Advanced Studies in Pure Mathematics 16, 1988
Conformal Field Theory and Solvable Lattice Models
pp. 373–426

Supersymmetric Extension of the Kadomtsev-Petviashvili Hierarchy and the Universal Super Grassmann Manifold

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To the memory of our teacher, Professor Motoo Kinoshita

Abstract

A supersymmetric extension of the Kadomtsev-Petviashvili hierarchy (SKP hierarchy) is proposed. It is shown that the solution space is identical with the universal super Grassmann manifold. An explicit formula of a solution to the SKP hierarchy is presented as a quotient of superdeterminants. Our arguments are based upon algebraic investigations of the structure of a linear supersymmetric differential equation.

Contents

Introduction

In this paper we treat a supersymmetric (SUSY) extension of certain completely integrable systems.

Received March 26, 1987.

Supersymmetry is a formalism for describing Bose fields and Fermi fields simultaneously, which is originated from the unification theory in particle physics. Recently study of supersymmetry has been exploited in mathematical contexts such as Lie theory [4, 7, 10, 11, 12, 21, 28, 31], differential geometry [1, 6, 13, 17], algebraic geometry [18, 19], and nonlinear integrable systems [2, 3, 8, 14, 15, 16, 20, 22, 29, 30, 32].

On the other hand, at the beginning of eighties, a hierarchy of nonlinear differential equations was discovered by M. Sato, which includes, as the reduced systems, the celebrated Korteweg-de Vries (KdV) equation, nonlinear Schrödinger (NLS) equation, Toda lattice and so on. It is called the Kadomtsev-Petviashvili (KP) hierarchy. Sato's fundamental theorem [24, 25, 27] says that the KP hierarchy is naturally interpreted as a dynamical system on the universal Grassmann manifold UGM. In his framework, the so-called Sato equation is the most important. The Sato equation for the KP hierarchy is a system of equations for a wave operator, and linearizes the original nonlinear problem. Namely, the nonlinear problem of the KP hierarchy reduces to a problem of linear algebra. At this stage, UGM naturally arises. We can obtain a general solution to the KP hierarchy by solving the Grassmann equation attached to a point of UGM, which is a linear algebraic equation of infinite dimensions. We call such scheme the Grassmann formalism. We mention that, in [26], Sato proposed another point of view that the Sato equation is regarded as describing the time evolution of nonlinear ordinary differential equations coming from the decomposition of a linear ordinary differential operator. This point of view is crucial in our framework.

So far, various schemes for SUSY extensions of classical soliton equations have been proposed by several authors mentioned above and their complete integrability has been verified. Yu. I. Manin and A.O. Radul [20] already formulated a SUSY extension of the KP hierarchy, and proved the complete integrability by means of the variational methods according to the Gel'fand-Dikii theory. However they did not clarify the structure of the solution space and the representation of the solutions. Therefore it is quite natural to make an attempt to establish the SUSY Grassmann formalism which should play the same role as the original one does in the classical soliton theory.

Now we give the definition of the super KP (SKP) hierarchy. Let θ be an abstract Grassmann number and put $\Theta=\partial_\theta+\theta\partial_x$. This is an odd derivation with the property that $\Theta^2=\partial_x$. Let L be a SUSY micro-differential operator, $L=\sum_{i=0}^{\infty} u_i\Theta^{1-i}$ with $u_0=1$, $\Theta(u_1)+2u_2=0$ (the parity of u_i is equal to i mod 2). The SKP hierarchy of type m ($m=0, 1$) is, by definition, a system of infinitely many Lax equations:

$$\Theta_{2l}(L)=(-)^l[B_{2l}, L], \qquad \Theta_{2l-1}(L)=(-)^{l+m}\{[B_{2l-1}, L]_+ - 2L^{2l}\},$$
$$(l=1, 2, \cdots),$$

where Θ_{2l}, Θ_{2l-1} are the operators of the time evolution and B_{2l}, B_{2l-1} are the SUSY differential operator parts of L^{2l}, L^{2l-1} respectively (see Section 3.1). We should notice the sign factors in the above equations. They did not appear in the definition of Manin and Radul's [20]. To make a formal integration of the SKP hierarchy, we find a SUSY microdifferential operator $W=\sum_{-\infty<j\le 0} w_j\Theta^j$ (a SUSY wave operator) satisfying the SUSY Sato equations. Then the nonlinear problem of the SKP hierarchy reduces to a problem of the linear algebra as follows: An operator $W=\sum_{-\infty<j\le 0} w_j\Theta^j$ is a SUSY wave operator of the SKP hierarchy if and only if the coefficients w_j's solve the SUSY Grassmann equation

$$(0.1) \qquad {}^t\vec{w}\Phi\Xi=0,$$

where ${}^t\vec{w}=(w_j)_{j\in Z}$ ($w_j=0$ for $j>0$), Φ is a certain SUSY Wronski matrix and Ξ is a superframe. This equation is solved explicitly by means of the superdeterminant. The SUSY Grassmann equation (0.1) is invariant under the right action of the general linear supergroup (see Section 3.2). Thus the super Grassmann manifolds arise naturally in the theory. Our conclusion is that the SKP hierarchy is a dynamical system on the universal super Grassmann manifold (USGM).

The organization of this paper is as follows: Chapter I reviews the theory of the KP hierarchy. In Chapter 2 we discuss the Grassmann hierarchy associated with the decomposition of a linear ordinary differential operator, and its SUSY extension. Especially Section 2.2 is devoted to the foundation of the theory of linear SUSY differential equations, based upon the SUSY Wronski matrix. In Chapter 3 we introduce the SKP hierarchy as the limiting case of the Grassmann hierarchy, and investigate the integrating procedure. In Section 3.2, a super τ (tau) field for the SKP hierarchy is defined as a superdeterminant of an infinite size matrix. We present a formula representing solutions to the SKP hierarchy by means of a super τ field, which is the main theorem in this paper. This main theorem is proved in Section 3.3 together with some relations enjoyed by a super τ field. The reduced SKP hierarchies are proposed in Section 3.4, which include a SUSY extension of the KdV equation as a special case. Chapter 4 concerns with the representation theory of infinite dimensional Lie superalgebras which emerge as the hidden symmetry algebras of the SKP hierarchy and its reduced versions. We give a representation of the Lie superalgebra $\mathfrak{gl}(\infty\,|\,\infty)$ and of its one-dimensional central extension $\mathfrak{gl}(\infty\,|\,\infty)^\sim$ via free field operators.

The super Virasoro algebras and the super Kac-Moody algebras are realized as subalgebras of $\mathfrak{gl}(\infty \mid \infty)^{\sim}$.

Acknowledgements. It is our great pleasure to thank Doctor M. Noumi for fruitful discussions on the ideas of Professor M. Sato. The first author (K.U.) is grateful to Professors P.P. Kulish and J. Hietarinta for their stimulating discussions at the Helsinki conference in 1986. Financial support from Research Aid of Inoue Foundation of Science is gratefully acknowledged. The second author (H.Y.) expresses his sincere gratitude to Professor K. Okamoto for constant encouragement. The authors thank Mrs. K. Wada, a secretary of Yokohama City University, for her beautiful typing.

Chapter 1. Review of the Theory of the KP Hierarchy

To introduce the KP hierarchy, we prepare a one-dimensional space variable x and infinite dimensional time variables $t=(t_1, t_2, t_3, \cdots)$. Let $\mathscr{R}$ be the algebra of formal power series over $\boldsymbol{C}$; $\mathscr{R}=\boldsymbol{C}[[x, t]]$, which has a structure of a differential algebra with a derivation $\partial_x=\partial/\partial x$: $\mathscr{R}\to\mathscr{R}$. The variables (x, t) are regarded as coordinates on an affine space $\boldsymbol{C}^N$, and $\mathscr{R}$ is the formal completion of the function algebra on the space. Let $\mathscr{K}$ be the quotient field of $\mathscr{R}$.

By $\mathscr{E}_{\mathscr{K}}$ (resp. $\mathscr{E}_{\mathscr{R}}$) is meant the algebra of microdifferential operator with coefficients in $\mathscr{K}$ (resp. in $\mathscr{R}$);

$$\begin{aligned}\mathscr{E}_{\mathscr{K}}&=\mathscr{K}((\partial_x^{-1}))\\ &=\{\sum_{-\infty<\nu\ll\infty} p_\nu(x, t)\partial_x^\nu \mid p_\nu(x, t)\in\mathscr{K}\}.\end{aligned}$$

The algebra structure of $\mathscr{E}_{\mathscr{K}}$ is prescribed by the generalized Leibniz rule:

$$\partial_x^k\cdot f=\sum_{\nu=0}^{\infty}\binom{k}{\nu}\partial_x^\nu(f)\cdot\partial_x^{k-\nu} \qquad (k\in\boldsymbol{Z}),$$

where $f\in\mathscr{K}$ and the dot stands for a product of two microdifferential operators. Through a direct sum decomposition

$$\mathscr{E}_{\mathscr{K}}=\mathscr{D}_{\mathscr{K}}\oplus\mathscr{E}_{\mathscr{K}}(-1),$$

where $\mathscr{D}_{\mathscr{K}}=\mathscr{K}[\partial_x]$ and $\mathscr{E}_{\mathscr{K}}(m)=\mathscr{K}[[\partial_x^{-1}]]\cdot\partial_x^m$ $(m\in\boldsymbol{Z})$, $X\in\mathscr{E}_{\mathscr{K}}$ is uniquely represented as $X=X_++X_-$ $(X_+\in\mathscr{D}_{\mathscr{K}}, X_-\in\mathscr{E}_{\mathscr{K}}(-1))$.

Let L be a monic microdifferential operator of the first order, $L=\sum_{i=0}^{\infty}u_i(x, t)\partial_x^{1-i}\in\mathscr{E}_{\mathscr{K}}(1)^{monic}$ with $u_0=1$, $u_1=0$. The KP hierarchy is, by definition, a system of the Lax equations

(1.1) $\partial L/\partial t_n = [B_n, L]$ with $B_n = (L^n)_+$ $(n=1, 2, \cdots)$,

or equivalently a system of the Zakharov-Shabat equations

(1.2) $\partial B_n/\partial t_m - \partial B_m/\partial t_n + [B_m, B_n] = 0$ $(n, m=1, 2, \cdots)$.

From (1.1) with $n=2, 3$ (or (1.2) with $n=2, m=3$), one gets the celebrated Kadomtsev-Petviashvili equation

$$\frac{3}{4}u_{t_2t_2} = \left(u_{t_3} - 3uu_x - \frac{1}{4}u_{xxx}\right)_x,$$

where $u_x = \partial u/\partial x$ etc.

Sato's fundamental theorem [24, 25, 27] states that the KP hierarchy is interpreted as a dynamical system on the universal Grassmann manifold UGM. Here we will describe the outline of the proof from a viewpoint of the Grassmann equation.

The first step is a passage from the Lax equations to the Sato equations. For a solution $L \in \mathscr{E}_{\mathscr{R}}(1)^{monic}$ to the KP hierarchy, there exists a monic microdifferential operator of the 0-th order $W = \sum_{-\infty < j \leq 0} w_j(x, t)\partial_x^j \in \mathscr{E}_{\mathscr{R}}(0)^{monic}$ $(w_0 = 1)$ satisfying the system of operator equations

(1.3) $\partial W/\partial t_n = B_n W - W\partial_x^n$ $(n=1, 2, \cdots)$.

We call W a wave operator. The Sato equations (1.3) can be viewed as a linearization for the KP hierarchy, and conversely the consistency condition of the Sato equations gives rise to the KP hierarchy. Notice that, in (1.3), $B_n = (W\partial_x^n W^{-1})_+$. Introducing a differential operator Ψ of infinite order,

$$\Psi(t, \partial_x) = \exp\left(\sum_{n=1}^{\infty} t_n \partial_x^n\right),$$

one easily sees that (1.3) leads to

(1.4) $\partial \tilde{W}/\partial t_n = B_n \tilde{W}$ $(n=1, 2, \cdots)$,

where $\tilde{W} = W\Psi$.

To transform the Sato equations to the Grassmann equation, we consider the following Cauchy problem of a system of operator equations:

(1.5) $\partial Y/\partial t_n = B_n Y$ $(n=1, 2, \cdots)$,

where $Y = \sum_{j=0}^{\infty} y_j(x, t)\partial_x^j$ $(y_j(x, t) \in \mathscr{R})$ is a differential operator of infinite order with the initial condition

(1.6) $$Y|_{t=0}=1.$$

Set $U=\tilde{W}^{-1}Y$. From (1.4) and (1.5), it follows that

$$\partial U/\partial t_n=0 \quad (n=1, 2, \cdots).$$

This tells us that U does not depend on the variables t, and furthermore the initial condition (1.6) shows that $U=(\tilde{W}^{-1}Y)|_{t=0}=(W|_{t=0})^{-1}$. Hence the operator U takes the form

(1.7) $$U=\sum_{-\infty<j\leq 0} u_j(x)\partial_x^j \qquad (u_0=1, u_j(x)\in \boldsymbol{C}[[x]]).$$

Since the operator W and Y are related through

$$Y=W\Psi U,$$

one obtains, projecting the both sides onto $\mathscr{E}_{\mathscr{R}}(-1)$,

(1.8) $$(W\Psi U)_-=0.$$

This equation substantially characterizes the operator W and turns out to be equivalent to the Grassmann equation as follows: Set $Z=\Psi U=\sum_{j\in Z} z_j\partial_x^j$. Introducing the quantity $z_j^{(\nu)}\in\mathscr{R}$ by

$$WZ=\sum_{\nu\in Z}\{\sum_{j\in Z} w_j z_j^{(\nu)}\}\partial_x^\nu,$$

one sees that (1.8) reads

(1.9) $${}^t\vec{w}\mathscr{Z}=0,$$

where ${}^t\vec{w}=(w_j)_{j\in Z}$ is a row vector ($w_j=0$ for $j>0$) and $\mathscr{Z}=(z_j^{(\nu)})_{j\in Z,\nu\in N^c}$ is a rectangular matrix of size $Z\times N^c$. After some calculation one knows that $\mathscr{Z}$ satisfies the linear differential equations below:

$$\partial\mathscr{Z}/\partial t_n=\Lambda^n\mathscr{Z}, \qquad \partial\mathscr{Z}/\partial x=\Lambda\mathscr{Z}-\mathscr{Z}\Lambda_{N^c},$$

where $\Lambda=(\delta_{\mu+1,\nu})_{\mu,\nu\in Z}$, $\Lambda_{N^c}=(\delta_{\mu+1,\nu})_{\mu,\nu\in N^c}$ are the shift matrices. Integrating these equations, we see that $\mathscr{Z}$ is represented as

(1.10) $$\mathscr{Z}=\exp\left(x\Lambda+\sum_{n=1}^{\infty} t_n\Lambda^n\right)\Xi\exp(-x\Lambda_{N^c})$$

with a constant matrix $\Xi=(\xi_{\mu\nu})_{\mu\in Z,\nu\in N^c}$. Furthermore the form of the operator U (1.7) guarantees that

(1.11) $$\xi_{\nu\nu}=1, \quad \xi_{\mu\nu}=0 \quad \text{for any } \nu\in N^c \text{ and } \mu<\nu.$$

Hence Ξ is of maximal rank, or a frame of size N^c of the vector space C^Z. Substituting (1.10) into (1.9), we obtain the following proposition.

Proposition 1.1. *Let $W \in \mathscr{E}_{\mathscr{R}}(0)^{monic}$ be a wave operator corresponding to a solution $L \in \mathscr{E}_{\mathscr{R}}(1)^{monic}$ to the* KP *hierarchy. Then the coefficients of W solve the Grassmann equation*

$$ {}^t\vec{w}\Phi\Xi=0, \tag{1.12}$$

where $\Phi=\exp(x\Lambda+\sum_{n=1}^{\infty}t_n\Lambda^n)$, and Ξ is an N^c-frame satisfying the condition (1.11).

By $\mathrm{FR}(N^c; C)$ we mean [25]

$$\begin{aligned}\mathrm{FR}(N^c; C)=\{\Xi=(\xi_{\mu\nu})_{\mu\in Z,\nu\in N^c}\in \mathrm{Mat}(Z\times N^c; C)\,|\,{}^\exists m\in N \text{ such that}\\ \xi_{\mu\nu}=\delta_{\mu\nu} \text{ for } \mu<-m,\ \mu\le\nu,\ =0 \text{ for } -m\le\nu<0,\\ \mu\le -m, \text{ and } \Xi \text{ is of maximal rank}\}.\end{aligned}$$

It is known [25] that, for an N^c-frame $\Xi\in\mathrm{FR}(N^c; C)$, the Grassmann equation (1.12) has a unique solution ${}^t\vec{w}=(w_j)_{j\in Z}$ ($w_0=1$, $w_j=0$ for $j>0$) in the quotient field $\mathscr{K}$. We can show that operator $W=\sum_{-\infty<j\le 0}w_j(x,t)\partial_x^j$ which comes from the solution ${}^t\vec{w}$ solve the Sato equations (1.3). Thus a general frame Ξ yields a solution $L\in\mathscr{E}_{\mathscr{K}}(1)^{monic}$ to the KP hierarchy via the Grassmann equation. All equivalence classes of frames modulo the right action of $\mathrm{GL}(N^c; C)$ constitute the universal Grassmann manifold UGM [24, 25, 27];

$$\mathrm{UGM}=\mathrm{FR}(N^c; C)/\mathrm{GL}(N^c; C),$$

where the group $\mathrm{GL}(N^c; C)$ is, by definition,

$$\begin{aligned}\mathrm{GL}(N^c; C)=\{g=(g_{\mu\nu})_{\mu,\nu\in N^c}\in \mathrm{Mat}(N^c; C)\,|\,{}^\exists m\in N \text{ such that}\\ g_{\mu\nu}=\delta_{\mu\nu} \text{ for } \nu<-m,\ \mu\le\nu,\ =0 \text{ for } -m\le\nu<0,\\ \mu<-m, \text{ and } (g_{\mu\nu})_{-m\le\mu,\nu<0} \text{ is invertible}\}.\end{aligned}$$

Since a solution of the Grassmann equation is invariant under a right multiplication $\Xi\to\Xi g$ by an element g of $\mathrm{GL}(N^c; C)$, we are led to Sato's fundamental theorem on the KP hierarchy.

Theorem 1.2 ([24, 25, 27]). *The solution space of the* KP *hierarchy is identified with* UGM, *and the time evolution of a solution*

$$L(x,t;\partial_x)\longrightarrow L(x+x',t+t';\partial_x)$$

is translated to the dynamical motion

$$\Xi \bmod \mathrm{GL}(N^c; \boldsymbol{C}) \longrightarrow \Phi(x', t')\Xi \bmod \mathrm{GL}(N^c; \boldsymbol{C})$$

on UGM.

Remark. For detailed discussions about UGM, readers may consult with [24, 25, 27].

Solving the Grassmann equation by means of Cramér's formula, one obtains an explicit representation of a wave operator in terms of a frame Ξ:

$$w_{-j} = (-)^j \det({}^t\Xi_j \Phi \Xi)/\det({}^t\Xi_0 \Phi \Xi) \quad (j=1, 2, \cdots),$$

where $\Xi_j = (\delta_{\mu\nu}\ (\mu \in \boldsymbol{Z};\ \nu < -j) \,|\, \delta_{\mu,\nu+1}\ (\mu \in \boldsymbol{Z};\ -j \leq \nu < 0))$. A τ function for the KP hierarchy is nothing but the quantity that appears in this denominator:

$$\tau(t, \Xi) = \det({}^t\Xi_0 \Phi \Xi)|_{x=0}.$$

The τ function plays a crucial role in the soliton theory and is relevant to the representation theory of the general linear group. In the sequel we will list up the main properties of τ functions.

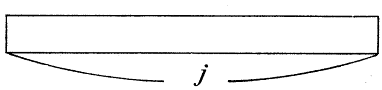
To this end we need the Schur polynomials. First $p_j(t)$ is introduced by

$$\exp\left(\sum_{n=1}^{\infty} t_n \lambda^n\right) = \sum_{j=0}^{\infty} p_j(t) \lambda^j.$$

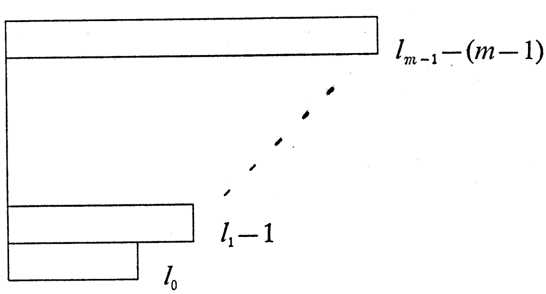
For example,

$$p_0(t)=1, \qquad p_1(t)=t_1, \qquad p_2(t)=t_2+t_1^2/2.$$

The Schur polynomial $p_j(t)$ corresponds to the Young diagram



The Schur polynomial for a general Young diagram



is given by

$$\chi_{l_0 l_1 \cdots l_{m-1}}(t) = \det (p_{l_i - j}(t))_{0 \le i, j < m}.$$

Next we define the Hirota bilinear differential operator. Let $f(t)$, $g(t) \in C[[t]]$ and $\partial_t{}^{\alpha} = \partial_{t_1}{}^{\alpha_1} \partial_{t_2}{}^{\alpha_2} \cdots$. Then the Hirota operator D^{α} acts on $f(t) \cdot g(t)$ as follows:

$$D^{\alpha} f(t) \cdot g(t) = \partial_s{}^{\alpha} f(t+s) g(t-s)|_{s=0}.$$

Proposition 1.3 ([24, 25, 27]). (1) *Let* $W = \sum_{-\infty < j \le 0} w_j(x, t) \partial_x^j$ *be a wave operator for the* KP *hierarchy and* $\tau(t)$ *be the* τ *function for* W. *Then the coefficient* $w_j(x, t)$ *is represented by* $\tau(t)$ *as*

$$w_{-j}(x, t) = p_j(-\tilde{\partial}) \tau(t) / \tau(t)|_{t \to x+t}, \tag{1.13}$$

where $\tilde{\partial} = (\partial_{t_1}, 2^{-1}\partial_{t_2}, 3^{-1}\partial_{t_3}, \cdots)$ *and* $x + t = (x + t_1, t_2, t_3, \cdots)$.

