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Exactly Solvable SOS Models II:

Proof of the star-triangle relation and combinatorial identities

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Dedicated to Professor Nagayoshi Iwahori on his 60th birthday

§ 1 Introduction

This is a continuation of the paper [1], hereafter referred to as Part I. As announced therein, we give detailed proofs of (i) the star-triangle relation (STR) for the restricted solid-on-solid (SOS) models [2], and (ii) the combinatorial identities used in the evaluation of the local height probabilities (LHPs) [1, 3]. We try to keep this paper self-contained so that the mathematical content is comprehensible without reading Part I. Below we shall outline the setting and the content of each section.

1.1. The fusion models

The STR is the following system of equations for functions $W(a,b,c,d|u)$ ($a,b,c,d \in \boldsymbol{Z}$, $u \in \boldsymbol{C}$), to be called Boltzmann weights:

$$\sum_g W(a,b,g,f|u)\ W(f,g,d,e|u+v)\ W(g,b,c,d|v)$$
$$=\sum_g W(f,a,g,e|v)\ W(a,b,c,g|u+v)\ W(g,c,d,e|u).$$

Section 2 deals with the construction of solutions to the STR by the fusion procedure. Using the Boltzmann weights of the eight vertex SOS (8VSOS) model [4] as an elementary block, we construct "composed blocks" satisfying the STR. As in the 8VSOS case the resulting weights depend on the elliptic "nome" p as well as the parameter u.

This construction given in section 2.1 is known as the block spin transformation in the renormalization group theory. Namely, we sum up the freedoms $\ell_i \in \boldsymbol{Z}$ associated with sites i of a given lattice $\mathscr{L}_1$, leaving free the ones in $\mathscr{L}_N = N\mathscr{L}_1$. (See Fig. 1.1.) The ℓ_i is called a height in the sequel. In general, the locality of the Hamiltonian is not preserved by this

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transformation. However, by starting from a specially chosen inhomogeneous 8VSOS Hamiltonian we can retain the locality. (See Fig. 1.2.) Here it is crucial that the 8VSOS weights satisfy the STR.

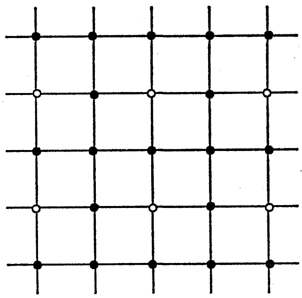


Fig. 1.1

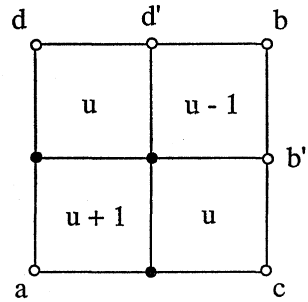


Fig. 1.2

Fig. 1.1. The block spin transformation with $N=2$. The solid circles are summed up.

Fig. 1.2. The locality of the renormalized weight. Upon summing over solid circles, the weight is independent of b' and d'.

In the 8VSOS model the adjacent heights ℓ_i, ℓ_j are restricted to $|\ell_i - \ell_j| = 1$, while the fusion procedure gives rise to the constraint

$$\ell_i - \ell_j = -N, \ -N+2, \ \cdots, \ N. \tag{1.1.1}$$

A pair of integers (ℓ_i, ℓ_j) satisfying (1.1.1) is called weakly admissible.

At this stage the models are not quite realistic (any more than the 8VSOS model is), for the ℓ_i ranges over all integers. Following Andrews, Baxter and Forrester (ABF) [5], we next pick up (section 2.2) for each positive integer L a finite subset of the weights such that the STR is satisfied among themselves and ℓ_i is restricted to

$$\ell_i = 1, \ \cdots, \ L-1.$$

In fact, this restriction process fetches us another constraint for adjacent heights

$$\ell_i + \ell_j = N+2, \ \cdots, \ 2L-N-2. \tag{1.1.2}$$

A weakly admissible pair (ℓ_i, ℓ_j) is called admissible if it satisfies (1.1.2). Although the block spin transformation alone produces models equivalent to the original one, the restricted models labelled by (L, N) are all inequivalent because of (1.1.2).

Section 2.3 is a side remark on the vertex-SOS correspondence first established by Baxter [4] for $N=1$. It is straightforward to generalize it to the correspondence between our restricted SOS models and the fusion vertex models [6]. Actually, it was this path that led us to the construction of section 2.1.

1.2. The corner transfer matrix (CTM) method

The CTM is a powerful tool in evaluating the local height probability $P(a)$

$$P(a)=\frac{1}{Z}\sum_{\text{the central height}=a}\prod W(\ell_1,\ell_2,\ell_3,\ell_4), \tag{1.2.1a}$$

$$Z=\sum\prod W(\ell_1,\ell_2,\ell_3,\ell_4). \tag{1.2.1b}$$

Here the product is taken over all faces, ℓ_1, ℓ_2, ℓ_3, ℓ_4 are four surrounding heights of a face, and the sum extends over all possible two dimensional configurations of heights in (1.2.1b) (with the restriction that the central height is fixed to a in (1.2.1a)). The method has recourse to the existence of limiting values of the weights such that the heights along diagonals lying northeast and southwest are frozen to be equal (see Fig. 1.3). In fact, such a limit is realized at $p=\pm 1$. To put it in a different way, the model shrinks to a sort of one dimensional system in this limit. The eigenvalues of the CTM therein are cast into one dimensional configuration sums. On the other hand, the STR implies a simple dependence of these eigenvalues with respect to the nome p. Therefore the evaluation of the LHPs for general p is reduced to that of the 1D configuration sums.

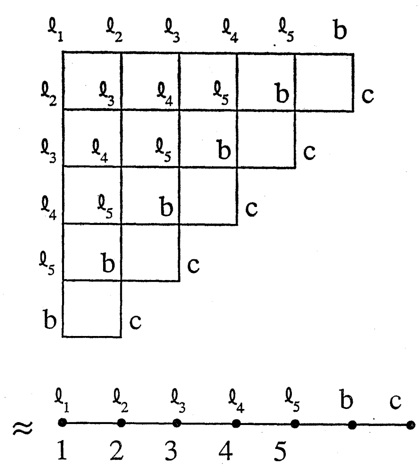


Fig. 1. 3 The equivalence to a one dimensional system in the frozen limit. The integer under ℓ_i signifies its multiplicities in the CTM.

Section 3 is devoted to the derivation of the 1D configuration sums. Since the matrix elements of the CTM are products of face weights, the task is to compute the weight $W(a, b, c, d|u)$ in the above mentioned limit. The result is written in terms of a parameter w (related to the parameter u) which survives the limit as

$$\lim W(a, b, c, d|u) \sim \delta_{ac} w^{\varepsilon H(d,a,b)}.$$

Here $\sim$ means that some scaling factor has been dropped. The weight function $H(a, b, c)$ reads as

(1.2.2) Regimes II, III ($\varepsilon = -$): $H(a, b, c) = \dfrac{|a-c|}{4}$,

Regimes I, IV ($\varepsilon = +$):

(1.2.3a) $$H(a, b, c) = \min\left(n-b, \frac{\min(a, c)-b+N}{2}\right) \quad \text{if } n \geq b,$$

(1.2.3b) $$= \min\left(b-n-1, \frac{b-\max(a, c)+N}{2}\right) \quad \text{if } n+1 \leq b.$$

Here $n=[L/2]$. (If L is even the argument in regimes I, IV is a little bit more complicated than this. See section 3.3 for details.) With the definition of $H(a, b, c)$ above, the 1D configuration sums assume the form

(1.2.4a) $$\sum_{\ell_1=a, \ell_{m+1}=b, \ell_{m+2}=c} q^{\phi_m(\ell_1, \cdots, \ell_{m+2})},$$

(1.2.4b) $$\phi_m(\ell_1, \cdots, \ell_{m+2}) = \sum_{j=1}^{m} jH(\ell_j, \ell_{j+1}, \ell_{j+2}).$$

The sum in (1.2.4a) extends over $\ell = (\ell_j)_{j=1,\cdots,m+2}$ such that $\ell_1 = a$, $\ell_{m+1} = b$, $\ell_{m+2} = c$ and that the pair (ℓ_j, ℓ_{j+1}) is admissible $(j=1, \cdots, m+1)$. It is intriguing to note that we thus encounter a one dimensional Brownian motion in a discrete time m with restrictions (1.1.1–2) and the weight (1.2.4b). Up to a power of $\sqrt{q}$, (1.2.4a) defines two kinds of polynomials in q depending on the form of $H(a, b, c)$ (1.2.2–3). We denote them by $X_m(a, b, c)$ or $Y_m(a, b, c)$ accordingly. The $X_m(a, b, c)$ is used in regimes II, III, while so is the $Y_m(a, b, c)$ in regimes I, IV. The b, c represent the boundary heights, to which the precise definition of the LHPs refers (see Part I).

1.3. Linear difference equations

In section 4, we rewrite the q-polynomial $X_m(a, b, c)$ or $Y_m(a, b, c)$ into series involving the Gaussian polynomials. This is done by a systematic use of linear difference equations.

Let us first consider the case where $H(a, b, c)$ is given by (1.2.2). The first step deals with the solution $f_m^{(N)}(b, c)$ of the linear difference equation

$$f_m^{(N)}(b, c)=\sum_d{}' f_{m-1}^{(N)}(d, b)q^{m|d-c|/4}. \tag{1.3.1}$$

Here the sum $\sum_d'$ extends over d such that (d, b) is weakly admissible. We fix the initial condition

$$f_0^{(N)}(b, c)=\delta_{b0}.$$

The $f_m^{(N)}(b, c)$ is called the fundamental solution. In the asymptotic region $1\ll a, b, c\ll L-1$ $X_m(a, b, c)$ coincides with $f_m^{(N)}(b-a, c-a)$. An explicit formula of $f_m^{(N)}(b, c)$ is given in Theorem 4.1.1. Using this we show that $f_m^{(N)}(b, c)$ also satisfies an equation at equal m.

$$\sum_{d=b-N, b-N+2, \cdots, N-b} q^{(a+md)/4} f_{m-1}^{(N)}(a+d, a-b)=(a\to -a). \tag{1.3.2}$$

This additional identity characteristic to the fundamental solution plays a key role as explained below. The second step is to represent $X_m(a, b, c)$ as a linear superposition of $f_m^{(N)}(b, c)$ (Theorem 4.4.1). The $X_m(a, b, c)$ is characterized by the linear difference equation

$$\begin{aligned} &X_m(a, b, c)=\sum_d{}'' X_{m-1}(a, d, b)q^{m|d-c|/4}. \\ &X_0(a, b, c)=\delta_{ab}. \end{aligned} \tag{1.3.3}$$

Here the sum $\sum_d''$ extends over d such that (d, b) is admissible. The equation (1.3.3) can be viewed as (1.3.1) supplemented by the "boundary condition"

$$(\sum_d{}'-\sum_d{}'')X_{m-1}(a, d, b)q^{m|d-c|/4}=0.$$

Note that the d appearing in $\sum_d'-\sum_d''$ are "close" to the boundaries $d=1$ or $d=L-1$ in the sense that $d+b\leq N$ or $d+b\geq 2L-N$. The extra equation (1.3.2) is responsible for this boundary condition.

The case when $H(a, b, c)$ is given by (1.2.3) follows a similar line (section 4.3). We denote the corresponding fundamental solution by $g_m^{(N)}(b, c)$. The expression for the $f_m^{(N)}(b, c)$ is piecewise analytic in (b, c), which reflects the piecewise analyticity of the function $|a-c|/4$. If $n\gg b$ or $n\ll b$, the behavior of $H(a, b, c)$ is essentially the same as (1.2.2). In fact,

$$H(a, b, c)=-\frac{|a-c|}{4}+\frac{a+c-2b+2N}{4}$$

for $n \gg b$, and $g_m^{(N)}(b, c)$ is expressed simply in terms of $f_m^{(N)}(b, c)$ with q replaced by q^{-1}. So is the case $n \ll b$. The expression of $g_m^{(N)}(b, c)$ is obtained throughout the intermediate regions of (b, c) as well by patchwork (see Fig. 4.1).

Section 4.2 is devoted to deriving several different expressions for $f_m^{(N)}(b, c)$. This is necessary in section 5 when we consider the $m \to \infty$ limit of $X_m(a, b, c)$ with q replaced by q^{-1}. (This is the 1D configuration sum relevant to Regime II.) The various identities proved in this section have emerged from computer experiments by using MACSYMA and FORTRAN.

1.4. The 1D configuration sums as modular functions

With the preliminaries in section 4, we proceed to the proofs of the main result of Part I; namely that the 1D configuration sums in the $m \to \infty$ limit give rise to modular functions. Except for the rather simple case of regime I, the modular functions are grasped as the branching coefficients appearing in theta function identities. (See also Appendix C.) Since they are fully exposed in Part I (Appendices A–B), we merely recall the basic definitions necessary in the proofs.

The result in regime III (resp. IV) is given in Theorem 5.1.1 (resp. Theorem 5.1.2). The branching coefficients $c_{j_1 j_2 j_3}^{(\varepsilon)}(\tau)$ ($\varepsilon = \pm$) used therein are defined through the theta function identity

$$\Theta_{j_1, m_1}^{(-,\varepsilon)}(z, q)\Theta_{j_2, m_2}^{(-,+)}(z, q)/\Theta_{1,2}^{(-,+)}(z, q) = \sum_{j_3} c_{j_1 j_2 j_3}^{(\varepsilon)}(\tau)\Theta_{j_3 m_3}^{(-,\varepsilon)}(z, q).$$

Here the sum extends over $j_3 \in \mathbf{Z}$ such that $0 < j_3 < m_3$ (resp. $0 < j_3 \leq m_3$) if $\varepsilon = +$ (resp. $\varepsilon = -$). The theta function $\Theta_{j,m}^{(-,\varepsilon)}(z, q)$ is defined by

$$\Theta_{j,m}^{(-,\varepsilon)}(z, q) = \sum_{n \in \mathbf{Z}, \gamma = n + j/2m} \varepsilon^n q^{m\gamma^2}(z^{-m\gamma} - z^{m\gamma}).$$

By residue calculus we obtain an expression of $c_{j_1 j_2 j_3}^{(\pm)}(\tau)$ as a threefold sum (see (5.1.1)). It is rather straightforward to identify $X_m(a, b, c)$ (or $Y_m(a, b, c)$) with this form of $c_{j_1 j_2 j_3}^{(\pm)}(\tau)$.

The difficulty arises when we consider the $m \to \infty$ limit of $X_m(a, b, c; q^{-1})$. For simplicity sake we explain it in the case $a = \langle b - mN \rangle$ (see eq. (B.1) for the definition of $\langle\ \rangle$) and $c = b + N$. The dominant contribution to this quantity comes from the configuration $(\bar{\ell}_j)_{j=1,\cdots,m+2}$, where $\bar{\ell}_j = \langle b + (j - m - 1)N \rangle$. Setting $k = -[(b - mN - 1)/(L-2)]$ we have

$$\phi_m(\bar{\ell}_1, \cdots, \bar{\ell}_{m+2}) = \frac{m(m+1)N + k(k-1)(L-2) - 2k(1 - b + mN)}{4}.$$

The trouble is that the term proportional to m^2 is fractional (note the implicit dependence on m through k). This fact suggests the occurrence of subtle cancellation in the expression of $X_m\ (a, b, c)$ as a series of the Gaussian polynomials. To overcome this difficulty is the highlight of our combinatorial analysis.

