $N \geq 1$, and every element of $\rho_{a,a}(\pi_1)$ is of this form. Now take any positive integer i which is $\equiv 1 \bmod N$, and consider the triple $(\boldsymbol{F}(i), s(i), \psi(i))$ defined by

$F(i) := $ the extension of F of degree i,
$s(i) := s$, viewed as an $F(i)$-valued point of S,
$\psi(i) := \psi \cdot$(the trace from $F(i)$ down to F).

Then for each such i we have

$$\rho_{a,a}(\mathrm{Frob}_{s(i),F(i)}) = \rho_{a,a}(\mathrm{Frob}_{s,F})^i = \rho_{a,a}(\mathrm{Frob}_{s,F}),$$

and so by Theorem 5.7 we have

$$|\mathrm{trace}(\rho_{a,a}(\mathrm{Frob}_{s,F})) - \mathrm{Moment}_{a,a}(L, F(i), s(i), \psi(i))| \leq M_{a,a}/\sqrt{q}^{\,i}.$$

Taking an increasing sequence of such i's, we obtain an expression for $\mathrm{trace}(\rho_{a,a}(\mathrm{Frob}_{s,F}))$ as the limit of the nonnegative real quantities $\mathrm{Moment}_{a,a}(L, F(i), s(i), \psi(i))$. This shows that trace $(\rho_{a,a})$ is real and everywhere nonnegative. Q.E.D.

Lemma 5.10. *In the axiomatic situation* 5.2, *if* $A=1$, *then all the representations* $\rho_{a,b}$ *are either one-dimensional or zero, and all the* $\rho_{a,a}$ *are one dimensional. If* $A \geq 2$, *then*

$$\dim(\rho_{1,1}) = 1.$$
$$A^4 \geq \dim(\rho_{2,2}) \geq 2.$$

Proof. For the moment, let (a, b) be any pair of nonnegative integers. Pick a triple (F, s, ψ) consisting of a finite field F of characteristic $\neq l$, an F-valued point s of $U_{a,b}$, and a nontrivial $\bar{\mathbf{Q}}_l$-valued additive character ψ of F. Let us denote by $\pi_1 := \pi_1((U^\vee)_s)$, by $\Lambda: \pi_1 \to Gl(W)$ the irreducible l-adic representation of π_1 corresponding to $\mathcal{G}(s, \psi)$, and by G the Zariski closure of $\Lambda(\pi_1^{\mathrm{geom}})$ in $Gl(W)$. Then

$$\dim(W) = A,$$
$$\dim(\rho_{a,b}) = \text{dimension of the } G\text{-coinvariants in } W^{\otimes a} \otimes (W^\vee)^{\otimes b}.$$

So for any (a, b), in particular for (2,2), we have the inequality

$$\dim(\rho_{a,b}) \leq A^{a+b},$$

which incidentally shows that if $A=1$, then all $\dim(\rho_{a,b}) \leq 1$, and $\dim(\rho_{a,a}) \leq 1$. Because G acts irreducibly on W, it acts semisimply on each tensor space $W^{\otimes a} \otimes (W^\vee)^{\otimes b}$.

Suppose now that $a=b$. Then

$$W^{\otimes a} \otimes (W^v)^{\otimes a} = \mathcal{E}nd(W^{\otimes a}),$$

so if we decompose $W^{\otimes a}$ as a sum of G-irreducibles V_i with multiplicities $n_{i,a}$,

$$W^{\otimes a} = \sum n_{i,a} V_i,$$

then we have

$$\dim(\rho_{a,a}) = \dim((\underline{\mathrm{End}}(W^{\otimes a}))_G) = \dim((\underline{\mathrm{End}}(W^{\otimes a}))^G) = \sum (n_{i,a})^2.$$

The G-irreducibility of W gives $\dim(\rho_{1,1}) = 1$. If $A \geq 2$, then the non-trivial direct-sum decomposition

$$W^{\otimes 2} = \mathrm{Symm}^2(W) + \Lambda^2(W)$$

shows that $\sum n_{i,2} \geq 2$, whence $\dim(\rho_{2,2}) = \sum (n_{i,2})^2 \geq 2$, as required. Q.E.D.

Speculation 5.11. In the axiomatic situation, suppose that $A \geq 2$. Then one might expect that the isomorphism class of the pair (G, W) consisting of the algebraic group G together with its representation W, which apparently depends on the choice of the triple $(\boldsymbol{F}, s, \psi)$ as well as on L, is actually independent of the choice of $(\boldsymbol{F}, s, \psi)$. One could further ask for some sort of a conjectural recipe for this isomorphism class. One might further expect that "in general", $\boldsymbol{G}_m \cdot G$ is either $GL(W)$, or open in $GO(W)$, or, if A is even, possibly $GSP(W)$. In such cases, $\dim(\rho_{2,2})$ would have the values 2, 3, 3 respectivelly. Can one produce examples where $\dim(\rho_{2,2}) \geq 4$?

Corollary 5.12. *In the axiomatic situation* 5.2, *if* $A \neq 0$ *then the one-dimensional representation* $\rho_{1,1}$ *is trivial.*

Proof. This follows immediate from Theorem 5.8 and Corollary 5.9, since the group $GL(1, \boldsymbol{R})^+$ of multiplicative positive reals contains no nontrivial finite subgroups. Q.E.D.

Lemma 5.13. *In the axiomatic situation* 5.2, *suppose that* $A \geq 1$. *Then for each integer* $a \geq 1$, $\mathrm{trace}(\rho_{a,a})$ *is everywhere* ≥ 1.

Proof. Fix a triple $(\boldsymbol{F}, s, \psi)$ consisting of a finite field $\boldsymbol{F}$ of characteristic $\neq l$, an $\boldsymbol{F}$-valued point s of $U_{a,a} \cap U_{1,1}$, and a nontrivial $\bar{\boldsymbol{Q}}_l$-valued additive character ψ of $\boldsymbol{F}$. Let us denote by X the finite set $(U^\vee)_s(\boldsymbol{F})$. Recall (Lemma 2.7) that

$$|\# X - q^r| \leq M_2 q^{r-1}.$$

Assume that $q > M_2$, so that X is nonempty. Endow X with the measure

μ which gives each point mass $1/\# X$. Consider the nonnegative function f on X whose value on $e^\vee$ in $(U^\vee)_s(F)=X$ is

$$\begin{aligned} f(e^\vee) &= |(\sqrt{q})^{-n} S(L, F, s, \psi, e^\vee)| \\ &= |(\sqrt{q})^{-n} \operatorname{trace}(\mathrm{Frob}_{e^\vee, F} \,|\, \mathcal{G}(s, \psi))| \end{aligned}$$

Then

$$\int_X f d\mu = (q^r/\# X)(\sqrt{q})^{-n} |(L, F, s, \psi)|, \tag{5.13.1}$$

and for any integer $a \geq 1$, we have

$$\int_X f^{2a} d\mu = (q^r/^\# X) \operatorname{Moment}_{a,a}(L, F, s, \psi). \tag{5.13.2}$$

One knows that on a measure space (X, μ) of total mass one, the L^p norms for $p \geq 1$ are increasing functions of p. Applying this to our function f, we find that for $a=1$

$$\int_X f^{2a} d\mu \geq \left(\int_X f^2 d\mu\right)^a.$$

In terms of moments, this gives

$$(q^r/\# X) \operatorname{Moment}_{a,a}(L, F, s, \psi) \geq ((q^r/\# X) \operatorname{Moment}_{1,1}(L, F, s, \psi))^a.$$

By combining Theorem 5.7 and Corollary 5.12, this gives

$$\operatorname{trace}(\rho_{a,a}(\mathrm{Frob}_{s,F})) \geq \operatorname{trace}(\rho_{1,1}(\mathrm{Frob}_{s,F})) + O(1/\sqrt{q}) \geq 1 + O(1/\sqrt{q}).$$

The result now follows by the same limit argument used in the proof of Corollary 5.9. Q.E.D.

Corollary 5.14. *In the axiomatic situation* 5.2, *if* $A=1$, *then the one dimensional representations* $\rho_{a,a}$ *are all trivial.*

Proof. Again, the group of multiplicative positive reals has no nontrivial finite subgroups. Q.E.D.

Corollary 5.15. *In the axiomatic situation* 5.2, *suppose* $A \geq 2$. *Then the mean value of* $\operatorname{trace}(\rho_{2,2})$ [*on the finite image of* $\rho_{2,2}$] *is* ≥ 2.

Proof. The mean value of the trace of a representation of a finite group is an integer, nemely the multiplicity of the trivial representation in the given one. By Lemma 5.13, every element has trace ≥ 1, and by

Lemma 5.10 the trace of the identity, namely $\dim(\rho_{2,2})$, is ≥ 2, so the mean trace is >1, and being an integer it is ≥ 2. Q.E.D.

Corollary 5.16. *In the axiomatic situation* 5.2, *suppose that* $A \geq 2$. *For any real nubmer* y *in the interval* $1 \leq y < 2$, *the fraction of the elements in* $\mathrm{Im}(\rho_{2,2})$ *whose trace is* $> y$ *is at least* $(2-y)/(\dim(\rho_{2,2})-y)$.