(2) *A* τ *function for the* KP *hierarchy satisfies an infinite number of the Hirota bilinear differential equations*

$$\sum_{i=0}^{m} (-)^i \chi_{l_0 \cdots l_{m-2} k_i}(\tilde{D}/2) \chi_{k_0 \cdots k_{i-1} k_{i+1} \cdots k_m}(-\tilde{D}/2) \tau \cdot \tau = 0. \tag{1.14}$$

($\chi_{l_0 \cdots l_{m-1}}(t)$ *are antisymmetric in indices.*) *Conversely, if* $\tau(t) \in C[[t]]$ *solves* (1.14), *it is a* τ *function for the* KP *hierarchy.*

We mention that there exists another formula that connects a τ function with a wave operator. For a wave operator $W = \sum_{-\infty < j \le 0} w_j \partial_x^j$, we introduce a formal power series by

$$\log \Big(\sum_{-\infty < j \le 0} w_j \lambda^j \Big) = \sum_{n=1}^{\infty} v_n \lambda^{-n}.$$

Then the τ function $\tau(t)$ for the operator W satisfies

$$-\frac{\partial}{\partial t_n} \log \tau = n v_n + \sum_{j=1}^{n-1} \frac{\partial}{\partial t_j} v_{n-j} \qquad (n = 1, 2, \cdots). \tag{1.15}$$

It is known [5] that this is equivalent to (1.13).

To conclude this section we discuss on the reduction of the KP hierarchy [9]. The l-reduced KP hierarchy is a system obtained by imposing a constraint

$$L^l \in \mathscr{D}_x \tag{1.16}$$

on the KP hierarchy. This condition means that a solution L does not depend on the time variables t_{ln} $(n=1, 2, \cdots)$; $\partial L/\partial t_{ln}=0$. In terms of geometry, the isotropy subgroup of the point $\Xi \bmod GL(N^c; C)$ in UGM, where the frame Ξ corresponds to the solution L, contains an abelian group $\{\exp(\sum_{n=1}^{\infty} c_{ln}\Lambda^{ln}) \mid c_{ln} \in C\}$. That is, the frame Ξ must satisfy $\Lambda^l \Xi = \Xi g$ for some $g \in GL(N^c; C)$. The KdV equation is included in the 2-reduced KP hierarchy. In fact, if we impose the condition $u_{t_2}=0$ on the KP equation, it reduces to

$$u_{t_3} - 3uu_x - \frac{1}{4}u_{xxx} = 0,$$

which is just the KdV equation.

Chapter 2.

§ 2-1. Linear differential equations and the Grassmann hierarchy

In this section we first construct nonlinear ordinary differential equations whose solution spaces are canonically identified with finite dimensional Grassmann manifolds. Such differential equations are obtained through algebraic study on linear ordinary differential equations [26].

Let $\mathscr{K}$ be a differential field of characteristic zero with a derivation $\partial = \partial_x : \mathscr{K} \to \mathscr{K}$. The particular element $x \in \mathscr{K}$ is defined as $\partial_x(x)=1$, and $\mathscr{C} = \{f \in \mathscr{K} \mid \partial_x(f)=0\}$ is the constant field of $\mathscr{K}$. Let $\mathscr{D}_{\mathscr{K}}$ be the ring of differential operators with coefficients in $\mathscr{K}$, namely, $\mathscr{D}_{\mathscr{K}} = \mathscr{K}[\partial_x]$. The following fact is well-known.

Proposition 2.1. *Elements f_j $(0 \le j < N)$ in $\mathscr{K}$ are $\mathscr{C}$ linearly independent if and only if the Wronski matrix*

$$\mathrm{WR}(f_0, \cdots, f_{N-1}) = (\partial_x^i(f_j))_{0 \le i, j < N}$$

is invertible.

We consider a linear differential equation

$$Pu = 0, \tag{2.1}$$

where P is a monic operator of order N, $P \in \mathscr{D}(N)^{monic}$, or more generally, a system of differential equations of the first order

$$\partial_x(\vec{u}) = A \cdot \vec{u},$$

where $A \in \mathrm{Mat}(N; \mathscr{K})$.

Proposition 2.2. *The solution spaces* $\mathrm{Sol}_{\mathscr{K}}(P)=\{u\in\mathscr{K}\mid Pu=0\}$ *and* $\mathrm{Sol}_{\mathscr{K}}(\partial_x-A)=\{\vec{u}\in\mathscr{K}\mid\partial_x(\vec{u})=A\cdot\vec{u}\}$ *are* $\mathscr{C}$ *linear spaces of finite dimension, and*

$$\dim_{\mathscr{C}}\mathrm{Sol}_{\mathscr{K}}(P),\ \dim_{\mathscr{C}}\mathrm{Sol}_{\mathscr{K}}(\partial_x-A)\leq N.$$

Motivated by this fact, one defines the $\mathscr{K}$-solvability of a differential equation (2.1) by

Definition. A linear differential equation (or an operator $P\in\mathscr{D}_{\mathscr{K}}(N)^{monic}$) is said to be $\mathscr{K}$-solvable if and only if

$$\dim_{\mathscr{C}}\mathrm{Sol}_{\mathscr{K}}(P)=N.$$

An operator $W\in\mathscr{D}_{\mathscr{K}}(m)^{monic}$ is called a right factor of P if P belongs to the left ideal $\mathscr{D}_{\mathscr{K}}\cdot W$, i.e., if one finds an operator $Z\in\mathscr{D}_{\mathscr{K}}(N-m)^{monic}$ satisfying

$$P=Z\cdot W. \tag{2.2}$$

Obviously $\mathrm{Sol}_{\mathscr{K}}(W)\subset\mathrm{Sol}_{\mathscr{K}}(P)$. Moreover the next statement on the solvability of W is deduced by making use of the preceding propositions.

Proposition 2.3. *Suppose that the operator* P *in* (2.2) *is* $\mathscr{K}$*-solvable. Then the operators* Z *and* W *are both* $\mathscr{K}$*-solvable.*

Write the operator W as

$$W=\sum_{j=0}^{m}w_j\partial_x^{m-j}\quad(w_0=1). \tag{2.3}$$

The condition (2.2) yields a system of nonlinear ordinary differential equations of the coefficients w_j's, which is referred to as $E(m;P)$.

Example. Let $P=\partial_x^2$, $Z=\partial_x+z$, $W=\partial_x+w$. From the condition (2.2) one obtains an equation $w_x-w^2=0$. This equation can be easily integrated. An arbitrary solution of the equation is represented as

$$w=\frac{-\xi_0}{\xi_0x+\xi_1},$$

where $(\xi_0,\xi_1)\in\mathscr{C}^2$ should be a non-zero vector. Since the same solution as above comes from $(a\xi_0,a\xi_1)$ $(a\neq0)$, the solution space is identified with the 1-dimensional projective space $\boldsymbol{P}^1(\mathscr{C})$.

In general, for a $\mathscr{K}$-solvable operator P, the solution space of $E(m;P)$ is shown to be isomorphic to the Grassmann manifold

$$\mathrm{GM}(N, m)=\mathrm{FR}(N, m; \mathscr{C})/\mathrm{GL}(m; \mathscr{C}),$$

where $\mathrm{FR}(N, m; \mathscr{C})=\{\Xi \in \mathrm{Mat}(N\times m; \mathscr{C})\,|\,\Xi$ is of maximal rank$\}$ is the set of m-frames in $\mathscr{C}^N$. Let P be a monic, $\mathscr{K}$-solvable operator of order N. Then Proposition 2.3 tells us that the solution space $\mathrm{Sol}_{\mathscr{X}}(W)$ of a right factor W (2.2) of P is an m dimensional subspace in the N dimensional $\mathscr{C}$ linear space $V=\mathrm{Sol}_{\mathscr{X}}(P)$. Hence one has a map

$$\begin{aligned}\mathrm{Sol}_{\mathscr{X}}(\cdot) : \{W \in \mathscr{D}_{\mathscr{X}}(m)^{monic}\,|\, W \text{ is a right factor of } P\}\\ \longrightarrow \mathrm{GM}(m; V)=\{U\subset V\,|\,\dim_{\mathscr{C}} U=m\}.\end{aligned}$$

This map is shown to be bijective as follows: Let $U \in \mathrm{GM}(m\colon V)$ and $\{\psi_j\}_{0\le j<m}$ be a basis of U. First an operator $W \in \mathscr{D}_{\mathscr{X}}(m)^{monic}$ is uniquely determined by the condition

$$W\psi_j=0 \quad (0\le j<m). \tag{2.4}$$

Notice that, writing the operator W as (2.3), the equations above read

$$(w_m, \cdots, w_1)\cdot \mathrm{WR}(\psi_0, \cdots, \psi_{m-1})=-(\partial_x^m(\psi_0), \cdots, \partial_x^m(\psi_{m-1})), \tag{2.5}$$

which is a system of linear algebraic equations with respect to w_j's. Futhermore the division theorem of differential operators deduces that the operator W is a right factor of P. Thus one concludes

Proposition 2.5 ([26]). *One has an isomorphism*

$$\mathrm{Sol}_{\mathscr{X}}(\cdot) : \{W \in \mathscr{D}_{\mathscr{X}}(m)^{monic}\,|\, W \textit{ is a right factor of } P\}\xrightarrow{\ \sim\ }\mathrm{GM}(m; V).$$

Therefore the solution space of $E(m; P)$ is canonically identified with the Grassmann manifold $\mathrm{GM}(m; V)\simeq \mathrm{GM}(N, m)$.

Let $\{\varphi_j\}_{0\le j<N}$ be a basis of V. Then ψ_j's are expressed as

$$\psi_j=\sum_{i=0}^{N-1}\varphi_i\xi_{ij} \quad (0\le j<m).$$

Setting $\Xi=(\xi_{ij})_{0\le i<N, 0\le j<m} \in \mathrm{FR}(N, m; \mathscr{C})$, the equation (2.5) reads

$${}^t\vec{w}\Phi\Xi=0, \tag{2.6}$$

where ${}^t\vec{w}=(w_m, \cdots, w_1, 1, 0, \cdots, 0) \in \mathscr{K}^N$, $\Phi=\mathrm{WR}(\varphi_0, \cdots, \varphi_{N-1})$. This is a finite dimensional analogue of the equation (1.12) in Chapter 1. In the case that P is an operator with constant coefficients, a solution of $\mathrm{E}(m; P)$ has $N-1$ independent time evolutions according to the KP hierarchy's flow. Such a hierarchy is referred to as the Grassmann hierarchy of the type $\mathrm{E}(m; P)$. For definiteness of the argument, let $\mathscr{K}$ be the quotient field of the ring of formal power series $\boldsymbol{C}[[x, t_1, \cdots t_{N-1}]]$ and $P \in \mathscr{D}_C(N)^{monic}$. Let V be the solution space of an enlarged system

$$(2.7) \qquad P\varphi=0, \quad \partial_{t_l}(\varphi)=\partial_x{}^l(\varphi) \quad (l=1, \cdots, N-1),$$

and consider the inverse correspondence in Proposition 2.5. Applying ∂_{t_l} to the equation, we see that the time evolution of the operator W corresponding to the subspace $U \in \mathrm{GM}(m; V)$ satisfies

$$\left(\frac{\partial W}{\partial t_l}+W\cdot\partial_x{}^l\right)\psi_j=0 \quad (0\leq j<m).$$

The division theorem again entails that

$$\frac{\partial W}{\partial t_l}+W\cdot\partial_x{}^l=B_l\cdot W$$

for some operator $B_l \in \mathscr{D}_{\mathscr{K}}(l)^{monic}$. Putting

$$W'=W\cdot\partial_x{}^{-m} \in \mathscr{E}_{\mathscr{K}}(0)^{monic},$$

the Sato equatons

$$\frac{\partial W'}{\partial t_l}=B_l W'-W'\partial_x{}^l \quad (l=1, \cdots, N-1)$$

$$\text{with} \quad B_l=(W\partial_x{}^l W'^{-1})_+$$

are derived. For an operator P with generic constant coefficients the Grassmann hierarchy of the type $\mathrm{E}(m; P)$ gives a soliton solution of the KP hierarchy, and, in the most degenerate case, i.e., $P=\partial_x{}^N$, it yields a rational solution.

Fianlly we remark that taking the limit of $N\to\infty$, $m\to\infty$ recovers the whole of the KP hierarchy, and that the equation (2.6) leads to the Grassmann equation for the KP hierarchy.

§ 2-2. Supersymmetric linear differential equations

Let $\mathscr{A}$ be a Grassmann algebra $\Lambda(\mathscr{C}^M)$, and e_i $(0\leq i<M)$ be the generators of $\mathscr{A}$ satisfying the Grassmann relation $e_ie_j+e_je_i=0$. The set Π is a set of multi-indices $\alpha=(\alpha_0, \cdots, \alpha_{l-1})$ such that $\alpha_0<\alpha_1<\cdots<\alpha_{l-1}$ $(0<l\leq M)$. For $\alpha=(\alpha_0, \cdots, \alpha_{l-1}) \in \Pi$, we define $|\alpha|=l$ and $e_\alpha=e_{\alpha_0}\cdots e_{\alpha_{l-1}}$. The index α with $|\alpha|=0$ stands for the void sequence ϕ,

and $e_\phi=1$. For an integer m, $\underline{m}$ denotes the residue class of m in $\boldsymbol{Z}_2$. The algebra $\mathscr{A}$ is supercommutative with the $\boldsymbol{Z}_2$-gradrtion, $\mathscr{A}=\mathscr{A}_{\underline{0}}\oplus\mathscr{A}_{\underline{1}}$

$$\mathscr{A}_{\underline{\nu}}=\bigoplus_{l\equiv\nu}\sum_{|\alpha|=l}\mathscr{C}e_\alpha.$$

The canonical projection $\varepsilon:\mathscr{A}\to\mathscr{C}=\mathscr{A}/(\mathscr{A}_{\underline{1}})$ is referred to as the body map, where $(\mathscr{A}_{\underline{1}})$ is the ideal generated by $\mathscr{A}_{\underline{1}}$.

Definition. Let V be a right $\mathscr{A}$ module. Then V is said to be a graded $\mathscr{A}$ module if and only if

(i) V is a $\boldsymbol{Z}_2$-graded module; $V=V_{\underline{0}}\oplus V_{\underline{1}}$,

(ii) $V_{\underline{n}}\cdot\mathscr{A}_{\underline{\nu}}\subset V_{\underline{n}+\underline{\nu}}$ $(n,\nu=0,1)$.

Definition. (a) Let V be a free right $\mathscr{A}$ module of rank N. Then it is said to be *pure* if and only if

(i) V is a graded $\mathscr{A}$ module,

(ii) there exists an $\mathscr{A}$ basis $\{v_j\}_{0\le j<N}$ of V such that $v_j\in V_{\underline{j}}$ for $0\le j<N$.

Such basis is called a pure basis, and $V_{\underline{n}}=\sum v_j\cdot\mathscr{A}_{\underline{j}+\underline{n}}$.

(b) Let V be a pure $\mathscr{A}$ module of rank N. An $\mathscr{A}$ submodule U of V is said to be a *pure $\mathscr{A}$ submodule* of V if and only if

(i) U is a pure $\mathscr{A}$ module,

(ii) U is a graded $\mathscr{A}$ submodule of V, i.e., $U_{\underline{n}}\subset V_{\underline{n}}$ $(n=0,1)$.

Lemma 2.5. *Let V be a pure $\mathscr{A}$ module of rank N, and $\{v_j\}_{0\le j<N}$, $\{u_j\}_{0\le j<N}$ be two pure bases of V. Then these are related by*

$$(u_0,\ \cdots,u_{N-1})=(v_0,\cdots,v_{N-1})\cdot g$$

for some $g\in\mathrm{SGL}(N;\mathscr{A})$. The supergroup $\mathrm{SGL}(N;\mathscr{A})$ is defined by

$$\mathrm{SGL}(N;\mathscr{A})=\{g=(g_{\mu\nu})_{0\le\mu,\nu<N}\in\mathrm{Mat}(N;\mathscr{A})\,|\,g_{\mu\nu}\in\mathscr{A}_{\underline{\mu}+\underline{\nu}}\ \text{and}\ \varepsilon(g)=(\varepsilon(g_{\mu\nu}))\ \text{is invertible}\}.$$

For a pure $\mathscr{A}$ module V of rank N we introduce a super Grassmann manifold by

$$\mathrm{SGM}(m;V)=\{U\subset V\,|\,U\ \text{is a pure}\ \mathscr{A}\ \text{submodule of rank}\ m\ \text{of}\ V\},$$

and the set of superframes by

$$\mathrm{SFR}(N,m;\mathscr{A})=\{\varXi=(\xi_{\mu\nu})_{\substack{0\le\mu<N\\0\le\nu<m}}\in\mathrm{Mat}(N\times m;\mathscr{A})\,|\,\xi_{\mu\nu}\in\mathscr{A}_{\underline{\mu}+\underline{\nu}}\ \text{and}\ \varepsilon(\varXi)\ \text{is of maximal rank}\}.$$

The supergroup $\mathrm{SGL}(m; \mathscr{A})$ acts on this set from the right, and the quotient set is a super Grassmann manifold $\mathrm{SGM}(N, m; \mathscr{A})$;

$$\mathrm{SGM}(N, m; \mathscr{A}) = \mathrm{SFR}(N, m; \mathscr{A})/\mathrm{SGL}(m; \mathscr{A}).$$

Lemma 2.6. *The super Grassmann manifold* $\mathrm{SGM}(m; V)$ *is identified with* $\mathrm{SGM}(N, m; \mathscr{A})$.

Proof. We fix a pure basis $\{v_j\}_{0 \le j < N}$ of V. Let $U \in \mathrm{SGM}(m; V)$ and $\{u_j\}_{0 \le j < m}$ be its pure basis. They are expressed as an $\mathscr{A}$ linear combination of v_j's

$$u_j = \sum v_i \xi_{ij} \qquad \xi_{ij} \in \mathscr{A}_{\underline{i}+\underline{j}}.$$

Set $\Xi = (\xi_{ij}) \in \mathrm{Mat}(N \times m; \mathscr{A})$. We show that $\varepsilon(\Xi)$ is of maximal rank. Suppose that $\varepsilon(\Xi)$ is not of maximal rank. Then one can find a non-zero vector $\vec{c} \in \mathscr{C}^m$ satisfying $\varepsilon(\Xi)\vec{c} = 0$. Since

$$(u_0, \cdots, u_{m-1})\vec{c} = (v_0, \cdots, v_{N-1})\Xi\vec{c},$$

multiplying the element $e_0 \cdots e_{M-1} \in \mathscr{A}$ to the both sides, and noting that $\varepsilon(\Xi\vec{c}) = 0$, one has $(u_0, \cdots, u_{m-1})(\vec{c}e_0 \cdots e_{M-1}) = 0$. This contradicts the fact that u_j's are $\mathscr{A}$ free. Thus it is proved that $\varepsilon(\Xi)$ is of maximal rank. From Lemma 2.5 it follows that change of a pure basis in the module U induces the right action of $\mathrm{SGL}(m; \mathscr{A})$ on the frame Ξ. Thus one has a canonical one to one correspondence

$$\mathrm{SGM}(m; V) \ni U \to \Xi \bmod \mathrm{SGL}(m; \mathscr{A}) \in \mathrm{SGM}(N, m; \mathscr{A}). \qquad \square$$

Now let us construct the ring of supersymmetric (abbreviated as SUSY henceforth) differential operators. Regard the differential field $\mathscr{K}$ as a $\boldsymbol{Z}_2$-graded algebra by $\mathscr{K} = \mathscr{K}_{\underline{0}} \oplus \mathscr{K}_{\underline{1}}$ ($\mathscr{K}_{\underline{0}} = \mathscr{K}$, $\mathscr{K}_{\underline{1}} = \{0\}$), and $\mathscr{C}[\theta]$ be the Grassmann algebra generated by θ over $\mathscr{C}$ ($\theta^2 = 0$). We define the algebra $\mathscr{Q}$ by

$$\mathscr{Q} = (\mathscr{C}[\theta] \otimes \mathscr{K}) \otimes \mathscr{A}. \tag{2.8}$$

Under the $\boldsymbol{Z}_2$-graded algebraic structure introduced in [13], [17] (we will refer to this structure as $\boldsymbol{Z}_2$-graded tensor, hereafter), the algebra $\mathscr{Q}$ is a supercommutative algebra with the $\boldsymbol{Z}_2$-gradation $\mathscr{Q} = \mathscr{Q}_{\underline{0}} \oplus \mathscr{Q}_{\underline{1}}$,

$$\mathscr{Q}_{\underline{\nu}} = \{f = \sum_{\alpha \in \Pi, |\alpha| \equiv \nu} f^{(\alpha)} e_\alpha + \theta \sum_{\alpha \in \Pi, |\alpha| \equiv \nu+1} f^{(\alpha)} e_\alpha \,|\, f^{(\alpha)} \in \mathscr{K}\}.$$

Elements in $\mathscr{Q}$ are called superfields. The body map ε extends to the algebra $\mathscr{Q}$ in a canonical way;

$$\varepsilon\colon \mathscr{Q}\to\mathscr{K}, \qquad \varepsilon(f)=f^{(\phi)}.$$

We introduce a SUSY differential operator Θ by

$$\Theta=\partial_\theta+\theta\partial_x.$$

Writing an element $\varphi\in\mathscr{Q}$ as $\varphi=f+\theta g$ $(f, g\in\mathscr{K}\otimes\mathscr{A})$, the action of Θ is prescribed by

$$\Theta(\varphi)=g+\theta\partial_x(f).$$

Note that $\Theta^2=\partial_x$. The set of SUSY differential operators is defined by

$$\mathscr{D}_{\mathscr{Q}}^{1|1}=\mathscr{Q}[\Theta]=\sum_{0\le j\ll+\infty} p_j\Theta^j\,|\,p_j\in\mathscr{Q}\},$$

which is endowed with a noncommutative ring structure via the super Leibniz rule:

$$\Theta^{2n}\cdot\varphi=\sum_{j=0}^{n}\binom{n}{j}\partial_x{}^j(\varphi)\cdot\Theta^{2n-2j},$$

$$\Theta^{2n+1}\cdot\varphi=\sum_{j=0}^{n}\binom{n}{j}\partial_x{}^j(\Theta(\varphi))\cdot\Theta^{2n-2j}+(-)^\nu\sum_{j=0}^{n}\binom{n}{j}\partial_x{}^j(\varphi)\cdot\Theta^{2n+1-2j}$$

$$(\varphi\in\mathscr{Q}_\nu).$$

The dot stands for a product of a SUSY operator Θ^m and a multiplication operator by an element φ. The $\boldsymbol{Z}_2$-gradation is given by

$$(\mathscr{D}_{\mathscr{Q}}^{1|1})_\nu=\{\sum_{0\le j\ll+\infty} p_j\Theta^j\,|\,p_j\in\mathscr{Q}_{j+\nu}\},$$

and a filtration compatible with this gradation by

$$\mathscr{D}_{\mathscr{Q}}^{1|1}=\bigcup_{N=0}^{\infty}\mathscr{D}_{\mathscr{Q}}^{1|1}(N),$$

where $\mathscr{D}_{\mathscr{Q}}^{1|1}(N)=\{\sum_{0\le j\le N}p_j\Theta^j\,|\,p_j\in\mathscr{Q}\}$. The order of a SUSY operator is defined by

$$\mathrm{ord}_\Theta(P)=N,$$

if $P=\sum_{0\le j\le N}p_j\Theta^j$ and $p_N\neq 0$.