Our goal is Theorem 5.2.1, by which the $m\to\infty$ limit of $X_m(a, \bar{\ell}_{m+1}, \bar{\ell}_{m+2}; q^{-1})$ is identified with the branching coefficient $e^{\ell}_{jk}(\tau)$. The $e^{\ell}_{jk}(\tau)$ are characterized by theta function identities of ℓ variables (see (B.5) of Part I). We refer to Appendix B of Part I as for the Lie algebra theoretic interpretation of $e^{\ell}_{jk}(\tau)$. (See also [3] as for $c^{(+)}_{j_1j_2j_3}(\tau)$.) In [7] it was found that the matrix inverse to $(e^{\ell}_{jk}(\tau))$ is given simply by theta series $(\tilde{p}_{kj}(\tau)/\eta(\tau))$ (see (5.2.4)). Upon adjusting the precise power of q, our task is reduced to showing

$$\lim_{\substack{m\to\infty \\ nN\equiv\rho \bmod 2(L-2)}} \sum_{0<a<L} q^{M(m,a,b)} X_m(a, b, b+N; q^{-1}) \tilde{p}^{L-2}_{a-1,b'-1}(\tau)$$
$$\begin{aligned} &= \eta(\tau) && \text{if } \langle b'\rangle = \langle b-\rho\rangle, \\ &= 0 && \text{otherwise,} \end{aligned}$$

where

$$M(m, a, b) = \frac{m(m+1)N}{4} - \frac{1}{4(L-2)}\left(mN + \frac{L}{2} - b\right)^2 + \frac{1}{4L}\left(\frac{L}{2} - a\right)^2 + \frac{1}{24}.$$

This is proved in section 5.2 exploiting the expressions of $f^{(N)}_m(b, c)$ given in section 4.2.

The text is followed by four Appendices A-D. Appendix A determines the lowest order of the branching coefficients $c^{(\pm)}_{j_1j_2j_3}(\tau)$. Appendix B is the proof of the fact that $(\bar{\ell}_j) = (\langle b+jN\rangle)$ is a ground state configuration. Appendix C expresses $c^{(\pm)}_{j_1j_2j_3}(\tau)$ with $m_2 = 3, 4$ in terms of theta zero values. Appendix D gives the free energy per site for the fusion models.

1.5. Notations

In sections 4,5, we retain the following notations in Part I.

(1.5.1)
$$\begin{aligned} \begin{bmatrix} m \\ j \end{bmatrix} &= (q)_m/(q)_{m-j}(q)_j && \text{if } 0\le j\le m, \\ &= 0 && \text{otherwise,} \end{aligned}$$

where

(1.5.2) $$(z)_m=(1-z)(1-zq)\cdots(1-zq^{m-1}).$$

(1.5.3) $$E(z, q)=\prod_{k=1}^{\infty}(1-zq^{k-1})(1-z^{-1}q^k)(1-q^k).$$

(1.5.4) $$\theta_1(u, p)=2|p|^{1/8}\sin u\prod_{k=1}^{\infty}(1-2p^k\cos 2u+p^{2k})(1-p^k).$$

(1.5.5) $$\varphi(q)=\prod_{k=1}^{\infty}(1-q^k).$$

(1.5.6) $$\eta(\tau)=q^{1/24}\varphi(q), \qquad q=e^{2\pi i\tau}.$$

Throughout this paper we shall fix this relation between q and τ.

(1.5.7) $$\begin{aligned}\varepsilon_j^{\ell}&=1/2 &&\text{if } j\equiv 0 \bmod \ell,\\ &=1 &&\text{otherwise.}\end{aligned}$$

In sections 2, 3, we use the following notations of [3]. Let $H(u)$ and $\Theta(u)$ denote the Jacobian elliptic theta functions with the half periods K and iK' (see [8]).

(1.5.8) $$[u]=\theta_1(\pi\lambda u/2K, p)=\zeta H(\lambda u)\Theta(\lambda u),$$

where $p=e^{-\pi K'/K}$, $\zeta=p^{-1/8}\varphi(p)/\varphi(p^2)^2$ and λ is a free parameter. (We have changed the definition of $[u]$ in [3] by the factor ζ.)

(1.5.9) $$[u]_m=[u][u-1]\cdots[u-m+1],$$

(1.5.10) $$\begin{bmatrix} u \\ m \end{bmatrix}=\frac{[u]_m}{[m]_m}.$$

The symbol (1.5.10) is used only in sections 2, 3 and in Appendix D. It is not to be confused with the Gaussian polynomial (1.5.1).

§ 2. The Fusion Models

2.1. Fusion of SOS models

Let $\mathscr{L}$ be a two dimensional square lattice. An SOS model on $\mathscr{L}$ consists of (i) an integer variable ℓ_i on each site (=lattice point) i of $\mathscr{L}$, and (ii) a function $W(a, b, c, d)$ of a quadruple of integers (a, b, c, d). We call ℓ_i a height variable and $W(a, b, c, d)$ a Boltzmann weight (or simply a weight). The integers a, b, c, d represent a configuration of heights round a face (=an elementary square), ordered *anticlockwise* from the southwest corner.

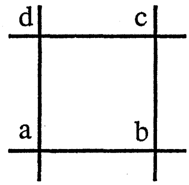


Fig. 2.1 A height configuration round a face.

Take three systems of Boltzmann weights W, W' and W'', all depending on a complex variable u. The STR for W, W' and W'' is the following set of functional equations.

$$\begin{aligned}&\sum_g W(a,b,g,f\,|\,u)W'(f,g,d,e\,|\,u+v)W''(g,b,c,d\,|\,v)\\ &=\sum_g W''(f,a,g,e\,|\,v)W'(a,b,c,g\,|\,u+v)W(g,c,d,e\,|\,u)\\ &\qquad\text{for any } a,b,c,d,e,f.\end{aligned} \tag{2.1.1}$$

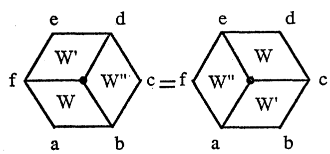


Fig. 2.2 The STR for the system of weights W, W', W''.

As for the significance of the STR in the theory of solvable lattice models, see [8]. Our aim here is to construct a class of solutions to the STR on the basis of a known one by the *fusion procedure.*

As a seed solution, we take the 8-vertex SOS model of Baxter [4]. By definition, its Boltzmann weight $W_{11}(a,b,c,d\,|\,u)$ is set to 0 unless

$$|a-b|=|b-c|=|c-d|=|d-a|=1. \tag{2.1.2}$$

The nonzero weights are parametrized in terms of the elliptic theta function (1.5.4)

$$[u]=\theta_1(\pi\lambda u/2K, p) \tag{2.1.3}$$

as follows.

$$W_{11}(\ell\pm1,\ \ell\pm2,\ \ell\pm1,\ \ell\,|\,u)=\frac{[u+1]}{[1]}, \tag{2.1.4a}$$

$$W_{11}(\ell\mp1,\ \ell,\ \ell\pm1,\ \ell\,|\,u)=\frac{[\xi+\ell\pm1][u]}{[\xi+\ell][1]}, \tag{2.1.4b}$$

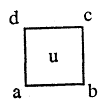
(2.1.4c) $$W_{11}(\ell \pm 1, \ell, \ell \pm 1, \ell \mid u) = \frac{[\xi + \ell \mp u]}{[\xi + \ell]},$$

Here λ and ξ are free parameters.

Remark. For convenience we have modified the weights $W(a, b \mid d, c)$ of [5], eq. (1.2.12b) (with $\rho' = 1/[1]$) as $\sqrt{[\xi + a]/[\xi + c]}\, W_{11}(a, b, c, d \mid u) = \sqrt{-1}^{\,c-a} W(a, b \mid d, c)$. The l.h.s. will be the symmetrized weight $S_{11}(a, b, c, d \mid u)$ defined below (2.1.24). Our variables are related to those in [5] through

(2.1.5) $$u = (v - \eta)/2\eta, \quad \lambda = 2\eta, \quad \xi = w_0/2\eta.$$

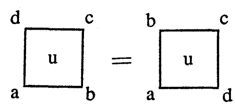
We shall often write the weights (2.1.4) graphically as



$= W_{11}(a, b, c, d \mid u).$

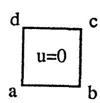
Besides the STR (2.1.1) (with $W = W' = W'' = W_{11}$), they satisfy the symmetry

(2.1.6)



For $u = 0$ and $u = -1$ they simplify to:

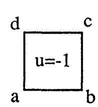
(2.1.7)



$= \delta_{ac},$

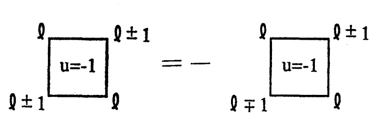
where (2.1.2) is implied,

(2.1.8a)



$= 0 \qquad \text{if } |b - d| = 2,$

(2.1.8b)



$\left(= \frac{[\xi + \ell \pm 1]}{[\xi + \ell]} \right).$

These properties will play a role in the following construction.

An elementary step of the fusion procedure is provided by

Lemma 2.1.1. (i) *Put*

(2.1.9) $$W'_{21}(a, b, c, d\,|\,u)=\sum_{a'} W_{11}(a, a', c', d\,|\,u+1)W_{11}(a', b, c, c'\,|\,u).$$

Then the r.h.s. is independent of the choice of c' *provided that* $|c-c'|=|c'-d|=1$.

(ii) *For all* a, b, c, d *we have* $W'_{21}(a, b, c, d\,|\,-1)=0$.

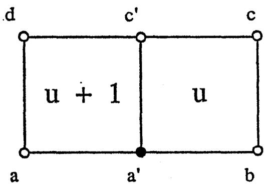
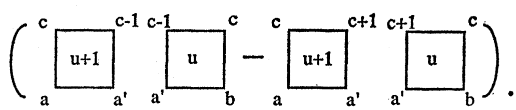


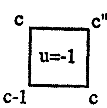
Fig. 2. 3 An elementary fusion step. The sum is taken over a' (solid circle), keeping the rest (open circles) fixed. It is independent of c'.

Proof. To see (i), it suffices to check the case $c=d$ (otherwise the possible choice of c' is unique). We are to show

(2.1.10) $0=\sum_{a'}$



Let $c''=c+1$ (or $c-1$), and multiply each summand of (2.1.10) by



$\neq 0$. Thanks to (2.1.8b) the result can be put into the form

(2.1.11)
$$\sum_{c'=c\pm 1} W_{11}(a, a', c', c\,|\,u+1)W_{11}(a', b, c, c'\,|\,u)W_{11}(c', c, c'', c\,|\,-1)$$
$$=\sum_{c'=c\pm 1} W_{11}(a', b, c', a\,|\,-1)W_{11}(a, c', c'', c\,|\,u)W_{11}(c', b, c, c''\,|\,u+1).$$

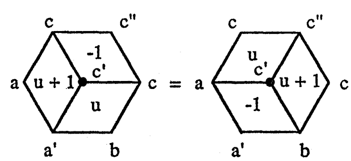


Fig. 2. 4 The STR used in the proof of the c'-independence.

To get the second line we have used the STR (2.1.1) with $u\to -1$ and $v\to u+1$. If $|a-b|=2$, then $W_{11}(a', b, c', a\,|\,-1)=0$ by (2.1.8a). If $a=b$, then take the sum over a' in (2.1.11). Owing to (2.1.8b), the summands contribute with opposite signs for each fixed c'. This proves (2.1.10).

To show (ii), choose $c'=b\pm 2$ in (2.1.9) and use (2.1.8a). This choice is not allowed if $c=b\mp 1$ and $d=b\pm 1$, in which case apply (2.1.7) or (2.1.8b). □

Now let M and N be positive integers. Define

$$W'_{MN}(a, b, c, d\,|\,u)$$

$$=\sum \prod_{\substack{0\le i\le M-1\\ 0\le j\le N-1}} W_{11}(\alpha_{ij}, \alpha_{i+1j}, \alpha_{i+1j+1}, \alpha_{ij+1}\,|\,u-i-j+M-1), \tag{2.1.12}$$

$$\alpha_{00}=a, \quad \alpha_{M0}=b, \quad \alpha_{MN}=c, \quad \alpha_{0N}=d,$$

where the sum is taken over all allowed configurations $\{\alpha_{ij}\}$ (*i.e.* the neighboring pairs must differ by 1), keeping fixed the corner heights a, b, c, d and the right/top boundary heights

$$\alpha_{M1}, \cdots, \alpha_{MN-1}, \quad \alpha_{1N}, \cdots, \alpha_{M-1N}. \tag{2.1.13}$$

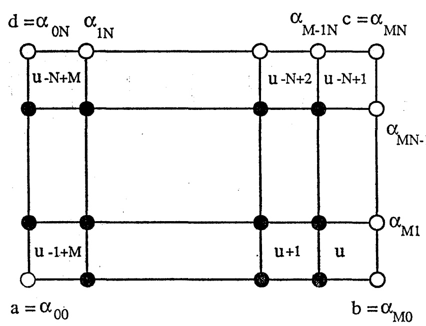


Fig. 2.5 Fused weight. The sum is taken over solid circles.

Repeated use of Lemma 2.1.1 shows that the result is independent of the choice of (2.1.13) on the condition that $|\alpha_{iN}-\alpha_{i+1N}|=1$ $(0\le i\le M-1)$ and $|\alpha_{Mj}-\alpha_{Mj+1}|=1$ $(0\le j\le N-1)$. Moreover, because of Lemma 2.1.1 (ii) the weights W'_{MN} have zeros independent of a, b, c, d. Factoring them out we define the (M, N)-weight by (see (1.5.9))

$$W_{MN}(a, b, c, d\,|\,u)$$

$$= W'_{MN}(a, b, c, d\,|\,u)[1]^{MN}/([N]_N \prod_{j=1}^{N} [u+M-j]_{M-1}), \quad (N\le M), \tag{2.1.14}$$

$$= W'_{MN}(a, b, c, d\,|\,u)[1]^{MN}/([M]_M \prod_{j=1}^{M} [u+M-j]_{N-1}), \quad (M\le N).$$

By the construction it is obvious that $W_{MN}(a, b, c, d\,|\,u)$ vanishes unless

(2.1.15)
$$a-b,\quad c-d=-M,\ -M+2,\ \cdots,\ M,$$
$$a-d,\quad b-c=-N,\ -N+2,\ \cdots,\ N.$$

They also inherit the symmetry (2.1.6) for W_{11} in the form

(2.1.16)
$$W_{MN}(a, b, c, d\,|\,u)=W_{NM}(a, d, c, b\,|\,u+M-N).$$

Theorem 2.1.2 (Theorem 1 of [2]). *For a triple of positive integers M, N, P, we have the following STR*

(2.1.17)
$$\sum_g W_{MN}(a, b, g, f\,|\,u)W_{MP}(f, g, d, e\,|\,u+v)W_{NP}(g, b, c, d\,|\,v)$$
$$=\sum_g W_{NP}(f, a, g, e\,|\,v)W_{MP}(a, b, c, g\,|\,u+v)W_{MN}(g, c, d, e\,|\,u).$$

Proof. It is sufficient to prove the STR for the W'_{MN}, since the normalization factors in (2.1.14) cancel out. We will show the STR for the case $M=2$, $N=P=1$. The general cases are proved similarly. From the definition of W'_{21} (2.1.9), the l.h.s. of (2.1.17) becomes

(2.1.18)
$$\sum_g \sum_{a',f'} (W_{11}(a, a', f'', f\,|\,u+1)W_{11}(a', b, g, f''\,|\,u))$$
$$\times(W_{11}(f, f', e', e\,|\,u+v+1)W_{11}(f', g, d, e'\,|\,u+v))$$
$$\times W_{11}(g, b, c, d\,|\,v).$$

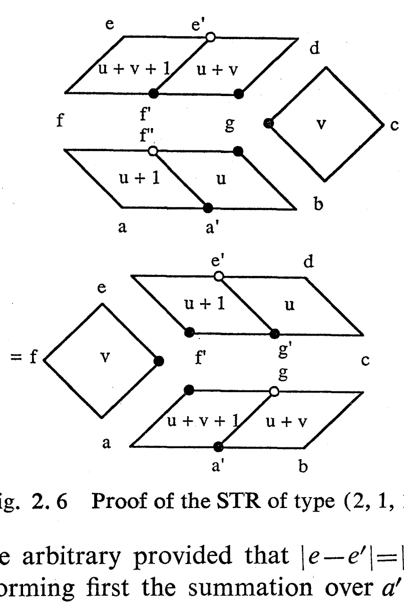


Fig. 2. 6 Proof of the STR of type (2, 1, 1).