Proof. Let us denote by μ the Haar measure on $\mathrm{Im}(\rho_{2,2})$, normalized to have total mass 1. Thus

$$1 = \text{total mass} = \mu(\text{trace} \leq y) + \textstyle\sum_{x>y} \mu(\text{trace} = x),$$

and

$$\begin{aligned} 2 \leq \text{mean trace} &= \textstyle\sum_{x \leq y} x \cdot \mu(\text{trace} = x) + \sum_{x>y} x \cdot \mu(\text{trace} = x) \\ &\leq y \cdot \mu(\text{trace} \leq y) + \dim(\rho_{2,2}) \cdot \mu(\text{trace} > y), \\ &\leq y \cdot (1 - \mu(\text{trace} > y)) + \dim(\rho_{2,2}) \cdot \mu(\text{trace} > y), \end{aligned}$$

whence

$$2 - y \leq (\dim(\rho_{2,2}) - y) \cdot \mu(\text{trace} > y),$$

as required. Q.E.D.

Remark 5.16.1. An example of a representation ρ of a finite group Γ whose trace is real, ≥ 1, and with mean trace ≥ 2, is provided by the direct sum of the trivial representation and the regular representation, a case in which the estimate in the above lemma is exact for $y=1$. Notice also that if $\dim(\rho_{2,2})=2$ then $\rho_{2,2}$ must be trivial, since it contains the trivial representation at least twice!

(5.16.2) In terms of the constant M_2 of 2.7, and the constants $M_{1,1}$ and $M_{2,2}$ of 5.6, we define the constant M_5 by

$$M_5 := \sup(M_{2,2}, 2M_{1,1} + 2M_2).$$

Corollary 5.17. *In the axiomatic situation* 5.2, *suppose* $A \geq 2$. *For each real number* y *in the interval* $1 \leq y < 2$, *denote by* $\Gamma(y)$ *the conjugation-stable subset of the finite group* $\Gamma := \mathrm{Image}(\rho_{2,2})$ *consisting of all elements of* Γ *of trace* $\geq y$. *Then we have*

(5.17.1) $\#\Gamma(y)/\#\Gamma \geq (2-y)/(A^4 - y)$. $\quad \#\Gamma(1)/\#\Gamma = 1$.

(5.17.2) *for any triple* $(\boldsymbol{F}, s, \psi)$ *consisting of a finite field* $\boldsymbol{F}$ *of cardinality* $q \geq 2M_2$ *and characteristic* $\neq l$, *an* $\boldsymbol{F}$*-valued point* s *of* $U_{2,2} \cap U_{1,1}$, *and a nontrivial* $\overline{\boldsymbol{Q}}_l$*-valued additive character* ψ *of* $\boldsymbol{F}$, *we have*:

If $\rho_{2,2}(\mathrm{Frob}_{s,F})$ *lies in* $\Gamma(y)$, *then*

$$(\sqrt{q})^{-n}|(L, F, s, \psi)|\leq 1-(y-1)/2(1+A^2)+2M_5/\sqrt{q}.$$

Proof. The lower bound for the mass of $\Gamma(y)$ is given by Corollaries 5.13 (if $y=1$) and 5.16. The idea is to apply Lemma 5.1 to the function f considered in the proof of Lemmas 5.13. The hypothesis $q\geq 2M_2$ insures that the finite space $X:=U^\vee(F)$ is nonempty. In view of the identity 5.13.1 and the trivial inequality $q^r/\# X\geq 1$, we see that

$$\int_X |f|\,d\mu\geq(\sqrt{q})^{-n}|(L, F, s, \psi)|,$$

so we need only show that f satisfies the hypotheses of Lemma 5.1 with the choice of constants $C=y-1$, $M=A$, $\varepsilon=M_5/\sqrt{q}$. This results, via 5.13.2 applied with $a=1$, 2 from Lemma 2.7, Theorem 5.7 and Corollary 5.12. Q.E.D.

Corollary 5.18. *In the axiomatic situation* 5.2, *suppose* $A\geq 2$, *For each closed point s of* $U_{1,1}\cap U_{2,2}$, *denote by* $F(s)$ *its residue field and by* $q(s)$ *the cardinality of* $F(s)$.

(5.18.1) *The inequality*

$$(\sqrt{q})^{-n}|(L, F(s), s, \psi)|\leq 1+2M_5/\sqrt{q}(s)$$

holds for every closed point s of $U_{1,1}\cap U_{2,2}$ *with* $q(s)\geq 2M_2$, *and every choice of nontrivial* ψ.

(5.18.2) *For any constant* M'', *the set of closed points s of* $U_{2,2}\cap U_{1,1}$ *such that the inequality*

$$(\sqrt{q})^{-n}|(L, F(s), s, \psi)|\leq 1-1/4(1+A^2)-M''/\sqrt{q(s)}$$

holds for any (*or equivalently for every*) *choice of nontrivial* ψ, *contains a subset which has Dirihclet density* $\geq 1/2A^4$.

Proof. The first assertion is just 5.17 with $y=1$. For the second, apply 5.17 with $y=1.5+\varepsilon$, which makes the assertion obvious if q is sufficiently large, i.e., provided we remove from $U_{2,2}\cap U_{1,1}$ some finite set of closed points. Q.E.D.

Section 6

(6.0) In this section, we will show that $A\geq 2$ in a wide variety of cases of the form

$X=$ an irreducible n-dimensional subvariety of $E=A^r$, over C

$\mathscr{F}=$ the constant sheaf $\bar{Q}_l$ on X, extended by zero to E.

In the discussion which follows, it will always be understood that $\mathscr{F}$ is the constant sheaf, and all references to cohomology groups and Euler characteristics will be with respect to constant coefficients $\bar{Q}_l$ unless the coefficient sheaf is mentioned explicitly. We will speak simply of the "A-number" of X. We will freely make use of the comparison theorems between l-adic and "classical" cohomology of complex varieties. This will allow us when convenient to compute using singular cohomology with constant coefficients $\bar{Q}$ rather than l-adic cohomology with coefficients in $\bar{Q}_l$.

(6.1) We begin with a review of the basic facts, due to Goresky-Macpherson, about complex Lefschetz-type theorems for hyperplane sections in the quasi-affine setting.

(6.1.1) Suppose that X is an irreducible n-dimensional subvariety of an r-dimensional complex vector space E, and that V is a dense open set of X which is smooth. Thus V is a connected smooth n-dimensional locally closed subvariety of E. Let us denote by $E^\vee$ the linear dual of E. Pick coordinates $x_1, \cdots, x_r$ in E, and dual coordinates $y_1, \cdots, y_r$ in $E^\vee$.

(6.1.2) To a point $y=(y_1, \cdots, y_r)$ in $E^\vee$, we associate the affine hyperplane H_y in E of equation $\sum_i x_i y_i=1$. Thus H_y, as y varies over $E^\vee-\{O\}$, runs over all the affine hyperplanes in E which do not contain the origin. We denote by W the locally closed subvariety of $E\times E^\vee$ consisting all points (x, y) such that x lies in V and $\sum_i x_i y_i=1$. We denote by π: $W\to E^\vee$ the projection, and call it the universal family of affine hyperplane sections of V.

(6.1.3) One verifies easily (Jacobian criterion) that W is smooth of dimension $n+r-1$, and that π is flat over any point y in $E^\vee$ for which V is not contained in H_y. (To see this flatness, it suffices to show that for any closed point $\alpha=(\alpha_1, \cdots, \alpha_r)$ of $E^\vee$ for which the function $\sum_i \alpha_i x_i-1$ is nonzero on V, the r functions $y_i-\alpha_i$ form an M-sequence in the local ring of every closed point of W of the form (x_0, α). If we pick $r-n$ functions $f_i(x)$ which locally near x_0 define V in E, then it is equivalent to show that the $1+(r-n)+r$ functions $\sum_i x_i y_i-1, f_1(x), \cdots, f_r(x), y_1-\alpha_1, \cdots, y_r-\alpha_r$, form an M-sequence in the local ring of $E\times E^\vee$ at (x_0, α). But forming an M-sequence is independent of the order in which one takes the functions, so we may check by taking the functions in the opposite order. This reduces us to the statement that in the local ring of V at x_0, the function $\sum_i \alpha_i x_i-1$ is nonzero, which is precisely our hypothesis.)

(6.2) According to a fundamental result (cf. [Gor-Mac], Theorem

4.1) of Goresky and MacPherson, there exists an open dense set U in $\boldsymbol{E}^\vee$ with the following three properties:

6.2.1. For any u in U, the intersection $V \cap H_u$ is smooth of dimension $n-1$: indeed the morphism $\pi | \pi^{-1}(U)$ is smooth, everywhere of relative dimension $n-1$.

6.2.2. The sheaves $R^i \pi_! \bar{Q}$ are all lisse on U.

6.2.3. For any u in U, the inclusion of $V \cap H_u$ into V induces on homotopy groups π_i (resp. on cohomology groups H^i) a bijection for $i < n-1$, and a surjection (resp. an injection) for $i = n-1$.

(6.3) Given V, we will say that an affine hyperplane H in $\boldsymbol{E}$ is sufficiently general (with respect to V) if H does not contain the origin and if, when viewed as a point of $\boldsymbol{E}^\vee$, it lies in the dense open set U. For such a sufficiently general H, the restriction map on cohomology

$$H^{n-1}(V) \longrightarrow H^{n-1}(V \cap H)$$

is injective. We denote by $Ev^{n-1}(V \cap H)$ its cokernel.