Proposition 2.7 (Division theorem). *Let W be a monic* SUSY *differential operator of order m, i.e., $W=\sum_{i=0}^{m}w_i\Theta^{m-i}$ $(w_0=1)$. For any $P\in\mathscr{D}_{\mathscr{Q}}^{1|1}$ there exist uniquely two operators Z and $R\in\mathscr{D}_{\mathscr{Q}}^{1|1}$ satisfying*

$$P=ZW+R$$

and $\mathrm{ord}_\Theta(R)<m$.

We consider a linear SUSY differential equation

$$Pu=0, \tag{2.9}$$

where $P=\sum_{j=0}^N p_j\Theta^{N-j}\in\mathscr{D}_{\mathscr{Q}}^{1|1}(N)^{monic}$ (i.e., $p_0=1$). The solution space of (2.9) $\mathrm{Sol}_{\mathscr{Q}}(P)=\{u\in\mathscr{Q}\,|\,Pu=0\}$ is canonically endowed with the structure of a right graded $\mathscr{A}$ module.

Theorem 2.8. *Let* $\mathscr{R}=\boldsymbol{C}[[x]]$ *and* $\mathscr{K}$ *be the quotient field of* $\mathscr{R}$, $\mathscr{K}=\boldsymbol{C}((x))$. *The algebra* $\mathscr{S}$ *of formally regular superfields is defined by* $\mathscr{S}=(\boldsymbol{C}[\theta]\otimes\mathscr{R})\otimes\mathscr{A}$ *and* $\mathscr{D}_{\mathscr{S}}^{1|1}$ *denotes the algebra of* SUSY *differential operators with coefficients in* $\mathscr{S}$.

(i) *Let* $P=\sum_{j=0}^N p_j(x,\theta)\Theta^{N-j}\in\mathscr{D}_{\mathscr{S}}^{1|1}(N)^{monic}$. *Then the solution space* $\mathrm{Sol}_{\mathscr{Q}}(P)$ *is included in* $\mathscr{S}$ *and is a free right* $\mathscr{A}$ *module of rank N.*

(ii) *Assume that* $P\in(\mathscr{D}_{\mathscr{S}}^{1|1}(N))_{\underline{N}}$. *Then* $\mathrm{Sol}_{\mathscr{Q}}(P)$ *is a pure* $\mathscr{A}$ *module of rank N with the* $\boldsymbol{Z}_2$*-gradation*

$$\mathrm{Sol}_{\mathscr{Q}}(P)=\mathrm{Sol}_{\mathscr{Q}}(P)_{\underline{0}}\oplus\mathrm{Sol}_{\mathscr{Q}}(P)_{\underline{1}},$$

where $\mathrm{Sol}_{\mathscr{Q}}(P)_{\underline{n}}=\{u\in\mathscr{Q}_{\underline{n}}\,|\,Pu=0\}$.

To prove this theorem, we should first show the following proposition.

Proposition 2.9. *Consider a system of* SUSY *differential equations of the first order*

$$\Theta(\vec{u})=A\vec{u}, \tag{2.10}$$

where $\vec{u}\in\mathscr{S}^N$ *and* $A=A(x,\theta)\in\mathrm{Mat}(N;\varphi)$. *This system has a fundamental solution matrix* $\Phi(x,\theta)\in\mathrm{Mat}(N,\mathscr{S})$ *with an initial condition* $\Phi(0,0)=I$ *(hence* $\Phi(x,\theta)\in\mathrm{GL}(N;\mathscr{S})$*).*

Proof. Writing $A(x,\theta)=F(x)+\theta G(x)$, $\vec{u}(x,\theta)=\vec{\varphi}(x)+\theta\vec{\psi}(x)$, the system (2.10) reads

$$\vec{\psi}=F\vec{\varphi},\qquad \partial_x(\vec{\varphi})=G\vec{\varphi}+F^*\vec{\psi},$$

where F^* is defined by $F^*=(f_{ij,\underline{0}}-f_{ij,\underline{1}})$ when $F=(f_{ij})$ with $f_{ij}=f_{ij,\underline{0}}+f_{ij,\underline{1}}$ ($f_{ij,\underline{\nu}}\in\mathscr{S}_{\underline{\nu}}$). The equation for φ reduces to

$$\partial_x(\vec{\varphi})=(G+F^*F)\vec{\varphi},$$

which has a fundamental solution matrix $U(x)$ with an initial condition $U(0)=I$. Then $\Phi(x, \theta)=e^{\theta F(x)}U(x)$ is a fundamental solution matrix satisfying the condition. □

Proof of Theorem 2.8.

(i) It is easy to show that $\mathrm{Sol}_{\mathscr{S}}(P)\subset\mathscr{S}$. The equation (2.9) is equivalent to a system of the first order

$$\Theta(\vec{u})=A\vec{u}, \tag{2.11}$$

where $\vec{u}=(\Theta^i(u))_{0\le i<N}$, and

$$A=\begin{bmatrix} 0 & 1 & & & \\ & \ddots & \ddots & 0 & \\ & & \ddots & \ddots & \\ & 0 & & \ddots & \ddots \\ & & & 0 & 1 \\ -p_N & \cdots & & -p_2 & -p_1 \end{bmatrix}.$$

Proposition 2.9 says that there exists a fundamental solution matrix $\Phi=(\Theta^i(\varphi_j))_{0\le i,j<N}$ of (2.11) satisfying $\Phi(0,0)=1$. It is obvious that $\{\varphi_j\}_{0\le j<N}$ provides $\mathscr{A}$ generators of $\mathrm{Sol}_{\mathscr{S}}(P)$. We show that these generators are $\mathscr{A}$ free. Suppose that $\sum\varphi_j\cdot a_j=0$ for some $\vec{a}\in\mathscr{A}^N$. Then $\Phi\vec{a}=0$. The invertibility of Φ implies $\vec{a}=0$. Thus φ_j's are $\mathscr{A}$ free.

(ii) The following lemma is easily proved by the Taylor expansion.

Lemma 2.10. *A formally regular superfield φ belongs to $\mathscr{S}_{\underline{\nu}}$ if and only if all the initially values $\Theta^j\varphi|_{x=\theta=0}$ belong to $\mathscr{A}_{\underline{j}+\underline{\nu}}$.*

The initial condition $\Phi(0,0)=I$ means that $\Theta^i(\varphi_j)|_{x=\theta=0}\in\mathscr{A}_{\underline{i}+\underline{j}}$ for $0\le i<N$. Using the equation $Pu=0$ and Lemma 2.10 one can verify that $\Theta^i(\varphi_j)|_{x=\theta=0}\in\mathscr{A}_{\underline{i}+\underline{j}}$ for any i. Hence $\varphi_{\underline{j}}\in\mathscr{S}_{\underline{j}}$. Obviously $\mathrm{Sol}_{\mathscr{S}}(P)_{\underline{\nu}}=\sum\varphi_j\mathscr{A}_{\underline{j}+\underline{\nu}}$. Thus the theorem is proved. □

Examples of linear SUSY differential equations

(i) $\Theta^N(u)=0$.

A pure basis of the solution space is given by $\varphi_{2j}=x^j/j!$, $\varphi_{2j+1}=\theta x^j/j!$ $(0\le j<N/2)$ for even N, or $\varphi_{2j}=x^j/j!$ $(0\le j<(N+1)/2)$, $\varphi_{2j+1}=\theta x^j/j!$ $(0\le j<(N-1)/2)$ for odd N.

(ii) Let $A\in\mathrm{Mat}(N;\mathscr{A}_{\underline{0}})$ and consider a system with the coefficient A:

$$\Theta(\vec{u})=A\vec{u}.$$

As a fundamental solution matrix one has $\Phi(x,\theta)=\exp(\theta A+xA^2)$.

(iii) $(\Theta^3+a\Theta)u=0$ $(a \in \mathscr{A}_0)$.

A pure basis of the solution space is given by $\varphi_0=1$, $\varphi_1=\theta e^{-ax}$, $\varphi_2=e^{-ax}$.

The following example tells us that the solution space $\mathrm{Sol}_{\mathscr{Q}}(P)$ is not always a free module.

(iv) Let us consider

$$\left(\partial_x-\frac{e_1e_2}{x}\right)u=0.$$

An arbitrary solution of this equation is represented by an $\mathscr{A}$ linear combination of $u_0=1+e_1e_2\log x$ and $u_1=\theta u_0$. Hence $\mathrm{Sol}_{\mathscr{Q}}(P)$ is generated by e_1u_0, e_2u_0, e_1u_1, e_2u_1. The solution space is not $\mathscr{A}$ free.

Theorem 2.8 and the above example (iv) suggest the following definition of linear SUSY differential equations.

Definition. A linear SUSY differential equation $Pu=0$ $(P \in \mathscr{D}_{\mathscr{Q}}^{1|1}(N)^{monic})$, or simply the operator P, is said to be $\mathscr{Q}$-solvable if and only if the solution space $\mathrm{Sol}_{\mathscr{Q}}(P)$ is a free right $\mathscr{A}$ module of rank N.

In what follows we will study the structure of the solution space $\mathrm{Sol}_{\mathscr{Q}}(P)$ when it is a free $\mathscr{A}$ module. For this end we must investigate, in more detail, the structure of a linear SUSY differential equation itself. Differential operators $P_{\underline{i},\underline{j};\alpha,\beta} \in \mathscr{D}_{\mathscr{X}}$ $(i,j=0,1,\ \alpha,\beta \in \Pi)$ are associated with a SUSY differential operators $P \in \mathscr{D}_{\mathscr{Q}}^{1|1}$ by the following prescription: When expanding a superfield $\varphi=\varphi_{\underline{0}}+\varphi_{\underline{1}}$ $(\varphi_{\underline{j}} \in \mathscr{Q}_{\underline{j}})$ in the Grassmann basis $e_\alpha(\alpha \in \Pi)$

$$\varphi_{\underline{j}}=\sum_{|\alpha|\equiv j}\varphi_{\underline{j}}^{(\alpha)}e_\alpha+\theta\sum_{|\alpha|\equiv j+1}\varphi_{\underline{j}}^{(\alpha)}e_\alpha \quad (\varphi_{\underline{j}}^{(\alpha)} \in \mathscr{K}),$$

$P_{\underline{i},\underline{j};\alpha,\beta}$ is introduced through

$$(P\varphi)_{\underline{i}}^{(\alpha)}=\sum P_{\underline{i},\underline{j};\alpha,\beta}(\varphi_{\underline{j}}^{(\beta)}).$$

A total order "$\prec$" is defined in the set Π as the lexicographical order, and the set Π is thought of as a totally ordered set $(\Pi, \prec)$ hereafter. The operators $P_{\underline{i},\underline{j};\alpha,\beta}$ form a matrix of differential operators

$$\tilde{P}=(\tilde{P}_{i,j})_{i,j=0,1} \in \mathrm{Mat}(2^{M+1};\mathscr{D}_{\mathscr{X}})$$

where $\tilde{P}_{i,j}=(P_{i,j;\alpha,\beta})_{\alpha,\beta\in\Pi} \in \mathrm{Mat}(2^M;\mathscr{D}_{\mathscr{X}})$ is a lower triangular matrix with respect to the order "$\prec$". The map $P\to\tilde{P}$ is an algebra homomorphism from $\mathscr{D}_{\mathscr{Q}}^{1|1}$ to $\mathrm{Mat}(2^{M+1};\mathscr{D}_{\mathscr{X}})$. A system of differential equations

$$\tilde{P}\tilde{u}=0 \tag{2.12}$$

is associated with a SUSY differential equation $Pu=0$, and the solution space $\mathrm{Sol}_{\mathscr{D}}(P)$ is isomorphic to the solution space of (2.12)

$$\mathrm{Sol}_{\mathscr{K}}(\tilde{P})=\{\vec{u}\in\mathscr{K}^{2^{M+1}}\,|\,\tilde{P}\vec{u}=0\}$$

as a $\mathscr{C}$ linear space via the correspondence

$$\mathrm{Sol}_{\mathscr{D}}(P)\ni u\to\vec{u}=(\vec{u}_j)_{j=0,1}\in\mathrm{Sol}_{\mathscr{K}}(\tilde{P}),$$

where $\vec{u}_j=(u_j^{(\alpha)})_{\alpha\in\Pi}$.

The order of the matrix entries in $\tilde{P}_{\underline{i},\underline{j}}$ is seen as follows: For even N, $\tilde{P}_{\underline{0},\underline{0};\alpha\alpha}$, $\tilde{P}_{\underline{1},\underline{1};\alpha\alpha}\in\mathscr{D}_{\mathscr{K}}(N/2)^{monic}$ and

$$\text{(2.13)}\qquad \begin{aligned}&\mathrm{ord}(\tilde{P}_{\underline{i},\underline{j};\alpha\beta})<\frac{N}{2}\quad(\beta\prec\alpha,\ i=0,1),\\ &\mathrm{ord}(\tilde{P}_{\underline{i},\underline{i};\alpha\beta})<\frac{N}{2}\quad(\beta\preceq\alpha,\ i\neq j).\end{aligned}$$

For odd N, $\tilde{P}_{\underline{1},\underline{0};\alpha\alpha}\in\mathscr{D}_{\mathscr{K}}\left(\frac{N+(-1)^{|\alpha|}}{2}\right)^{monic}$, and

$$\mathrm{ord}(\tilde{P}_{\underline{1},\underline{0};\alpha\beta})<\frac{N+(-1)^{|\beta|}}{2}\quad(\beta\prec\alpha),$$

$$\mathrm{ord}(\tilde{P}_{\underline{0},\underline{0};\alpha\beta})<\frac{N+(-1)^{|\beta|}}{2}\quad(\beta\preceq\alpha),$$

and $\tilde{P}_{\underline{0},\underline{1};\alpha\alpha}\in\mathscr{D}_{\mathscr{K}}\left(\frac{N-(-1)^{|\alpha|}}{2}\right)^{monic}$,

$$\mathrm{ord}(\tilde{P}_{\underline{0},\underline{1};\alpha\beta})<\frac{N-(-1)^{|\beta|}}{2}\quad(\beta\prec\alpha),$$

$$\mathrm{ord}(\tilde{P}_{\underline{1},\underline{1};\alpha\beta})<\frac{N-(-1)^{|\beta|}}{2}\quad(\beta\preceq\alpha).$$

Such condition on the order of the differential operators gives the next theorem, which justifies the definition of the $\mathscr{Q}$-solvability.

Theorem 2.11. *Let* $P\in\mathscr{D}_{\mathscr{D}}^{1|1}(N)^{monic}$ *and* $\mathrm{Sol}_{\mathscr{D}}(P)$ *be a free right* $\mathscr{A}$ *module. Then one has*

$$\mathrm{rank}_{\mathscr{A}}\,\mathrm{Sol}_{\mathscr{D}}(P)\leq N.$$

Proof. For simplicity of the argument, we assume N to be an even integer. Notice that, when the solution space $\mathrm{Sol}_{\mathscr{D}}(P)$ is a free $\mathscr{A}$ module,

$$\dim_{\mathscr{C}}\mathrm{Sol}_{\mathscr{D}}(P)=\dim_{\mathscr{C}}\mathrm{Sol}_{\mathscr{D}}(\tilde{P})=2^M\times\mathrm{rank}_{\mathscr{A}}\,\mathrm{Sol}_{\mathscr{D}}(P).$$

The condition (2.13) enables one to rewrite equivalently the equation (2.12) to a system of equations of the first order

$$\partial_x([\vec{u}])=[\tilde{P}]\cdot[\vec{u}],$$

where $[\vec{u}]=([\vec{u}_{\underline{j}}])_{j=0,1}$ with $[\vec{u}_{\underline{j}}]=(\partial_x{}^k(u_j^{(\alpha)}))_{0\leq k<N/2,\alpha\in\Pi}\in\mathscr{K}^{2^{M-1}\cdot N}$ and the matrix $[\tilde{P}]\in\mathrm{Mat}(2^M\cdot N;\mathscr{K})$ is determined by $\tilde{P}$. By virtue of Proposition 2.2, we see that $\dim_{\mathscr{C}}\mathrm{Sol}_{\mathscr{Q}}(\tilde{P})\leq 2^M\cdot N$, and the theorem is proved. □

Let us call an operator $P\in\mathscr{D}_{\mathscr{Q}}^{1|1}$ a pure operator of order N when $P\in(\mathscr{D}_{\mathscr{Q}}^{1|1}(N)^{monic})_N$. Now we will discuss on the structure of the solution space $\mathrm{Sol}_{\mathscr{Q}}(P)$ when P is a pure operator. In such a case, as was mentioned before, $\mathrm{Sol}_{\mathscr{Q}}(P)$ is a graded $\mathscr{A}$ module with the $\boldsymbol{Z}_2$-gradation

$$\mathrm{Sol}_{\mathscr{Q}}(P)=\mathrm{Sol}_{\mathscr{Q}}(P)_{\underline{0}}\oplus\mathrm{Sol}_{\mathscr{Q}}(P)_{\underline{1}}, \tag{2.14}$$

where $\mathrm{Sol}_{\mathscr{Q}}(P)_{\underline{n}}=\{u\in\mathscr{Q}_{\underline{n}}\,|\,Pu=0\}$. The system of equations (2.12) splits to the two systems

$$\tilde{P}_{\underline{N},\underline{0}}\vec{u}_{\underline{0}}=0,\qquad \tilde{P}_{\underline{N}+\underline{1},\underline{1}}\vec{u}_{\underline{1}}=0. \tag{2.15}$$

One has a direct sum decomposition

$$\mathrm{Sol}_{\mathscr{K}}(\tilde{P})=\mathrm{Sol}_{\mathscr{K}}(\tilde{P}_{\underline{N},\underline{0}})\oplus\mathrm{Sol}_{\mathscr{K}}(\tilde{P}_{\underline{N},\underline{1}}),$$

corresponding to the $\boldsymbol{Z}_2$-gradation (2.14), and each direct summand $\mathrm{Sol}_{\mathscr{Q}}(P)_{\underline{n}}$ is isomorphic to $\mathrm{Sol}_{\mathscr{K}}(\tilde{P}_{\underline{N}+\underline{n},\underline{n}})$ as a $\mathscr{C}$ linear space. The condition (2.13) gives a supremum of the dimension of these solution spaces: For even N,

$$\dim_{\mathscr{C}}\mathrm{Sol}_{\mathscr{K}}(\tilde{P}_{\underline{0},\underline{0}}),\ \dim_{\mathscr{C}}\mathrm{Sol}_{\mathscr{K}}(\tilde{P}_{\underline{1},\underline{1}})\leq 2^{M-1}\cdot N, \tag{2.16}$$

and for odd N,

$$\dim_{\mathscr{C}}\mathrm{Sol}_{\mathscr{K}}(\tilde{P}_{\underline{1},\underline{0}}),\ \dim_{\mathscr{C}}\mathrm{Sol}_{\mathscr{K}}(\tilde{P}_{\underline{0},\underline{1}})\leq 2^{M-1}\cdot N.$$

The next two lemmas will play an essential role in the proof of Theorem 2.14.