Here f'' and e' are arbitrary provided that $|e-e'|=|e'-d|=|f-f''|=|f''-g|=1$. Performing first the summation over a' and using Lemma

2.1.1 we can set $f''=f'$ in (2.1.18). Then by applying twice the STR for the weight W_{11}, (2.1.18) is transformed into

$$\sum_{f'}\sum_{a',g} W_{11}(f,a,f',e\,|\,v)(W_{11}(a,a',g,f'\,|\,u+v+1)W_{11}(a',b,c,g\,|\,u+v))$$
$$\times(W_{11}(f',g,e',e\,|\,u+1)W_{11}(g,c,d,e'\,|\,u)).$$

Again by Lemma 2.1.1 and the definition of W'_{21} (2.1.9), this is the r.h.s. of (2.1.17). □

We shall refer to (2.1.17) as the STR of type (M, N, P).

Below we list explicit formulas of the W_{MN}. In the course of the derivation we will use the identity

(2.1.19)
$$[x+z][x-z][y+w][y-w]-[x+w][x-w][y+z][y-z]$$
$$=[x+y][x-y][z+w][z-w].$$

First consider the case $N=1$.

Lemma 2.1.3 (eq. (5) of [2]). *The $(M, 1)$-weight is given by*

(2.1.20a)
$$W_{M1}(\ell+1,\ell'+1,\ell',\ell\,|\,u)$$
$$=\left[\xi+\frac{\ell+\ell'-M}{2}\right]\left[u+\frac{\ell'-\ell+M}{2}\right]\Big/[1][\xi+\ell],$$

(2.1.20b)
$$W_{M1}(\ell+1,\ell'-1,\ell',\ell\,|\,u)$$
$$=\left[\xi-u+\frac{\ell+\ell'-M}{2}\right]\left[\frac{\ell'-\ell+M}{2}\right]\Big/[1][\xi+\ell],$$

(2.1.20c)
$$W_{M1}(\ell-1,\ell'+1,\ell',\ell\,|\,u)$$
$$=\left[\xi+u+\frac{\ell+\ell'+M}{2}\right]\left[\frac{\ell-\ell'+M}{2}\right]\Big/[1][\xi+\ell],$$

(2.1.20d)
$$W_{M1}(\ell-1,\ell'-1,\ell',\ell\,|\,u)$$
$$=\left[\xi+\frac{\ell+\ell'+M}{2}\right]\left[u+\frac{\ell-\ell'+M}{2}\right]\Big/[1][\xi+\ell].$$

Proof. To derive (2.1.20a), we choose the sequence $\{\alpha_{i1}\}$ (2.1.13) in the definition of W'_{M1} (2.1.12) as

$$\alpha_{i1}=\ell-i \qquad \text{if } 0\le i\le M-\frac{\ell'-\ell+M}{2},$$
$$=\ell'-M+i \qquad \text{if } M-\frac{\ell'-\ell+M}{2}\le i\le M.$$

For such a choice of $\{\alpha_{i1}\}$, the sequence $\{\alpha_{i0}\}$ in (2.1.13) is uniquely determined to be $\alpha_{i0}=\alpha_{i1}+1$. Then (2.1.20a) follows immediately. We show the formula (2.1.20b) by an induction on M. Assume that it is true for M. By the definition and the induction hypothesis we have

$$
\begin{aligned}
&W_{M+1,1}(\ell+1, \ell'-1, \ell', \ell \mid u)\\
&\quad=(W_{M1}(\ell+1, \ell'-2, \ell'-1, \ell \mid u+1)W_{11}(\ell'-2, \ell'-1, \ell', \ell'-1 \mid u)\\
&\qquad+W_{M1}(\ell+1, \ell', \ell'-1, \ell \mid u+1)W_{11}(\ell', \ell'-1, \ell', \ell'-1 \mid u))[1]/[u+1]\\
&\quad=\left(\left[\xi-u-1+\frac{\ell+\ell'-1-M}{2}\right]\left[\frac{\ell'-1-\ell+M}{2}\right]\Big/[1][\xi+\ell]\right.\\
&\qquad\times\frac{[\xi+\ell'][u]}{[1][\xi+\ell'-1]}+\left[\xi+\frac{\ell+\ell'-1-M}{2}\right]\left[u+1+\frac{\ell'-1-\ell+M}{2}\right]\\
&\qquad\left.\Big/[1][\xi+\ell]\times\frac{[\xi+\ell'-1-u]}{[\xi+\ell'-1]}\right)[1]/[u+1].
\end{aligned}
$$

Using the identity (2.1.19) with

$$
2x=\xi+\frac{\ell+\ell'-3-M}{2},\qquad 2y=\xi+\frac{3\ell'-\ell-1+M}{2},
$$

$$
2z=\xi-2u-1+\frac{\ell+\ell'-1-M}{2},\qquad 2w=\xi+\frac{\ell+\ell'+1-M}{2},
$$

we obtain (2.1.20b) for $M+1$. The equalities (2.1.20c) and (2.1.20d) are shown similarly. □

Now we turn to the general (M, N)-weights. The definition (2.1.12) can be viewed as defining the W'_{MN} in terms of the W'_{M1}:

$$
W'_{MN}(a, b, c, d \mid u)=\sum\prod_{i=1}^{N} W'_{M1}(a_{i-1}, b_{i-1}, b_i, a_i \mid u-i+1),\tag{2.1.21}
$$
$$
a_0=a,\quad a_N=d,\quad b_0=b,\quad b_N=c.
$$

Here the sum extends over $a_1, \cdots, a_{N-1}$ such that $|a_i-a_{i+1}|=1$ for $0\leq i\leq N-1$. Using (2.1.21) we show the

Lemma 2.1.4 (Appendix of [2]). *Assuming* $N\leq M$, *we have* (*see* (1.5.10))

$$
\begin{aligned}
&W_{MN}(\ell+2j-N, \ell+2i-M+N, \ell+2i-M, \ell \mid u)\\
&\quad=\begin{bmatrix}M-i\\N-j\end{bmatrix}\begin{bmatrix}\xi+\ell+i+j-M-1\\j\end{bmatrix}\begin{bmatrix}i+u\\j\end{bmatrix}\begin{bmatrix}\xi+\ell+i+u\\N-j\end{bmatrix}\\
&\qquad\Big/\begin{bmatrix}\xi+\ell+j\\N-j\end{bmatrix}\begin{bmatrix}\xi+\ell+2j-N-1\\j\end{bmatrix},
\end{aligned}\tag{2.1.22a}
$$

$$W_{MN}(\ell+2j-N,\ \ell+2i-M-N,\ \ell+2i-M,\ \ell\,|\,u)$$

$$(2.1.22b)\qquad =\begin{bmatrix} i \\ j \end{bmatrix}\begin{bmatrix} \xi+\ell+i+j-M-1-u \\ j \end{bmatrix}\begin{bmatrix} M-i+u \\ N-j \end{bmatrix}\begin{bmatrix} \xi+\ell+i \\ N-j \end{bmatrix}$$

$$\Big/\begin{bmatrix} \xi+\ell+j \\ N-j \end{bmatrix}\begin{bmatrix} \xi+\ell+2j-N-1 \\ j \end{bmatrix}.$$

Proof. These can be checked by an induction on N. The case $j=0$ or N is straightforward, since in (2.1.21) the choice of $\{a_j\}$ is unique. The induction proceeds in a similar way as in the derivation of (2.1.20b). □

Choosing the sequence in $\{\alpha_{Mj}\}$ (2.1.13) suitably, the general (M, N)-weight is expressed as a sum of products of weights of type (2.1.22). In fact, we get

$$W_{MN}(\ell+2j-N,\ \ell'+2i-N,\ \ell',\ \ell\,|\,u)\begin{bmatrix} N \\ i \end{bmatrix}$$

$$(2.1.23a)\qquad =\sum_{k=\max(0,i+j-\cdots}^{\min(i,j)} W_{Mi}(l+2k-i,\ \ell'+i,\ \ell',\ \ell\,|\,u-N+i)$$

$$\times W_{M,N-i}(\ell+2j-N,\ \ell'+2i-N,\ \ell'+i,\ \ell+2k-i\,|\,u),$$

$$(2.1.23b)\qquad =\sum_{k=\max(0,j-i)}^{\min(N-i,j)} W_{M,N-i}(\ell+2k-N+i,\ \ell'-N+i,\ \ell',\ \ell\,|\,u-i)$$

$$\times W_{Mi}(\ell+2j-N,\ l'+2i-N,\ \ell'-N+i,\ \ell+2k-N+i\,|\,u).$$

Let us modify the weight W_{MN} as

$$(2.1.24a)\qquad S_{MN}(a,b,c,d\,|\,u)=\left(\frac{(a,b)_M(d,a)_N}{(d,c)_M(c,b)_N}\right)^{1/2} W_{MN}(a,b,c,d\,|\,u),$$

$$(\ell,\ell')_M=(\ell',\ell)_M$$

$$(2.1.24b)\qquad =\begin{bmatrix} M \\ \dfrac{\ell-\ell'+M}{2} \end{bmatrix}^{-1} \frac{\left[\xi+\dfrac{\ell+\ell'-M}{2},\ \xi+\dfrac{\ell+\ell'+M}{2}\right]}{\sqrt{[\xi+\ell][\xi+\ell']}},$$

where we set

$$(2.1.24c)\qquad [A,B]=[A][A+1]\cdots[B],\qquad [A,A-1]=1.$$

(We have changed the definition of $(\ell,\ell')_M$ from that of ref. 2 by the common factor $[M]_M$). The STR remains valid for S_{MN}, because the factor in front of the r.h.s. of (2.1.24a) cancels out in the STR.

As a result of this modification S_{MN} aquires the following symmetry.

Theorem 2.1.5 (Theorem 2 of [2]).

(2.1.25a) $S_{MN}(a, b, c, d|u) = S_{NM}(a, d, c, b|M-N+u)$

(2.1.25b) $= S_{NM}(c, b, a, d|M-N+u)$

(2.1.25c) $= S_{MN}(c, d, a, b|u)$

(2.1.25d) $= (g_a g_c/g_b g_d) S_{MN}(b, a, d, c|-M+N-1-u)$,

where $g_\ell = \varepsilon_\ell \sqrt{[\xi+\ell]}$, $\varepsilon_\ell = \pm 1$ *and* $\varepsilon_\ell \varepsilon_{\ell+1} = (-)^\ell$.

For the proof we prepare

Lemma 2.1.6.

(2.1.26) $S_{MN}(a, b, c, d|u) = (g_a g_c/g_b g_d) S_{MN}(d, c, b, a|-M+N-1-u)$.

Proof. Direct calculations using explicit formulas (2.1.22) give

$$W_{MN}(a, b, c, d|u) = \frac{(d, c)_M}{(a, b)_M} \frac{g_a g_c}{g_b g_d} W_{MN}(d, c, b, a|-M+N-1-u) \tag{2.1.27}$$

for the case $|b-c| = N$. Next using (2.1.23a) and the definition (2.1.24) of S_{MN}, we have

$$\begin{aligned} &S_{MN}(\ell+2j-N, \ell'+2i-N, \ell', \ell|u) \begin{bmatrix} N \\ i \end{bmatrix} \\ &= \left(\frac{(\ell+2j-N, \ell'+2i-N)_M (\ell, \ell+2j-N)_N}{(\ell, \ell')_M (\ell', \ell'+2i-N)_N} \right)^{1/2} \\ &\quad \times \sum_{k=\max(0,i+j-N)}^{\min(i,j)} W_{Mi}(\ell+2k-i, \ell'+i, \ell', \ell|u-N+i) \\ &\quad\quad \times W_{M,N-i}(\ell+2j-N, \ell'+2i-N, \ell'+i, \ell+2k-i|u). \end{aligned}$$

By applying the equality (2.1.27) for the extreme case ($|b-c|=N$), the r.h.s. becomes

$$\begin{aligned} &\frac{g_{\ell'} g_{\ell+2j-N}}{g_\ell g_{\ell'+2i-N}} \times \left(\frac{(\ell, \ell')_M (\ell, \ell+2j-N)_N}{(\ell+2j-N, \ell'+2i-N)_M (\ell', \ell'+2i-N)_N} \right)^{1/2} \\ &\quad \times \sum_{k=\max(0,i+j-N)}^{\min(i,j)} W_{M,N-i}(\ell+2k-i, \ell'+i, \ell'+2i-N, \ell+2j-N| \\ &\qquad\qquad -M+N-i-1-u) \\ &\quad\quad \times W_{Mi}(\ell, \ell', \ell'+i, \ell+2k-i|-M+N-1-u). \end{aligned}$$

This is just $\begin{bmatrix} N \\ i \end{bmatrix}$ times the r.h.s. of (2.1.26). □

Proof of Theorem 2.1.5. We know already (2.1.25a) by (2.1.16). Using this and Lemma 2.1.6 alternately, we have

$$\begin{aligned} S_{MN}(a, b, c, d \mid u) &= S_{NM}(a, d, c, b \mid M-N+u) \\ &= \frac{g_a g_c}{g_b g_d} S_{NM}(b, c, d, a \mid -u-1) \\ &= \frac{g_a g_c}{g_o g_d} S_{MN}(b, a, d, c \mid -M+N-1-u) \\ &= S_{MN}(c, d, a, b \mid u), \end{aligned}$$

which proves (2.1.25c-d). Eq. (2.1.25b) follows from (2.1.25a, c). □

2.2. Restricted SOS models

Hereafter we consider exclusively the SOS models with $M=N$. We shall also specialize the parameters in (2.1.3–4) to

(2.2.1a) $$\lambda = 2K/L,$$

(2.2.1b) $$\xi = 0,$$

where L is a positive integer satisfying

(2.2.2) $$L \geq N+3.$$

The condition (2.2.1a) gives rise to the symmetry

(2.2.3) $$[L-u] = [u].$$

Let ℓ_i, ℓ_j be adjacent heights. In addition to the restriction (2.1.15) (with $M=N$)

(2.2.4) $$\ell_i - \ell_j = -N, -N+2, \cdots, N,$$

we impose further the constraint

(2.2.5) $$\ell_i + \ell_j = N+2, N+4, \cdots, 2L-N-2.$$

These two conditions imply in particular that each height variable can assume at most $L-1$ states

(2.2.6) $$\ell_i = 1, 2, \cdots, L-1.$$

We remark that if $N=1$, then conversely (2.2.5) follows from (2.2.4) and (2.2.6).