Theorem 6.4. *Notations as above, if X is nonsingular then we have the inequality*

$$A \geq \dim H^n(X).$$

More precisely, if H is any affine hyperplane in $\boldsymbol{E}$ which is sufficiently general with respect to X, we have the formula

$$A = \dim H^n(X) + \dim Ev^{n-1}(X \cap H).$$

Proof. For X nonsingular, the middle extension K is just $\mathscr{F}[n]$ itself, so by Lemma 4.6, we have

$$A = (-1)^n [\chi(X) - \chi(X \cap H)].$$

Since X (resp. $X \cap H$) is affine of dimension n (resp. $n-1$), its H^i vanishes for $i > n$ (resp. for $i > n-1$), so the asserted exact formula for A follows immediately from the Lefschetz-style theorem of Goresky-MacPherson recalled in 6.2 above. The exact formula for A trivially implies the asserted inequality for A. Q.E.D.

Corollary 6.5. *Suppose that $r \geq 2$, and that $F(x_1, \cdots, x_r)$ is a nonzero weighted homogeneous polynomial whose unique critical point is the origin $\boldsymbol{O}$ in $\boldsymbol{E}$. Then for any nonzero constant α, the variety X in $\boldsymbol{E}$ of equation $F = \alpha$ has $A \geq 2$.*

Proof. For any nonzero weighted homogeneous F, and any nonzero α, the variety $F=\alpha$ is a nonsingular (by the generalized Euler identity) hypersurface of dimension $n=r-1$. This variety is irreducible if and only if $\alpha^{-1}F$, or equivalently F, is not a k'th power in $\boldsymbol{C}[x]:=\boldsymbol{C}[x_1, \cdots, x_r]$ for any $k\geq 2$. [To see this, we argue as follows. For some choice of weights for the variables x_i, F is isobaric of some weight $d\geq 1$. Introducing a new variable z of weight 1, we see that $F-\alpha$ factors in $\boldsymbol{C}[x]$ if and only if $F-\alpha z^d$ factors in $\boldsymbol{C}[x, z]:=\boldsymbol{C}[x_1, \cdots, x_r, z]$, i.e., if and only if $z^d-\alpha^{-1}F$ factors in $\boldsymbol{C}[x, z]$, or equivalently (thanks to Gauss's Lemma) in $\boldsymbol{C}(x)[z]$. By standard galois theory, $z^d-\alpha^{-1}F$ factors in $\boldsymbol{C}(x)[z]$ if and only if $\alpha^{-1}F=G^k$ for some $k\geq 2$ dividing d, and some G in $\boldsymbol{C}(x)$ (by normality any such G will lie in $\boldsymbol{C}[x]$). If the gradient $\mathrm{grad}(F)$ does not vanish at every $\boldsymbol{C}$-valued point of $F=0$, then F cannot be such a power, and consequently $F=\alpha$ will be irreducible. This last criterion applies in particular if $r\geq 2$ and F has only isolated critical points (in general, it applies if the critical points of F have codimension ≥ 2 in $\boldsymbol{C}^r$) to give the irreducibility of $F-\alpha$ for α nonzero.

If F has $\boldsymbol{O}$ as its unique critical point, then by [Mil, 9.2, 7.2, 6.5] it follows that for any nonzero α, $H^n(F=\alpha)$ has dimension $\mu(F)$, where

$$\mu(F)=\dim(\boldsymbol{C}[x_1, \cdots, x_r]/\text{ the ideal generated by the } \partial F/\partial x_i).$$

Moreover, $H^i(F=\alpha)$ vanishes for $i\neq 0$, n, and the H^0 is one-dimensional.

In view of Theorem 3.4, we have $A\geq\mu(F)$. Because $\boldsymbol{O}$ is a critical point of F, we must have $\mu(F)\geq 1$. If $\mu(F)\geq 2$, we are done.

If $\mu(F)=1$, then the ideal $\mathfrak{I}$ generated by the $\partial F/\partial x_i$, which by hypothesis defines a nonempty artinian scheme concentrated at the origin, must be the entire ideal $\mathfrak{m}=(x_1, \cdots, x_r)$. By looking at what this means modulo $\mathfrak{m}^2$, we see that after a linear change of coordinates, F must have the form

$$F=\textstyle\sum_{1\leq i\leq r}(x_i)^2+\text{higher terms}.$$

By the weighted homogeneity of F, there can be no higher terms, i.e., we are dealing with

$$F=\textstyle\sum(x_i)^2.$$

We claim that in this case we have, for any sufficiently general affine hyperplane H in $\boldsymbol{E}$,

$$\dim Ev^{n-1}((F=\alpha)\cap H)=1,$$

so that in fact $A=2$.

We first treat the case $r=2$. In this case, $n-1=0$, and $(F=\alpha)\cap H$ consists of two points, so $\dim Ev^{n-1}((F=\alpha)\cap H)=1$, as required. Suppose now that $r\geq 3$. Then $H^{n-1}(F=\alpha)$ vanishes, and so we are assertnig that $\dim H^{n-1}((F=\alpha)\cap H)=1$. To see this, pass to the r-dimensional projective space $\boldsymbol{P}^r$ with homogeneous coordinates $z, x_1, \cdots, x_r$, and consider the smooth quadric $\bar{X}$ in $\boldsymbol{P}^r$ of homogeneous equation

$$\alpha z^2=\sum (x_i)^2,$$

which is the closure of $F=\alpha$ in $\boldsymbol{P}^r$, and the smooth hyperplane $\bar{H}$ in $\boldsymbol{P}^r$ which is the closure of H. Clearly the topology of $(F=\alpha)\cap H$ does not change as H varies over hyperplanes such that $\bar{H}$ is transverse both to $\bar{X}$ and to its "part at infinity" $\bar{X}\cap(z=0)$. As the particular H of affine equation $x_r=0$ is such a hyperplane, we may use it to calculate with. But for this H, $(F=\alpha)\cap H$ is just $\sum (x_i)^2=\alpha$ in one fewer variable, and so just as above its H^{n-1} has dimension $\mu(F|x_r=0)=1$. Q.E.D.

Remark 6.6. The reader will find a number of calculations of the A number of "sufficiently nice" smooth varieties in [Ka, 5.1.1, 5.1.2, 5.4.1].

(6.7) We next turn to the calculation of A for some slightly singular X's. Before we begin, it will be convenient to give a general semicontinuity result for the middle Betti number in a smooth family of smooth affine varieties. This is based on an argument of Deligne in [De-3, 7.10.1 and 7.10.2].

Proposition 6.7.1. *Let k be an algebraically field, S a smooth connected affine k-scheme of dimension $d\geq 1$, $f: X\to S$ an affine morphism of finite type which is smooth, and everywhere of some relative dimension $n\geq 0$. Let l be a prime number which is invertible in k, and $\mathcal{G}$ a lisse $\bar{\boldsymbol{Q}}_l$-sheaf on X. For each point s in S, pick a geometric point $\bar{s}$ in S lying over it, and denote by $X_{\bar{s}}$ the geometric fiber $f^{-1}(\bar{s})$. Then the two (constructible) functions on S defined by*

$$s\longrightarrow \dim H^n(X_{\bar{s}}, \mathcal{G}\,|\,X_{\bar{s}})$$

and by

$$s\longrightarrow \dim H^n_c(X_{\bar{s}}, \mathcal{G}\,|\,X_{\bar{s}})$$

both decrease under specialization.

Proof. By Pioncare duality fibre by fibre, replacaing $\mathcal{G}$ by its dual interchanges the two functions, so it suffices to treat the second. Its constructibility, mise pour memoire, is a very special case of the proper mapping theorem and the proper base change theorem. To show that it decreases under specialization, it suffices, by a standard reduction, to treat the case when S is a smooth affine connected *curve.*

In this case, let us denote by U a nonvoid open set of S over which $R^n f_! \mathcal{G}$ is lisse. We will show that the canonical adjunction map

$$R^n f_! \mathcal{G} \longrightarrow j_* j^*(R^n f_! \mathcal{G})$$

is injective, which renders obvious the asserted decrease under specialization. The kernel of this map is a skyscraper sheaf, whose global sections are the group $H^0_c(S, R^n f_! \mathcal{G})$. So it suffices to show that $H^0_c(S, R^n f_! \mathcal{G})$ vanishes. Because the fibres of f are all smooth and affine of relative dimension n (or empty!), and $\mathcal{G}$ is lisse, it follows from the Lefschetz affine theorem, Poincare duality and proper base change, that $R^i f_! \mathcal{G}$ vanishes for $i < n$.

The Leray spectral sequence

$$E^2_{p,q} = H^p_c(S, R^q f_! \mathcal{G}) \Longrightarrow H^{p+q}_c(X, \mathcal{G})$$

then shows that $E^2_{0,n} = E^\infty_{0,n}$, i.e., that $H^0_c(S, R^n f_! \mathcal{G})$ is a quotient of $H^n_c(X, \mathcal{G})$. But X is smooth affine, purely of dimension $n+1$, and $\mathcal{G}$ is lisse, so the groups $H^i_c(X, \mathcal{G})$ vanish for $i < n+1$, in particular for $i = n$.

Q.E.D.

Corollary 6.7.2. *Hypotheses and notations as above, the sheaf $R^n f_! \mathcal{G}$ is lisse on S if and only if the function*

$$s \longrightarrow \dim H^n_c(X_{\bar{s}}, \mathcal{G} \mid X_{\bar{s}})$$

is constant on S.

Proof. Reduce first to the case when S is a curve. Then the injectivity of $R^n f_! \mathcal{G} \to j_* j^*(R^n f_! \mathcal{G})$ makes the assertion clear. Q.E.D.

(6.8) We now return to the estimation of A-numbers, this time for a special class of slightly singular hypersurfaces X in $\boldsymbol{E}$.