Lemma 2.12. *Let P be a pure operator of order N. Then for a solution $\varphi_{\underline{n}}\in\mathrm{Sol}_{\mathscr{Q}}(P)_{\underline{n}}$ $(n=0,1)$, the body part of $\varphi_{\underline{0}}$ is a solution of the equation $\tilde{P}_{\underline{N},\underline{0};\phi\phi}(f)=0$, and the body part of $\Theta(\varphi_{\underline{1}})$ is a solution of the equation $\tilde{P}_{\underline{N},\underline{1};\phi\phi}(f)=0$.*

This lemma is easily proved from (2.15). □

Lemma 2.13. (i) *Superfields* $\varphi_{2j} \in \mathcal{Q}_{\underline{0}}$ $(0 \leq j < l)$ *are* $\mathcal{A}$ *free if and only if the body parts* $\varepsilon(\varphi_{2j})$ *are* $\mathcal{C}$ *linearly independent.*

(ii) *Superfields* $\varphi_{2j+1} \in \mathcal{Q}_{\underline{1}}$ $(0 \leq j < k)$ *are* $\mathcal{A}$ *free if and only if the body parts of* $\Theta(\varphi_{2j+1})$ *are* $\mathcal{C}$ *linearly independent.*

(iii) *Both sets of superfields* $\{\varphi_{2j}\}_{0\leq j<l}$ *and* $\{\varphi_{2j+1}\}_{0\leq j<k}$ *are* $\mathcal{A}$ *free if and only if the combined superfields* $\{\varphi_{2j}\ (0 \leq j < l),\ \varphi_{2j+1}\ (0 \leq j < k)\}$ *are* $\mathcal{A}$ *free.*

(iv) *Superfields* $\varphi_j \in \mathcal{Q}_{\underline{j}}$ $(0 \leq j < N)$ *are* $\mathcal{A}$ *free if and only if*

$$\mathrm{SWR}(\varphi_0, \cdots, \varphi_{N-1}) = (\Theta^i(\varphi_j))_{0\leq i, j<N}$$

is an invertible matrix.

The matrix $\mathrm{SWR}(\varphi_0, \cdots, \varphi_{N-1})$ is referred to as a SUSY Wronski matrix of $\varphi_0, \cdots, \varphi_{N-1}$. The proofs of (i) and (ii) are so easy that we omit them.

Proof of (iii). We assume that $\{\varphi_{2j}\}_{0\leq j<l}$ and $\{\varphi_{2j+1}\}_{0\leq j<k}$ are $\mathcal{A}$ free respectively. Set $\vec{\varphi}_{\underline{0}} = (\varphi_{2j})_{0\leq j<l} \in \mathcal{Q}_{\underline{0}}^l$ and $\vec{\varphi}_{\underline{1}} = (\varphi_{2j+1})_{0\leq j<k} \in \mathcal{Q}_{\underline{1}}^k$, and suppose that, for vectors $\vec{a}_{\underline{0}} \in \mathcal{A}^l$, $\vec{a}_{\underline{1}} \in \mathcal{A}^k$,

$$\sum_{n=0,1} {}^t\vec{\varphi}_{\underline{n}} \cdot \vec{a}_{\underline{n}} = 0. \tag{2.17}$$

Expanding $\vec{a}_{\underline{n}}$ in the Grassmann basis as $\vec{a}_{\underline{n}} = \sum_{\alpha\in\Pi} \vec{a}_{\underline{n}}^{(\alpha)} e_\alpha$ $(\vec{a}_{\underline{n}}^{(\alpha)} \in \mathcal{C}^l$ or $\mathcal{C}^k)$, and multiplying the both sides of (2.17) by the element $e_0 \cdots e_{M-1} \in \mathcal{A}$, one has

$$\varepsilon({}^t\vec{\varphi}) \cdot \vec{a}_{\underline{0}}^{(\phi)} + \theta\varepsilon(\Theta({}^t\vec{\varphi}_{\underline{1}})) \cdot \vec{a}_{\underline{1}}^{(\phi)} = 0.$$

The statements (i), (ii) say that $\varepsilon(\varphi_{2j})$ $(0 \leq j < l)$ and $\theta\varepsilon(\Theta(\varphi_{2j+1}))$ $(0 \leq j < k)$ are $\mathcal{C}$ linearly independent. Hence one sees that $\vec{a}_{\underline{0}}^{(\phi)} = 0$, $\vec{a}_{\underline{1}}^{(\phi)} = 0$. In such a way one can verify $\vec{a}_{\underline{n}}^{(\alpha)} = 0$ for any $\alpha \in \Pi$ by induction. Thus the superfields $\{\varphi_{2j}\ (0 \leq j < l),\ \varphi_{2j+1}\ (0 \leq j < k)\}$ are $\mathcal{A}$ free. The converse statement is obvious.

Proof of (iv). If the superfields $\{\varphi_j\}_{0\leq j<m}$ $(\varphi_j \in \mathcal{Q}_{\underline{j}})$ are $\mathcal{A}$ free, the Wronski matrices $\mathrm{WR}_{\underline{0}} = \mathrm{WR}(\varepsilon(\varphi_0), \varepsilon(\varphi_2), \cdots)$ and $\mathrm{WR}_{\underline{1}} = \mathrm{WR}(\varepsilon(\Theta(\varphi_1)), \varepsilon(\Theta(\varphi_3)), \cdots)$ are invertible, because of the statements (i), (ii) in this lemma and Proposition 2.1. It is easy to show that $\varepsilon(\mathrm{SWR}(\varphi_0, \cdots, \varphi_{m-1}))$ is similar to $\mathrm{WR}_{\underline{0}} + \mathrm{WR}_{\underline{1}}$. Hence $\mathrm{SWR}(\varphi_0, \cdots, \varphi_{m-1})$ is invertible. □

Theorem 2.14. *Let P be a pure operator of order N, and suppose that the solution space* $\mathrm{Sol}_{\mathcal{Q}}(P)$ *is a free* $\mathcal{A}$ *module. Then one has*

(i) *There exists an* $\mathcal{A}$ *basis* $\{\varphi_j\}$ *of* $\mathrm{Sol}_{\mathcal{Q}}(P)$ *such that*

$$\varphi_{2j} \in \mathscr{Q}_{\underline{0}} \ (0 \leq j < l), \qquad \varphi_{2j+1} \in \mathscr{Q}_{\underline{1}} \ (0 \leq j < k),$$

where $l+k = \mathrm{rank}_{\mathscr{A}} \mathrm{Sol}_{\mathscr{Q}}(P)$.

(ii) *In* (i), $l, k \leq N/2$ *for even* N *or*, $l \leq (N+1)/2$, $k \leq (N-1)/2$ *for odd* N.

Proof. (i) Let $u_j = u_{j,\underline{0}} + u_{j,\underline{1}}$ $(0 \leq j < l+k$, $u_{j,\underline{n}} \in \mathrm{Sol}_{\mathscr{Q}}(P)_{\underline{n}})$ be an $\mathscr{A}$ basis of $\mathrm{Sol}_{\mathscr{Q}}(P)$. One can assume without loss of generality that $\{u_{j,\underline{0}}\}_{0 \leq j < l}$ is a maximal subset of $\mathscr{A}$ free fields in the set of even fields $\{u_{j,\underline{0}}\}_{0 \leq j < l+k}$. From Lemma 2.13 (i) it follows that $\{\varepsilon(u_{j,\underline{0}})\}$ are $\mathscr{C}$ linearly independent, but that $\{\varepsilon(u_{j,\underline{0}})\}_{0 \leq j < i}$ and $\varepsilon(u_{i,\underline{0}})$ $(i \geq l)$ are not $\mathscr{C}$ linearly independent. Hence one has

$$\varepsilon(u_{i,\underline{0}}) = \sum_{0 \leq j < l} \varepsilon(u_{j,\underline{0}}) c_{ij} \quad (l \leq i,\ c_{ij} \in \mathscr{C}).$$

Set $v_i \in \mathrm{Sol}_{\mathscr{Q}}(P)$ $(l \leq i < l+k)$ by

$$v_i = u_i - \sum_{0 \leq j < l} u_j c_{ij}.$$

Then $\{v_i\}_{l \leq i < l+k}$ are $\mathscr{A}$ free and $\varepsilon(v_i) = 0$. Writing $v_i = v_{i,\underline{0}} + v_{i,\underline{1}}$ $(v_{i,\underline{n}} \in \mathrm{Sol}_{\mathscr{Q}}(P)_{\underline{n}}$, one sees that $\{v_{i,\underline{1}}\}_{l \leq i < l+k}$ are $\mathscr{A}$ free. In fact, if they are not $\mathscr{A}$ free, it follows from Lemma 2.13 (ii) that $\{\varepsilon(\Theta(v_{i,\underline{1}}))\}_{l \leq i < l+k}$ are not $\mathscr{C}$ linearly independent. Hence one finds a non-zero vector $\vec{c} = (c_i) \in \mathscr{C}^k$ satisfying

$$\sum_{l \leq i < l+k} \varepsilon(\Theta(v_{i,\underline{1}})) c_i = 0.$$

Noting that $\varepsilon(v_i) = 0$, one has

$$\sum v_i (c_i e_0 \cdots e_{M-1}) = \theta \{\sum \varepsilon(\Theta(v_{i,\underline{1}})) c_i\} e_0 \cdots e_{M-1} = 0,$$

which contradicts the fact that $\{v_i\}_{l \leq i < l+k}$ are $\mathscr{A}$ free. Thus one obtains $\mathscr{A}$ free systems $\{u_{j,\underline{0}}\}_{0 \leq j < l} \in \mathrm{Sol}_{\mathscr{Q}}(P)_{\underline{0}}$, $\{v_{i,\underline{1}}\}_{l < i < l+k} \in \mathrm{Sol}_{\mathscr{Q}}(P)_{\underline{1}}$. Lemma 2.13 (iii) ensures that the mixed fields $\{u_{j,\underline{0}} (0 \leq j < l),\ v_{i,\underline{1}}\ (l \leq i < l+k)$ are $\mathscr{A}$ free.

(ii) The $\mathscr{A}$ free generators $\{\varphi_{2j}\}_{0 \leq j < l}$ (resp. $\{\varphi_{2j+1}\}_{0 \leq j < k}$) of $\mathrm{Sol}_{\mathscr{Q}}(P)$ give $\mathscr{C}$ linearly independent elements $\{\varepsilon(\varphi_{2j})\}_{0 \leq j < l}$ (resp. $\{\varepsilon(\Theta(\varphi_{2j+1}))\}_{0 \leq j < k}$) of $\mathrm{Sol}_{\mathscr{K}}(\tilde{P}_{\underline{N},\underline{0};\phi\phi})$ (resp. $\mathrm{Sol}_{\mathscr{K}}(\tilde{P}_{\underline{N}+\underline{1},\underline{1};\phi\phi})$ because of Lemma 2.12. Hence the statement follows from the condition (2.16) and Proposition 2.2. □

When the operator P in Theorem 2.14 is $\mathscr{Q}$-solvable, l and k should be $l = k = N/2$ for even N, or $l = (N+1)/2$ and $k = (N-1)/2$ for odd N. Thus one gets

Corollary 2.15. *Let P be a pure, $\mathscr{Q}$-solvable operator of order N. Then the solution space* $\mathrm{Sol}_{\mathscr{Q}}(P)$ *is a pure* $\mathscr{A}$ *free module of rank N. Namely there exists an* $\mathscr{A}$ *basis* $\{\varphi_j\}_{0\leq j<N}$ *of* $\mathrm{Sol}_{\mathscr{Q}}(P)$ *such that* $\varphi_j\in \mathrm{Sol}_{\mathscr{Q}}(P)_{\underline{j}}$. *Furthermore the* SUSY *Wronski matrix* $\mathrm{SWR}(\varphi_0, \cdots, \varphi_{N-1})$ *is invertible.*

§ 2-3. Supersymmetric Grassmann hierarchy

In this section we will discuss a supersymmetric extension of the Grassmann hierarchy based upon the argument in the previous section. As is the case of the Grassmann hierarchy, our framework in the limiting case naturally leads to a supersymmetric extension of the KP hierarchy.

For a given pure operator P of order N ($P\in(\mathscr{D}_{\mathscr{Q}}^{1|1}(N))_{\underline{N}}^{monic}$), we consider a right factor W of P. Namely, W satisfies

$$P=Z\cdot W \tag{2.18}$$

for some operator Z. We restrict ourselves to the case that W is a pure operator of order m

$$W=\sum_{j=0}^{m} w_j\Theta^{m-j} \quad (w_0=1,\ w_j\in\mathscr{Q}_{\underline{j}}). \tag{2.19}$$

Then the left factor Z is automatically a pure operator of order $N-m$.

Proposition 2.16. *If P is $\mathscr{Q}$-solvable, then Z and W in* (2.18) *are also $\mathscr{Q}$-solvable.*

Proof. For simplicity of the argument, we assume that N is even. From (2.18) it follows that the matrix differential operators $\tilde{P}$, $\tilde{Z}$, $\tilde{W}$, corresponding to P, Z, W, respectively, satisfy $\tilde{P}=\tilde{Z}\cdot\tilde{W}$, namely,

$$\tilde{P}_{\underline{\nu},\underline{\nu}}=\tilde{Z}_{\underline{\nu},\underline{m}+\underline{\nu}}\cdot\tilde{W}_{m+\underline{\nu},\underline{\nu}} \quad (\nu=0,\ 1). \tag{2.20}$$

When P is $\mathscr{Q}$-solvable, (2.16) in Section 2.2 shows that

$$\dim_{\mathscr{C}}\mathrm{Sol}_{\mathscr{X}}(\tilde{P}_{\underline{\nu},\underline{\nu}})=N\cdot 2^{M-1} \quad (\nu=0,\ 1). \tag{2.21}$$

On the other hand, an inequality

$$\dim_{\mathscr{C}}\mathrm{Sol}_{\mathscr{X}}(\tilde{P}_{\underline{\nu},\underline{\nu}})\leq\dim_{\mathscr{C}}\mathrm{Sol}_{\mathscr{X}}(\tilde{Z}_{\underline{\nu},\underline{m}+\underline{\nu}})+\dim_{\mathscr{C}}\mathrm{Sol}_{\mathscr{X}}(\tilde{W}_{\underline{m}+\underline{\nu},\underline{\nu}}) \tag{2.22}$$

holds because of (2.20). Hence, taking into account (2.16), one deduces that the equality in (2.22) must hold and $\dim_{\mathscr{C}}\mathrm{Sol}_{\mathscr{X}}(\tilde{W}_{\underline{m}+\underline{\nu},\underline{\nu}})=m\cdot 2^{M-1}$ ($\nu=0$, 1). Futhermore, since the matrix $\tilde{W}_{\underline{m}+\underline{\nu},\underline{\nu}}$ is lower triangular, one can find $\mathscr{C}$ linearly independent solutions with the next properties;

$$\vec{\psi}_{2j}=(\psi_{2j}^{(\alpha)})_{\alpha\in\Pi}\in \mathrm{Sol}_{\mathscr{X}}(\tilde{W}_{\underline{m},\underline{0}})$$

($0\leq j<m/2$ for even m, or $0\leq j<(m+1)/2$ for odd m) and $\psi_{2j}^{(\phi)}$ are $\mathscr{C}$ linearly independent,

$$\vec{\psi}_{2j+1}=(\psi_{2j+1}^{(\alpha)})_{\alpha\in\Pi}\in \mathrm{Sol}_{\mathscr{X}}(\tilde{W}_{\underline{m+1},\underline{1}})$$

($0\leq j<m/2$ for even m, or $0\leq j<(m-1)/2$ for odd m) and $\psi_{2j+1}^{(\phi)}$ are $\mathscr{C}$ linearly independent. Let $\{\psi_j\}_{0\leq j<m}$ ($\psi_j\in \mathrm{Sol}_{\mathscr{Q}}(W)_{\underline{j}}$) be corresponding to these solution vectors. Then they are $\mathscr{A}$ free, because

$$\varepsilon(\psi_{2j})=\psi_{2j}^{(\phi)},\ \varepsilon(\Theta(\psi_{2j+1}))=\psi_{2j+1}^{(\phi)},$$

are $\mathscr{C}$ linearly independent, respectively. Thus the module $\mathrm{Sol}_{\mathscr{Q}}(W)$ has m $\mathscr{A}$-free elements and $\dim_{\mathscr{C}}\mathrm{Sol}_{\mathscr{Q}}(W)=m\cdot 2^M$. Therefore it is a free $\mathscr{A}$ module of rank m. A similar discussion is applicable to the $\mathscr{Q}$-solvability of the operator Z. □

The condition (2.18) for a pure operator W to be a right factor of P provides a system of SUSY nonlinear differential equations for the coefficients of W, which is denoted by $\mathrm{SE}(m;P)$.

Example. In (2.18), set $P=\Theta^3$, $Z=\Theta^2+z_1\Theta+z_2$ ($z_j\in\mathscr{Q}_{\underline{j}}$) and $W=\Theta+w$ ($w\in\mathscr{Q}_{\underline{1}}$). The equation $\mathrm{SE}(1;\Theta^3)$ reads

$$w_x-w\cdot\dot{w}=0$$

($\dot{w}=\Theta(w)$). Putting $w=f(x)+\theta g(x)$ ($f(x)\in\mathscr{Q}_{\underline{1}}$, $g(x)\in\mathscr{Q}_{\underline{0}}$), one has

$$g_x=g^2,\quad f_x-f\cdot g=0.$$

Integrating these equations, one obtains a general solution

$$w(x,\theta)=\frac{\xi_1-\theta\xi_0}{\xi_0 x+\xi_2},$$

where $\xi_j\in\mathscr{A}_{\underline{j}}$ and $(\varepsilon(\xi_0),\varepsilon(\xi_2))\neq(0,0)$. Thus solutions of $\mathrm{SE}(1;\Theta^3)$ are parametrized by the set

$$\{(\xi_0,\xi_1,\xi_2)\,|\,\xi_j\in\mathscr{A}_{\underline{j}},\ (\varepsilon(\xi_0),\varepsilon(\xi_2))\neq(0,0)\}/\mathrm{GL}(1;\mathscr{A}_{\underline{0}}),$$

which is just the $(1\,|\,1)$-dimensional super projective space $\boldsymbol{P}^{1|1}(\mathscr{A})$.

We can show that the solution space of the system $\mathrm{SE}(m;P)$ is identical with a super Grassmann manifold in the following manner.

Theorem 2.17. *Let P be a $\mathscr{Q}$-solvable, pure operator of order N, and*

$V=\mathrm{Sol}_{\mathcal{Q}}(P)$. *Then the map*

$$\mathrm{Sol}_{\mathcal{Q}}(\cdot): \{W \in (\mathcal{D}_{\mathcal{Q}}^{1|1}(m))_{\underline{m}}^{monic} \mid W \text{ is a right factor of } P\} \xrightarrow{\sim} \mathrm{SGM}(m; V)$$

is a bijection.

Proof. Let $W \in (\mathcal{D}_{\mathcal{Q}}^{1|1}(m))_{\underline{m}}^{monic}$ be a right factor of P. Then Proposition 2.16 and Corollary 2.15 say that $\mathrm{Sol}_{\mathcal{Q}}(W) \in \mathrm{SGM}(m; V)$. Now let $U \in \mathrm{SGM}(m; V)$ and $\{\psi_j\}_{0\le j<m}$ a pure basis of U. We will show that there exists a unique, pure, right factor W of P with $\mathrm{Sol}_{\mathcal{Q}}(W)=U$. Consider a system of linear equations

$$(w_m, \cdots, w_1, 1)(\Theta^i(\psi_j))_{\substack{0\le i\le m \\ 0\le j<m}}=0. \tag{2.23}$$

Since the SUSY Wronski matrix $\mathrm{SWR}(\psi_0, \cdots, \psi_{m-1})$ is invertible (Lemma 2.13 (iv)), this system has a unique solution $(w_m, \cdots, w_1)$ with $w_j \in \mathcal{Q}_{\underline{j}}$. Defining a pure operator by (2.19), one gets

$$W\psi_j=0 \quad (0\le j<m). \tag{2.24}$$

Here we remark that the operator W is determined only by the pure submodule U, and is not dependent on the choice of a pure basis. Let us show that W is a right factor of P. By Proposition 2.7, one sees that the operator is uniquely represented as

$$P=Z\cdot W+R,$$

where Z, $R \in \mathcal{D}_{\mathcal{Q}}^{1|1}$ and $\mathrm{ord}_{\Theta}(R)<m$. Since $P\psi_j=W\psi_j=0$, one gets $R\psi_j=0$. Writing $R=\sum_{j=1}^{m} r_j\Theta^{m-j}$, the coefficients must satisfy

$$(r_m, \cdots, r_1)\cdot \mathrm{SWR}(\psi_0, \cdots, \psi_{m-1})=0.$$

The invertibility of $\mathrm{SWR}(\psi_0, \cdots, \psi_{m-1})$ shows that $R=0$. Consequently the map $\mathrm{Sol}_{\mathcal{Q}}(\cdot)$ is surjective. From the unique solvability of (2.23), it is easy to prove that the map $\mathrm{Sol}_{\mathcal{Q}}(\cdot)$ is injective. □

Super time evolution is defined on the solutions of $\mathrm{SE}(m; P)$ for an operator P with constant coefficients. We call the resulting system the SUSY Grassmann hierarchy of the type $\mathrm{SE}(m; P)$ [29]. Let $(t_j)_{1\le j<N}$ be SUSY time variables, with t_{2l} even, t_{2l+1} odd. Let $C[t_1, t_3 \cdots]$ be a Grassmann algebra generated by the odd time variables, and $\mathcal{K}$ be the quotient field of $C[[x, t_2, t_4, \cdots]]$. The supercommutative algebra $\mathcal{Q}$ is now introduced as

$$\mathcal{Q}=(C[\theta]\otimes\mathcal{K})\otimes(C[t_1, t_3, \cdots]\otimes\mathcal{A}),$$

where all the tensor products are taken to be Z_2-graded. Note that, replacing $\mathscr{A}$ by $C[t_1, t_3, \cdots]\otimes\mathscr{A}$, our arguments which have been exploited so far remain valid.