The condition (2.2.5) naturally enters for the following reason. As an illustration, take the following (2, 2)-weight

$$S_{22}(\ell, \ell, \ell, \ell \,|\, u) = \frac{[\ell-1-u][\ell+u]}{[\ell-1][\ell]} + \frac{[\ell-1][\ell+2]}{[\ell][\ell+1]} \frac{[u][1+u]}{[1][2]}.$$

Because of the specialization (2.2.1), this now has poles at $\ell=1$ or $L-1$. We forbid such configurations to occur by requiring (2.2.5).

In Part I, we called a pair of heights (a, b) *admissible* if it satisfies (2.2.4–5). By abuse of language, we call a weight $S_{NN}(a, b, c, d\,|\, u)$ admissible if the pairs (a, b), (b, c), (c, d) and (d, a) are all admissible.

In this paragraph we shall show that admissible weights are well defined (Theorem 2.2.1), and that they satisfy the STR among themselves (Theorem 2.2.4). (The latter statement does not follow directly from the STR (2.1.17) for the unspecialized weights, since non-admissible configurations can occur in the summand even if $a, b, \cdots, f$ are all admissible.) Following ABF [5], we call the resulting models *restricted SOS models*.

In order to prove their existence we make use of the explicit formulas in the previous paragraph for the symmetrized weights $S_{NN}(a, b, c, d\,|\, u)$. We shall frequently use the parametrization

$$(2.2.7) \qquad a=\ell+N-2r, \quad b=\ell+2(N-k), \quad c=\ell+N-2s, \quad d=\ell.$$

Thanks to the 180° rotational symmetry (2.1.24c), we can assume without loss of generality

$$(2.2.8a) \qquad 0 \le k \le N.$$

Then the weight $S_{NN}(\ell+N-2r, \ell+2(N-k), \ell+N-2s, \ell\,|\, u)$ is admissible if and only if

$$(2.2.8b) \qquad \max(0, \ell+2N-L-k+1) \le r, s \le \min(\ell-1, k).$$

In terms of (2.2.7) the formulas (2.1.22–24) read as follows.

$$S_{NN}(\ell+N-2r, \ell+2(N-k), \ell+N-2s, \ell\,|\, u)$$

$$(2.2.9a) \qquad = \sqrt{S} \sum_{i=\max(0, s-r)}^{\min(k-r, s)} U(i)$$

$$(2.2.9b) \qquad = \sqrt{S} \sum_{i=\max(k-N, s-r)}^{\min(k-r, s)} D(i),$$

where

$$(2.2.10) \quad S = \left(\frac{(\ell, \ell+N-2r)_N (\ell+N-2r, \ell+2(N-k))_N}{(\ell, \ell+N-2s)_N (\ell+N-2s, \ell+2(N-k))_N} \right) \Big/ \begin{bmatrix} N \\ k-s \end{bmatrix}^2$$

$$=\frac{1}{\begin{bmatrix}N\\k-r\end{bmatrix}\begin{bmatrix}N\\k-s\end{bmatrix}}$$

$$\times\frac{\begin{bmatrix}N\\s\end{bmatrix}\begin{bmatrix}\ell+N-r\\N+1\end{bmatrix}\begin{bmatrix}\ell+2N-k-r\\N+1\end{bmatrix}\begin{bmatrix}\ell+N-2s\\1\end{bmatrix}}{\begin{bmatrix}N\\r\end{bmatrix}\begin{bmatrix}\ell+N-s\\N+1\end{bmatrix}\begin{bmatrix}\ell+2N-k-s\\N+1\end{bmatrix}\begin{bmatrix}\ell+N-2r\\1\end{bmatrix}},$$

$$\begin{aligned}U(i)= & W_{N,N-k+s}(\ell+N-k-s+2i,\ \ell+2N-k-s,\\ & \qquad \ell+N-2s,\ \ell\,|\,u-k+s)\\ & \times W_{N,k-s}(\ell+N-2r,\ \ell+2(N-k),\ \ell+2N-k-s,\\ & \qquad \ell+N-k-s+2i\,|\,u)\end{aligned}\tag{2.2.11}$$

$$=\frac{\begin{bmatrix}s\\s-i\end{bmatrix}\begin{bmatrix}N-i\\k-r-i\end{bmatrix}\begin{bmatrix}\ell+N-k-s+i-1\\N-k+i\end{bmatrix}\begin{bmatrix}\ell+2N-k-s+i\\r-s+i\end{bmatrix}}{\begin{bmatrix}\ell+N-k+i\\s-i\end{bmatrix}\begin{bmatrix}\ell+N-2r-1\\k-r-i\end{bmatrix}\begin{bmatrix}\ell+N-k-s+2i-1\\N-k+i\end{bmatrix}}$$

$$\times\begin{bmatrix}\ell+N-r-s+i\\r-s+i\end{bmatrix}^{-1}$$

$$\times\begin{bmatrix}\ell+N-k+u\\s-i\end{bmatrix}\begin{bmatrix}\ell+N-r-s-1-u\\k-r-i\end{bmatrix}\begin{bmatrix}N-k+u\\N-k+i\end{bmatrix}\begin{bmatrix}i+u\\r-s+i\end{bmatrix},$$

$$\begin{aligned}D(i)= & W_{N,k-s}(\ell+k-2r+s-2i,\ \ell+N-k-s,\ \ell+N-2s,\ \ell\\ & \qquad |\,u-N+k-s)\\ & \times W_{N,N-k+s}(\ell+N-2r,\ \ell+2(N-k),\ \ell+N-k-s,\\ & \qquad \ell+k-2r+s-2i\,|\,u)\end{aligned}$$

$$=\frac{\begin{bmatrix}k-r+s-i\\s-i\end{bmatrix}\begin{bmatrix}N-s\\k-r-i\end{bmatrix}\begin{bmatrix}\ell+N-k-r-1\\N-k+i\end{bmatrix}\begin{bmatrix}\ell+N-s\\r-s+i\end{bmatrix}}{\begin{bmatrix}\ell+N-2r+s-i\\s-i\end{bmatrix}\begin{bmatrix}\ell+k-2r+s-2i-1\\k-r-i\end{bmatrix}\begin{bmatrix}\ell+N-2r-1\\N-k+i\end{bmatrix}}\tag{2.2.12}$$

$$\times\begin{bmatrix}\ell+k-r-i\\r-s+i\end{bmatrix}^{-1}$$

$$\times\begin{bmatrix}\ell+N-r-i+u\\s-i\end{bmatrix}\begin{bmatrix}\ell+N-r-i-1-u\\k-r-i\end{bmatrix}$$

$$\times\begin{bmatrix}N-k+r-s+i+u\\N-k+i\end{bmatrix}\begin{bmatrix}-N+k+u\\r-s+i\end{bmatrix}.$$

Theorem 2.2.1. *An admissible weight is well defined.*

Proof. Let A be a real number and B a non-negative integer. Call a term $\begin{bmatrix} A \\ B \end{bmatrix}$ of type I if $B \leq A < L$, of type II if $B < L$. We can easily check the following:

$$0 < \begin{bmatrix} A \\ B \end{bmatrix} < \infty \qquad \text{if } \begin{bmatrix} A \\ B \end{bmatrix} \text{ is of type I,}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} < \infty \qquad \text{if } \begin{bmatrix} A \\ B \end{bmatrix} \text{ is of type II.}$$

Now consider the expressions (2.2.9). Using (2.2.7–8) and $\max(k-N, s-r) \leq i \leq \min(k-r, s)$, we can verify that (i) all the factors appearing in (2.2.10) and in the denominator of (2.2.11–12) are of type I, and that (ii) those in the numerator of (2.2.11–12) are of type II. The theorem follows from these facts. □

In what follows a weight $S_{NN}(a, b, c, d \mid u)$ is always understood to be admissible.

Lemma 2.2.2.

(2.2.13) $$S_{NN}(L-a, L-b, L-c, L-d \mid u) = S_{NN}(a, b, c, d \mid u).$$

Proof. We consider first the specialization (2.2.1a), regarding ξ as yet arbitrary. From (2.1.4) and (2.2.3), we see that each unsymmetrized (1, 1)-weight $W_{11}(a, b, c, d \mid u)$ is invariant under the transformation

(2.2.14) $$\xi \longrightarrow -\xi, \quad a \longrightarrow L-a, \quad b \longrightarrow L-b, \quad c \longrightarrow L-c, \quad d \longrightarrow L-d.$$

Therefore the unsymmetrized (N, N)-weights have the same nature by the definition. Moreover we can see from (2.1.23b) that the symmetrizing factor $(a, b)_N (d, a)_N / (d, c)_N (c, b)_N$ is also invariant under (2.2.14). Letting ξ tend to 0 we get (2.2.13). □

Lemma 2.2.3. *Put*

(2.2.15) $$\bar{N} = N + \ell - k - 1, \quad \bar{\ell} = k + 1, \quad \bar{k} = \ell - 1.$$

If $N \geq \bar{N}$, *then we have*

(2.2.16) $$\begin{aligned} &S_{NN}(\ell + N - 2r, \ell + 2(N-k), \ell + N - 2s, \ell \mid u) \\ &= \frac{\begin{bmatrix} N-u \\ N-\bar{N} \end{bmatrix}}{\begin{bmatrix} N \\ \bar{N} \end{bmatrix}} S_{\bar{N}\bar{N}}(\bar{\ell} + \bar{N} - 2r, \bar{\ell} + 2(\bar{N} - \bar{k}), \bar{\ell} + \bar{N} - 2s, \bar{\ell} \mid u). \end{aligned}$$

Proof. First we show that $D(i+1)/D(i)$ is invariant under the change

$$\ell \longrightarrow \bar{\ell}, \quad N \longrightarrow \bar{N}, \quad k \longrightarrow \bar{k}. \tag{2.2.17}$$

A direct calculation by using (2.2.12) shows that

$$D(i+1)/D(i) = D_1(i)D_2(i)$$

where

$$D_1(i) = \frac{[\ell+N-2r+s-i][\ell+N-r-i][\ell+k-2r+s-2i-2]\cdot[\ell+k-r-i][s-i]}{[\ell+k-2r+s-2i][\ell+k-2r-i-1][N-k+r-s+i+1]\cdot[N-k+i+1][r-s+i+1]}$$
$$\times \frac{[N-k+r-s+i+1+u][-N+k-r+s-i+u]}{[\ell+N-r-i-1-u][\ell+N-r-i+u]},$$

$$D_2(i) = \frac{[\ell-r-i-1][k-r-i]}{[\ell-r+s-i-1][k-r+s-i]}.$$

Note that $\ell+N=\bar{\ell}+\bar{N}$, $\ell+k=\bar{\ell}+\bar{k}$, $N-k=\bar{N}-\bar{k}$. Therefore under (2.2.17) all factors in $D_1(i)$ are invariant, while in $D_2(i)$ the two factors get interchanged both in the numerator and in the denominator.

Because of the symmetry (2.1.25b), we may assume $r \leq s$. Since $D_2(\ell-r-1)=0$, it suffices to consider $D(i)$ with $i \leq l-r-1$. We thus have

$$s-r \leq i \leq \min(\ell-r-1, k-r, s).$$

The transformation (2.2.17) does not change both ends of this interval. Therefore we get

$$\frac{S_{NN}(\ell+N-2r, \ell+2(N-k), \ell+N-2s, \ell \mid u)}{S_{\bar{N}\bar{N}}(\bar{\ell}+\bar{N}-2r, \bar{\ell}+2(\bar{N}-\bar{k}), \bar{\ell}+\bar{N}-2s, \bar{\ell} \mid u)} = \left(\frac{S}{\bar{S}}\right)^{1/2} \frac{D(s-r)}{\bar{D}(s-r)} \tag{2.2.18}$$

where $\bar{S}$, $\bar{D}$ are obtained from S, D by applying (2.2.17). Using (2.2.10, 12) we find

$$\frac{S}{\bar{S}} = \left(\frac{[\ell-r, k-r][N-s+1, N-r]}{[\bar{N}+1, N][\bar{N}-s+1, \bar{N}-r]}\right)^2,$$

$$\frac{D(s-r)}{\bar{D}(s-r)} = \frac{[\bar{N}-s+1, \bar{N}-r][\ell+N-k-u\ N-u]}{[\ell-r, k-r][N-s+1, N-r]},$$

where $[A, B]$ is defined in (2.1.24c). So

$$\text{the r.h.s. of (2.2.18)} = \frac{[\bar{N}+1-u,\ N-u]}{[\bar{N}+1,\ N]} = \frac{\begin{bmatrix} N-u \\ N-\bar{N} \end{bmatrix}}{\begin{bmatrix} N \\ \bar{N} \end{bmatrix}}.$$

This completes the proof of (2.2.16). □

Now let us proceed to the proof of the STR. We prepare several lemmas. We call (a, b) lower (resp. upper) non-admissible if $a+b \leq N$ (resp. $a+b \geq 2L-N$). A weakly admissible pair (a, b) cannot be both lower and upper non-admissible.

Lemma 2.2.4. *Assume that the pairs (d, c) and (c, b) are admissible and that (d, a), (a, b) are weakly admissible but not both admissible. The symmetrized weight $S_{NN}(a, b, c, d|u)$ is then finite-valued. It is vanishing if one of the following occurs:*

(i) $a=0$,
(ii) *either (d, a) or (a, b) is admissible,*
(iii) *(d, a) is lower non-admissible and (a, b) is upper non-admissible,*
(iv) *(d, a) is upper non-admissible and (a, b) is lower non-admissible.*

Proof. By virtue of the symmetry (2.1.25a), we can assume without loss of generality that

$$(2.2.19) \qquad b \geq d.$$

We consider the following three cases for (d, a)

(1) $N+2 \leq d+a \leq 2L-N-2$,
(2) $d+a \leq N$,
(3) $d+a \geq 2L-N$,

and for (a, b)

(1′) $N+2 \leq a+b \leq 2L-N-2$,
(2′) $a+b \leq N$,
(3′) $a+b \geq 2L-N$.

The case (1)–(1′) is excluded by the assumption of the lemma. The cases (1)–(2′), (3)–(1′) and (3)–(2′) do not occur because, under the assumption (2.2.19), (2′) implies (2) and (3) implies (3′).

The symmetries (2.1.25a) and (2.2.13) allows us to reduce (3)–(3′) to (2)–(2′) and (1)–(3′) to (2)–(1′). So we have only three cases to check:

Case 1: (2)–(1′). *Case* 2: (2)–(2′). *Case* 3: (2)–(3′).

Let us use the parametrization (2.2.7). From the assumptions we have

(2.2.20a) $$0 \leq r \leq k \leq N,$$

(2.2.20b) $$\max(0, \ell+2N-L-k+1) \leq s \leq \min(\ell-1, k).$$

The condition (2) is nothing but

(2.2.21) $$\ell - r \leq 0,$$

while (1′), (2′), (3′) can be written respectively as follows:

(2.2.22a) $$1 \leq \ell+N-k-r \leq L-N-1,$$

(2.2.22b) $$\ell+N-k-r \leq 0,$$

(2.2.22c) $$\ell+N-k-r \geq L-N.$$

Use the formula (2.2.9a) for $S_{NN}(a, b, c, d|u)$, in which i is restricted to

(2.2.23) $$\max(0, s-r) \leq i \leq \min(k-r, s).$$

From (2.2.20–21) and (2.2.23) we can write

(2.2.24) $$\sqrt{S}\, U(i) = \frac{\sqrt{AB[a]}}{C} \times (\text{a non-zero finite-valued factor}),$$

where

$$A = [\ell-r, \ell+N-r], \qquad B = [\ell+N-k-r, \ell+2N-k-r],$$
$$C = [\ell+N-k-r+i, \ell+N-r-s+i].$$

For $A=[I, J]$ let A_{ini} and A_{fin} signify respectively I and J. We write $A \subset B$ if $A_{ini} \geq B_{ini}$ and $A_{fin} \leq B_{fin}$. We find from (2.2.20–21,23) that

(2.2.25a) $$C \subset B,$$

(2.2.25b) $$[0] \subset A,$$

(2.2.25c) $$C_{fin} < L.$$

In Case 1 and Case 3, $C_{ini} > 0$ follows respectively from (2.2.22a) and (2.2.22c). Therefore (2.2.24) is equal to 0 from (2.2.25b-c). In Case 2 the finiteness is a direct consequence of (2.2.24–25) because $-L < C_{ini}$.