(6.8.1) Suppose that $r \geq 3$, and that $F(x_1, \cdots, x_r)$ is a nonzero weighted homogeneous polynomial whose unique critical point is the origin $\boldsymbol{O}$ in $\boldsymbol{E}$. Let X denote the hypersurface in $\boldsymbol{E}$ of equation $F=0$. Such an X is necessarily irreducible, because it is connected (being contractible, thanks to the weighted homogeneity) and normal (being a hypersurface of dimension ≥ 2 which is nonsingular outside of the point $\boldsymbol{O}$). Therefore the variety

$$V = X - \{\boldsymbol{O}\}$$

is a smooth connected variety of dimension $n=r-1$ in $\boldsymbol{E}$.

Theorem 6.9. *Suppose that $r\geq 3$, $n=r-1$, and that $F(x_1,\cdots,x_r)$ is a nonzero weighted homogeneous polynomial whose unique critical point is the origin $\boldsymbol{O}$ in $\boldsymbol{E}$. Let X denote the hypersurface in $\boldsymbol{E}$ of equation $F=0$, V the open set $X-\{\boldsymbol{O}\}$, and H any affine hyperplane in $\boldsymbol{E}$ which is sufficiently general with respect to V. The A-number for X is given by the formula*

$$A=\dim Ev^{n-1}(V\cap H)=\dim H^{n-1}(V\cap H)-\dim H^{n-1}(V).$$

Proof. Let us denote by $j\colon V\hookrightarrow X$ the inclusion. The middle extension K of $\bar{\boldsymbol{Q}}_{l,V}[n]$, the constant sheaf on V placed in degree $-n$, is (the extension by zero to all of $\boldsymbol{E}$ of) the complex $\tau^{\boldsymbol{O}}_{<0}(Rj_*\bar{\boldsymbol{Q}}_{l,V}[n])$ on X. The formula for A is

$$A=\chi(X,K)-\chi(X\cap H,K).$$

We have a short exact sequences of complexes

$$0\longrightarrow j_!\bar{\boldsymbol{Q}}_{l,V}[n]\longrightarrow Rj_*\bar{\boldsymbol{Q}}_{l,V}[n]\longrightarrow W\longrightarrow 0,$$

in which the complex W is supported at the origin. Applying the exact functor $\tau^{\boldsymbol{O}}_{<0}$, we obtain a short exact sequence

$$0\longrightarrow j_!\bar{\boldsymbol{Q}}_{l,V}[n]\longrightarrow K\longrightarrow \tau^{\boldsymbol{O}}_{<0}W\longrightarrow 0.$$

Because H does not contain the origin, the intersection $X\cap H$ is equal to $V\cap H$, and the restriction to $V\cap H$ of K is just $\bar{\boldsymbol{Q}}_{l,V\cap H}[n]$, so the formula for A reduces to

$$A=\chi(X,K)-(-1)^n\chi(V\cap H).$$

The above exact sequence for K gives

$$\chi(X,K)=(-1)^n\chi(X,j_!\bar{\boldsymbol{Q}}_{l,V})+\chi(X,\tau^{\boldsymbol{O}}_{<0}W).$$

The exact sequence

$$0\longrightarrow j_!\bar{\boldsymbol{Q}}_{l,V}[n]\longrightarrow \bar{\boldsymbol{Q}}_{l,X}[n]\longrightarrow \bar{\boldsymbol{Q}}_{l,O}[n]\longrightarrow 0$$

on X and the fact that X is contractible (by the weighted homogeneity of F) shows that

$$H^i(X,j_!\bar{\boldsymbol{Q}}_{l,V}[n])=0 \qquad \text{for } all\ i.$$

In particular, $\chi(X,j_!\bar{\boldsymbol{Q}}_{l,V})=0$, and hence

$$A=\chi(X, \tau^{0}_{<0}W)-(-1)^{n}\chi(V\cap H).$$

The complex W is supported at the origin, so its cohomology sheaves are its global cohomology groups on X, viewed as skyscraper sheaves at the origin. Taking the global cohomology on X in the exact sequence

$$0\longrightarrow j_{!}\bar{Q}_{l,V}[n]\longrightarrow Rj_{*}\bar{Q}_{l,V}[n]\longrightarrow W\longrightarrow 0$$

gives an isomorphism

$$H^{i+n}(V)\xrightarrow{\sim}\mathscr{H}^{i}(W)_{0} \qquad \text{for every } i.$$

Thus the formula for A boils down to

$$\begin{aligned} A&=\textstyle\sum_{i<0}(-1)^{i}\dim H^{i+n}(V)-(-1)^{n}\chi(V\cap H)\\ &=\textstyle\sum_{i<n}(-1)^{i+n}\dim H^{i}(V)-\sum_{i}(-1)^{i+n}\dim H^{i}(V\cap H). \end{aligned}$$

Since a general H does not contain the origin $\boldsymbol{O}$, the intersection $V\cap H$ is equal to the intersection $X\cap H$. Therefore $V\cap H$ is affine, of dimension $n-1$, so $H^{i}(V\cap H)$ vanishes for $i>n-1$. Applying the Goresky-Macpherson result 6.2 now gives the asserted formula for A. Q.E.D.

(6.10) We now turn to the detailed examination of a particular class of weighted homogeneous polynomials F in $r\geq 3$ variables with $\boldsymbol{O}$ as unique critical point, namely those which are sums of $m\geq 1$ nonzero *homogeneous* forms F_{b} of various distinct degrees d in *disjoint* sets of variables. That such an F has $\mathbf{0}$ as unique critical point means percisely that each nonzero F_{d}, viewed as a function of "its" variables, has $\boldsymbol{O}$ as unique critical point. In particular, each nonzero F_{d} has degree $d\geq 2$. Writing such an F as the sum of its nonzero homogeneous components F_{d}, we see that a given variable x_{i} occurs in precisely one nonzero homogeneous component $F_{d(i)}$, and that the integer $d(i)$ is characterized by the fact that $\partial F/\partial x_{i}$ is nonzero homogeneous of degree $d(i)-1$. We say that x_{i} is of co-weight $d(i)$. We denote by D the least common multiple of the $d(i)$'s, and by $e(i)$ the integer $D/d(i)$. We say that x_{1} has weight $e(i)$. Permuting the variables if necessary, we may and will assume that

$$d(1)\leq d(2)\leq\cdots\leq d(r).$$

For want of a better terminology, we will call such F's "mixed homogeneous nonsingular of type $(d(1)\leq\cdots\leq d(r))$". The archtypical example of such an F is the mixed-type Waring's problem polynomial $\sum_{i}(x_{i})^{d(i)}$, with each $d(i)\geq 2$.

Theorem 6.11. *Suppose that F is mixed homogeneous nonsingular of type $(d(1) \leq \cdots \leq d(r))$, with $r \geq 3$. If $d(r) \geq 3$, then the variety X of equation $F=0$ has $A \geq 2$.*

Proof. Our proof is unfortunately rather computational. It is based on combining the formula for A given by Theorem 6.9,

$$A = \dim H^{n-1}(V \cap H) - \dim H^{n-1}(V),$$

with the exact determination of $\dim H^{n-1}(V)$ (Lemma 6.12 below) and a lower bound for $\dim H^{n-1}(V \cap H)$ (Lemma 6.13 below).

(6.11.1) Given $d(1) \leq \cdots \leq d(r)$, we denote by $N(d(1), \cdots, d(r))$ the number of r-tuples $(w(1), \cdots, w(r))$ of integers satisfying the inequlities

$$0 < w(i) < d(i), \qquad \text{for } i = 1, \cdots, r$$

and the congruence

$$\textstyle\sum_i w(i)/d(i) \equiv 0 \qquad \text{mod } \boldsymbol{Z}.$$

(6.11.2) We denote by $P(d(1), \cdots, d(r))$ the product $\prod_i (d(i)-1)$.

Lemma 6.12. *Hypotheses and notations as in Theorem* 6.11 *above,*

$$\dim H^{n-1}(V) = N(d(1), \cdots, d(r)).$$

Lemma 6.13. *Hypotheses and notations as in Theorem* 6.11 *above,*

$$\dim H^{n-1}(V \cap H) \geq P(d(2), d(3), \cdots, d(r)).$$

Before giving the proofs of Lemmas 6.12 and 6.13, we will explain how the imply Theorem 6.11. In view of the formula for A, this amounts to showing that

$$P(d(2), d(3), \cdots, d(r)) - N(d(1), \cdots, d(r)) \geq 2$$

provided that $2 \leq d(1) \leq \cdots \leq d(r)$ and $d(r) \geq 3$.

We first treat the special case $d(1) = d(2) = \cdots = d(r-1) = 2$, $d(r) = 3$. In this case, one sees by inspection that $P(d(2), d(3), \cdots, d(r)) = 2$, while $N(d(1), \cdots, d(r)) = 0$.