Remark. Let $\mathscr{S}=(C[\theta]\otimes C[[x, t_2, t_4, \cdots]])\otimes(C[t_1, t_3, \cdots]\otimes\mathscr{A})$ and define Z_2-gradation in a natural way. Then we have

$$\mathscr{Q}=\{f\cdot g^{-1}\,|\,f\in\mathscr{S},\ g\in\mathscr{S}_{\underline{0}},\ \varepsilon(g)\neq 0\}.$$

Introduce SUSY vector fields Θ_l $(1\leq l<N)$ [20] by

$$\Theta_{2l}=\frac{\partial}{\partial t_{2l}},$$

$$\Theta_{2l-1}=\frac{\partial}{\partial t_{2l-1}}+\sum_{1\leq 2k-1<N} t_{2k-1}\frac{\partial}{\partial t_{2l+2k-2}}.$$

It is easily checked that they satisfy the following commutation/anti-commutation relations:

$$\begin{aligned}&[\Theta_{2l}, \Theta_{2k}]=[\Theta_{2l}, \Theta_{2k-1}]=0,\\ &[\Theta_{2l-1}, \Theta_{2k-1}]_+(=\Theta_{2l-1}\Theta_{2k-1}+\Theta_{2k-1}\Theta_{2l-1})=2\Theta_{2l+2k-2},\\ &[\Theta_{2l}, \Theta]=[\Theta_{2l-1}, \Theta]_+=0.\end{aligned}\tag{2.25}$$

Let $P=\sum_{j=0}^{N} p_j\Theta^{N-j}$ $(p_0=1,\ p_j\in\mathscr{A}_{\underline{j}})$ and consider an enlarged system

$$\begin{aligned}&P\varphi=0,\quad \Theta_{2l}(\varphi)=(-)^l\Theta^{2l}(\varphi)\quad (2\leq 2l<N),\\ &\Theta_{2l-1}(\varphi)=(-)^l\Theta^{2l-1}(\varphi)\quad (1\leq 2l-1<N).\end{aligned}\tag{2.26}$$

The bracket relations (2.25) ensure this system to be consistent, and the solution space V to be a pure $\mathscr{A}$ module of rank N. Let $U\in \mathrm{SGM}(m;\, V)$, and $W\in(\mathscr{D}_{\mathscr{Q}}^{1|1}(m))_{\underline{m}}^{monic}$ correspond to U via the inverse map of $\mathrm{Sol}_{\mathscr{Q}}(\cdot)$. Applying Θ_{2l-1} to the both sides of (2.24) and taking the parity of W into account, one gets

$$\{\Theta_{2l-1}(W)+(-)^{m+l}W\Theta^{2l-1}\}\psi_j=0.$$

The division theorem, as is used in the preceding argument, implies that there exists an operator $B_{2l-1}\in\mathscr{D}_{\mathscr{Q}}^{1|1}$ satisfying

$$\Theta_{2l-1}(W)=(-)^{m+l}\{B_{2l-1}W-W\Theta^{2l-1}\}.\tag{2.27}$$

Similarly one gets

$$\Theta_{2l}(W)=(-)^l\{B_{2l}W-W\Theta^{2l}\}.\tag{2.28}$$

Here the operators B_l's are given by

$$B_l=(W\Theta^l W^{-1})_+.$$

($W^{-1} \in \mathscr{E}_{\mathscr{Q}}^{1|1}(-m)^{monic}$. $\mathscr{E}_{\mathscr{Q}}^{1|1}$ is the ring of SUSY microdifferential operators. For details, see the next chapter.) A set of the equations (2.27), (2.28) is a SUSY extension of the Sato equations (1.3).

Finally we discuss a SUSY analogue of the Grassmann equation (1.12). In the system (2.26), we take $P=\Theta^N$. As a fundamental solution matrix Φ we can choose

$$\Phi=\mathrm{SWR}(\varphi_0, \cdots, \varphi_{N-1})=\exp(\theta\Lambda_N+x\Lambda_N{}^2+\sum_{1\leq n<N} t_n\Gamma_N{}^n),$$

where $\Lambda_N=(\delta_{\mu+1,\nu})_{0\leq\mu,\nu<N}$, $\Gamma_N=((-)^\nu\delta_{\mu+1,\nu})_{0\leq\mu,\nu<N}$. Write the basis ψ_j as $\psi_j=\sum\varphi_i\xi_{ij}$ and set $\Xi=(\xi_{ij}) \in \mathrm{SFR}(N, m; \mathscr{A})$. Then the equation (2.23) reads

$$ {}^t\vec{w}\Phi\Xi=0, \tag{2.29}$$

where ${}^t\vec{w}=(w_m, \cdots, w_1, 1, 0, \cdots, 0) \in \mathscr{Q}^N$.

Fixing the parity of m, take the limit of $N\to\infty$, $m\to\infty$ in (2.27), (2.28) or (2.29). Then we reach a picture of SUSY extensions of the KP hierarchy of two different types according to the parity.

Chapter 3.

§ 3.1. The supersymmetric KP hierarchy

Let $\mathscr{A}$ be an infinite dimensional Grassmann algebra, $\varinjlim_N \Lambda(C^N)$. The space $B_{\mathscr{A}}^{N|N}$ is a superaffine space of dimension $(N|N)$ over $\mathscr{A}$ whose coordinate system is denoted by (x, θ, t), where $t=(t_1, t_2, t_3, \cdots)$. Here x, t_{2l} are even variables and θ, t_{2l-1} are odd ones. This superaffine space plays the role of the affine space C^N in the theory of the KP hierarchy. In fact, (x, θ) is regarded as the coordinate system on the superaffine space $B_{\mathscr{A}}^{1|1}$ of dimension $(1|1)$ over $\mathscr{A}$, and t as supertime variables.

Let $\mathscr{S}_C$ be a supercommutative algebra of formal series generated by (x, θ, t). Namely,

$$\mathscr{S}_C=\{\sum_{0\leq l+|\mu|+|\alpha|<\infty} c_{l\mu\alpha}x^l t^\mu t_\alpha+\theta\sum_{0\leq l'+|\mu'|+|\alpha'|<\infty} c_{l'\mu'\alpha'}x^{l'}t^{\mu'}t_{\alpha'} \mid \text{a formal series with } c_{l\mu\alpha}, c_{l'\mu'\alpha'} \in C\},$$

where l, $l' \in N$, and the indices $\mu=(\mu_2, \cdots, \mu_{2N})$, $\mu'=(\mu'_2, \cdots, \mu'_{2N}) \in$

$\bigcup_{N=0}^{\infty} \boldsymbol{N}^N$, and the indices $\alpha=(\alpha_1, \cdots, \alpha_{2M-1})$, $\alpha'=(\alpha'_1, \cdots, \alpha'_{2M-1})$ run over the set of increasing sequences of finite length with $\alpha_{2j-1} \in 2\boldsymbol{N}+1$. For an index $\mu=(\mu_2, \cdots, \mu_{2N})$, set $t^\mu=t_2^{\mu_2}\cdots t_2^{\mu_{2M}}$, $|\mu|=\sum_{l=1}^{N}\mu_{2l}$, and for an index $\alpha=(\alpha_1, \cdots, \alpha_{2M-1})$, set $t_\alpha=t_{\alpha_1}\cdots t_{\alpha_{2M-1}}$, $|\alpha|=2M-1$ (for the void sequence, ϕ, $t_\phi=1$, $|\phi|=0$). The $\boldsymbol{Z}_2$-gradation of $\mathscr{S}_C$ is defined in an obvious way. The supercommutative algebra $\mathscr{S}$ of (formally regular) superfields is, by definition,

$$\mathscr{S}=\mathscr{S}_C\otimes\mathscr{A}$$

where the tensor product is taken to be $\boldsymbol{Z}_2$-graded. This algebra may be regarded as a formal completion of the function algebra on $B_{\mathscr{A}}^{N|N}$. The quotient algebra $\mathscr{Q}$ of $\mathscr{S}$ is defined by

$$\mathscr{Q}=\{f\cdot g^{-1}\,|\,f\in\mathscr{S},\ g\in\mathscr{S}_{\bar{0}},\ \varepsilon(g)\neq 0\},$$

where ε is the canonical projection; $\varepsilon\colon \mathscr{S}\to\boldsymbol{C}[[x, t_2, t_4, \cdots]]$ (namely, the body map). This is isomorphic to $\mathscr{Q}_C\otimes\mathscr{A}$, where $\mathscr{Q}_C$ is the quotient algebra of $\mathscr{S}_C$;

$$\mathscr{Q}_C=\{f\cdot g^{-1}\,|\,f\in\mathscr{S}_C,\ g\in(\mathscr{S}_C)_{\bar{0}},\ \varepsilon(g)\neq 0\}.$$

The SUSY differential operator Θ (see Section 2.2 for the definition) acts on $\mathscr{Q}$. The formal inverse operator of Θ is introduced by

$$\Theta^{-1}=\theta+\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial x}\right)^{-1}.$$

The set $\mathscr{E}_{\mathscr{Q}}^{1|1}$ (resp. $\mathscr{E}_{\mathscr{S}}^{1|1}$) of SUSY microdifferential operators with coefficients in $\mathscr{Q}$ (resp. in $\mathscr{S}$) is defined by

$$\mathscr{E}_{\mathscr{Q}}^{1|1}=\mathscr{Q}_C((\Theta^{-1}))\otimes\mathscr{A} \qquad (\text{resp. } \mathscr{E}_{\mathscr{S}}^{1|1}=\mathscr{S}_C((\Theta^{-1}))\otimes\mathscr{A}),$$

which is endowed with a structure of a non-supercommutative algebra via the generalized SUSY Leibniz rule

$$\Theta^{2k}\cdot f=\sum_{j=0}^{\infty}\binom{k}{j}\partial_x{}^j(f)\cdot\Theta^{2k-2j},$$

$$\Theta^{2k+1}\cdot f=\sum_{j=0}^{\infty}\binom{k}{j}\partial_x{}^j(\Theta(f))\cdot\Theta^{2k-2j}+(-)^\nu\sum_{j=0}^{\infty}\binom{k}{j}\partial_x{}^j(f)\cdot\Theta^{2k-2j+1},$$

where $f\in\mathscr{Q}_{\bar{\nu}}$ and k is an arbitrary integer. The $\boldsymbol{Z}_2$-gradation of $\mathscr{E}_{\mathscr{Q}}^{1|1}$; $\mathscr{E}_{\mathscr{Q}}^{1|1}=(\mathscr{E}_{\mathscr{Q}}^{1|1})_{\bar{0}}\oplus(\mathscr{E}_{\mathscr{Q}}^{1|1})_{\bar{1}}$, is given by

$$(\mathscr{E}_{\mathscr{Q}}^{1|1})_{\bar{\nu}}=\{\sum_{-\infty<j\ll\infty} p_j(x,\theta,t)\Theta^j\in\mathscr{E}_{\mathscr{Q}}^{1|1}\,|\,p_j(x,\theta,t)\in\mathscr{Q}_{\overline{\nu+j}} \text{ for any } j\}.$$

Furthermore a filtration compatible with the Z_2-gradation is introduced through

$$\mathscr{E}_{\mathscr{Q}}^{1|1}=\bigcup_{m\in Z}\mathscr{E}_{\mathscr{Q}}^{1|1}(m),$$

where $\mathscr{E}_{\mathscr{Q}}^{1|1}(m)=\mathscr{Q}_C[[\Theta^{-1}]]\Theta^m\otimes\mathscr{A}$. Through a direct sum decomposition

$$\mathscr{E}_{\mathscr{Q}}^{1|1}=\mathscr{D}_{\mathscr{Q}}^{1|1}\oplus\mathscr{E}_{\mathscr{Q}}^{1|1}(-1),$$

where $\mathscr{D}_{\mathscr{Q}}^{1|1}$ is the algebra of SUSY differential operators with coefficients in $\mathscr{Q}$, any element $P\in\mathscr{E}_{\mathscr{Q}}^{1|1}$ is uniquely represented as

$$P=P_++P_-$$

with $P_+\in\mathscr{D}_{\mathscr{Q}}^{1|1}$, $P_-\in\mathscr{E}_{\mathscr{Q}}^{1|1}(-1)$. An operator $P=\sum_{-\infty<j\le m}p_j(x,\theta,t)\Theta^j$ $\in\mathscr{E}_{\mathscr{Q}}^{1|1}$ $(p_m\neq 0)$ is invertible if and only if the body part $\varepsilon(p_m)$ does not vanish in $\mathscr{D}$, and then $P^{-1}\in\mathscr{E}_{\mathscr{Q}}^{1|1}(-m)$.

Now introduce the following even and odd vector fields:

$$\Theta_{2l}=\frac{\partial}{\partial t_{2l}},$$
$$\Theta_{2l-1}=\frac{\partial}{\partial t_{2l-1}}+\sum_{k=1}^{\infty}t_{2k-1}\frac{\partial}{\partial t_{2l+2k-2}}.$$

Though Θ_{2l-1} is an infinite series of odd derivations, its action on $\mathscr{Q}$ is well-defined. They satisfy the following commutation/anti-commutation relations:

$$[\Theta,\Theta_{2l}]=[\Theta,\Theta_{2l-1}]_+=0,\qquad [\Theta_{2l},\Theta_{2k}]=[\Theta_{2l},\Theta_{2k-1}]=0.$$
$$[\Theta_{2l-1},\Theta_{2k-1}]_+=2\Theta_{2l+2k-2}.$$

Taking into account the consideration for the SUSY Grassmann hierarchy, a supersymmetric extension of the KP hierarchy (the *SKP hierarchy*) should be formulated as follows: Let L be a SUSY micro-differential operator

(3.1) $$L=\sum_{i=0}^{\infty}u_i\Theta^{1-i}\in(\mathscr{E}_{\mathscr{Q}}^{1|1}(1))_{\bar{1}}\quad\text{with}\quad u_0=1,\ \Theta(u_1)+2u_2=0.$$

The SKP hierarchy of type m $(m=0, 1)$ is, by definition, a system of the Lax equations:

(3.2) $$\Theta_{2l}(L)=(-)^l[B_{2l},L],$$
$$\Theta_{2l-1}(L)=(-)^{l+m}\{[B_{2l-1},L]_+-2L^{2l}\}\qquad(l=1,2,\cdots),$$

where $B_l=(L^l)\in(\mathscr{D}_{\mathscr{Z}}^{1|1}(l))_l^{monic}$ and $\Theta_l(L)=\sum_{i=0}^{\infty}\Theta_l(u_i)\Theta^{1-i}$. Note that, from the condition in (3.1), $B_2=\Theta^2=\partial_x$. Hence the (x, t)-dependence of a solution L emerges as $(t_1, -x+t_2, t_3, t_4, \cdots)$. The system (3.2) turns out to be equivalent to a system of the Zakharov-Shabat equations:

$$
\begin{aligned}
&(-)^k\Theta_{2k}(B_{2l})-(-)^l\Theta_{2l}(B_{2k})+[B_{2l}, B_{2k}]=0,\\
(3.3)\quad &(-)^k\Theta_{2k}(B_{2l-1})-(-)^{l+m}\Theta_{2l-1}(B_{2k})+[B_{2l-1}, B_{2k}]=0,\\
&(-)^{k+m}\Theta_{2k-1}(B_{2l-1})+(-)^{l+m}\Theta_{2l-1}(B_{2k-1})\\
&\qquad -[B_{2l-1}, B_{2k-1}]_+ +2B_{2l+2k-2}=0 \qquad (l, k=1, 2, \cdots).
\end{aligned}
$$

The type 1 hierarchy is obtained by reversing the sign of all odd time variables $(t_{2k-1}\to -t_{2k-1})$ in the type 0 hierarchy. Henceforth we will deal only with the type 0 hierarchy.

The first equation in (3.3) with $k=2$, $l=3$ gives rise to the SKP equation, which is recognized as a supersymmetric extension of the single KP equation in Chapter 1: Set

$$
\begin{aligned}
B_4&=\Theta^4+2u_3\Theta+2u_4,\\
B_6&=\Theta^6+3u_3\Theta^3+3u_4\Theta^2+v_5\Theta+v_6.
\end{aligned}
$$

Then the SKP equation reads

$$
\begin{aligned}
&2u_{3,t_4}=-3u_{3,xx}+2v_{5,x},\\
(3.4)\quad &3u_{4,t_4}=-3u_{4,xx}+6u_3u_{3,x}-4u_3v_5+2v_{6,x},\\
&v_{5,t_4}+2u_{3,t_6}=v_{5,xx}-2u_{3,xxx}-6u_3\dot{u}_{3,x}-6(u_3u_4)_x-2(u_3v_5)^{\cdot},\\
&v_{6,t_4}+2u_{4,t_6}=v_{6,xx}+2u_3\dot{v}_6-2u_{4,xxx}-6u_3\dot{u}_{4,x}-6u_4u_{4,x}+2\dot{u}_4v_5,
\end{aligned}
$$

where the dot stands for the Θ derivative. The body parts of the second and the fourth equations yield the KP equation.

§ 3.2. The universal super Grassmann manifold

We will discuss about the procedure of integrating a solution $L=\sum_{i=0}^{\infty}u_i(x, \theta, t)\Theta^{1-i}$ $(u_0=1,\ \Theta(u_1)+2u_2=0)$ to the SKP hierarchy in the case that they belong to $\mathscr{E}_{\mathscr{S}}^{1|1}$, namely, $u_i\in\mathscr{S}_{\underline{i}}$.

As is the case of the KP hierarchy, one first finds a SUSY micro-differential operator of order 0,

$$
W=\sum_{-\infty<j\le 0}w_j(x, \theta, t)\Theta^j \quad \text{with} \quad w_0=1,\ w_j\in\mathscr{S}_{\underline{j}}
$$

satisfying

(3.5) $$L=W\Theta W^{-1},$$

(3.6) $$\Theta_{2l}(W)=(-)^l\{B_{2l}W-W\Theta^{2l}\},$$
$$\Theta_{2l-1}(W)=(-)^l\{B_{2l-1}W-W\Theta^{2l-1}\}.$$

Such an operator is referred to as a *SUSY wave operator*, and the equations (3.6) as the SUSY Sato equations. Introduce a SUSY differential operator Ψ of the infinite order by

$$\Theta_{2l}(\Psi)=(-)^l\Theta^{2l}\cdot\Psi,\qquad \Theta_{2l-1}(\Psi)=(-)^l\Theta^{2l-1}\cdot\Psi$$

with an initial condition $\Psi|_{t=0}=1$ (the identity operator). It is explicitly written down as

$$\Psi=\exp\left\{\sum_{l=1}^{\infty}(-)^l t_{2l}\Theta^{2l}+\sum_{l=1}^{\infty}(-)^l t_{2l-1}\Theta^{2l-1}\right\}.$$

One readily sees that the operator $\tilde{W}=W\cdot\Psi$ solves

(3.7) $$\Theta_{2l}(\tilde{W})=(-)^l B_{2l}\tilde{W},\quad \Theta_{2l-1}(\tilde{W})=(-)^l B_{2l-1}\tilde{W},\quad l=1,2,\cdots.$$

Apart from this, consider the following equations.

(3.8) $$\Theta_{2l}(Y)=(-)^l B_{2l}Y,\quad \Theta_{2l-1}(Y)=(-)^l B_{2l-1}Y,\quad l=1,2,\cdots,$$

where Y is a SUSY differential operator of the infinite order

$$Y=\sum_{m=0}^{\infty}y_m(x,\theta,t)\Theta^m \qquad (y_m\in\mathcal{S}_{\underline{m}}),$$

with an initial condition $Y|_{t=0}=1$. Setting a formal power series $v(x,\theta,t;\lambda)$ in λ by

$$v(x,\theta,t;\lambda)=Y(e^{\theta\lambda+x\lambda^2})$$
$$=\sum_{m=0}^{\infty}v_m(x,\theta,t)\lambda^m,$$

one can easily verify that the Cauchy problem (3.8) is equivalent to

$$\Theta_{2l}(v_m)=(-)^l B_{2l}(v_m),\qquad \Theta_{2l-1}(v_m)=(-)^l B_{2l-1}(v_m),$$
$$\Theta^n(v_m)|_{x=\theta=t=0}=\delta_{n,m},$$

which has a unique solution because of the Zakharov-Shabat equations (3.3). Since both $\tilde{W}$ nad Y satisfy the same equations (3.7), (3.8), introducing an operator U through

(3.9) $$U=\tilde{W}^{-1}\cdot Y,$$

one finds that $\Theta_n(U)=0$ for any n. Consequently the coefficients of U does not depend on t, that is,

$$\begin{aligned} U=U(x,\theta;\Theta)&=\tilde{W}^{-1}\cdot Y|_{t=0} \\ &=W^{-1}|_{t=0}, \end{aligned}$$

so that it is expanded to

(3.10) $$U=\sum_{-\infty<j\le 0} u_j(x,\theta)\Theta^j \qquad (u_0=1).$$

The relation (3.9) is rewritten into

(3.11) $$Y=W\cdot Z,$$

where the operator Z is, by definition,

$$Z=\Psi\cdot U=\sum_{j\in Z} z_j(x,\theta,t)\Theta^j,$$

a SUSY microdifferential operator of the infinite order. Taking the $(-)$ parts of the both sides of (3.11), one gets

(3.12) $$(W\cdot Z)_-=0.$$

Introduce the fields $z_j^{(\nu)}\in\mathcal{S}_{j+\nu}$ through

$$W\cdot Z=\sum_{\nu\in Z}\Big(\sum_{j\in Z} w_j z_j^{(\nu)}\Big)\Theta^\nu,$$

and the rectangular matrix $\mathscr{Z}$ through

$$\mathscr{Z}=(z_j^{(\nu)})_{j\in Z,\nu\in N^c}\in \mathrm{Mat}\,(Z\times N^c;\mathcal{S}).$$

The equation (3.12) reads

(3.13) $${}^t\vec{w}\cdot\mathscr{Z}=0,$$

where ${}^t\vec{w}=(w_j)_{j\in Z}$, ($w_j=0$ for $j>0$).