Finally we note that $S_{NN}(0, b, c, d|u) = 0$ follows immediately from (2.2.24). □

Lemma 2.2.5. *Assume that the pairs* (a, d) *and* (d, c) *are admissible and that* (a, b) *and* (b, c) *are weakly admissible, then* $S_{NN}(a, b, c, d|u)$ *is finite-valued.*

Proof. From Theorem 2.1.5, we have

$$S_{NN}(a, b, c, d\,|\,u) = \frac{g_a g_c}{g_b g_d} S_{NN}(b, c, d, a\,|\,-1-u).$$

Here $S_{NN}(b, c, d, a\,|\,-1-u)$ is finite-valued from Lemma 2.2.4. We have $g_d \neq 0$ because of the admissibility of the pair (a, d). As for g_b the factor $\sqrt{[b]}$ in the numerator of (2.2.24) cancels it out. This proves the lemma. □

Lemma 2.2.6. *Consider Case* 2 *in the proof of Lemma* 2.2.4. *We assume* (2.2.19) *and that* $[a] \neq 0$. *Setting*

(2.2.26) $$\bar{r} = \ell + N - r$$

we have

(2.2.27a) $$\sqrt{S}\, W_{NN}(a, b, c, d\,|\,u) = \begin{bmatrix} N \\ \frac{c-b+N}{2} \end{bmatrix}^{-1} \sum_i \sqrt{S}\, U(i),$$

where

(2.2.27b) $$\max(0, s-r, s-\bar{r}) \leq i \leq \min(k-r, k-\bar{r}, s)$$

Each summand is then non-vanishing.

Proof. From (2.2.24–25), we find that

$$\begin{aligned} \sqrt{S}\, U(i) &\neq 0 \qquad \text{if } C_{ini} \leq 0 \leq C_{fin}, \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

The first condition is rewritten as

$$s - \bar{r} \leq i \leq k - \bar{r}.$$

The lemma follows immediately from this. □

Lemma 2.2.7. *Let* (d, c), (c, b) *be admissible. Furthermore we suppose that* $a \neq 0$ *and the pairs* (d, a), (a, b), $(d, -a)$ $(-a, b)$ *are all weakly admissible. Then*

(2.2.28a) $$W_{NN}(a, b, c, d\,|\,u)/W_{NN}(-a, b, c, d\,|\,u) = -1.$$

(2.2.28b) $$W_{NN}(d, a, b, c\,|\,u)/W_{NN}(d, -a, b, c\,|\,u) = 1,$$

(2.2.28c) $$W_{NN}(c, d, a, b\,|\,u)/W_{NN}(c, d, -a, b\,|\,u) = 1,$$

(2.2.28d) $$W_{NN}(b, c, d, a\,|\,u)/W_{NN}(c, b, d, -a\,|\,u) = 1.$$

Proof. First we show (2.2.28a). Note that the weak admissibility of $(d, -a)$, $(-a, b)$ implies that (d, a), (a, b) are lower non-admissible. We may assume (2.2.19). Retaining the parametrization (2.2.7), we have $-a = \ell + N - 2\bar{r}$ where $\bar{r}$ is given by (2.2.26). In Lemma 2.2.6, the range of i (2.2.27b) is invariant under the change $r \to \bar{r}$. So it is sufficient to show that

$$U(i)/\bar{U}(i) = -1$$

for all i satisfying (2.2.27b), where $\bar{U}(i)$ is given by (2.2.11) with $\bar{r}$ in place of r. For the calculation we note the simple facts:

$$[A_1, B_1][A_2, B_2]/[A_1, B_2][A_2, B_1] = 1 \quad \text{if } \max(A_1, A_2) \le \min(B_1, B_2) + 1. \tag{2.2.29}$$

$$\begin{aligned} [A, B] &= (-)^{B-A}[-B, -A] && \text{if } A, B \in \mathbb{Z} \text{ and } A \le 0 \le B, \\ &= (-)^{B-A+1}[-B, -A] && \text{otherwise.} \end{aligned} \tag{2.2.30}$$

Writing down the ratio we have

$$\begin{aligned} \frac{U(i)}{\bar{U}(i)} &= \frac{[N-k+\bar{r}+1, \ell+2N-k-s+i]_1 [N-k+r+1, N-i]_1 \cdot [-k+r+i, r-\bar{r}-1]_{\bar{2}}}{[-k+\bar{r}+i, \bar{r}-r-1]_{\bar{3}} [\bar{r}-r+1, \bar{r}-s+i]_2 \cdot [N-k+r+1, \ell+2N-k-s+i]_1} \\ &\times \frac{[r-\bar{r}+1, r-s+i]_3 [\ell+N-k-s+i-u, \bar{r}-s-1-u]_{\bar{4}} \cdot [-r+s+1+u, i+u]_4}{[N-k+\bar{r}+1, N-i]_1 [\ell+N-k-s+i-u, r-s-1-u]_{\bar{4}} \cdot [-\bar{r}+s+1+u, i+u]_4} \\ &\times \frac{[1, k-\bar{r}-i]_3 [1, \bar{r}-s+i]_2}{[1, k-r-i]_2 [1, r-s+i]_3}. \end{aligned}$$

We grouped together the members to which we apply (2.2.29) by putting the suffix $j = 1, 2, 3, 4$. For those with barred suffix, we apply also (2.2.30). We thus find

$$U(i)/\bar{U}(i) = (-)^{(k-\bar{r}-i)+(k-r-i)+(\bar{r}-r)-1} = -1$$

as desired. This completes the proof of (2.2.28a).

Next we proceed to (2.2.28b). The rest are shown similarly. First note the following formula:

$$\frac{(d, a)_N}{(d, -a)_N} = (-)^a \frac{\sqrt{[-a]}}{\sqrt{[a]}} = \frac{g_{-a}}{g_a}. \tag{2.2.31}$$

This is a direct consequence of (2.1.24b). From (2.1.24a), Theorem 2.1.5, (2.2.31) and (2.2.30a), we have

$$\frac{W_{NN}(a, b, c, d\,|\,u)}{W_{NN}(a, -b, c, d\,|\,u)} = \frac{\sqrt{(c, b)_N(a, -b)_N}}{\sqrt{(a, b)_N(c, -b)_N}} \frac{S_{NN}(a, b, c, d\,|\,u)}{S_{NN}(a, -b, c, d\,|\,u)}$$

$$= \frac{\sqrt{(c, b)_N(a, -b)_N}}{\sqrt{(a, b)_N(c, -b)_N}} \frac{g_{-b}}{g_b} \frac{S_{NN}(b, c, d, a\,|\,-1-u)}{S_{NN}(-b, c, d, a\,|\,-1-u)}$$

$$= \frac{(c, b)_N}{(c, -b)_N} \frac{g_{-b}}{g_b} (-) = 1. \qquad \square$$

So much for the preparation. Now it is rather straightforward to show

Theorem 2.2.8 (Theorem 3 of [2]). *The set of the admissible weights $S_{NN}(a, b, c, d\,|\,u)$ satisfy the STR among themselves.*

Proof. The symmetrized weights for the unrestricted models satisfy the STR. We are to show that if the exterior pairs (a, b), (b, c), (c, d), (d, e), (e, f), (f, a) are all admissible, the terms with non-admissible inner heights cancel out among themselves in each side of the STR.

Let us consider the l.h.s. of Fig. 2.2. Set

$$R(g) = S_{NN}(a, b, g, f\,|\,u) S_{NN}(f, g, d, e\,|\,u+v) S_{NN}(g, b, c, d\,|\,v).$$

If admissible pairs and non-admissible pairs coexist among the inner pairs (g, b), (g, d) and (g, f), then $R(g)$ vanishes because of Lemma 2.2.4. The same is true if lower and upper non-admissible pairs coexist.

From Lemma 2.2.2, it is enough to consider the case that the inner pairs are lower non-admissible.

If a pair (a, b) is admissible, it is clear from (2.1.24b) that the factor $(a, b)_N$ is strictly positive. So Lemma 2.2.4–5 and Theorem 2.1.5 allow us to prove the cancellation of the unsymmetrized weights

$$R'(g) = W_{NN}(a, b, g, f\,|\,u) W_{NN}(f, g, d, e\,|\,u+v) W_{NN}(g, b, c, d\,|\,v)$$

instead of the symmetrized ones.

Under the condition that $a > 0$, if the pair (a, g) is weakly admissible and lower non-admissible then so is $(a, -g)$. Therefore we have

$$\text{the summation of the lower non-admissible terms} = R'(0) + \sum_{g>0} (R'(g) + R'(-g)),$$

where

$$R'(g)=W_{NN}(a,b,g,f|u)W_{NN}(f,g,d,e|u+v)W(g,b,c,d|v).$$

$R'(0)$ vanishes from Lemma 2.2.4 while $R'(g)+R'(-g)=0$ follows from Lemma 2.2.7. We have now proved that non-admissible terms in the l.h.s. cancel out.

The proof is the same for the r.h.s. □

2.3. Vertex-SOS correspondence

In [4] through an attempt to obtain the eigenvectors of its row-to-row transfer matrix, Baxter found an equivalence of the eight vertex model to an SOS model. We shall extend this equivalence to the fusion models. The result of this paragraph is not used in the rest of this paper.

First we recall the fusion of the eight vertex model [6]. We denote by $R^{\alpha\beta}_{\gamma\delta}$ the Boltzmann weight of the eight vertex model associated with a vertex as indicated in Fig. 2.7.

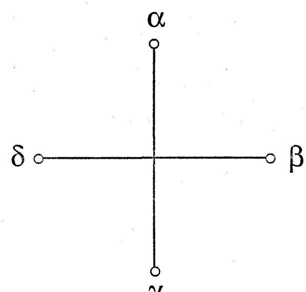


Fig. 2. 7 A vertex configuration.

We use the following parametrization ($\lambda\neq 0$):

$$\begin{aligned} R^{\alpha\alpha}_{\alpha\alpha}(u)&=\rho_0\Theta(\lambda)\Theta(\lambda u)H(\lambda(u+1)),\\ R^{\alpha\beta}_{\alpha\beta}(u)&=\rho_0\Theta(\lambda)H(\lambda u)\Theta(\lambda(u+1)),\\ R^{\alpha\beta}_{\beta\alpha}(u)&=\rho_0 H(\lambda)\Theta(\lambda u)\Theta(\lambda(u+1)),\\ R^{\alpha\alpha}_{\beta\beta}(u)&=\rho_0 H(\lambda)H(\lambda u)H(\lambda(u+1)),\\ \rho_0&=1/\Theta(0)H(\lambda)\Theta(\lambda). \end{aligned} \tag{2.3.1}$$

Here $\alpha,\beta=\pm 1$ and $\alpha\neq\beta$.

Let $V=Cv_+\oplus Cv_-\simeq C^2$. We define $E_{\alpha\beta}\in\mathrm{End}(V)$ $(\alpha,\beta=\pm 1)$ by $E_{\alpha\beta}v_\gamma=v_\alpha\delta_{\beta\gamma}$, and set $R(u)=\sum R^{\alpha\beta}_{\gamma\delta}(u)E_{\gamma\alpha}\otimes E_{\delta\beta}\in\mathrm{End}(V\otimes V)$. Let $V_1,\cdots,V_M$ be copies of V. Given $T\in\mathrm{End}(V\otimes V)$ we define $T^{jk}\in\mathrm{End}(V_1\otimes\cdots\otimes V_M)$ by $T^{jk}=\iota_{jk}T\pi_{jk}$, where ι_{jk} is the natural injection $\iota_{jk}\colon V_j\otimes V_k\to$

$V_1\otimes\cdots\otimes V_M$ and π_{jk} is the natural projection $\pi_{jk}\colon V_1\otimes\cdots\otimes V_M\to V_j\otimes V_k$. In this notation the Yang-Baxter equation reads as

$$R^{12}(u)R^{13}(u+v)R^{23}(v)=R^{23}(v)R^{13}(u+v)R^{12}(u). \tag{2.3.2}$$

We denote by $I\in\mathrm{End}(V\otimes V)$ the identity, and by $C\in\mathrm{End}(V\otimes V)$ the transposition: $Cv_1\otimes v_2=v_2\otimes v_1$. We set $P=(C+I)/2$. We have

$$R(0)=C, \tag{2.3.3a}$$

$$R(-1)=C-I. \tag{2.3.3b}$$

Set $u=-1$ in (2.3.1) and multiply it by P^{12}. Then, because of (2.3.3b) and $P(C-I)=0$, the l.h.s. is zero. Therefore we have

$$P^{12}R^{23}(u)R^{13}(u-1)C^{12}=P^{12}R^{23}(u)R^{13}(u-1). \tag{2.3.4}$$

Now we denote by $P_{1\ldots M}$ the projection on the space of the symmetric tensors in $V_1\otimes\cdots\otimes V_M$:

$$P_{1\ldots M}=\frac{1}{M!}(C^{1M}+\cdots+C^{M-1M}+I)\cdots(C^{12}+I).$$

We call $P_{1\ldots M}$ a symmetrizer for short. We prepare further copies $V_{\bar{1}},\cdots,V_{\bar{N}}$ of V and use $T^{j\bar{k}}$ in the sense similar to T^{jk} with V_k replaced by $V_{\bar{k}}$. We define an operator $R'_{1\ldots M\bar{j}}(u)\in\mathrm{End}(V_1\otimes\cdots\otimes V_M\otimes V_{\bar{1}}\otimes\cdots\otimes V_{\bar{N}})$ by

$$R'_{1\ldots M\bar{j}}(u)=P_{1\ldots M}R^{1\bar{j}}(u+M-1)\cdots R^{M\bar{j}}(u).$$

Lemma 2.3.1.

(2.3.5a) (i) $R'_{1\ldots M\bar{j}}(u)P_{1\ldots M}=R'_{1\ldots M\bar{j}}(u)$.

(ii) $R'_{1\ldots M\bar{j}}(u)=0$

(2.3.5b) *for* $u=-1,\cdots,-M+1,\ -1+iK'/\lambda,\cdots,$
$-M+1+iK'/\lambda$.

Proof. (i) This follows immediately from (2.3.4).