We now turn to the general case. Notice first that if $(w(1), \cdots, w(r))$ is one of the r-tuples counted by $N(d(1), \cdots, d(r))$, then $w(r)$ is uniquely determined by $(w(1), \cdots, w(r-1))$, and any $(w(1), \cdots, w(r-1))$ which so occurs does *not* satisfy

$$\textstyle\sum_{1 \leq i \leq r-1} w(i)/d(i) \equiv 0 \qquad \text{mod } \boldsymbol{Z}.$$

Therefore we have the inequality

$$N(d(1), \cdots, d(r)) \leq P(d(1), d(2), \cdots, d(r-1)) - N(d(1), \cdots, d(r-1)).$$

(We remark for use below that this inequlity is an equality in the case that all the $d(i)$ are equal.) Therefore we have

$$\begin{aligned} &P(d(2), d(3), \cdots, d(r)) - N(d(1), \cdots, d(r)) \\ &\quad \geq P(d(2), d(3), \cdots, d(r)) - P(d(1), d(2), \cdots, d(r-1)) \\ &\qquad + N(d(1), \cdots, d(r-1)) \\ &\quad \geq P(d(2), d(3), \cdots, d(r-1))[d(r)-d(1)] + N(d(1), \cdots, d(r-1)). \end{aligned}$$

We must now distinguish several cases. Notice that $P(d(2), d(3), \cdots, d(r-1))$ is trivially $\geq d(r-1)-1$, so in particular is ≥ 1. Therefore if $d(r)-d(1) \geq 2$, then already the first term $P(d(2), d(3), \cdots, d(r-1)) \times [d(r)-d(1)]$ above is ≥ 2. If $d(r)-d(1)=1$, but $d(r-1)$ is ≥ 3, then $P(d(2), d(3), \cdots, d(r-1))[d(r)-d(1)]$ is ≥ 2. If $d(r)-d(1)=1$, and $d(r-1)=2$, then $(d(1), d(2), \cdots, d(r))$ is $(2, 2, \cdots, 2, 3)$, a case already treated above.

It remains to treat the case when $d(r)=d(1)$, whence all the $d(i)$ are equal to $D=d(r) \geq 3$. In this case we claim that the term $N(d(1), \cdots, d(r-1))$ is $\geq D-1 \geq 2$. In fact, $N(D, D, \cdots, D$; repeated $r-1$ times$)=N(D; r-1)$ is easily computed exactly by using the recurrence relation

$$N(D; r) + N(D; r-1) = (D-1)^{r-1}$$

together with the initial condition $N(D; 1)=0$. One finds

$$N(D; r-1) = [(D-1)/D][(D-1)^{r-2} + (-1)^{r-1}].$$

For fixed $D \geq 3$, $N(D; r-1)$ is increasing in the integer variable $r \geq 3$. So the worst case is $r=3$, in which case $N(D; 2)=D-1$. This concludes the reduction of Theorem 6.11 to Lemma 6.12 and 6.13.

We now turn to the proof of

Lemma 6.12. *Hypotheses and notations as in Theorem* 6.11 *above,*

$$\dim H^{n-1}(V) = N(d(1), \cdots, d(r)).$$

Proof. In terms of the weights $e(i)=D/d(i)$ of the variables x_i, we have the identity of weighted homogeneity

$$F(t^{e(1)}x_1, \cdots, t^{e(r)}x_r) = t^D F(x_1, \cdots, x_r)$$

for any scalar t. In particular, the group $\boldsymbol{\mu}=\boldsymbol{\mu}_D$ of D'th roots of unity operates on the variety of equation $F=1$. We will first construct a canonical isomorphism

$$H^{n-1}(V)=H^n(F=1)^{\mu}.$$

For this we first view $V=(F=0)-\{\boldsymbol{O}\}$ as a smooth divisor in $\boldsymbol{E}-\{O\}$ whose complement is the open set $\boldsymbol{E}[1/F]$ where F is invertible, and use the residue sequence. Now $\boldsymbol{E}-\{\boldsymbol{O}\}$ has cohomology only in degrees 0 and $2r-1=2n+1$, so we find an isomorphism, induced by residue,

$$H^n(\boldsymbol{E}[1/F])=H^{n-1}(V).$$

Now consider the D-fold finite etale galois $\boldsymbol{\mu}$-covering Z of $\boldsymbol{E}[1/F]$ defined by extracting the D'th root of $1/F$. Thus Z is the subscheme of $\boldsymbol{E}\times\boldsymbol{G}_m$ of equation $t^D F(x_1, \cdots, x_r)=1$. By means of the identity of weighted homogeneity for F above, we see that Z is isomorphic to the product $(F=1)\times\boldsymbol{G}_m$, with ζ in $\boldsymbol{\mu}$ acting on this product by

$$(x_1, \cdots, x_r, t)\longrightarrow(\zeta^{e(1)}x_1, \cdots, \zeta^{e(r)}x_r, \zeta t).$$

The Kunneth formula gives $H^*(Z)=H^*(F=1)\otimes H^*(\boldsymbol{G}_m)$. Because the action of $\boldsymbol{G}_m$ on itself by translation acts trivially on its cohomology, we find

$$H^*(\boldsymbol{E}[1/F])=H^*(Z)^{\mu}=H^*(F=1)^{\mu}\otimes H^*(\boldsymbol{G}_m).$$

Since $F=1$ has cohomology only in dimensions 0 and n, we find

$$H^{n-1}(V)=H^n(\boldsymbol{E}[1/F])=H^n(Z)^{\mu}=H^n(F=1)^{\mu}.$$

We next claim that in the universal family of F's which are mixed homogeneous nonsingular of type $(d(1)\leq\cdots\leq d(r))$, the dimension of $H^n(F=1)^{\mu}$, or equivalently of $H^n_c(F=1)^{\mu}$ remains constant. Let us denote by

$$f\colon (F_{\mathrm{univ}}=1)\longrightarrow S_{\mathrm{univ}}$$

this family. The $\boldsymbol{\mu}$ action defines an S-automorphism of f, so $(R^n f_! \overline{\boldsymbol{Q}}_l)^{\mu}$ is a direct factor of $R^n f_! \overline{\boldsymbol{Q}}_l$. Since S_{univ} is visibly connected, it suffices to show that $R^n f_! \overline{\boldsymbol{Q}}_l$ is lisse, for then its direct factor $(R^n f_! \overline{\boldsymbol{Q}}_l)^{\mu}$ is itself necessarily lisse, and so of constant fibre-rank by the connectedness of S_{univ}. By Corollary 6.7.2, to show that $R^n f_! \overline{\boldsymbol{Q}}_l$ is lisse on S_{univ}, it suffices to show that the dimension of $H^n(F=1)$ is constant. But this dimension is equal $\mu(F)$, itself the product of the $\mu(F_d)$'s for the nonzero homogeneous components F_d of F, each viewed as functions of its own variables. Reinterpreting $\mu(F_d)$ as the reduced middle Betti number of the variety $F_d=1$, we are reduced to proving the constancy of the Betti numbers universally in the case when F itself varies over all "homogeneous nonsingular" forms of some degree degree d in some number $k\geq 1$ of variables. But such a variety $F=1$ is the complement in a smooth projective hypersurface of degree d in $\boldsymbol{P}^k$ (the one of homogeneous equation $F=Z^d$) of a smooth hyperplane section (the one of equation $z=0$), so the constancy of the ranks of the cohomology groups of $F=1$ as F so varies is clear In fact, one finds that μ of such an F is $(d-1)^\vee$, by looking at the special case when F is of Fermat type

Now that we know that $\dim H_c^n(F=1)^\mu$ remains constant as F varies over mixed homogeneous nonsingulars of given type, we may compute it by taking for F the Waring polynomial $\sum_i (x_i)^{d(i)}$ In this case, the Pham-Brieskorn description (cf. [Mil, 9.1, 9.2]) of the cohomology of $F=1$ shows that as a representation of the product group $\boldsymbol{\mu}_{d(1)}\times\cdots\times\boldsymbol{\mu}_{d(r)}$, with $(\zeta_1,\cdots,\zeta_r)$ mapping $(\cdots,x_i,\cdots)$ to $(\cdots,\zeta_i x_i,\cdots)$, $H^n(F=1)$ is the tensor product of the augmentation representations of the $\boldsymbol{\mu}_{d(i)}$. Concretely, this means that $H^n(F=1)$ is $\boldsymbol{\mu}_{d(1)}\times\cdots\times\boldsymbol{\mu}_{d(r)}$-isomorphic to the span of monomials X^W in $x_1,\cdots,x_r$ whose exponents satisfy

$$0<w(i)<d(i).$$

Such a monomial is invariant under the action of $\boldsymbol{\mu}=\boldsymbol{\mu}_D$, with ζ mapping $(\cdots,x_i,\cdots)$ to $(\cdots,\zeta^{e(i)}x_i,\cdots)$, if and only if $\sum_i e(i)w(i)\equiv 0 \bmod D$. Dividing through by D, this becomes the condition $\sum_i w(i)/d(i)\equiv 0 \bmod \boldsymbol{Z}$. The number of such $\boldsymbol{\mu}$-invariant monomials is precisely the number $N(d(1),\cdots,d(r))$. This concludes the proof of Lemma 6.12. Q.E.D.

We now turn to the proof of

Lemma 6.13. *Hypotheses and notations as in Theorem* 6.11 *above,*

$$\dim H^{n-1}(V\cap H)\geq P(d(2),d(3),\cdots,d(r)).$$

Proof. Let H_0 be *any* affine hyperplane in $\boldsymbol{E}$ which does not contain $\boldsymbol{O}$ and for which the intersection $V\cap H_0=X\cap H_0$ is smooth of dimension $n-1$. Then we claim that

$$\dim H^{n-1}(V\cap H)\geq\dim H^{n-1}(V\cap H_0).$$

To see this, we argue as follows. Consider first the universal family of sections of X by hyperplanes of this form. In terms of the dual coordinates $y_1,\cdots,y_r$ on $\boldsymbol{E}^\vee$, denote by W the subvariety W of $\boldsymbol{E}\times\boldsymbol{E}^\vee$

consisting of points (x, y) which satisfy the two equations

$$F(x)=0, \qquad \sum_i x_i y_i=1.$$

Because F is weighted homogeneous with $\boldsymbol{O}$ as unique critical point, one sees easily by the Jacobian criterion that W is smooth of codimension two in $\boldsymbol{E}\times\boldsymbol{E}^\vee$.