Proposition 3.1. *The rectangular matrix* $\mathscr{Z}$ *solves*

(3.14) $$\Theta_n(\mathscr{Z})=\Gamma^n\mathscr{Z} \qquad n=1,2,\cdots,$$

(3.15) $$\Theta(\mathscr{Z})=\Lambda\mathscr{Z}-\mathscr{Z}^*\Lambda_{N^c},$$

where $\Lambda=(\delta_{\mu+1,\nu})_{\mu,\nu\in Z}$, $\Gamma=((-)^\nu\delta_{\mu+1,\nu})_{\mu,\nu\in Z}$, *and* $\Lambda_{N^c}=(\delta_{\mu+1,\nu})_{\mu,\nu\in N^c}$, $\mathscr{Z}^*=(z_j^{(\nu)*})$. *(For the* $*$ *operator, see Section 2.2.)*

Proof. One obtains, by virtue of the super Leibniz rule,

$$z_{2j}^{(\nu)}=\sum_{k=0}^{\infty}\binom{j}{k}\partial_x^{\,k}(z_{\nu-2j+2k}),$$
(3.16)
$$z_{2j+1}^{(\nu)}=\sum_{k=0}^{\infty}\binom{j}{k}\{\partial_x^{\,k}(\dot{z}_{\nu-2j+2k})-(-)^{\nu}\partial_x^{\,k}(z_{\nu-2j-1+2k})\},$$

and from the fact that $\Theta_{2n+1}(\mathscr{Z})=(-)^{n+1}\Theta^{2n+1}(\mathscr{Z})$, one has

$$\Theta_{2n+1}(z_i)=(-)^{n+1}\sum_{k=0}^{\infty}\binom{n}{k}\{\partial_x^{\,k}(\dot{z}_{i-2n+2k})-(-)^{i}\partial_x^{\,k}(z_{i-2n+2k-1})\}.$$

Hence one gets

$$\begin{aligned}\Theta_{2n+1}(z_{2j}^{(\nu)})&=(-)^{n+1}\sum_{l,k=0}^{\infty}\binom{j}{k}\binom{n}{l}\partial_x^{\,l+k}\{\dot{z}_{\nu-(2j+2n)+(2l+2k)}\\&\qquad\qquad-(-)^{\nu}z_{\nu-(2j+2n+1)+(2l+2k)}\}\\&=(-)^{n+1}\sum_{\mu=0}^{\infty}\binom{j+n}{\mu}\partial_x^{\,\mu}\{\dot{z}_{\nu-(2j+2n)+2\mu}-(-)^{\nu}z_{\nu-(2j+2n+1)+2\mu}\}\\&=(-)^{n+1}z_{2j+2n+1}^{(\nu)}.\end{aligned}$$

To calculate $\Theta_{2n+1}(z_{2j+1}^{(\nu)})$, first one observes that, by means of the anti-commutation relation $[\Theta_{2n+1},\Theta]_+=0$,

$$\begin{aligned}\Theta_{2n+1}\Big(\sum_{k=0}^{\infty}\binom{j}{k}\partial_x^{\,k}(\dot{z}_{\nu-2j+2k})\Big)&=-\Theta(\Theta_{2n+1}(z_{2j}^{(\nu)}))\\&=(-)^{n}\Theta(z_{2j+2n+1}^{(\nu)}).\end{aligned}$$

Hence one gets

$$\begin{aligned}\Theta_{2n+1}(z_{2j+1}^{(\nu)})&=(-)^{n}\sum_{\mu=0}^{\infty}\binom{j+n}{\mu}\partial_x^{\,\mu+1}(z_{\nu-(2j+2n)+2\mu})\\&\quad+(-)^{n}\sum_{\mu=0}^{\infty}\binom{j+n}{\mu}\partial_x^{\,\mu}(z_{\nu-(2j+2n+2)+2\mu})\\&=(-)^{n}\sum_{\mu=0}^{\infty}\binom{j+n+1}{\mu}\partial_x^{\,\mu}(z_{\nu-(2j+2n+2)+2\mu}).\\&=-(-)^{n+1}z_{2j+2n+2}^{(\nu)}.\end{aligned}$$

Thus one obtains $\Theta_{2n+1}(\mathscr{Z})=\Gamma^{2n+1}\mathscr{Z}$. The equation $\Theta_{2n}(\mathscr{Z})=\Gamma^{2n}\mathscr{Z}$ can be verified likewise. Next we compute $\Theta(z_j^{(\nu)})$. The well-known formula

$\binom{j+1}{k+1}=\binom{j}{k+1}+\binom{j}{k}$ applies to the calculation of $\Theta(z_{2j+1}^{(\nu)})$, and one obtains

$$\Theta(z_{2j+1}^{(\nu)})=z_{2j+2}^{(\nu)}-(-)^{\nu}z_{2j+1}^{(\nu-1)}.$$

The definition (3.16) leads to

$$\Theta(z_{2j}^{(\nu)})=z_{2j+1}^{(\nu)}+(-)^{\nu}z_{2j}^{(\nu-1)}.$$

Noting the parity of $z_j^{(\nu)}$, one sees that these identities are encapsulated to

$$\Theta(z_j^{(\nu)})=z_{j+1}^{(\nu)}-(z_j^{(\nu+1)})^*,$$

which implies the equation (3.15). □

From the equation (3.14), (3.15), the matrix $\mathscr{Z}$ is represented as

(3.17) $$\mathscr{Z}=\Phi\cdot\Xi\cdot\exp(-\theta\Lambda_{N^c}-x(\Lambda_{N^c})^2),$$

where

(3.18) $$\Phi=\exp\left(\theta\Lambda+x\Lambda^2+\sum_{n=1}^{\infty}t_n\Gamma^n\right),$$

and Ξ is a constant matrix

$$\Xi=(\xi_{\mu,\nu})_{\mu\in Z,\,\nu\in N^c}\in\mathrm{Mat}(Z\times N^c;\mathscr{A})\quad\text{with}\quad\xi_{\mu\nu}\in\mathscr{A}_{\underline{\mu+\nu}}.$$

(By $\mathrm{Mat}(Z\times N^c;\mathscr{A})$, $\mathrm{Mat}(N^c;\mathscr{A})$ we mean $\mathrm{Mat}(Z\times N^c;C)\otimes\mathscr{A}$, $\mathrm{Mat}(N^c;C)\otimes\mathscr{A}$, respectively.) The matrix Ξ is the initial value of $\mathscr{Z}$, namely

$$\Xi=\mathscr{Z}|_{x=\theta=t=0},$$

and is of maximal rank (the C matrix $\varepsilon(\Xi)$ of maximal rank). More precisely we have

Lemma 3.2. *The matrix entries $\xi_{\mu\nu}$ of Ξ satisfy*

$$\xi_{\nu\nu}=1 \textit{ for any } \nu\in N^c, \textit{ and } \xi_{\mu\nu}=0 \textit{ for } \mu<\nu.$$

Proof. Expand Ψ to

$$\Psi=\sum_{n=0}^{\infty}q_n(t)\Theta^n,$$

where $q_n(t)\in\mathscr{S}_{\underline{n}}$ and $q_n(0)=\delta_{n,0}$. Then the coefficients z_j are expressed by

$$z_j = \sum_{l,m,n}' \binom{n}{l} q_{2n}(t)\partial_x{}^l(u_m(x,\theta))$$
$$+ \sum_{l,m,n}' \binom{n}{l} q_{2n+1}(t)\partial_x{}^l(\dot{u}_m(x,\theta))$$
$$+ \sum_{l,m,n}' (-)^m \binom{n}{l} q_{2n+1}(t)\partial_x{}^l(u_m(x,\theta)),$$

where the summation $\sum'_{l,m,n}$ means that the indices l, m, n range over the set $\{(l, m, n) \mid l \geq 0,\ n \geq 0,\ m \leq 0,\ m+2n-2l=j\}$. Hence, for an even index ν, one obtains

$$\xi_{\nu+2j,\nu} = z^{(\nu)}_{\nu+2j}|_{x=\theta=t=0}$$
$$= \sum_{k=0}^{\infty} \binom{\nu/2+j}{k} \sum_{-\infty<m\leq 0} \partial_x{}^k(u_m(x,\theta))|_{x=\theta=0} \cdot \delta_{m,2k-2j}$$
$$= \delta_{j,0}.$$

The other cases are similarly verified. □

Substituting (3.17) to (3.13), one obtains the following proposition.

Proposition 3.3. *The coefficients* $w_j(x, \theta, t)$ $(j<0)$ *of a* SUSY *wave operator* $W \in (\mathscr{E}_{\mathscr{S}}^{1|1}(0))_0^{monic}$ *satisfy a system of an infinite number of linear equations*

$$(3.19) \qquad {}^t\vec{w}\Phi\Xi = 0,$$

where ${}^t\vec{w} = (w_j)_{j\in Z}$ $(w_0=1,\ w_j=0$ *for* $j>0)$, *for a matrix*

$$\Xi = (\xi_{\mu\nu})_{\mu\in Z, \nu\in Z^c} \in \mathrm{Mat}\,(Z \times N^c; \mathscr{A}) \quad with \quad \xi_{\mu\nu} \in \mathscr{A}_{\underline{\mu}+\underline{\nu}}, \quad \xi_{\mu\nu} = \delta_{\mu\nu} \quad (\mu \leq \nu).$$

The equation (3.19) is referred to as the SUSY Grassmann equation for the SKP hierarchy, which is an analogue of (1.12). Though we do not discuss here in detail, the SUSY Grassmann equation has a unique solution for a matrix Ξ in the set of superframes:

$$\begin{aligned}\mathrm{SFR}\,(N^c; \mathscr{A}) = \{\Xi = (\xi_{\mu\nu})_{\mu\in Z,\,\nu\in N^c} \in \mathrm{Mat}\,(Z\times N^c; \mathscr{A}) \mid \\ \xi_{\mu\nu} \in \mathscr{A}_{\underline{\mu}+\underline{\nu}},\ {}^\exists m \in N \text{ such that } \xi_{\mu\nu} = \delta_{\mu\nu} \\ \text{for } \mu < -m,\ \mu \leq \nu,\ \xi_{\mu\nu} = 0 \text{ for } -m \leq \nu < 0,\ \mu \leq -m, \\ \text{and } \varepsilon(\Xi) \text{ is of maximal rank}\}.\end{aligned}$$

The resulting solutions w_j belong to the quotient algebra $\mathscr{Q}$.

Proposition 3.4. *Let us consider the* SUSY *Grassmann equation for* $\Xi \in \mathrm{SFR}(N^c; \mathscr{A})$, *and set* $W=\sum_{-\infty<j\leq 0} w_j(x, \theta, t)\Theta^j \in (\mathscr{E}_{\mathscr{Q}}^{1|1}(0))_0^{monic}$ *for the solution* ${}^t\vec{w}$. *The operator* W *solves the* SUSY *Sato equations* (3.6) *with* $B_n=(W\Theta^n W^{-1})_+$ $(n=1, 2, \cdots)$.

Proof. First one observes that the matrix Φ (3.18) satisfies the linear equations

$$\Theta(\Phi)=\Lambda\Phi, \quad \Theta_n(\Phi)=\Gamma^n\Phi, \quad (n=1, 2, \cdots).$$

Notice that $\Gamma^{2l}=(-)^l\Lambda^{2l}$ and $[\Lambda, \Gamma]_+=0$. Differentiating (3.19) with respect to the odd derivations Θ_{2l-1}, one has, taking into account these observations and the parity of w_j,

$$(\Theta_{2l-1}({}^t\vec{w}))\Phi\Xi+(-)^l\, {}^t\vec{w}[2l-1]\Phi\Xi=0, \tag{3.20}$$

where ${}^t\vec{w}[l]=(w_{j+l})_{j\in Z}$. On the other hand, one finds that there uniquely exist fields $b_j^{(2l-1)}$ $(0\leq j\leq 2l-1,\ b_0^{(2l-1)}=1,\ b_j^{(2l-1)} \in \mathscr{Q}_j)$ satisfying

$$\begin{aligned}\text{L.H.S. of (3.20)}-(-)^l\sum_{j=0}^{2l-1} b_j^{(2l-1)}\Theta^{2l-1-j}({}^t\vec{w}\Phi\Xi)\\ ={}^t\vec{r}^{(2l-1)}\Phi\Xi,\end{aligned}$$

where the vector ${}^t\vec{r}^{(2l-1)}$ takes the form ${}^t\vec{r}^{(2l-1)}=(r_j^{(2l-1)})_{j\in Z}$ $(r_j^{(2l-1)}=0$ for $j\geq 0$, $r_j^{(2l-1)} \in \mathscr{Q}_j)$. This relation reads

$$\Theta_{2l-1}(W)+(-)^l W\Theta^{2l-1}-(-)^l B_{2l-1}W=R_{2l-1},$$

where

$$\begin{aligned}B_{2l-1}&=\sum_{j=0}^{2l-1} b_j^{(2l-1)}\Theta^{2l-1-j},\\ R_{2l-1}&=\sum_{-\infty<j<0} r_j^{(2l-1)}\Theta^j.\end{aligned}$$

From (3.19), (3.20) one has

$${}^t\vec{r}^{(2l-1)}\Phi\Xi=0.$$

Furthermore the uniqueness of a solution to the SUSY Grassmann equation for $\Xi \in \mathrm{SFR}(N^c; \mathscr{A})$ entails that ${}^t\vec{r}^{(2l-1)}=0$. Thus the operator W satisfies the SUSY Sato equation for Θ_{2l-1}. The equation for Θ_{2l} is similarly obtained. □

We introduce the supergroup $\mathrm{SGL}(N^c; \mathscr{A})$ by

$$\begin{aligned}\mathrm{SGL}(N^c;\mathscr{A})=\{g=(g_{\mu\nu})_{\mu,\nu\in N^c}\in \mathrm{Mat}(N^c;\mathscr{A})\,|\,g_{\mu\nu}\in\mathscr{A}_{\underline{\mu+\nu}},\\ {}^{\exists}m\in N \text{ such that } g_{\mu\nu}=\delta_{\mu\nu} \text{ for } \mu\le\nu,\ \mu<-m,\\ g_{\mu\nu}=0 \text{ for } -m\le\nu<0,\ \mu\le -m, \text{ and}\\ (\varepsilon(g_{\mu\nu}))_{-m\le\mu,\nu<0} \text{ is invertible}\}.\end{aligned}$$

This supergroup acts on the space $\mathrm{SFR}(N^c;\mathscr{A})$ from the right, and the universal super Grassmann manifold USGM is, by definition, the quotient space of $\mathrm{SFR}(N^c;\mathscr{A})$:

$$\mathrm{USGM}=\mathrm{SFR}(N^c;\mathscr{A})/\mathrm{SGL}(N^c;\mathscr{A}).$$

Proposition 3.4 says that USGM parametrizes general solutions in $\mathscr{Q}$ of the SKP hierarchy via the SUSY Grassmann equation, and especially that the subset

$$\mathrm{USGM}^{\phi}=\{\Xi=(\xi_{\mu\nu})\in \mathrm{SFR}(N^c;\mathscr{A})\,|\,\xi_{\mu\nu}=\delta_{\mu\nu} \text{ for } \mu\le\nu\}/\mathrm{SGL}(N^c;\mathscr{A})$$

provides formally regular solutions.

To study the time evolution of solutions to the SKP hierarchy, introduce an infinite number of supersymmetric derivations:

$$\bar{\Theta}=\frac{\partial}{\partial\theta}-\theta\frac{\partial}{\partial x},$$

$$\bar{\Theta}_{2l}=\frac{\partial}{\partial t_{2l}},\qquad \bar{\Theta}_{2l-1}=\frac{\partial}{\partial t_{2l-1}}-\sum_{k=1}^{\infty}t_{2k-1}\frac{\partial}{\partial t_{2l+2k-2}}.$$

Consider an even derivation

$$X=a\frac{\partial}{\partial x}+\zeta\bar{\Theta}+\sum_{n=1}^{\infty}c_n\bar{\Theta}_n,$$

where $a\in\mathscr{A}_{\underline{0}}$, $\zeta\in\mathscr{A}_{\underline{1}}$, $c_n\in\mathscr{A}_{\underline{n}}$. Then one easily sees that X commutes with the derivations Θ and Θ_n. Therefore it acts infinitesimally on the solution space of the SKP hierarchy. For a superfield $f\in\mathscr{Q}$, one has

$$(e^X f)(x,\theta,t)=f(x',\theta',t'), \tag{3.21}$$

where $x'=x+a+\theta\zeta$, $\theta'=\theta+\zeta$, $t'_{2l-1}=t_{2l-1}+c_{2l-1}$, $t'_{2l}=t_{2l}+c_{2l}+\sum_{k=1}^{\infty}t_{2k-1}c_{2l-2k+1}$. This formula implies that the derivation X induces an infinitesimal supersymmetric time evolution of the superfield f. Furthermore, since the fundamental solution matrix Φ (3.18) has the multiplicative property with respect to the SUSY time evolution (3.21), i.e.,

$$(e^X\Phi)(x,\theta,t)=\Phi(x,\theta,t)\Phi(a,\zeta,c),$$

the SKP hierarchy is translated to a dynamical system on USGM with the super time evolution,

$$\Xi \bmod \mathrm{SGL}(N^c; \mathscr{A}) \longrightarrow \Phi(x, \theta, t)\Xi \bmod \mathrm{SGL}(N^c; \mathscr{A}).$$

In order to solve explicitly the SUSY Grassmann equation, we need some algebraic concepts. With a matrix $X=(x_{ij})_{i,j\in Z}$,

$$\check{X}=(X_{\underline{\alpha},\underline{\beta}})_{\alpha,\beta=0,1}$$

is associated where the blocks are set as $X_{\underline{\alpha},\underline{\beta}}=(x_{ij})_{i\in 2Z+\alpha, j\in 2Z+\beta}$. Applying this rearrangement to the SUSY Grassmann equation, it is rewritten into the form

$$(3.22)\qquad (\cdots, w_{-4}, w_{-2}, 1, 0, \cdots; \cdots, w_{-3}, w_{-1}, 0, \cdots)\check{\Phi}\check{\Xi}=0.$$

Let $A=(A_{\underline{\alpha},\underline{\beta}})_{\alpha,\beta=0,1}$ be an invertible matrix with $A_{\underline{\alpha},\underline{\beta}} \in \mathrm{Mat}(m_\alpha \times m_\beta; \mathscr{A}_{\underline{\alpha}+\underline{\beta}})$. The invertibility of such matrices is equivalent to that of the matrices $\varepsilon(A_{\underline{0},\underline{0}})$ and $\varepsilon(A_{\underline{1},\underline{1}})$. A *superdeterminant* (or the *Berezinian*) [6, 17] of the matrix A is, by definition,

$$\mathrm{sdet}\, A=\det(A_{\underline{0},\underline{0}}-A_{\underline{0},\underline{1}}A_{\underline{1},\underline{1}}{}^{-1}A_{\underline{1},\underline{0}})/\det A_{\underline{1},\underline{1}}.$$

The inverse of the superdeterminant is given by

$$\mathrm{s}^{-1}\mathrm{det}\, A=\det(A_{\underline{1},\underline{1}}-A_{\underline{1},\underline{0}}A_{\underline{0},\underline{0}}{}^{-1}A_{\underline{0},\underline{1}})/\det A_{\underline{0},\underline{0}}.$$

We should remark that a superdeterminant is multiplicative with respect to the product of matrices. By virtue of Cramer's formula in linear algebra, one sees that the even unknowns w_{-2j} in (3.22) are expressed in the form of a quotient of superdeterminants. To get the formula representing the odd unknowns w_{-2j-1}, we first look for the formula for the first one w_{-1}, and consider the first SUSY Sato equation $\Theta_1(W)= -(B_1W-W\Theta)$. Finally we obtain the main theorem in this paper.

Theorem 3.5. *The coefficients of a* SUSY *wave operator attached to a superframe* $\Xi \in \mathrm{SFR}(N^c; \mathscr{A})$ *are given by*

$$(3.23)\qquad w_{-1}=\Theta\{\log(\mathrm{sdet}({}^t\check{\Xi}_0\check{\Phi}\check{\Xi}))\}$$

$$(3.24)\qquad =\Theta_1\{\log(\mathrm{sdet}({}^t\check{\Xi}_0\check{\Phi}\check{\Xi}))\},$$

and

$$(3.25)\qquad w_{-2j}=(-)^j\,\mathrm{sdet}({}^t\check{\Xi}_{2j}\check{\Phi}\check{\Xi})/\mathrm{sdet}({}^t\check{\Xi}_0\check{\Phi}\check{\Xi}),$$

(3.26) $$w_{-2j-1}=(-)^{j}(\Theta+\Theta_1)(\mathrm{sdet}\,({}^t\check{\Xi}_{2j}\check{\Phi}\check{\Xi}))/2\mathrm{sdet}\,({}^t\check{\Xi}_0\check{\Phi}\check{\Xi}),$$

for $j=1,2,\cdots$. *The frame* $\check{\Xi}_{2j}$ *is defined by*

$$\check{\Xi}_{2j}=\begin{pmatrix}\Xi_j & 0\\ 0 & \Xi_0\end{pmatrix},$$

where the definition of Ξ_j *is found in Chapter* 1.

The superdeterminant appeared in the denominator of (3.23)–(3.26) is regarded as a superanalogue of a τ function of the KP hierarchy, and is referred to as a *super* τ *field* of the SKP hierarchy. We will give a proof of Theorem 3.5 and details of a super τ field in the next section.