(ii) For $u=-1,\cdots,-M+1$ $R'_{1\ldots M\bar{j}}(u)=0$ because of (2.3.3) and the identity $P^{12}C^{1\bar{j}}(C^{2\bar{j}}-I)=0$. The latter half is proved similarly. □

This lemma tells that

$$R_{1\ldots M\bar{j}}(u)=R'_{1\ldots M\bar{j}}(u)[1]^{M-1}/[u+M-1]_{M-1}$$

is holomorphic. For $M\geq N$ we define the (M,N)-weight

$$R_{MN}(u) \in \mathrm{End}\,(\underbrace{V\otimes\cdots\otimes V}_{M}\otimes\underbrace{V\otimes\cdots\otimes V}_{N})$$
$$\simeq \mathrm{End}\,(V_1\otimes\cdots\otimes V_M\otimes V_{\bar{1}}\otimes\cdots\otimes V_{\bar{N}})$$

by

$$R_{MN}(u)=P_{\bar{1}\dots\bar{N}}R_{1\dots M\bar{N}}(u)\cdots R_{1\dots M\bar{1}}(u-N+1)[1]^N/[N]_N.$$

For $M<N$ we define it through the following commutative diagram.

$$\begin{array}{ccc} \underbrace{V\otimes\cdots\otimes V}_{M}\otimes\underbrace{V\otimes\cdots\otimes V}_{N} & \xrightarrow{C_{MN}} & \underbrace{V\otimes\cdots\otimes V}_{N}\otimes\underbrace{V\otimes\cdots\otimes V}_{M} \\ {\scriptstyle R_{MN}(u)}\downarrow & & \downarrow{\scriptstyle R_{NM}(u+M-N)} \\ \underbrace{V\otimes\cdots\otimes V}_{M}\otimes\underbrace{V\otimes\cdots\otimes V}_{N} & \xrightarrow[C_{MN}]{} & \underbrace{V\otimes\cdots\otimes V}_{N}\otimes\underbrace{V\otimes\cdots\otimes V}_{M} \end{array}$$

The C_{MN} is the transposition of $\underbrace{V\otimes\cdots\otimes V}_{M}$ and $\underbrace{V\otimes\cdots\otimes V}_{N}$. As (2.3.5a) we have from (2.3.4)

$$R_{MN}(u)=R_{MN}(u)P_{1\dots M}=R_{MN}(u)P_{\bar{1}\dots\bar{N}}. \tag{2.3.6}$$

Theorem 2.3.2 ([6]). *Fix a triple of integers* (M, N, P), *and set* $V_1=\underbrace{V\otimes\cdots\otimes V}_{M}$, $V_2=\underbrace{V\otimes\cdots\otimes V}_{N}$ *and* $V_3=\underbrace{V\otimes\cdots\otimes V}_{P}$. *We define* R^{12}_{MN}, R^{13}_{MP}, $R^{23}_{NP}\in\mathrm{End}\,(V_1\otimes V_2\otimes V_3)$ *as in* (2.3.1). *Then they satisfy the Yang-Baxter equation*:

$$R^{12}_{MN}(u)R^{13}_{MP}(u+v)R^{23}_{NP}(v)=R^{23}_{NP}(v)R^{13}_{MP}(u+v)R^{12}_{MN}(u). \tag{2.3.7}$$

Proof. If we discard all the symmetrizers appearing in (2.3.7), the equality follows by a repeated use of (2.3.1). Multiplying this identity by the symmetrizers from the left and using (2.3.6) we obtain (2.3.7). □

Now we establish an equivalence between R_{MN} and W_{MN}. Choose arbitrary constants s^+ and s^-, and fix ξ in (2.1.4) by

$$\xi=\frac{s^++s^-}{2}-\frac{K}{\lambda}.$$

We set

$$\phi_{ab}(u)=\begin{pmatrix} H(\lambda(s^\varepsilon+a-\varepsilon u)) \\ \Theta(\lambda(s^\varepsilon+a-\varepsilon u)) \end{pmatrix} \quad \text{if } \varepsilon=b-a=\pm 1,$$
$$=0 \quad \text{otherwise.}$$

We define a vector $\phi_{M,ab}(u)$ in $\underbrace{V\otimes\cdots\otimes V}_{M}$ by

$$\phi_{M,ab}(u)=P_{1\ldots M}(\phi_{a_0a_1}(u+M-1)\otimes\cdots\otimes\phi_{a_{M-1}a_M}(u)) \quad (a_0=a,\ a_M=b). \tag{2.3.8}$$

The a_i are integers satisfying $|a_i-a_{i+1}|=1$, The definition (2.3.8) is independent of the choice of these integers. We note that $W_{MN}(a, b, c, d|u)$ are invariant under the change of $(a, b, c, d, \xi, s^+, s^-)$ to $(-a, -b, -c, -d, -\xi, -s^-+2K/\lambda, -s^++2K/\lambda)$. This is useful when we check the following identity due to Baxter [4] (see Fig. (2.8)):

$$R(u-v)(\phi_{dc}(u)\otimes\phi_{cb}(v))=\sum_a W_{11}(a, b, c, d|u-v)\phi_{ab}(u)\otimes\phi_{da}(v). \tag{2.3.9}$$

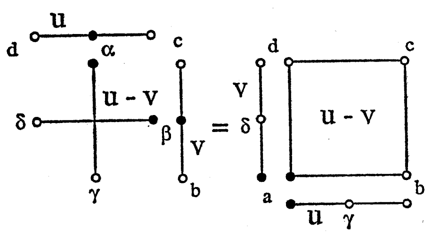


Fig. 2. 8 The vertex-SOS correspondence.

By a similar argument as in Theorem 2.3.2 the identity (2.3.9) is generalized to

Theorem 2.3.3 ([2]).

$$R_{MN}(u-v)(\phi_{M,dc}(u)\otimes\phi_{N,cb}(v))=\sum_a W_{MN}(a, b, c, d|u-v)\phi_{M,ab}(u)\otimes\phi_{N,da}(v).$$

We omit the proof, which is similar to that of Theorem 2.3.2. A simple case is schematically shown in Fig. 2.9.

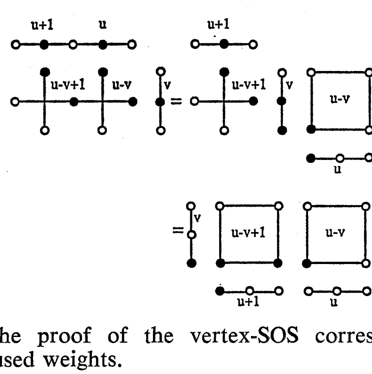


Fig. 2. 9 The proof of the vertex-SOS correspondence for fused weights.

Remark. The first formula in section 4 of [2] should be coorrected as

$$R(u)=\frac{1}{\Theta(0)}\sum_{a=0}^{3} w_a(u)\sigma_a\otimes\sigma_a.$$

§ 3. One Dimensional Configuration Sums

In this section we transform the LHPs into 1D configuration sums by means of the CTM method. As in the case $N=1$ [5], we consider the following four regimes.

Regime I:	$-1<p<0,$	$0<u<L/2-1,$
Regime II:	$0<p<1,$	$0<u<L/2-1,$
Regime III:	$0<p<1,$	$-1<u<0,$
Regime IV:	$-1<p<0,$	$-1<u<0.$

3.1. Boltzmann weights in the conjugate modulus

In order to evaluate the LHPs we appeal to Baxter's corner transfer matrix (CTM) trick [8]. The method is summarized in Appendix A of [5] for $N=1$. Apart from the change of notation (2.1.5), the reasoning therein applies equally well for general N. As a result, the LHP $P(a)$ is given in terms of a 1D configuration sum (in an appropriate limit $m\to\infty$)

$$X_m(a,b,c)=\sum q^{\phi_m(\ell_1,\cdots,\ell_{m+2})},\qquad \ell_1=a,\quad \ell_{m+1}=b,\quad \ell_{m+2}=c, \tag{3.1.1}$$

where the sum is taken over sequences $\ell_2,\cdots,\ell_m$ which are admissible in the sense that (ℓ_j,ℓ_{j+1}) satisfies (2.2.4–5) for $1\leq j\leq m+1$. (For the definition of q, see Table 1 in Part I). The goal of section 3 is to determine the function $\phi_m(\ell_1,\cdots,\ell_{m+2})$. The precise definition of the LHPs and the expressions in terms of the 1D configuration sums are given in section 2 of Part I.

The working depends on the sign of the "nome" p: $p>0$ (regime II, III), or $p<0$ (regime I, IV). As in Part I, let $|p|=e^{-\varepsilon/L}$ and define the variable x by

$$\begin{aligned} x&=e^{-4\pi^2/\varepsilon} && \text{in regime II, III,}\\ &=e^{-2\pi^2/\varepsilon} && \text{in regime I, IV.}\end{aligned}$$

We shall consider the modified weight

(3.1.2) $$\mathscr{S}_N(a,b,c,d)=S_{NN}(a,b,c,d)\frac{G_aG_c}{G_bG_d}F^{-N},$$

where G_a and F will be specified below (3.1.6). The LHPs are unaffected by such a modification.

The line of argument to obtain $\phi_m(\ell_1,\cdots,\ell_{m+2})$ goes as follows [5, 8]. Let $A(u)$ denote the CTM corresponding to the southeast quadrant of the lattice. It is an operator acting on the subspace $\mathscr{H}_{b,c}$ of $\boldsymbol{C}^{L-1}\otimes\cdots\otimes\boldsymbol{C}^{L-1}$ ($m+2$ fold tensor product) spanned by the vectors $e_{\ell_1}\otimes\cdots\otimes e_{\ell_{m+2}}$ such that $\ell_{m+1}=b$, $\ell_{m+2}=c$ and $\{\ell_j\}_{j=1}^{m+2}$ is admissible. Here e_i signifies the standard basis of $\boldsymbol{C}^{L-1}$, and (b,c) is a fixed admissible pair. Define the face operators $\mathscr{U}_i$ $(2\leq i\leq m+1)$ by ([5], eq. (A2))

(3.1.3) $$\begin{aligned}&\mathscr{U}_ie_{\ell_1}\otimes\cdots\otimes e_{\ell_{m+2}}\\&\quad=\sum_{\ell_i'}\mathscr{S}_N(\ell_i,\ell_{i+1},\ell_i',\ell_{i-1})e_{\ell_1}\otimes\cdots\otimes e_{\ell_i'}\otimes\cdots\otimes e_{\ell_{m+2}}.\end{aligned}$$

(For $i=m+1$ the sum is confined to only one term $\ell_i'=\ell_i=b$.) The CTM is defined to be $A(u)=\mathscr{F}_2\mathscr{F}_3\cdots\mathscr{F}_{m+1}$, $\mathscr{F}_j=\mathscr{U}_{m+1}\mathscr{U}_m\cdots\mathscr{U}_j$ ([5], eq. (A14)). Then, in the large lattice limit, (i) $P_a=\mu_a/\sum_{a'}\mu_{a'}$ with $\mu_a=g_a^2\,\mathrm{trace}_{\ell_1=a}(A(-t))$, where $t=2$ or $2-L$ and g_a is given in Theorem 2.1.5 (cf. [5], eqs (A26, 28)); (ii) the eigenvalues of $A(u)$ have the form x^{n_iu} with $2n_i\in\boldsymbol{Z}$ (up to a common factor). Finally (iii) the n_i's are calculated by considering the limit

(3.1.4) $$x\longrightarrow 0,\quad u\longrightarrow 0,\quad w=x^u\quad\text{fixed.}$$

In this limit the face operators $\mathscr{U}_i$, and hence $A(u)$ also, become diagonal. (For even L the last step requires a slight modification. See the discussion at the end of section 3.3.) In what follows we mean by lim the limit (3.1.4).

To study the limit of (3.1.2) in this sense, we exploit the conjugate modulus transformation $p\to x$:

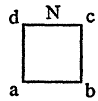
(3.1.5) $$[u]=\theta_1\left(\frac{\pi u}{L},p\right)=\kappa\ x^{\mu}(u).$$

The quantities κ, μ, (u), $G_a=G_{L-a}$ and F are given as follows.

(3.1.6)

	Regime II, III	Regime I, IV
κ	$\sqrt{2\pi L/\varepsilon}\,x^{L/8}$	$\sqrt{\pi L/\varepsilon}\,x^{L/16}$
μ	$u(u-L)/2L$	$u(2u-L)/2L$
(u)	$E(x^u,x^L)$	$E(x^u,-x^{L/2})$
G_a	$w^{a(a-L)/4L}$	$w^{a(a-L)/2L}$
F	$x^{u(u+1)/2L}$	$x^{u(2u+2-L)/2L}$

The symmetries (2.1.25a-d) (with $M=N$) and (2.2.13) remain valid for (3.1.2). As in section 2 we find it convenient to use the graphical notation

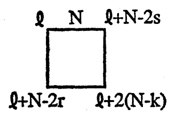


$$=\mathscr{S}_N(a, b, c, d).$$

Unless otherwise stated, we shall always assume $a \le c$, $d \le b$ and use the parametrization (2.2.7)

$$\begin{aligned} &a=\ell+N-2r, \quad b=\ell+2(N-k), \quad c=\ell+N-2s, \quad d=\ell, \\ &0\le k\le N, \quad \max(0, \ell+2N-L-k+1)\le s\le r\le \min(\ell-1, k). \end{aligned} \tag{3.1.7}$$

Rewriting (2.2.11) we get the expression for the modified weight



$$= \sum_{i=0}^{\min(s,k-r)} w^{\alpha} x^{\beta(i)} \mathscr{V}(i), \tag{3.1.8}$$

where

$$\begin{aligned} \alpha &= \frac{r-s-N+k}{2} && \text{in regime II, III,} \\ &= r && \text{in regime I, IV,} \end{aligned} \tag{3.1.9a}$$

$$\beta(i) = \frac{i(i+1)}{2} + i(r-s+N-k) + \frac{(r-s)(N-k+1)}{2}. \tag{3.1.9b}$$

The $\mathscr{V}(i)$ has the same form as $\sqrt{S}\,U(i)$ (2.2.9–11) with the symbol $[\ell]$ therein replaced by (ℓ) (3.1.6). We shall show that

$$\phi_m(\ell_1, \cdots, \ell_{m+2}) = \sum_{j=1}^{m} jH(\ell_j, \ell_{j+1}, \ell_{j+2}) \tag{3.1.10}$$

where in regimes II and III

$$H(a, b, c) = |a-c|/4, \tag{3.1.11}$$

and in regimes I, IV

$$\begin{aligned} H(a, b, c) &= \min\left(n-b, \frac{\min(a, c)-b+N}{2}\right) && \text{if } b\le n, \\ &= \min\left(b-n-1, \frac{b-\max(a, c)+N}{2}\right) && \text{if } b\ge n+1, \\ & && n=[L/2]. \end{aligned} \tag{3.1.12}$$

Note that $H(a, b, c)$ (3.1.12) has the symmetry:

(3.1.13) $$H(2n+1-a, 2n+1-b, 2n+1-c)=H(a, b, c).$$

3.2. Regime II, III

In this case we have by the definition (3.1.6)

(3.2.1) $$(\ell \pm u)=1+O(x) \qquad \text{if} \quad 0<\ell<L.$$

Theorem 3.2.1

(3.2.2) $$\lim \begin{array}{|c|} \overset{d \quad N \quad c}{\square} \\ a \qquad b \end{array} = \delta_{ac} w^{-|b-d|/4}.$$

Proof. From the representation (3.1.8), (2.2.9–11) together with (3.2.1) it follows that $\mathscr{V}(i)=O(1)$ for all i in the sum. The power $\beta(i)$ (3.1.9b) is non-negative, and vanishes if and only if $i=0$ and $r=s$. In this case one can check that $\lim \mathscr{V}(0)=1$, so the r.h.s. of (3.1.8) tends to $w^{-(N-k)/2}=w^{-|b-d|/4}$. This completes the proof. □

From Theorem 3.2.1 follows the expression (3.1.10–11) of $\phi_m(\ell_1, \cdots, \ell_{m+2})$.