Now consider the projection of W to $\boldsymbol{E}^\vee$. We claim that this projection is flat. By the argument of 6.1.3, it suffices to show that $\sum_i \alpha_i x_i-1$ is not a zero divisor on X. But $\sum_i \alpha_i x_i-1$ can be a zero divisor on X only if, as a function on $\boldsymbol{E}$, it lies in the ideal of F. But this is impossible, since F vanishes at $\boldsymbol{O}$ and $\sum_i \alpha_i x_i-1$ does not.

Now that we know that W is flat over $\boldsymbol{E}^\vee$, let S denote the line in $\boldsymbol{E}^\vee$ which joins H to H_0, and denote by W_S the inverse image of S in W. Then the projection π of W_S to S is flat. Therefore, π is smooth at point w in W_S if and only if w is smooth in its fibre. The set S' of points s in S whose fibre is smooth contains both H_0 and the open set $U\cap S$, which is nonempty because it contains H. Therefore S' is open dense in S.

Therefore $\pi^{-1}(S')$ is smooth and affine over S', with fibres purely of dimension $n-1$ (or empty). Because H is sufficiently general with respect to V, the general value of the middle Betti number of the fibre is $\dim H^{n-1}(V\cap H)$, and so the asserted inequality

$$\dim H^{n-1}(V\cap H)\geq \dim H^{n-1}(V\cap H_0)$$

results from the semicontinuity result Proposition 6.7.1.

Armed with this inequality, we argue as follows. By a linear change of variables, we may assume that each of the nonzero homogeneous components F_b of F is in "general position" with respect to its coordinates. [By this we mean that for each nonzero F_d, the $(x_i\partial/\partial x_i)(F_d)$, as the x_i run over the variables which occur in F_d, have no common zero other than the origin.] Take of H_0 the hyperplane of equation $x_1=1$. The intersection $V\cap H_0$ is easily checked to be smooth, again by the Jacobian criterion.

Concretely, $V\cap H_0$ is defined in the n-dimensional affine space with coordinates $x_2, \cdots, x_r$ by the equation

$$F(1, x_2, \cdots, x_r)=0.$$

It remains only to verify that

$$\dim H^{n-1}(F(1, x_2, \cdots, x_r)=0)=P(d(2), d(3), \cdots, d(r)).$$

This would be clear if F were itself a Waring polynomial $\sum_i (x_i)^{d(i)}$. We will reduce to this case by showing that the dimension in question does not change as the original F varies in the universal family of mixed homogeneous nonsingulars of given tyep all of whose F_d's are in general position with respect to their variables.

Given F, let us denote by $\tilde{F}$ the form of degree D in r variables defined by

$$\tilde{F}(\cdots, x_i, \cdots) = F(\cdots, (x_i)^{e(i)}, \cdots).$$

The point is that $\tilde{F}$ is itself "homogeneous nonsingular" of degree D, and in general position with respect to its variables. The group $\Gamma = \boldsymbol{\mu}_{e(2)} \times \cdots \times \boldsymbol{\mu}_{e(r)}$ acts on the variety

$$\tilde{F}(1, x_2, \cdots, x_r) = 0,$$

and the quotient is the variety

$$F(1, x_2, \cdots, x_r) = 0.$$

Therefore we have

$$H^{n-1}(\tilde{F}(1, x_2, \cdots, x_r) = 0)^{\Gamma} = H^{n-1}(F(1, x_2, \cdots, x_r) = 0).$$

The construction $F \to \tilde{F}$ and the action of the finite group Γ both pass to the universal family, whose parameter space is smoth and connected. Exactly as in the proof of Lemma 6.12 we are reduced to verifying the constancy of

$$\dim H^{n-1}(\tilde{F}(1, x_2, \cdots, x_r) = 0)$$

as F varies in the universal family. Once again this constancy is obvious, because the variety $\tilde{F}(1, x_2, \cdots, x_r) = 0$ is the complement of a transverse hyperplane section of a smooth projective hypersurface of given degree D and dimension $n-1$. Q.E.D.

Section 7

(7.0) This section is devoted to formulating explicitly some applications of the results of the earlier sections in entirely elementary terms.

(7.1) We fix an integer $r \geq 1$, and a closed subscheme X of $\boldsymbol{A}^r_{\boldsymbol{Z}}$ Spec$(\boldsymbol{Z}[x_1, \cdots, x_r])$, the r-dimensional affine space over $\boldsymbol{Z}$. We make the hypothesis

(7.1.1) The complex variety $X_{\boldsymbol{C}}$ is reduced and irreducible, of dimension $n \geq 0$.

(7.2.1) For each prime number p, and each r-tuple $(a)=(a_1, \cdots, a_r)$ of elements of $\boldsymbol{F}_p$, we denote by $S(p;(a))$ the exponential sum

$$S(p;(a)) := \sum_{(x) \text{ in } X(\boldsymbol{F}_p)} \exp((2\pi i/p)(\textstyle\sum_i a_i x_i)^n).$$

(7.2.2) For each prime number p we denote by $M(p)$ the mean absolute value of the "normalized" sums $S(p;(a))/(\sqrt{p})$;

$$M(p) := (\sqrt{p})^{-n-2r} \sum_{(a) \text{ in } (\boldsymbol{F}_p)^r} |S(p;(a))|.$$

(7.2.3) We denote by A the nonnegative integer which is the "A-number" of $X_{\boldsymbol{C}}$.

Theorem 7.3. *Hypotheses and notations as in* 7.1 *and* 7.2 *above, there exists a constant C such that the means $M(p)$ behave in the following way.*

(7.3.1) *If $A=0$, then for all primes p,*

$$M(p) \leq C/\sqrt{p}.$$

(7.3.2) *If $A=1$, then for all primes p,*

$$|M(p)-1| \leq C/\sqrt{p},$$

(7.3.3) *If $A \geq 2$, then for all primes p,*

$$M(p) \leq 1 + C/\sqrt{p}.$$

and the inequality

$$M(p) \leq 1 - 1/4(1+A^2)$$

holds for all p in a set of primes of Dirichlet density $\geq 1/2A^4$. In fact, there exists a galois extension $K/\boldsymbol{Q}$ such that $M(p) \leq 1-1/4(1+A^2)$ holds for all sufficiently large primes p which split completely in K.

Proof. This is the spelling out of 4.4, 4.8, 4.9, 4.10, 5.17 and 5.18 in the case when $\mathscr{F}$ is the constant sheaf and the base S is Spec$(\boldsymbol{Z})$.

Q.E.D.

(7.4) Here is a variant on the above theorem where we allow "twisting" of our sums by multiplicative characters. Fix an invertible function g on X, and an integer $N \geq 1$. For each prime p which satisfies

$$p \equiv 1 \bmod N,$$

fix a multiplicative character

$$\chi_{p,N}\colon (\boldsymbol{F}_p)^\times \longrightarrow \boldsymbol{C}^\times$$

of *exact* order N. For each r-tuple $(a)=(a_1, \cdots, a_r)$ of elements of $\boldsymbol{F}_p$, we denote by $S(p; g, \chi_{p,N}, (a))$ the exponential sum

$$S(p; g, \chi_{p,N}, (a)) := \sum_{(x) \text{ in } X(\boldsymbol{F}_p)} \exp((2\pi i/p)(\textstyle\sum_i a_i x_i))\chi_{p,N}(g(x)),$$

and by $M(p; g, \chi_{p,N})$ the mean absolute value of the "normalized" sums $S(p; g, \chi_{p,N}, (a))/(\sqrt{p})^n$;

$$M(p; g, \chi_{p,N}) := (\sqrt{p})^{-n-2r} \sum_{(a) \text{ in } (\boldsymbol{F}_p)^r} |S(p; g, \chi_{p,N}, (a))|.$$

The behavior of these means, as p varies over primes $p \equiv 1 \bmod N$, is also governed by A in exactly the same way. More precisely, we have

Theorem 7.5. *Hypotheses and notations as* 7.1, 7.2 *and* 7.4 *above, there exists a constant* C *such that the means* $M(p; g, \chi_{p,N})$ *behave in the following way.*

(7.5.1) *If* $A=0$, *then for any invertible function* g *on* X, *any integer* $N \geq 1$, *any prime* $p \equiv 1 \bmod N$, *and any multiplicative character* $\chi_{p,N}$,

$$M(p; g, \chi_{p,N}) \leq C/\sqrt{p}.$$

(7.5.2) *If* $A=1$, *then for any invertible function* g *on* X, *any integer* $N \geq 1$, *any prime* $p \equiv 1 \bmod N$, *and any multiplicaitve character* $\chi_{p,N}$,

$$|M(p; g, \chi_{p,N}) - 1| \leq C/\sqrt{p}.$$

(7.5.3) *If* $A \geq 2$, *then for any invertible function* g *on* X, *any integer* $N \geq 1_{\varepsilon}$ *any prime* $p \equiv 1 \bmod N$, *and any multiplicative character* $\chi_{p,N}$,

$$M(p; g, \chi_{p,N}) \leq 1 + C/\sqrt{p}.$$

(7.5.4) *If* $A \geq 2$, *there exists for each pair* (N, g) *a finite galois extension* $K_{g,N}$ *of* $\boldsymbol{Q}$ *such that* $K_{g,N}$ *contains the cyclotomic field* $\boldsymbol{Q}(\zeta_N)$ *and such that inequality*

$$M(p; g, \chi_{p,N}) \leq 1 - 1/4(1+A^2)$$

holds for all sufficiently large primes p *which split completely in* K.