§ 3-3. Proof of the main theorem and a super τ field as a superdeterminant

Define $a_{\mu\nu}=a_{\mu\nu}(x,\theta,t)\in\mathscr{S}_{\underline{\mu+\nu}}$ by

$$(a_{\mu\nu})_{\mu\in Z,\nu\in N^c}=\Phi\cdot\Xi,$$

where Φ is the fundamental solution matrix (3.18) and $\Xi\in\mathrm{SFR}\,(N^c;\mathscr{A})$. The equation satisfied by Φ gives rise to the following identities:

(3.27) $$\Theta(a_{\mu\nu})=a_{\mu+1,\nu},\qquad \partial_x(a_{\mu\nu})=a_{\mu+2,\nu},$$
$$\Theta_{2l-1}(a_{\mu\nu})=(-)^{\mu+l}a_{\mu+2l-1,\nu},\qquad \Theta_{2l}(a_{\mu\nu})=(-)^{l}a_{\mu+2l,\nu}.$$

Set the rectangular matrices A, B, C, D and $\tilde{B}$, $\tilde{C}$ by

$$A=(a_{ij})_{-\infty<i,j\le-2},\qquad B=(a_{i\beta})_{\substack{-\infty<i\le-2\\ -\infty<\beta\le-1}},$$
$$C=(a_{\alpha j})_{\substack{-\infty<\alpha\le-1\\ -\infty<j\le-2}},\qquad D=(a_{\alpha\beta})_{-\infty<\alpha,\beta\le-1},$$

and

$$\tilde{B}=(a_{i\beta})_{\substack{-\infty<i<+\infty\\ -\infty<\beta\le-1}},\qquad \tilde{C}=(a_{\alpha j})_{\substack{-\infty<\alpha<+\infty\\ -\infty<j\le-2}}.$$

Hereafter i,j,k stand for even integers and α,β,γ for odd integers. The SUSY Grassmann equation (3.22) is rewritten into

(3.28) $$(\cdots w_{-4},w_{-2};\cdots w_{-3},w_{-1})\begin{bmatrix}A & B\\ C & D\end{bmatrix}$$
$$=-((a_{0j})_{-\infty<j\le-2};(a_{0\beta})_{-\infty<\beta\le-1}).$$

Now let us define for every integer μ

$$d_{\mu j}=a_{\mu j}-\sum_{-\infty<\beta\leq-1}a_{\mu\beta}c_{\beta j}\qquad(-\infty<j\leq-2),$$

where $(c_{\alpha j})_{\substack{-\infty<\alpha\leq-1\\-\infty<j\leq-2}}=D^{-1}C$, and

$$d_{\mu\beta}=a_{\mu\beta}-\sum_{-\infty<j\leq-2}a_{\mu j}b'_{j\beta}\qquad(-\infty<\beta\leq-1),$$

where $(b'_{j\beta})_{\substack{-\infty<j\leq-2\\-\infty<\beta\leq-1}}=A^{-1}B$. Set

$$\Delta=\det(d_{ij})_{-\infty<i,j\leq-2},\qquad\bar{\Delta}=\det(d_{\alpha\beta})_{-\infty<\alpha,\beta\leq-1}.$$

Let $\Delta\begin{pmatrix}i_0,\cdots,i_{n-1}\\\mu_0,\cdots,\mu_{n-1}\end{pmatrix}$ denote the determinant which is obtained by replacing i_0-th row by $(d_{\mu_0 j})_{-\infty<j\leq-2}$ and so on in the determinant Δ. The determinant $\bar{\Delta}\begin{pmatrix}\alpha_0,\cdots,\alpha_{n-1}\\\mu_0,\cdots,\mu_{n-1}\end{pmatrix}$ is defined as well.

A super τ field attached to a superframe $\Xi\in\mathrm{SFR}(N^c;\mathscr{A})$ is given by

$$\tau=\mathrm{s}\det\begin{pmatrix}A&B\\C&D\end{pmatrix}=\Delta/d,$$

where $d=\det D$. Its inverse is

$$\tau^{-1}=\mathrm{s}^{-1}\det\begin{pmatrix}A&B\\C&D\end{pmatrix}=\bar{\Delta}/a,$$

where $a=\det A$. (We should notice that, in the notation of Theorem 3.5, the super τ field is given by $\tau=\mathrm{s}\det({}^t\check{\Xi}_0\check{\Phi}\check{\Xi})$.) By means of the Cramér formula we can solve the equation (3.27) and get

$$w_{2l}=f_{2l}/\tau,\tag{3.29}$$

$$w_{2l+1}=f_{2l+1}/\tau^{-1}\qquad(l\in N^c),\tag{3.30}$$

where $f_{2l}=(-)^l\Delta\begin{pmatrix}2l\\0\end{pmatrix}\Big/d$, $f_{2l+1}=(-)^l\bar{\Delta}\begin{pmatrix}2l+1\\0\end{pmatrix}\Big/a$. The expression (3.29) just coincides with (3.25) in Theorem 3.5. We establish some lemmas to prove Theorem 3.5 and to study relations satisfied by a super τ field.

Lemma 3.6. *Let the indices j, β run over the range $-\infty<j\leq-2$, $-\infty<\beta\leq-1$, and define $b_{i\beta}$, $c'_{\alpha j}$ by $(b_{i\beta})_{\substack{-\infty<i<+\infty\\-\infty<\beta\leq-1}}=\tilde{B}\cdot D^{-1}$, $(c'_{\alpha j})_{\substack{-\infty<\alpha<+\infty\\-\infty<j\leq-2}}=\tilde{C}\cdot A^{-1}$.*

Then one has the following identities:

(i) $\Theta(d_{ij})=\delta_{0i}d_{1j}+\sum_{-\infty<\beta\leq -1} b_{i\beta}d_{\beta+1,j}$ $(-\infty<i\leq 0)$.

(ii) $\Theta_1(d_{ij})=-\delta_{0i}d_{1j}+\sum_{-\infty<\beta\leq -1} b_{i\beta}d_{\beta+1,j}$ $(-\infty<i\leq 0)$.

(iii) $\partial_x(d_{ij})=d_{i+2,j}-b_{i,-1}d_{1j}$ $(-\infty<i\leq -2)$.

(iv) $\Theta_3(d_{ij})=\delta_{i,-2}d_{1j}-\sum_{-\infty<\beta\leq -1} b_{i\beta}d_{\beta+3,j}$ $(-\infty<i\leq -2)$.

(v) $\Theta(d_{\alpha\beta})=\delta_{\alpha,-1}d_{0\beta}+\sum_{-\infty<j\leq -1} c'_{\alpha j}d_{j+1,\beta}$ $(-\infty<\alpha\leq -1)$.

Proof. (i) Since

$$(d_{\alpha j})_{\substack{-\infty<\alpha\leq -1\\ -\infty<j\leq -2}}=C-D(D^{-1}C)=0,$$

one has

$$\begin{aligned}0&=\Theta(d_{\alpha j})\\ &=d_{\alpha+1,j}-\sum_{-\infty<\beta\leq -1} a_{\alpha\beta}\cdot\Theta(c_{\beta j})\end{aligned}$$

for $-\infty<\alpha\leq -1$. Hence

$$(\Theta(c_{\alpha j}))_{\substack{-\infty<\alpha\leq -1\\ -\infty<j\leq -2}}=D^{-1}\cdot(d_{i+2,j})_{-\infty<i,j\leq -2}. \tag{3.31}$$

In the same manner one gets

$$\Theta(d_{ij})=d_{i+1,j}+\sum_{-\infty<\beta\leq -1} a_{i\beta}\cdot\Theta(c_{\beta j}), \tag{3.32}$$

and the second term is computed as follows:

$$\begin{aligned}&\Big(\sum_{-\infty<\beta\leq -1} a_{i\beta}\cdot\Theta(c_{\beta j})\Big)_{\substack{-\infty<i<+\infty\\ -\infty<j\leq -2}}\\ &=\tilde{B}\cdot(\Theta(c_{\alpha j}))_{\substack{-\infty<\alpha\leq -1\\ -\infty<j\leq -2}}\\ &=(\tilde{B}\cdot D^{-1})\cdot(d_{k+2,j})_{-\infty<k,j\leq -2}\\ &=\Big(\sum_{-\infty<\beta\leq -1} b_{i\beta}d_{\beta+1,j}\Big)_{\substack{-\infty<i<+\infty\\ -\infty<j\leq -2}},\end{aligned}$$

where (3.31) was applied. Inserting this result to (3.32) and noting that $d_{i+1,j}=0$ for $-\infty<i\leq -2$, one gets (i).

(ii) One can similarly prove as in the case of (i).

(iii) One gets in the same way as in (i) that

$$\partial_x(d_{ij})=d_{i+2,j}-\sum_{-\infty<\beta\leq -1} a_{i\beta}\cdot\partial_x(c_{\beta j}),$$

and the second term is computed as follows:

$$\Big(\sum_{-\infty<\beta\leq -1} a_{i\beta}\cdot\partial_x(c_{\beta j})\Big)_{-\infty<i,j\leq -2}$$
$$=B\cdot(\partial_x(D^{-1}C))$$
$$=BD^{-1}\{\partial_x(C)-\partial_x(D)\cdot D^{-1}C\}.$$

From the definition of the matrix D it follows that

$$\partial_x(D)\cdot D^{-1}=\begin{bmatrix} I_{N^c_{\rm odd}} \\ (a_{1\beta})_{-\infty<\beta\leq -1}\cdot D^{-1}\end{bmatrix},$$

where $N^c_{\rm odd}=\{\cdots, -5, -3, -1\}$ and $I_{N^c_{\rm odd}}$ denotes the identity matrix of the size $N^c_{\rm odd}$. Hence one gets

$$\partial_x(C)-\partial_x(D)\cdot D^{-1}C=\begin{pmatrix} 0_{N^c_{\rm odd}} \\ (d_{1j})_{-\infty<j\leq -2}\end{pmatrix},$$

which shows (iii). We omit the proofs of (iv) and (v) because they can be verified in a completely similar way as (i). □

Lemma 3.7. *Set* $D^{-1}=(\tilde{a}_{\alpha\beta})_{-\infty<\alpha,\beta\leq -1}$ *and* $\pi_{1\beta}=\sum_{-\infty<\gamma\leq -1} a_{1\gamma}\tilde{a}_{\gamma\beta}$. *Then one gets the following identities*:

(i) $\Theta(\log d)=\sum_{-\infty<i\leq 0} b_{i,i-1}$.

(ii) $\partial_x(\log d)=\pi_{1,-1}$.

(iii) $\Theta(d_{1j})=d_{2j}-\sum_{-\infty<\beta\leq -1}\pi_{1\beta}d_{\beta+1,j}$.

(iv) $\partial_x\big(\sum_{-\infty<i\leq 0} b_{i,i-1}\big)=b_{2,-1}-\sum_{-\infty<i\leq 0} b_{i,-1}\pi_{1,i-1}$.

(v) $\Theta(b_{i,-1})=\delta_{i+1,-1}+\sum_{-\infty<\gamma\leq 1} b_{i\gamma}b_{\gamma+1,-1}\quad (-\infty<i\leq -2)$.

Proof. (i) By the definition of $b_{i\beta}$ one sees that

$$\Theta(D)\cdot D^{-1}=(b_{i\beta})_{\substack{-\infty<i\leq 0\\ -\infty<\beta\leq -1}}. \tag{3.33}$$

Hence

$$\Theta(\log d)=\mathrm{tr}\,(\Theta(D)\cdot D^{-1})=\sum_{-\infty<i\leq 0} b_{i,i-1}.$$

(ii) By the definition of $\pi_{1\beta}$ one sees that

$$\partial_x(D)\cdot D^{-1}=\begin{pmatrix} I_{N^c_{\rm odd}} \\ (\pi_{1\beta})_{-\infty<\beta\leq -1}\end{pmatrix}. \tag{3.34}$$

Hence

$$\partial_x(\log d)=\mathrm{tr}\,(\partial_x(D)\cdot D^{-1})=\pi_{1,-1}.$$

(iii) This can be verified by a simple calculation.

(iv) From the definition of $b_{i\beta}$ it follows that

$$(\partial_x(b_{i\beta}))_{\substack{-\infty<i<+\infty \\ -\infty<\beta\leq -1}}=\partial_x(\tilde{B}\cdot D^{-1})$$
$$=\partial_x(\tilde{B})\cdot D^{-1}-\tilde{B}D^{-1}\cdot\partial_x(D)\cdot D^{-1}.$$

Inserting (3.34) into the equation above, one sees that

$$\partial_x(b_{i,i-1})=b_{i+2,i-1}-b_{i,i-3}-b_{i,-1}\pi_{1,i-1}.$$

Summing up this identity over $-\infty<i\leq 0$, (iv) is proved.

(v) Since

$$\Theta(BD^{-1})=\Theta(B)\cdot D^{-1}-B\cdot\Theta(D^{-1})$$
$$=I_{N^c_{\mathrm{odd}}}+BD^{-1}\cdot\Theta(D)\cdot D^{-1},$$

applying (3.33) to the above line, one has

$$\Theta(b_{i\beta})=\delta_{i+1,\beta}+\sum_{-\infty<\gamma\leq -1} b_{i\gamma}b_{\gamma+1,\beta},$$

where $-\infty<i\leq -2$, $-\infty<\beta\leq -1$. □

Lemma 3.8. *One has the following identities*:

(i) $\Theta(\tau)=\frac{1}{d}\left\{\sum_{-\infty<i\leq 2} b_{i,-1}\Delta\binom{i}{0}-b_{0,-1}\Delta\right\}.$

(ii) $\partial_x(\tau)+f_{-2}=-\partial_x(\log d)\cdot\tau-\frac{1}{d}\sum_{-\infty<i\leq -2} b_{i,-1}\Delta\binom{i}{1}.$

(iii) $\frac{1}{2}\{\Theta(f_{-2})+\Theta_1(f_{-2})\}=\frac{1}{d}\left\{\sum_{-\infty<i\leq -2} b_{i,-3}\Delta\binom{i}{0}-b_{0,-3}\Delta\right\}.$

(iv) $\Theta_3(\tau)=\frac{1}{d}\left\{-\sum_{-\infty<i\leq -2}\left(b_{i,-3}\Delta\binom{i}{0}+b_{i,-1}\Delta\binom{i}{2}\right)\right.$
$\left.+(b_{0,-3}+b_{2,-1})\Delta\right\}+\frac{1}{d}\Delta\binom{-2}{1}.$

(v) $2(f_{-2}+\partial_x(\tau))\cdot\Theta(\log\tau)-\Theta^3(\tau)$
$=\Theta(f_{-2})+\Theta^3(\log d)\cdot\tau+\frac{1}{d}\Delta\binom{-2}{1}-\frac{1}{d}\sum_{-\infty<i\leq -2} b_{i,-1}\Delta\binom{i}{2}.$

Proof. By $\Delta\binom{i}{(f_j)}$ is meant the determinant obtained by replacing the i-th row in Δ by a row vector $(f_j)_{-\infty<j\leq -2}$.

(i) $$\Theta(\tau)=\frac{1}{d}\sum_{-\infty<i\le -2}\Delta\begin{pmatrix} i \\ \Theta(d_{ij})\end{pmatrix}-\Theta(\log d)\cdot\tau$$
$$=\frac{1}{d}\Big\{\sum_{-\infty<i\le -2} b_{i,-2}\Delta\begin{pmatrix} i \\ 0\end{pmatrix}-b_{0,-1}\Delta\Big\}$$
$$+\Big(\sum_{-\infty<i\le 0} b_{i,i-1}\Big)\cdot\tau-\Theta(\log d)\cdot\tau.$$

Applying Lemma 3.7 (i) to the above line shows (i).

(ii) This is verified in a similar way by making use of Lemma 3.7 (iii).

(iii) From Lemma 3.6 (i) and Lemma 3.7 (i), it follows that

$$\Theta(f_{-2})=-\frac{1}{d}\Big\{\sum_{-\infty<i\le -4} b_{i,3}\Delta\begin{pmatrix} i & -2 \\ -2 & 0\end{pmatrix}-b_{0,-3}\Delta-b_{-2,-3}\Delta\begin{pmatrix} -2 \\ 0\end{pmatrix}\Big\}$$
$$-\frac{1}{d}\Delta\begin{pmatrix} -2 \\ 1\end{pmatrix}.$$

Noting Lemma 3.6 (ii) and that $\Theta(\log d)=\Theta_1(\log d)$, one has (iii).

(iv) This follows from Lemma 3.6 (iv) and

$$\Theta_3(\log d)=-\sum_{-\infty<i\le -2} b_{i,i-3}.$$

(v) Differentiating the both sides of Lemma 3.6 (iii), one has, by virtue of Lemma 3.7 (v),

$$\Theta^3(\tau)=-\Theta(f_{-2})-\Theta^3(\log d)\cdot\tau-2\partial_x(\log d)\cdot\tau$$
$$-\Big(\sum_{-\infty<i\le 0} b_{i,-1}\pi_{1,i-1}\Big)\cdot\tau+\frac{\Theta(d)}{d^2}\sum_{-\infty<i\le -2} b_{i,-1}\Delta\begin{pmatrix} i \\ 1\end{pmatrix}$$
$$-\frac{1}{d}\Delta\begin{pmatrix} -2 \\ 1\end{pmatrix}-\sum_{\substack{-\infty<i\le -2\\ -\infty<\gamma\le -1}} b_{i\gamma}b_{\gamma+1,-1}\Delta\begin{pmatrix} i \\ 1\end{pmatrix}$$
$$-\frac{1}{d}\sum_{\substack{-\infty<i,k\le -2\\ i\ne k}} b_{i,-1}b_{k,-1}\Delta\begin{pmatrix} k \\ -1\end{pmatrix}$$
$$+\frac{1}{d}\sum_{-\infty<i\le -2} b_{i,-1}\Delta\begin{pmatrix} i \\ 2\end{pmatrix}+\frac{1}{d}\sum_{-\infty<i,k\le -2} b_{i,-1}b_{k,-1}\Delta\begin{pmatrix} k & i \\ 0 & 1\end{pmatrix}.$$

Furthermore, noting that

$$\Theta(\log\tau)=\frac{1}{\Delta}\sum_{-\infty<i\le -2} b_{i,-1}\Delta\begin{pmatrix} i \\ 0\end{pmatrix}-b_{0,-1},$$

and applying the Plücker relation

$$\varDelta\binom{i}{1}\varDelta\binom{k}{0}-\varDelta\binom{k}{1}\varDelta\binom{i}{0}-\varDelta\cdot\varDelta\begin{pmatrix}k & i\\ 0 & 1\end{pmatrix}=0$$

to Lemma 3.8 (ii), one finally obtains (v). □

Proof of Theorem 3.5. First we prove (3.23). We only have to show that $f_{-1}=-\Theta(\tau^{-1})$. By virtue of Lemma 3.6 (v), one has

$$\Theta(\tau^{-1})=\frac{1}{a}\bar{\varDelta}\binom{-1}{0}+\Big(\sum_{-\infty<\alpha\leq -1}c'_{\alpha,\alpha-1}\Big)\tau^{-1}-\Theta(\log a)\cdot\tau^{-1}.$$

Since $\Theta(A)=C$,

$$\begin{aligned}\Theta(\log a)&=\operatorname{tr}(\Theta(A)\cdot A^{-1})\\&=\sum_{-\infty<\alpha\leq -1}c'_{\alpha,\alpha-1}.\end{aligned}$$

Thus one proves $f_{-1}=-\Theta(\tau^{-1})$. The identity (3.24) follows from Lemma 3.6 (i), (ii). What we remain to prove is the identity (3.26). Consider the Sato equation

$$\Theta_1(W)=-B_1W+W\Theta,$$

where $B_1=\Theta+2w_{-1}$. It is easy to show that

$$\begin{aligned}&\Theta_1(w_{-1})=-\Theta(w_{-1}),\\ &\Theta_1(w_{2j})=-\Theta(w_{2j})-2w_{-1}w_{2j}+2w_{2j-1},\\ &\Theta_1(w_{2j-1})=-\Theta(w_{2j-1})-2w_{-1}w_{2j-1}\qquad (j\in N^c).\end{aligned}\tag{3.35}$$

The first equation above is already satisfied because of (3.24). Setting $w_j=g_j/\tau$ ($j\leq -2$), the second equation yields

$$g_{2j-1}=\frac{1}{2}(\Theta+\Theta_1)(g_{2j}),\tag{3.36}$$

which consequently leads to the third one. The relation (3.36) entails (3.26). Thus the theorem is proved. □

It seems likely that a super τ field of the SKP hierarchy satisfies many relations. However on this point, our analysis has not been completed yet. We only propose here a set of infinitely many relations that connects a super τ field with the coefficients of the corresponding SUSY wave operator. Consider the residue term of the Sato equations (the residue term of a SUSY microdifferential operator is, by definition, the coefficient of Θ^{-1} in the operator [16]). By a bruteforce we can verify that the residue terms in the first few equations of (3.6) take the exact form with respect to Θ derivation. Integrating them we obtain the following relations:

$$\Theta(\log\tau)=\Theta_1(\log\tau)=w_{-1}. \tag{3.37}$$

$$\Theta_2(\log\tau)=-\dot{w}_{-1}. \tag{3.38}$$

$$\Theta_3(\log\tau)=-\left\{\frac{1}{2}(3\Theta+\Theta_1)(w_{-2})+w_{-1,x}-w_{-1}\dot{w}_{-1}\right\}. \tag{3.39}$$

$$\Theta_4(\log\tau)=2\dot{w}_{-3}+\dot{w}_{-1,x}-(\dot{w}_{-1})^2-2(w_{-1}w_{-2})^{\cdot}. \tag{3.40}$$

$$\begin{aligned}\Theta_5(\log\tau)=&\frac{1}{2}(5\Theta+\Theta_1)(w_{-4})+2w_{-3,x}+\dot{w}_{-2,x}+w_{-1,xx}\\&+(w_{-1}w_{-1,x})^{\cdot}-w_{-1}w_{-2,x}+(w_{-1}\dot{w}_{-2})^{\cdot}-2w_{-1}\dot{w}_{-3}\\&-\frac{1}{2}(\dot{w}_{-1}^2)^{\cdot}-\{(\dot{w}_{-1}+w_{-2})^2\}^{\cdot}+w_{-1}(\dot{w}_{-1})^2\\&+2w_{-1}\dot{w}_{-1}w_{-2}+w_{-1}(w_{-2})^2-w_{-2}w_{-3}.\end{aligned} \tag{3.41}$$

These relations may be regarded as an analogy of (1.15) that connects a τ function of the KP hierarchy with the corresponding wave operator. The relation (3.37) has already been proved in Theorem 3.5. Here we show (3.39).