3.3. Regime I, IV

In this case, the factor $(\ell)=E(x^\ell, -x^{L/2})$ shows a behavior different from that of regime II, III:

(3.3.1a) $$\begin{aligned} (\ell)&=1+O(\sqrt{x}) && \text{if } 1\le \ell<L/2, \\ &=2(1+O(x)) && \text{if } \ell=L/2 \text{ for } L \text{ even}, \end{aligned}$$

(3.3.1b) $$(\ell)=x^{L/2-\ell}(L-\ell)=-x^\ell(-\ell)$$

Note that if $\ell \ge L/2$, then (ℓ) gives rise to an extra power $L/2-\ell$. To avoid technical complexity we assume throughout regimes I, IV that

(3.3.2) $$L\ge 2N+1.$$

This implies

(3.3.3) $$(\ell), (\ell \pm u)=1+O(\sqrt{x}) \qquad \text{provided } 1\le \ell \le N.$$

We shall carry out the computation by following the four steps.

$$\text{(3.3.4)}\qquad \textit{Step } 1:\quad \lim \begin{array}{ccc} d & N & c \\ & \square & \\ a & & b \end{array} = \lim \begin{array}{ccc} d+1 & N-1 & c \\ & \square & \\ a & & b-1 \end{array} \qquad \text{if } d<b.$$

$$\text{(3.3.5)}\qquad \textit{Step } 2:\quad \lim \begin{array}{ccc} b & N & c \\ & \square & \\ a & & b \end{array} = 0 \quad \text{unless } a=c,$$

$$\text{or} \quad a+c=L \quad \text{with} \quad L \text{ even.}$$

$$\text{(3.3.6)}\qquad \textit{Step } 3:\quad \lim \begin{array}{ccc} b & N & a \\ & \square & \\ a & & b \end{array} = \prod_{j=(a+b-N)/2}^{a-1} \mathscr{W}(j) \times \prod_{j=a+1}^{(a+b+N)/2} \mathscr{W}(L-j),$$

where

$$\text{(3.3.7)}\qquad \begin{aligned} \mathscr{W}(j) &= 1 && \text{if } 0<j<L/2, \\ &= \frac{1+w}{2} && \text{if } j=L/2, \\ &= w && \text{if } L/2<j<L. \end{aligned}$$

$$\text{(3.3.8)}\qquad \textit{Step } 4:\quad \lim \begin{array}{ccc} b & N & L\text{-}a \\ & \square & \\ a & & b \end{array} = (-1)^{N+b+1-L/2} \frac{1-w}{1+w} \lim \begin{array}{ccc} b & N & a \\ & \square & \\ a & & b \end{array}$$

$$\text{if } 2a \neq L \quad \text{and} \quad L \text{ even.}$$

For $N=1$ these assertions (3.3.4–8) can be verified directly by using the representation (3.1.7–9) of the modified weight and the fact that

$$\text{(3.3.9)}\qquad \lim \frac{(\ell-u)}{(\ell)} = \mathscr{W}(\ell), \qquad \lim \frac{(\ell+u)}{(\ell)} = w^{-1}\mathscr{W}(L-\ell).$$

(The weight $\mathscr{S}_0(a, b, c, d)$ in (3.3.4) is understood as 1 if $a=b=c=d$, 0 otherwise.) In what follows we assume (3.3.4–8) to be true for $N-1$ in place of N.

Step 1.

First note that in the representations (3.1.8–9) of $\mathscr{S}_N(a, b, c, d)$ and $\mathscr{S}_{N-1}(a, b-1, c, d+1)$ in (3.3.4) the suffix i ranges over the common interval $0 \leq i \leq \min(s, k-r)$. Let $\mathscr{A}(i) = w^r x^{\beta(i)} \mathscr{V}(i)$, $\mathscr{A}'(i) = w^r x^{\beta'(i)} \mathscr{V}'(i)$ stand for the corresponding summands. In the following Lemmas 3.3.1–2 we show that for all i either $\lim \mathscr{A}(i)/\mathscr{A}'(i) = 1$ or else $\lim \mathscr{A}(i) = \lim \mathscr{A}'(i)$

$=0$ holds, thereby proving (3.3.4). Without loss of generality we shall assume $\ell \le L/2$.

Lemma 3.3.1. *We have*

(3.3.10) $$x^{\beta(i)-\beta'(i)}\mathscr{V}(i)/\mathscr{V}'(i)=O(1) \qquad \text{for all } 0\le i\le \min(s, k-r).$$

Moreover the l.h.s. tends to 1 *if* $2N+\ell-k-r>L/2$, *or* $i=0$ *and* $r=s$.

Proof. Dropping the factors of the form (3.3.3) we find that the l.h.s. of (3.3.10) behaves like $A(1+O(\sqrt{x}))$ with

$$A=x^{i+(r-s)/2}\left(\frac{(\ell-r)(\ell-s)}{(\ell+2N-k-r)(\ell+2N-k-s)}\right)^{1/2}\frac{(\ell+2N-s-k+i)}{(\ell-s+i)}.$$

By taking into account the contributions from the factors (ℓ) (cf. (3.3.1)) it can be shown that $A=O(x^{\nu})$, where $0\le\nu\le i+(r-s)/2$, and that $A=1+O(\sqrt{x})$ if $2N+\ell-k-r>L/2$ (see Fig. 3.1).

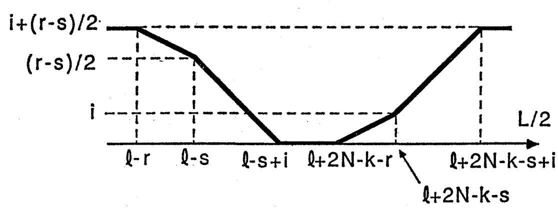


Fig. 3.1 Power estimate of the l.h.s. of (3.3.9).

Finally when $i=0$ and $r=s$ the l.h.s. of (3.3.10) becomes $(N-k+u)/(N-k)$, which tends to 1 in the limit by (3.3.3). This completes the proof of Lemma 3.3.1. □

Lemma 3.3.2. *Assume* $L/2\ge\ell, 2N+\ell-k-r+1$. *Then we have*

$$\lim x^{\beta(i)}\mathscr{V}(i)=0$$

except when $i=0$ *and* $r=s$.

Proof. Proceeding in the same way as in Lemma 3.3.1, we have $\mathscr{V}(i)=B\times O(1)$ where

$$B=\left(\frac{(2N+\ell-k-r+1,\ 2N+\ell-k-s)(N+\ell-2s)}{(N+\ell-r+1,\ N+\ell-s)}\right)^{1/2}$$
$$\times(2N+\ell-k-s+1,\ 2N+\ell-k-s+i).$$

Here we have set $(a, b)=(a)(a+1)\cdots(b)$ for $a\leq b$ and $(a, a-1)=1$.

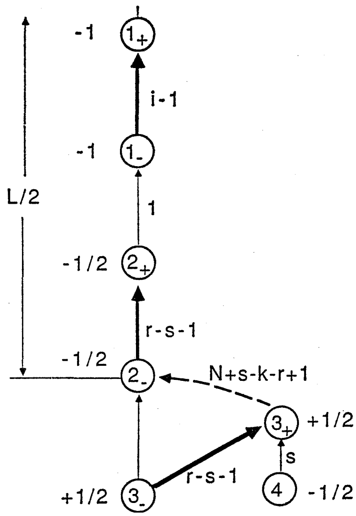


Fig. 3.2 Power estimate of $\mathcal{V}(i)$. The order relation is depicted for the factors appearing in B.

(1_+) = 2N+ℓ-k-s+i,

(1_-) = 2N+ℓ-k-s+1, (2_+) = 2N+ℓ-k-s,

(2_-) = 2N+ℓ-k-r+1, (3_+) = N+ℓ-s,

(3_-) = N+ℓ-r+1 and (4) = N+ℓ-2s.

Counting the total power of x (see Fig. 3.2), we obtain the estimate $x^{\beta(i)}B = O(x^{\gamma(i)})$ with

$$\gamma(i)=i(N-k+1)+\frac{1}{4}(r-s)(N+s+1-k-r+N-k+2) \qquad \text{if } N+s+1\geq k+r,$$

$$=i(N-k+1)+\frac{1}{2}\min(r-s,\ s+N-k+1) \quad \text{otherwise.}$$

Under the assumption of the lemma, $\gamma(i)>0$ in either case as desired. □

Step 2.

Here we make use of the STR of type $(N, 1, N)$ as in Fig. 3.3: The variables are so chosen that one of the summands in the r.h.s. vanishes. This provides us with the relation

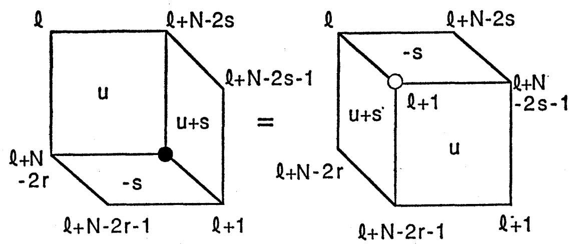
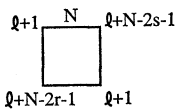
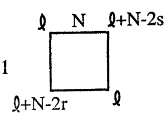
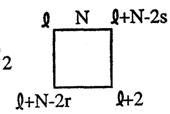


Fig. 3.3 The STR of type $(N, 1, N)$ used in deriving (3.3.10). In the r.h.s. the term with the center spin $\ell-1$ vanishes.

(3.3.11)



$= w C_1$ 

$+ \; x^{(r+s)/2+1} C_2$ 

,

(3.3.12)

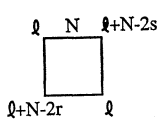
$$C_1 = \left(\frac{(\ell+N-2r-1)(\ell+N-2s-1)}{(\ell+N-2r)(\ell+N-2s)} \right)^{1/2} \times \frac{(\ell+N-r-s)(\ell+1+u)}{(\ell+N-r-s-1-u)(\ell+1)},$$

$$C_2/C_1 = \left(\frac{(\ell-r)(\ell-s)(\ell+N-r+1)(\ell+N-s+1)}{(N-r)(N-s)(r+1)(s+1)} \right)^{1/2} \times \frac{(N-r-s-1)(u)}{(\ell+N-r-s)(\ell+1+u)}.$$

Eq. (3.3.5) is an immediate consequence of the following lemma.

Lemma 3.3.3. *Assuming $r>s$, we have*

(3.3.13)



$= O(x^{\min(r-s, |\ell+N-r-s-L/2|)/2}).$

Proof. Using (3.3.2) and the symmetry (2.2.13) we may assume further $\ell+N<L$, so that $0 \le s < r \le \min(\ell-1, N)$. We prove (3.3.13) by an induction on s.

Suppose $s=0$. Then there is only one term in the sum (3.1.8), and we have

$$x^{\beta(0)}\mathscr{V}(0)=x^{r/2}\frac{\sqrt{(\ell+N)(\ell+N-2r)}}{(\ell+N-r)}\times O(1),$$

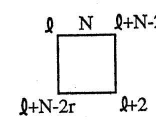
from which follows (3.3.13) in this case.

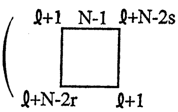
To the next step of the induction we use the following estimates for the quantities in (3.3.11–12):

(3.3.14)
$$\begin{aligned} C_1 &= O(\sqrt{x}) && \text{if } \ell+N-r-s\le L/2<\ell+N-2s,\\ &= O(\sqrt{x}^{-1}) && \text{if } \ell+N-2r\le L/2<\ell+N-r-s,\\ &= O(1) && \text{otherwise.}\end{aligned}$$

(3.3.15)
$$x^{(r+s)/2+1}C_2/C_1=O(1).$$

(3.3.16)



$= O\Big($  $\Big)$

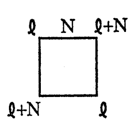
$$=O(x^{\min(r-s,\,|\ell+N-r-s-L/2|)/2}).$$

In (3.3.16) we used (3.3.4) and the induction hypothesis. The assertion (3.3.13) follows from (3.3.14–16) by virtue of (3.3.11). □

Step 3.

To prove (3.3.6) we may assume that $b\le L/2$, for it is invariant under the change $a\to L-a$, $b\to L-b$.

In the case $a=b+N$ (*i.e.* $r=0$), we have



$$=\prod_{j=0}^{N-1}\frac{(\ell+j-u)}{(\ell+j)},$$

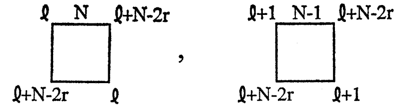
so (3.3.6) is obvious from (3.3.9).

Next suppose that (3.3.6) is true for r. Setting $r=s$ in (3.3.12) and noting $\ell+1\le L/2$, we have

$$\lim wC_1=\mathscr{W}(L-\ell-1)/\mathscr{W}(\ell+N-2r-1),$$

$$\lim x^{r+1}C_2=\varepsilon_{\ell+1}^{L/2}\frac{1-w}{\mathscr{W}(\ell+N-2r-1)}\lim x^{r+1}\frac{(\ell+N-r+1)}{(\ell+N-2r)}(N-2r-1).$$

Substitute these equations into (3.3.11), apply (3.3.4) and use the induction hypothesis for



Upon simplifying the result, we are left with the proof of

Lemma 3.3.4. *Assuming* $0 \le r \le N-1$ *and* $\ell+1 \le L/2$ *we have*

$$\mathscr{W}(L-N-\ell+2r)$$
$$=\mathscr{W}(L-\ell-1)+\varepsilon_{\ell+1}^{L/2}(1-w)\lim x^{r+1}\frac{(\ell+N-r+1)}{(\ell+N-2r)}(N-2r-1),$$

where $\varepsilon_\ell^{L/2}$ *is defined by* (1.5.7).

The proof can be done by case checking.

Step 4.

Here we utilize another STR of type $(N, 1, N)$ as shown below.

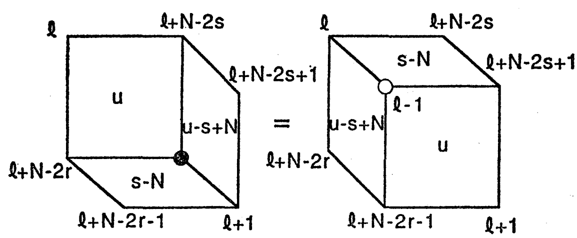
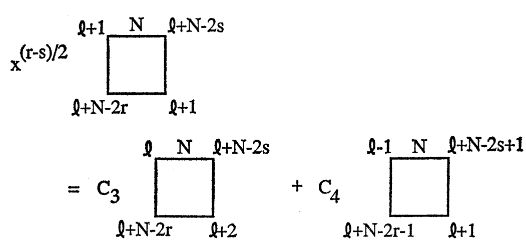


Fig. 3.4 The STR of type $(N, 1, N)$ used in deriving (3.3.17).

Explicitly we have

(3.3.17)



(3.3.18) $$C_3=\left(\frac{(s+1)(N-s)(\ell-r)(\ell+N-r+1)}{(r+1)(N-r)(\ell-s)(\ell+N-s+1)}\right)^{1/2}\frac{(r-s+1)(\ell-u)}{(\ell-r+s)(1+u)},$$

$$C_4=\left(\frac{(\ell+N-2r)(\ell+N-2s)(s)(N-s+1)(\ell-r-1)\cdot(\ell+N-r)}{(\ell+N-2r-1)(\ell+N-2s+1)(r+1)(N-r)(\ell-s)\cdot(\ell+N-s+1)}\right)^{1/2}\times\frac{(\ell+1)(r-s+1+u)}{(\ell-r+s)(1+u)}.$$

Because of (3.3.4), we can apply (3.3.8) to the weights in the r.h.s. of (3.3.17). Doing this and proceeding in the same way as in step 3, we find that the proof is reduced to the following.