Proof. Given g and N, extend scalars from $\boldsymbol{Z}$ to $\boldsymbol{Z}[\zeta_N, 1/N]$, pick a prime l dividing N, and a faithful $\bar{\boldsymbol{Q}}$-valued character χ of the group

$\mu_N(\boldsymbol{Z}[\zeta_N, 1/N])$. On the scheme $X \otimes \boldsymbol{Z}[\zeta_N, 1/N]$, we have the rank one lisse sheaf $\mathscr{L}_{\chi(g)}$ which is pure of weight zero. The key point is that its A-number (cf. 4.6) is the *same* as for the constant sheaf. (Indeed, for any lisse $\mathscr{F}$ on V, and for any lisse $\mathscr{G}$ on X, the middle extensions of $\mathscr{F}[n]$ and of $\mathscr{F}[n] \otimes j^*\mathscr{G}$ are related by the formula

$$j_{!*}(\mathscr{F}[n] \otimes j^*\mathscr{G}) = j_{!*}(\mathscr{F}[n]) \otimes \mathscr{G}.$$

Therefore the A-number for $\mathscr{F}[n] \otimes j^*\mathscr{G}$ is related to that for $\mathscr{F}[n]$ by

$$A(\mathscr{F}[n] \otimes j^*\mathscr{G}) = \operatorname{rank}(\mathscr{G}) \times A(\mathscr{F}[n]),$$

simply because the A-number of a middle extension K depends only on its local Euler characteristic function $\chi(K)$ (cf. [K-L, 3.0]). Applying this with $\mathscr{F}$ the constant sheaf, and $\mathscr{G} = \mathscr{L}_{\chi(g)}$ gives the equality of A's). Now apply 4.8, 4.9 and 4.4 over $S = \operatorname{Spec}(\boldsymbol{Z}[\zeta_N, 1/N])$ to the closed points of S whose residue fields are the prime field, i.e., those lying over primes $p \equiv 1 \bmod N$. For such p, bohth the set of $\chi_{p,N}$'s and the set of closed points of S lying over p are $(\boldsymbol{Z}/N\boldsymbol{Z})^\times$-torsors, so we obtain assertions 7.5.1-2-3. The constant C can be chosen uniformly, because the constants C_2, M_3, and M_4 which occur in 4.4, 4.8, and 4.9 are invariant under etale base change $S' \to S$. For 7.5.4, we apply 5.17 over $S = \operatorname{Spec}(\boldsymbol{Z}[\zeta_N, 1/N])$. This gives a finite galois extension $\tilde{K}_{g,N}$ of $\boldsymbol{Q}(\zeta_n)$ such that the asserted estimate certainly holds for every $p \equiv 1 \bmod N$ which splits completely in $\tilde{K}_{g,N}$. Replacing $\tilde{K}_{g,N}$ by its galois closure over $\boldsymbol{Q}$ gives the desired $K_{g,N}$. Q.E.D.

Theorem 7.6. *Suppose that $r \geq 2$, and that $F(x_1, \cdots, x_r)$ is a nonzero weighted homogeneous polynomial with integral coefficients whose unique critical point (when viewed as a complex polynomial) is the origin $\boldsymbol{O}$ in $\boldsymbol{A}^r_{\boldsymbol{C}}$. Let k be an integer, and consider the variety X of equation*

$$F(x_1, \cdots, x_r) = k.$$

Suppose either that k is nonzero, or that both $r \geq 3$ and F is mixed homogeneous nonsingular (cf. 6.10) of type $(d(1) \leq \cdots \leq d(r))$ with $d(r) = 3$. Then $X_{\boldsymbol{C}}$ is reduced and irreducible of dimension $n = r-1$, and $A \geq 2$.

Proof. This is 6.5, 6.8.1, and 6.11. Q.E.D.

Appendix. Variations on the Brylinski-Radon transform

Let S be an arbitrary (separated and noetherian) scheme, $r \geq 1$ an integer, $\tau \colon E \to S$ a vector bundle of rank r, and $\tau^\vee \colon E^\vee \to S$ the dual vector bundle. We denote by Λ: $PE \to S$ the projective bundle of *lines* in E, and by q: $V \to PE$ the tautological line bundle over PE (i.e., a point of V is a pair (L, P) consisting of a line L in E together with a point P in L, and $q(L, P) = L$). The complement $V_* := V - PE$ of the zero-section in V (i.e., viewing V as a line bundle over PE) *is* the complement $E_* := E - S$ of the zero-section in E (i.e., viewing E as a vector bundle over S). This identification $V_* = E_*$ extends to a morphism π: $V \to E$ which identifies V with the blowing up of E along its zero-section. We denote by j: $E - S \to E$ and by α: $V_* \to V$ the inclusions. We denote by $\rho := q \mid V_*$: $V_* \to PE$ the tautological $\boldsymbol{G}_m$-bundle over PE. We denote by $q^\vee$: $V^\vee \to PE$ the line bundle over PE which is dual to the tautological line bundle q: $V \to PE$, by $V_*^\vee$ the complement of the zero section in this line bundle, and by $\alpha^\vee$: $V_*^\vee \to V^\vee$ the inclusion. Thus a point of $V^\vee$ is a pair (L, φ) consisting of a line L in E and a linear form φ on L. This description makes clear that after pulling back the line bundle $V^\vee$ on PE to the product $PE \times_S E^\vee$, it admits a canonical section (over a point $(L, e^\vee)$ of $PE \times_S E^\vee$, one takes for φ the linear form on L induced by $e^\vee$). We denote this canonical section as

$$[\text{can sec}]\colon PE \times_S E^\vee \longrightarrow V^\vee \times_S E^\vee.$$

We denote by pr_1 the projection of $V^\vee \times_S E^\vee$ onto its first factor $V^\vee$, and by p_1 and p_2 the projections of $PE \times_S E^\vee$ onto its first and second factors respectively.

Suppose now that a prime number l is invertible on S. For any object K in $D_c^b(V^\vee, \bar{\boldsymbol{Q}}_l)$, we define a generalized Brylinski-Radon transform

$$\mathrm{BR}\langle K\rangle\colon D_c^b(PE, \bar{\boldsymbol{Q}}_l) \longrightarrow D_c^b(E^\vee, \bar{\boldsymbol{Q}}_l)$$

by the recipe

$$\mathrm{BR}\langle K\rangle(\mathscr{F}) = R(p_2)_!((p_1)^*(\mathscr{F}) \otimes [\text{can sec}]^*(pr_1)^*(K)).$$

(Since p_2 is proper, it would be the same to use $R(p_2)_*$ instead.) By proper base change, formation of $\mathrm{BR}\langle K\rangle(\mathscr{F})$ commutes with arbitrary change of base on S.

Now suppose that S is an $\boldsymbol{F}$-scheme, where $\boldsymbol{F}$ is a finite field of characteristic $\neq l$, and ψ is a nontrivial $\bar{\boldsymbol{Q}}_l$-valued additive character of $\boldsymbol{F}$. Then for any object L in $D_c^b(V, \bar{\boldsymbol{Q}}_l)$, we may, by viewing V as a line bundle over PE, for the Fourier transform $FT_\psi(L)$ in $D_c^b(V^\vee, \bar{\boldsymbol{Q}}_l)$. Having formed this, we may then form the Brylinski-Radon transform $\mathrm{BR}\langle \mathrm{FT}_\psi(L)\rangle(\mathscr{F})$ of an object $\mathscr{F}$ in $D_c^b(PE, \bar{\boldsymbol{Q}}_l)$.

On the other hand, with the same data L in $D_c^b(V, \bar{\boldsymbol{Q}}_l)$ and $\mathscr{F}$ in $D_c^b(PE, \boldsymbol{Q}_l)$, there is another natural way to obtain an object in $D_c^b(E^\vee, \bar{\boldsymbol{Q}}_l)$.

Namely, one first forms $q^*\mathscr{F}$ on V, one tensors it with L, one forms $R\pi_*(q^*\mathscr{F}\otimes L)$ on E, and then, viewing E as a vector bundle on S, one forms the Fourier transform $\mathrm{FT}_\psi(R\pi_*(q^*\mathscr{F}\otimes L))$ on $E^\vee$.

Proposition A1. *Hypotheses and notations as above, if S an $\boldsymbol{F}$-scheme, where $\boldsymbol{F}$ is a finite field of characteristic $\neq l$, and ψ is a nontrivial $\bar{\boldsymbol{Q}}_l$-valued additive character of $\boldsymbol{F}$, we have a canonical isomorphism of bifunctors*

$$\mathrm{FT}_\psi(R\pi_*(q^*\mathscr{F}\otimes L))[-r]\approx \mathrm{BR}\langle \mathrm{FT}_\psi(L)\rangle(\mathscr{F})[-1].$$

Proof. For L the constant sheaf, this is proven (though not quite stated) in [Bry, 9.13]. The proof in the general case goes along exactly the same lines, and is left to the reader. Q.E.D.

The case of interest to us is when L is the constant sheaf on V_*, extended by zero to V, i.e., the case $L=\alpha_!\alpha^*\bar{\boldsymbol{Q}}_l$.