Proposition 3.9. *A super τ field satisfies the relation* (3.39). *That is, w_j being the coefficients of the* SUSY *wave operator for the superfield τ, it enjoys the following relation which is equivalent to* (3.39).

$$\Theta_3(\tau)=-\frac{1}{2}\{3\Theta(f_{-2})+\Theta_1(f_{-2})\}+2\{f_{-2}+\partial_x(\tau)\}\cdot\Theta(\log\tau)-\Theta^3(\tau), \tag{3.42}$$

where the superfield f_{-2} is defined in (3.29).

Proof. Notice Lemma 3.8 (iii), (iv), (v). Remaining task is to show that

$$b_{2,-1}=\Theta^3(\log d)-\sum_{-\infty<i\le 0}b_{i,-1}\pi_{1,i-1}.$$

This follows from Lemma 3.7 (iv). □

§ 3.4. The reduced hierarchy

Let l be a positive integer. In the SKP hierarchy, consider the following condition:

(3.43) $$L^{2l}=B_{2l},$$

or the equivalent condition for a SUSY wave operator

(3.44) $$\Theta_{2ln}(W)=0 \qquad (n=1, 2, \cdots).$$

Geometrically the condition above is interpreted as follows: Let Ξ be a superframe associated with the SUSY wave operator W. Then the condition (3.44) says that the point in USGM represented by Ξ is stable under the action of the group generated by the Lie algebra $\sum_{n=1}^{\infty} \boldsymbol{C}\Gamma^{2ln}$. That is, for arbitrary $c_{2ln} \in \mathscr{A}_{\underline{0}}$, there exists an element $g \in \mathrm{SGL}(N^c; \mathscr{A})$ such that

$$\exp\left(\sum_{n=1}^{\infty} c_{2ln}\Gamma^{2ln}\right)\cdot\Xi=\Xi\cdot g.$$

The system which comes from the condition (3.43) is a SUSY extension of the l-reduced KP hierarchy, and is referred to as the l-reduced SKP hierarchy. The symmetry of this hierarchy is the Kac-Moody superalgebra without a center $sl(l|l; \boldsymbol{C}[\lambda, \lambda^{-1}])$ (Cf. Chapter 4). The 2-reduced SKP hierarchy gives rise to the SUSY KdV equation [14, 15, 16]:

$$u_{3,t_6}=\left(\frac{-1}{4}u_{3,xx}-\frac{3}{2}u_3\dot{u}_3-3u_3u_4\right)_x,$$

$$u_{4,t_6}=\left(\frac{-1}{4}u_{4,xx}-\frac{3}{2}u_4^2-\frac{3}{2}u_3\dot{u}_4\right)_x.$$

Chapter 4. Lie Superalgebras of Infinite Dimensions

In this chapter we construct realizations of the Lie superalgebra $\mathfrak{gl}(\infty\,|\,\infty)^{\sim}$ and of its subalgebras by making use of free field operators.

We first define Lie superalgebras [4,10]. A $\boldsymbol{Z}_2$-graded vector space $\mathfrak{g}=\mathfrak{g}_{\underline{0}}\oplus\mathfrak{g}_{\underline{1}}$ is called a *Lie superalgebra* if there is a bilinear bracket product $[\,,]$ on $\mathfrak{g}$ which satisfies the following conditions: If $x \in \mathfrak{g}_{\alpha}$ and $y \in \mathfrak{g}_{\beta}$, then 1) $[x, y] \in g_{\alpha+\beta}$, 2) $[x, y]=-(-)^{\alpha\beta}[y, x]$ and 3) $[x, [y, z]]=[[x, y], z]+(-)^{\alpha\beta}[y, [x, z]]$ for $z \in \mathfrak{g}$. The third relation is referred to as the super Jacobi identity.

An example of Lie superalgebras is constructed in the following manner. Let $N=m+n$ be a positive integer, and let

$$\mathfrak{g}_{\underline{0}}=\left\{\begin{pmatrix} A & 0 \\ 0 & D\end{pmatrix}\middle| A \in \mathrm{Mat}(m; \boldsymbol{C}),\ D \in \mathrm{Mat}(n; \boldsymbol{C})\right\},$$

$$\mathfrak{g}_{\underline{1}}=\left\{\begin{pmatrix} 0 & B \\ C & 0\end{pmatrix}\middle| B \in \mathrm{Mat}(m\times n; \boldsymbol{C}),\ C \in \mathrm{Mat}(n\times m; \boldsymbol{C})\right\},$$

so that $\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}$ is the space of all $N\times N$ complex matrices. For $X\in\mathfrak{g}_{\alpha}$, $Y\in\mathfrak{g}_{\beta}$, we define $[X, Y]=XY-(-)^{\alpha\beta}YX$. Then the space $\mathfrak{g}$ is a Lie superalgebra which is denoted by $\mathfrak{gl}(m|n)$. For $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\mathfrak{gl}(m|n)$, we define the supertrace by $\operatorname{str} X=\operatorname{tr} A-\operatorname{tr} D$. All supertraceless elements of $\mathfrak{gl}(m|n)$ constitute the Lie subsuperalgebra $\mathfrak{sl}(m|n)$.

Introduce the infinite dimensional Lie superalgebra $\mathfrak{gl}(\infty|\infty)=\mathfrak{gl}(\infty|\infty)_{\bar{0}}\oplus\mathfrak{gl}(\infty|\infty)_{\bar{1}}$ by

$$\mathfrak{gl}(\infty|\infty)_{\bar{0}}=\{\sum_{\substack{i,j\in\boldsymbol{Z}\\ \alpha=0,1}} a_{ij}^{(\alpha\alpha)}E_{ij}^{(\alpha\alpha)}\,|\,a_{ij}^{(\alpha\alpha)}=0 \text{ for } |i-j|\gg 1\},$$

and

$$\mathfrak{gl}(\infty|\infty)_{\bar{1}}=\{\sum_{\substack{i,j\in\boldsymbol{Z}\\ \alpha,\beta=0,1;\ \alpha\neq\beta}} a_{ij}^{(\alpha\beta)}E_{ij}^{(\alpha\beta)}\,|\,a_{ij}^{(\alpha\beta)}=0 \text{ for } |i-j|\gg 1\},$$

where $(E_{ij}^{(\alpha\beta)})_{kl}=\delta_{ik}\delta_{jl}$ for α, $\beta=0, 1$. The bracket product is defined by

$$[E_{ij}^{(\alpha\beta)}, E_{kl}^{(\gamma\varepsilon)}]=\delta^{\beta\gamma}\delta_{jk}E_{il}^{(\alpha\varepsilon)}-(-)^{(\alpha+\beta)(\gamma+\varepsilon)}\delta^{\alpha\varepsilon}\delta_{il}E_{kj}^{(\gamma\beta)}.$$

Now we construct the Lie superalgebra $\mathfrak{gl}(\infty|\infty)^{\sim}$ which is the one-dimensional central extension of $\mathfrak{gl}(\infty|\infty)$, by making use of free field operators. Let $\boldsymbol{A}$ be the Clifford algebra over $\boldsymbol{C}$ with generators $\psi_j^{(\alpha)}$, $\psi_j^{(\alpha)*}$ ($\alpha=0, 1; j\in\boldsymbol{Z}$) satisfying the defining relations:

$$[\psi_i^{(0)}, \psi_j^{(0)}]_+=[\psi_i^{(0)*}, \psi_j^{(0)*}]_+=0, \quad [\psi_i^{(0)}, \psi_j^{(0)*}]_+=\delta_{ij},$$
$$[\psi_i^{(1)}, \psi_j^{(1)}]=[\psi_i^{(1)*}, \psi_j^{(1)*}]=0, \quad [\psi_i^{(1)}, \psi_j^{(1)*}]=-\delta_{ij},$$
$$[\psi_i^{(0)}, \psi_j^{(1)}]=[\psi_i^{(0)}, \psi_j^{(1)*}]=[\psi_i^{(0)*}, \psi_j^{(1)}]=[\psi_i^{(0)*}, \psi_j^{(1)*}]=0.$$

An element of

$$W^{(0)}=(\sum_{j\in\boldsymbol{Z}}\boldsymbol{C}\psi_j^{(0)})\oplus(\sum_{j\in\boldsymbol{Z}}\boldsymbol{C}\psi_j^{(0)*})$$
$$(\text{resp. } W^{(1)}=(\sum_{j\in\boldsymbol{Z}}\boldsymbol{C}\psi_j^{(1)})\oplus(\sum_{j\in\boldsymbol{Z}}\boldsymbol{C}\psi_j^{(1)*}))$$

is referred to as a free fermion (resp. free boson). It is easy to check the following proposition.

Proposition 5.1. *The product $\psi_i^{(\alpha)}\psi_j^{(\beta)*}$ satisfies the same bracket relations as the matrix element $E_{ij}^{(\alpha\beta)}$.*

Next we consider the one-dimensional central extension $\mathfrak{gl}(\infty|\infty)^{\sim}=\mathfrak{gl}(\infty|\infty)\oplus\boldsymbol{C}z$. The even part is $\mathfrak{gl}(\infty|\infty)_{\bar{0}}^{\sim}=\mathfrak{gl}(\infty|\infty)_{\bar{0}}\oplus\boldsymbol{C}z$ and the odd part is $\mathfrak{gl}(\infty|\infty)_{\bar{1}}^{\sim}=\mathfrak{gl}(\infty|\infty)_{\bar{1}}$. The bracket product is defined by

$$[E_{ij}^{(\alpha\beta)}, E_{kl}^{(\gamma\varepsilon)}]^{\sim}=[E_{ij}^{(\alpha\beta)}, E_{kl}^{(\gamma\varepsilon)}]+(-)^{\alpha\beta}\delta^{\alpha\varepsilon}\delta^{\beta\gamma}\delta_{il}\delta_{jk}(Y_+(j)-Y_+(i)),$$
$$[\mathfrak{gl}(\infty\,|\,\infty)^{\sim}, z]^{\sim}=\{0\}, \quad \text{where } Y_+(i)=1 \ (i\geq 0), \quad =0 \ (i<0).$$

In other words,

$$[\sum a_{ij}^{(\alpha\beta)}E_{ij}^{(\alpha\beta)}, \quad \sum b_{ij}^{(\alpha\beta)}E_{ij}^{(\alpha\beta)}]^{\sim}=\sum c_{ij}^{(\alpha\beta)}E_{ij}^{(\alpha\beta)}+cz,$$

where

$$c_{ij}^{(\alpha\beta)}=\sum a_{ik}^{(\alpha\gamma)}b_{kj}^{(\gamma\beta)}-\sum(-)^{(\alpha+\gamma)(\beta+\gamma)}a_{kj}^{(\gamma\beta)}b_{ik}^{(\alpha\gamma)},$$
$$c=\sum(-)^{\alpha\beta}a_{ij}^{(\alpha\beta)}b_{ji}^{(\beta\alpha)}(Y_+(j)-Y_+(i)).$$

To construct a representation of $\mathfrak{gl}(\infty\,|\,\infty)^{\sim}$ via the free field operators, we introduce the vacuum expectation values for the quadratic elements in $\boldsymbol{A}$. Set the linear form $\langle\ \rangle$ by

$$\langle\psi_i^{(\alpha)}\psi_j^{(\beta)}\rangle=\langle\psi_j^{(\alpha)*}\psi_i^{(\beta)*}\rangle=0 \quad \text{for } \alpha, \beta=0, 1,$$
$$\langle\psi_i^{(\alpha)}\psi_j^{(\beta)*}\rangle=\langle\psi_j^{(\alpha)*}\psi_i^{(\beta)}\rangle=0 \quad \text{for } \alpha, \beta=0, 1, \quad \alpha\neq\beta,$$
$$\langle\psi_i^{(0)}\psi_j^{(0)*}\rangle=\delta_{ij}Y_-(i), \quad \langle\psi_j^{(0)*}\psi_i^{(0)}\rangle=\delta_{ij}Y_+(i),$$
$$\langle\psi_i^{(1)}\psi_j^{(1)*}\rangle=-\delta_{ij}Y_-(i), \quad \langle\psi_j^{(1)*}\psi_i^{(1)}\rangle=\delta_{ij}Y_+(i),$$

where $Y_-(i)=1-Y_+(i)$. We normalize the linear form by $\langle 1\rangle=1$. The normal product is defined by

$$:\psi_i^{(\alpha)}\psi_j^{(\beta)*}:=\psi_i^{(\alpha)}\psi_j^{(\beta)*}-\langle\psi_i^{(\alpha)}\psi_j^{(\beta)*}\rangle.$$

From Proposition 5.1 we obtain the following proposition.

Proposition 5.2. *The mapping* $E_{ij}^{(\alpha\beta)}\rightarrow :\psi_i^{(\alpha)}\psi_j^{(\beta)*}:$, $z\rightarrow 1$ *defines a realization of the Lie superalgebra* $\mathfrak{gl}(\infty\,|\,\infty)^{\sim}$.

We write $Z_{ij}^{(\alpha\beta)}=:\psi_i^{(\alpha)}\psi_j^{(\beta)*}:$. Define the elements

$$L_m^{(0)}=-\sum_{j\in\boldsymbol{Z}}jZ_{j+m,j}^{(00)} \quad \text{for} \quad m\in\boldsymbol{Z}.$$

Then we have the commutation relation

$$[L_m^{(0)}, L_n^{(0)}]=(m-n)L_{m+n}^{(0)}+\frac{1}{6}(m^3-m)\delta_{m+n,0}.$$

Hence $\sum_{m\in\boldsymbol{Z}}\boldsymbol{C}L_m^{(0)}\oplus\boldsymbol{C}\cdot 1$ is a Lie subalgebra of $\mathfrak{gl}(\infty\,|\,\infty)^{\sim}$, which is isomorphic to the celebrated Virasoro algebra. As is known, there are two manners for the supersymmetric extension of the Virasoro algebra, namely, the Ramond algebra and the Neveu-Schwarz algebra. The Ramond (resp.

the Neveu-Schwarz) algebra is the complex Lie superalgebra $\mathscr{L}=\mathscr{L}_{\underline{0}}\oplus\mathscr{L}_{\underline{1}}$, where the even part $\mathscr{L}_{\underline{0}}$ has the basis $\{l_m, z \mid m \in \boldsymbol{Z}\}$, and the odd part $\mathscr{L}_{\underline{1}}$ has the basis $\{g_k \mid k \in \boldsymbol{Z}\}$ (resp. $\{g_k \mid k \in \boldsymbol{Z}+1/2\}$) satisfying the following bracket relations:

$$[l_m, l_n]=(m-n)l_{m+n}+\frac{1}{8}(m^3-m)\delta_{m+n,0}z,$$

$$[l_m, g_k]=\left(\frac{1}{2}m-k\right)g_{m+k},$$

$$[g_j, g_k]=2l_{j+k}+\frac{1}{2}\left(j^2-\frac{1}{4}\right)\delta_{j+k,0}z, \quad \text{and}$$

$$[\mathscr{L}, z]=\{0\}.$$

We can realize these Lie superalgebras as subalgebras of $\mathfrak{gl}(\infty \mid \infty)^{\sim}$.

Proposition 5.3. *Let* $(\mu, \nu) \in \boldsymbol{C}^2$. *Define the elements*

$$L_m^{(\mu\nu)}=-\sum_{j\in Z} j(Z_{j+m,j}^{(00)}+Z_{j+m,j}^{(11)})+(\mu+m\nu)\sum_{j\in Z}(Z_{j+m,j}^{(00)}+Z_{j+m,j}^{(11)})$$
$$-\frac{1}{2}m\sum_{j\in Z}Z_{j+m,j}^{(11)}+\frac{1}{8}(1+4(\mu+\nu))\delta_{m,0}, \quad \textit{and}$$
$$G_k^{(\mu\nu)}=-\sqrt{-1}\sum_{j\in Z}(Z_{k+j,j}^{(01)}+jZ_{k+j,j}^{(10)})+\sqrt{-1}(\mu+2k\nu)\sum_{j\in Z}Z_{k+j,j}^{(10)},$$

for $m, k \in \boldsymbol{Z}$. *Then the mapping* $l_m \to L_m^{(\mu\nu)}$, $g_k \to G_k^{(\mu\nu)}$, $z \to 2+8\nu$ *gives a realization of the Ramond algebra.*

Proposition 5.4. *Let* $(\mu, \nu) \in \boldsymbol{C}^2$. *Define the elements*

$$L_m^{(\mu\nu)}=-\sum_{j\in Z} j(Z_{j+m,j}^{(00)}+Z_{j+m,j}^{(11)})+(\mu+m\nu)\sum_{j\in Z}(Z_{j+m,j}^{(00)}+Z_{j+m,j}^{(11)})$$
$$-\frac{1}{2}(m-1)\sum_{j\in Z}Z_{j+m,j}^{(11)}+\frac{1}{2}(1+\mu+\nu)\delta_{m,0}, \quad \textit{and}$$
$$G_{k+1/2}^{(\mu\nu)}=-\sqrt{-1}\sum_{j\in Z}(Z_{j+k,j}^{(01)}+jZ_{j+k+1,j}^{(10)})+\sqrt{-1}(\mu+(2k+1)\nu)\sum_{j\in Z}Z_{j+k+1,j}^{(10)},$$

for $m, k \in \boldsymbol{Z}$. *Then the mapping* $l_m \to L_m^{(\mu\nu)}$, $g_{k+1/2} \to G_{k+1/2}^{(\mu\nu)}$, $z \to 2+8\nu$ *gives a realization of the Neveu-Schwarz algebra.*

Now let us consider the Lie superalgebra

$$\mathfrak{g}=\mathfrak{sl}(m \mid n)\otimes \boldsymbol{C}[\lambda, \lambda^{-1}]\oplus \boldsymbol{C}z,$$

where $\boldsymbol{C}[\lambda, \lambda^{-1}]$ is the ring of Laurent polynomials of the indeterminate λ. The bracket relations are

$$[X\otimes\lambda^k, Y\otimes\lambda^l]=[X, Y]\otimes\lambda^{k+l}+k\delta_{k+l,0}\,\mathrm{str}\,(XY)z,$$
$$[\mathfrak{g}, z]=\{0\},$$

for $X, Y\in\mathfrak{sl}(m|n)$. Put

$$e_0=E_{m+n,1}\otimes\lambda,\quad e_i=E_{i,i+1}\otimes 1\quad (1\leq i\leq m+n),$$
$$f_0=E_{1,m+n}\otimes\lambda^{-1},\quad f_i=E_{i+1,i}\otimes 1\quad (1\leq i\leq m+n),$$
$$h_0=(E_{11}+E_{m+n,m+n})\otimes 1-z,\quad h_m=(E_{mm}+E_{m+1,m+1})\otimes 1,$$
$$h_i=(E_{ii}-E_{i+1,i+1})\otimes 1\quad (1\leq i\leq m+n,\ i\neq m),$$

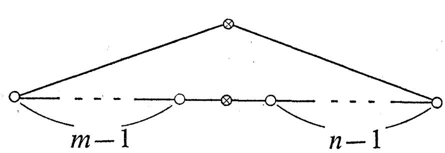
where $(E_{ij})_{kl}=\delta_{ik}\delta_{jl}$. Then we see that $\{e_j, f_j\,|\,1\leq j\leq m+n, j\neq m\}$ is the system of even Chevalley generators, and $\{e_0, e_m, f_0, f_m\}$ is the system of odd Chevalley generators. If we put

$$A=(a_{ij})=\begin{pmatrix}
0 & 1 & 0 & \cdots & & & & & 0 & -1\\
-1 & 2 & -1 & & & & & & & \\
0 & -1 & 2 & -1 & & & & & & \\
\vdots & & \ddots & \ddots & \ddots & & & \mathbf{0} & & \\
\vdots & & & -1 & 2 & -1 & & & & \\
\vdots & & & & -1 & 0 & 1 & & & \\
\vdots & & \mathbf{0} & & & -1 & 2 & -1 & & \\
\vdots & & & & & & \ddots & \ddots & \ddots & \\
0 & & & & & & & -1 & 2 & -1\\
-1 & & & & & & & & -1 & 2
\end{pmatrix},$$

then we see that the following bracket relations hold.

$$[h_i, h_j]=0,\qquad [e_i, f_j]=\delta_{ij}h_i,$$
$$[h_i, e_j]=a_{ij}e_j,\qquad [h_i, f_j]=-a_{ij}f_j,$$
$$(\mathrm{ad}\,e_i)^{1+|a_{ij}|}e_j=(\mathrm{ad}\,f_i)^{1+|a_{ij}|}f_j=0.$$

Hence the Lie superalgebra $\mathfrak{g}$ is an example of the *contragredient* Lie superalgebra (cf. [10]). The matrix A is called the Cartan matrix. The corresponding Dynkin diagram is



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Here ○ denotes the even simple root and ⊗ denotes the odd simple root of length zero.

The Lie superalgebra $\mathfrak{g}$ is realized in $\mathfrak{gl}(\infty\mid\infty)^{\sim}$ as follows: Elements of $\mathfrak{gl}(\infty\mid\infty)^{\sim}$ are written as $\sum a_{ij}^{(\alpha\beta)}Z_{ij}^{(\alpha\beta)}+c$. Consider the following conditions for the coefficients $a_{ij}^{(\alpha\beta)}$;

1) $a_{i+m,j+n}^{(\alpha\beta)}=a_{ij}^{(\alpha\beta)}$,
2) $\sum_{i=0}^{m-1} a_{i,i+km}^{(00)}-\sum_{i=0}^{n-1} a_{i,i+kn}^{(11)}=0$ for any $k\in \boldsymbol{Z}$.

Proposition 5.5. *The linear combinations with coefficients satisfying* 1) *and* 2) *construct a realization of the Lie superalgebra* $\mathfrak{sl}(m\mid n)\otimes \boldsymbol{C}[\lambda,\lambda^{-1}]\oplus \boldsymbol{C}z$.

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