Proposition A2. *Hypotheses and notations as above, we have a canonical isomorphism in $D^b_c(V^\vee, \bar{\boldsymbol{Q}}_l)$:*

$$\mathrm{FT}_\psi(\alpha_!\alpha^*(\bar{\boldsymbol{Q}}_l))=R(\alpha^\vee)_*(\alpha^\vee)^*(\bar{\boldsymbol{Q}}_l).$$

Proof. This is the special case $r=1$, $S=PE$, $E=V$, $j=\alpha$, of Proposition A4 below. Q.E.D.

Proposition A3. *Notations as above, suppose that S is a scheme of finite type over $\boldsymbol{Z}[1/l]$. Given any object $\mathscr{F}$ in $D^b_c(PE, \bar{\boldsymbol{Q}}_l)$, consider the object $\mathscr{G}$ in $D^b_c(E^v, \bar{\boldsymbol{Q}}_l)$ defined by*

$$\mathscr{G}:=\mathrm{BR}\langle R(\alpha^v)_*(\alpha^v)^*(\bar{\boldsymbol{Q}}_l)\rangle(\mathscr{F})(r)[r-1].$$

Then for any triple $(\boldsymbol{F}, s, \psi)$ consisting of a finite field $\boldsymbol{F}$ of characteristic $\neq l$, an $\boldsymbol{F}$-valued point s of S, and a nontrivial $\bar{\boldsymbol{Q}}_l$-valued additive character ψ of $\boldsymbol{F}$, we have a canonical isomorphims in $D^b_c(E_s, \bar{\boldsymbol{Q}}_l)$

$$\mathrm{FT}_{s,\psi}(\mathscr{G}\,|\,(E^\vee)_s)=j_!\rho^*\mathscr{F}\,|\,(E^\vee)_s.$$

Proof. The formation of $R(\alpha^\vee)_*(\alpha^\vee)^*(\bar{\boldsymbol{Q}}_l)$ commutes with arbitrary change of base on S, and consequently so does the formation of $\mathscr{G}$. So we may reduce to the case that S is the spectrum of $\boldsymbol{F}$. We have a canonical isomorphism on E

$$j_!\rho^*\mathscr{F}=R\pi_*(q^*\mathscr{F}\otimes\alpha_!\alpha^*\bar{\boldsymbol{Q}}_l).$$

Now apply Fourier transform. By Proposition A1, we have

$$\mathrm{FT}_{\bar\psi}(R\pi_*(q^*\mathscr{F}\otimes L))[r] \approx \mathrm{BR}\langle \mathrm{FT}_{\psi}(L)\rangle(\mathscr{F})[-1].$$

Taking $L=\alpha_!\alpha^*\bar{\boldsymbol{Q}}_l$ on V, and applying Proposition A2, this yields

$$\mathrm{FT}_{\bar\psi}(j_!\rho^*\mathscr{F})[-r]=\mathrm{BR}\langle R(\alpha^\vee)_*(\alpha^\vee)^*(\bar{\boldsymbol{Q}}_l)\rangle(\mathscr{F})[-1],$$

from which the the assertion follows by Fourier inversion. Q.E.D.

It remains only to give the general fact of which Proposition A2 is a special case.

Proposition A4. *Suppose that* $\boldsymbol{F}$ *is a finite field of characteristic* $\neq l$, *and* ψ *is a nontrivial* $\bar{\boldsymbol{Q}}_l$*-valued additive character of* $\boldsymbol{F}$. *Let* S *be an* $\boldsymbol{F}$*-scheme,* $r\geq 1$ *be an integer,* $\tau\colon E\to S$ *a vector bundle of rank* r, *and* $\tau^\vee\colon E^\vee\to S$ *the dual vector bundle. Denote by* $E-S$ *the complement of the zero section, by* $j\colon E-S\to E$ *its inclusion, and by* $i\colon S\to E$ *the inclusion of the zero section. Denote by* $j^\vee$ *and* $i^\vee$ *the corresponding maps for the dual vector bundle* $E^\vee$. *Then we have a canonical isomorphism in* $D_c^b(E^\vee,\bar{\boldsymbol{Q}}_l)$:

$$\mathrm{FT}_\psi(j_!j^*\bar{\boldsymbol{Q}}_l)=R(j^\vee)_*(j^\vee)^*(\bar{\boldsymbol{Q}}_l)[r-1].$$

Proof. We have an exact sequence on E

$$0\longrightarrow j_!j^*\bar{\boldsymbol{Q}}_l\longrightarrow \bar{\boldsymbol{Q}}_l\longrightarrow i_*\bar{\boldsymbol{Q}}_l\longrightarrow 0.$$

Applying Fourier transform, we get an exact triangle

$$\longrightarrow \mathrm{FT}_\psi(j_!j^*\bar{\boldsymbol{Q}}_l)\longrightarrow (i^\vee)_*(\bar{\boldsymbol{Q}}_l(-r))[-r]\longrightarrow \bar{\boldsymbol{Q}}_l[r]\longrightarrow.$$

The middle term vanishes on $E^\vee-S$, so we obtain an isomorphism

$$(j^\vee)^*(\bar{\boldsymbol{Q}}_l[r-1])=(j^\vee)^*(\mathrm{FT}_\psi(j_!j^*\bar{\boldsymbol{Q}}_l)).$$

Applying $R(j^\vee)_*$, we obtain an isomorphism

$$R(j^\vee)_*(j^\vee)^*(\bar{\boldsymbol{Q}}_l[r-1])=R(j^\vee)_*(j^\vee)^*(\mathrm{FT}_\psi(j_!j^*\bar{\boldsymbol{Q}}_l)).$$

We have a natural morphism of adjunction

$$\mathrm{FT}_\psi(j_!j^*\bar{\boldsymbol{Q}}_l)\longrightarrow R(j^\vee)_*(j^\vee)(\mathrm{FT}_\psi(j_!j^*\bar{\boldsymbol{Q}}_l)),$$

which it suffices to show is an isomorphism. This adjunction map sits in an exact triangle

$$\begin{aligned}\longrightarrow \underline{R\Gamma}_S(\mathrm{FT}_\psi(j_!j^*(\bar{\boldsymbol{Q}}_l)))\longrightarrow \mathrm{FT}_\psi(j_!j^*(\bar{\boldsymbol{Q}}_l))&\\ \longrightarrow R(j^\vee)_*(j^\vee)^*(\mathrm{FT}_\psi(j_!j^*(\bar{\boldsymbol{Q}}_l)))&\longrightarrow,\end{aligned}$$

so it suffices to show that $\underline{R\Gamma}_S(\mathrm{FT}_\psi(j_!j^*(\bar{\mathbf{Q}}_\ell)))$ vanishes.

To show this vanishing, apply $\underline{R\Gamma}_S$ to the exact triangle above

$$\longrightarrow \mathrm{FT}_\psi(j_!j^*\bar{\mathbf{Q}}_\ell) \longrightarrow (i^\vee)_*(\bar{\mathbf{Q}}_\ell(-r))[-r] \longrightarrow \bar{\mathbf{Q}}_\ell[r] \longrightarrow,$$

to obtain an exact triangle

$$\longrightarrow \underline{R\Gamma}_S\mathrm{FT}_\varphi(j_!j^*\bar{\mathbf{Q}}_\ell) \longrightarrow \underline{R\Gamma}_S(i^\vee)_*(\bar{\mathbf{Q}}_\ell(-r))[-r] \longrightarrow \underline{R\Gamma}_S\bar{\mathbf{Q}}_\ell[r] \longrightarrow.$$

So now it suffices to show that the map

$$\underline{R\Gamma}_S(i^\vee)_*(\bar{\mathbf{Q}}_\ell(-r))[-r] \longrightarrow \underline{R\Gamma}_S\bar{\mathbf{Q}}_\ell[r]$$

is an isomorphism. Both source and target are separately isomorphic to $(i^\vee)_*(\bar{\mathbf{Q}}_\ell(-r))[-r]$, so over each connected component of S, this map is either an isomorphism or it is zero. Passing to the maximal points of S^{red} to examine this question, we reduce to the case when S is itself the spectrum of a field. We must show that the above map is nonzero. Recaling its provenance, we must show that the map

$$(*) \qquad \underline{R\Gamma}_S\mathrm{FT}_\psi(\bar{\mathbf{Q}}_\ell) \longrightarrow \underline{R\Gamma}_S\mathrm{FT}_\psi(i_*\bar{\mathbf{Q}}_\ell)$$

is nonzero. As $\mathrm{FT}_\psi(\bar{\mathbf{Q}}_\ell)$ is supported in the origin, the canonical map

$$\underline{R\Gamma}_S\mathrm{FT}_\psi(\bar{\mathbf{Q}}_\ell) \longrightarrow \mathrm{FT}_\psi(\bar{\mathbf{Q}}_\ell)$$

is an isomorphism. The commutative square

$$\begin{tikzcd}
\underline{R\Gamma}_S\mathrm{FT}_\psi(\bar{\mathbf{Q}}_\ell) \arrow[r] \arrow[d, "\approx"] & \underline{R\Gamma}_S\mathrm{FT}_\psi(i^*\bar{\mathbf{Q}}_\ell) \arrow[d] \\
\mathrm{FT}_\psi(\bar{\mathbf{Q}}_\ell) \arrow[r] & \mathrm{FT}_\psi(i_*\bar{\mathbf{Q}}_\ell)
\end{tikzcd}$$

shows that if (*) is the zero map, then so is the map

$$\mathrm{FT}_\psi(\bar{\mathbf{Q}}_\ell) \longrightarrow \mathrm{FT}_\psi(i_*\bar{\mathbf{Q}}_\ell).$$

But this map is *not* zero, because it is the Fourier transform of a nonzero map! Q.E.D.

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