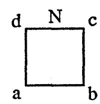
Lemma 3.3.5. *Let* $C_\nu^0=\lim x^{(s-r)/2}C_\nu$, $\nu=3, 4$. *Assuming* $s<r$ *we have*

$$\mathscr{W}(\ell-r)=C_3^0-C_4^0\frac{\mathscr{W}(\ell-r)\mathscr{W}(L-\ell-N+2r)}{\mathscr{W}(\ell+N-2r-1)\mathscr{W}(L-\ell-N+r)}. \tag{3.3.19}$$

This can also be verified by case checking, so we omit the proof.

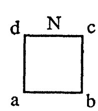
Summarizing (3.3.4–8) we obtain the following result.

Theorem 3.3.6. *If L is odd, then*

(3.3.20) lim 

$=\delta_{ac}w^{H(d,a,b)}$,

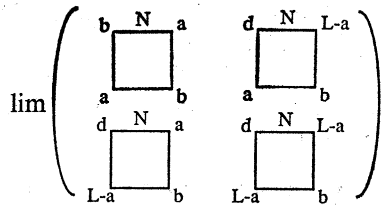
where $H(a, b, c)$ *is defined in* (3.1.12). *If L is even, then* (3.3.20) *holds except when the triple* (d, a, b) *and* $(d, L-a, b)$ *are both admissible* (*with* $2a\neq L$). *In these cases we have instead*

(3.3.21) lim 

$=\delta_{ac}w^{|a-n|-1}\frac{1+w}{2}-\delta_{a,L-c}(-1)^{b-n+N}w^{|a-n|-1}\frac{1-w}{2}.$

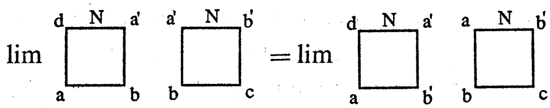
Remark. The discrepancy of the signs (of H in the exponent of w) in regime II, III (3.1.11), (3.2.2) and in regime I, IV (3.3.20) is due to the difference of the values of σ in Table 1 of Part I.

We thus find that (3.1.10, 12) is true for odd L. For even L the face operator $\mathscr{U}_i^0$ (3.1.3) in the limit is non-diagonal and contains 2 by 2 blocks of the form



Lemma 3.3.7. *The face operators $\mathscr{U}_i^0$ are mutually commutative.*

Proof. It is sufficient to prove $\mathscr{U}_i^0\mathscr{U}_{i+1}^0=\mathscr{U}_{i+1}^0\mathscr{U}_i^0$. This amounts to showing that



for all a, b, c, d, a', b'.

We may assume that (a, b) and $(a, L-b)$ are both admissible, for otherwise we must have $a=a'$, $b=b'$. Suppose for instance $a'=L-a\neq a$ and $b'=L-b\neq b$. From (3.3.21), both sides are then equal to

$$w^{|a-n|+|b-n|-2}((1-w)/2)^2.$$

The remaining cases can be verified similarly. □

Thus the face operators can be diagonalized simultaneously. For an admissible sequence (ℓ_j), define the vectors $|\ell_1, \cdots, \ell_{m+2}\rangle$ inductively by

$$|\ell_1, \ell_2\rangle=e_{\ell_1}\otimes e_{\ell_2},$$
$$|\ell_1, \cdots, \ell_{k+2}\rangle=|\ell_1, \cdots, \ell_{k+1}\rangle\otimes e_{\ell_{k+2}}+\varepsilon\,|\ell_1, \cdots, L-\ell_{k+1}\rangle\otimes e_{\ell_{k+2}}$$
$$\text{if } (\ell_k, \ell_{k+1}, \ell_{k+2}),\quad (\ell_k, L-\ell_{k+1}, \ell_{k+2}) \text{ are admissible,}$$
$$=|\ell_1, \cdots, \ell_{k+1}\rangle\otimes e_{\ell_{k+2}} \qquad \text{otherwise.}$$

Here $\varepsilon=\pm(-1)^{\ell_k-n+N}$ or 0 according as $\ell_k<n$, $>n$ or $=n$. Then we have

$$(3.3.22) \qquad \mathscr{U}_i|\ell_1, \cdots, \ell_{m+2}\rangle=w^{H(\ell_{i-1},\ell_i,\ell_{i+1})}|\ell_1, \cdots, \ell_{m+2}\rangle$$

for $2\leq i\leq m$, where $H(a, b, c)$ is given by the same equation (3.1.12). In general, the vectors $|\ell_1, \cdots, \ell_{m+2}\rangle$ may not belong to the space $\mathscr{H}_{b,c}$ (see 3.1) where the boundary heights $\ell_{m+1}=b$, $\ell_{m+2}=c$ are specified. Suppose further that they satisfy the additional conditions

$$(3.3.23) \qquad (b+c-N)/2<n-N \quad \text{or} \quad (b+c-N)/2\geq n+1.$$

Then (b, c) and $(L-b, c)$ cannot both be admissible. This guarantees that $|\ell_1, \cdots, \ell_{m+2}\rangle \in \mathscr{H}'_{b,c}$, and (3.3.22) is true for $i=m+1$ as well. Thus the CTM can be diagonalized without violating the boundary conditions. In the cases other than (3.3.23) the meaning of the configuration sum (3.1.1), (3.1.10, 12) for even L is obscure.

Remark. Eq. (3.3.23) coincides with the condition for the sequence $\cdots bcbc \cdots$ to be a ground state configuration in regime IV, with the exception $(b+c-N)/2=n-N$. See eq. (2.9b) of Part I.

§ 4. Combinatorial Identities

This section is devoted to the study of the linear difference equations that appeared in the combinatorial analysis of the 1D configuration sums in Part I. We derive explicit expressions for the fundamental solutions and rewrite them in several forms. The result contains a series of new combinatorial identities.

As in sections 2–3, we call a pair of integers (a, b) *weakly admissible* if the following relation holds

(4.0.1) $$a-b=-N, -N+2, \cdots, N.$$

A weakly admissible pair (a, b) is called *admissible* if it further satisfies

(4.0.2) $$a+b=N+2, N+4, \cdots, 2L-N-2.$$

4.1. Fundamental solution for the linear difference equation

For a weakly admissible pair (b, c) and integers $m, N\geq 0$, let $f_m^{(N)}(b, c)$ denote the solution to the following linear difference equation.

(4.1.1a) $$f_m^{(N)}(b, c)=\sum_d{}' f_{m-1}^{(N)}(d, b)q^{m|d-c|/4},$$

(4.1.1b) $$f_0^{(N)}(b, c)=\delta_{b0}.$$

Here the sum $\sum'$ is taken over d such that the pair (d, b) is weakly admissible. We set $f_m^{(N)}(b, c)=0$ if (b, c) is not weakly admissible.

One may consider m and b as discrete time and space variables, respectively. There are $N+1$ possible values of c for a given b. In this sense the equation (4.1.1) is a system of $N+1$ simultaneous equations in $1+1$ dimensions. It is of order 1 with respect to m.

Remark. By the definition $f_m^{(N)}(b, c)$ enjoys the following. Reflection symmetry:

(4.1.2) $$f_m^{(N)}(b,c)=f_m^{(N)}(-b,-c), \qquad \text{for } m\geq 0.$$

Support property:

(4.1.3) $$f_m^{(N)}(b,c)=0 \quad \text{unless } |b|\leq mN, \quad |c|\leq (m+1)N \\ \text{and } b\equiv mN \bmod 2.$$

In particular, for $N=0$, (4.1.3) asserts that

(4.1.4) $$f_m^{(0)}(b,c)=\delta_{b0}\delta_{c0}.$$

We seek for the solution to (4.1.1) in the form of a double sum

$$\sum_{j,k} A(m,j,k)q^{B(m,j,k)N+jb+kc}.$$

The coefficients A and B are polynomials in m, j, k. The sum extends over different regions of (j,k) according as the regions of (b,c) specified below.

Given an integer μ set

(4.1.5) $$R_\mu=\{b\in Z\,|\,(\mu-1)N\leq b\leq(\mu+1)N\},$$

where the left (resp. right) equality sign is taken if $\mu-1\leq 0$ (resp. $\mu+1\geq 0$). We also set $R_{\mu,\nu}=R_\mu\times R_\nu$. For a weakly admissible pair (b,c) and integer $m(\geq 1)$, let μ, ν $(=\mu\pm 1)$ be integers such that

(4.1.6a) $$(b,c)\in R_{\mu,\nu},$$

(4.1.6b) $$\mu\equiv m+1, \quad \nu\equiv m+2 \bmod 2.$$

Equation (4.1.6) uniquely determines μ and ν except when $b=0$ $(\mu=\pm 1)$ of $c=0$ $(\nu=\pm 1)$. In these cases either choice is allowed.

Theorem 4.1.1 ((3.4) of Part I). *For all weakly admissible pairs $(b,c)\in R_{\mu,\nu}$ and integers m, $N\geq 1$,*

(4.1.7a) $$(q)_{m-1}f_m^{(N)}(b,c) \\ =\Big(\sum_{\substack{k\leq(m+\mu-1)/2\\ j\geq(m+\nu)/2}}-\sum_{\substack{j\leq(m+\nu)/2-1\\ k\geq(m+\mu+1)/2}}\Big)(-1)^{j-k}q^{Q_{j,k}^{(m)}(b,c)}\begin{bmatrix} m-1\\ j\end{bmatrix}\begin{bmatrix} m\\ k\end{bmatrix},$$

(4.1.7b) $$Q_{j,k}^{(m)}(b,c)=\frac{1}{2}(j-k)(j-k+1)-\Big(j-\frac{m-1}{2}\Big)\Big(k-\frac{m}{2}\Big)N \\ +\frac{b}{2}\Big(j-\frac{m-1}{2}\Big)+\frac{c}{2}\Big(k-\frac{m}{2}\Big).$$

Proof. First we check the properties (4.1.2) and (4.1.3) for $f_m^{(N)}(b, c)$ given by (4.1.7). The former immediately follows from $Q_{j,k}^{(m)}(-b, -c)=Q_{m-1-j,m-k}^{(m)}(b, c)$. To show the latter suppose for instance $b>mN$. Then we have $\mu\geq m+1$ and $\nu\geq m$, so that $j\geq m$ or $k\geq m+1$. Therefore the product of the Gaussian polynomials identically vanishes. The other cases are similar.

For $m=1$ (4.1.7) can be directly verified by using (4.1.1). In the following we assume $m\geq 2$. In view of (4.1.2–3) we may assume

$$1\leq\mu\leq m-1 \quad \text{if } m \text{ is even } (m\geq 2), \tag{4.1.8a}$$

$$0\leq\mu\leq m-1 \quad \text{if } m \text{ is odd } (m\geq 3). \tag{4.1.8b}$$

There are two cases to consider: (i) $\nu=\mu+1$, (ii) $\nu=\mu-1$. Here we prove the case (i). The case (ii) can be verified similarly. Since we have $c>(\nu-1)N=\mu N$, the r.h.s. of (4.1.1a) reads as

$$\begin{aligned}
&\sum_d{}' f_{m-1}^{(N)}(d, b)q^{m|d-c|/4}=K_1+K_2+K_3,\\
&K_1=\sum_{d=b-N,b-N+2,\cdots,\mu N} f_{m-1}^{(N)}(d, b)q^{m(c-d)/4},\\
&K_2=\sum_{d=\mu N+2,\mu N+4,\cdots,c} f_{m-1}^{(N)}(d, b)q^{m(c-d)/4},\\
&K_3=\sum_{d=c+2,c+4,\cdots,b+N} f_{m-1}^{(N)}(d, b)q^{m(d-c)/4}.
\end{aligned} \tag{4.1.9}$$

In K_1, K_2 and K_3 the pair (d, b) belongs to $R_{\mu-1,\mu}$, $R_{\mu+1,\mu}$ and $R_{\mu+1,\mu}$, respectively. We substitute the expression (4.1.7) into $K_1\sim K_3$ and perform the d-summations. By using the identities

$$\begin{bmatrix} m-2 \\ j \end{bmatrix}\frac{1}{1-q^{m-j-1}}=\begin{bmatrix} m-1 \\ j \end{bmatrix}\frac{1}{1-q^{m-1}}, \tag{4.1.10a}$$

$$\begin{bmatrix} m-2 \\ j \end{bmatrix}\frac{1}{1-q^{j+1}}=\begin{bmatrix} m-1 \\ j+1 \end{bmatrix}\frac{1}{1-q^{m-1}}, \tag{4.1.10b}$$

we obtain the following:

$$\begin{aligned}
&K_i=K_i^{(+)}+K_i^{(-)}, \qquad i=1, 2, 3,\\
&K_i^{(\pm)}=\pm\Big(\sum_{\substack{j\geq(m+\mu-1)/2\\ k\leq(m+\mu-3)/2+\varepsilon_i}}-\sum_{\substack{j\leq(m+\mu-3)/2\\ k\geq(m+\mu-1)/2+\varepsilon_i}}\Big)(-1)^{j-k}q^{Q_{j,k}^{(m-1)}(0,b)+Q_i^{(\pm)}(j)}\\
&\qquad\times\begin{bmatrix} m-1 \\ j+1-\varepsilon_{4-i} \end{bmatrix}\begin{bmatrix} m-1 \\ k \end{bmatrix}\frac{1}{(q)_{m-1}},
\end{aligned} \tag{4.1.11}$$

where

$$\varepsilon_i = [i/2],$$

$$Q_1^{(+)}(j) = Q_2^{(-)}(j) = \frac{m}{4}(c - 2\mu N) + \frac{1}{2}\mu N(j+1), \tag{4.1.12a}$$

$$Q_1^{(-)}(j) = \frac{m}{4}(c - 2b + 2N + 4) + \frac{1}{2}(b - N - 2)(j+1), \tag{4.1.12b}$$

$$Q_2^{(+)}(j) = -\frac{m}{4}c + \frac{c}{2}(j+1), \tag{4.1.12c}$$

$$Q_3^{(+)}(j) = -\frac{m}{4}c + \frac{1}{2}(c+2)(j+1), \tag{4.1.12d}$$

$$Q_3^{(-)}(j) = -\frac{m}{4}c + \frac{1}{2}(b + N + 2)(j+1). \tag{4.1.12e}$$

First consider $K_1^{(-)}$ (resp. $K_3^{(-)}$). Under the replacement $(j, k) \to (k, j)$ (resp. $(j, k) \to (k-1, j+1)$) in (4.1.11), the sums $(\sum - \sum)$ exchange the sign while the summands are invariant due to the property

$$Q_{k,j}^{(m-1)}(0, b) + Q_1^{(-)}(k) = Q_{j,k}^{(m-1)}(0, b) + Q_1^{(-)}(j),$$
$$Q_{k-1,j+1}^{(m-1)}(0, b) + Q_3^{(-)}(k-1) = Q_{j,k}^{(m-1)}(0, b) + Q_3^{(-)}(j).$$

Thus we have $K_1^{(-)} = K_3^{(-)} = 0$. Next we observe from (4.1.12a) that

$$\begin{aligned} &K_1^{(+)} + K_2^{(-)} \\ &= -\sum_{\substack{k=(m+\mu-1)/2 \\ j \in \mathbf{Z}}} (-1)^{j-k} q^{Q_{j,k}^{(m-1)}(0,b) + Q_1^{(+)}(j)} \begin{bmatrix} m-1 \\ j \end{bmatrix} \begin{bmatrix} m-1 \\ k \end{bmatrix} \frac{1}{(q)_{m-1}} \\ &= \frac{(-1)^{k-1}}{(q)_{m-1}} (q^{(3-\mu-m)/2})_{m-1} \begin{bmatrix} m-1 \\ k \end{bmatrix} \\ &\qquad \times q^{m^2/8 - \{((\mu-1)N - c + 2)/4\} m + (\mu^2 + (2b-4)\mu + 3)/8}. \end{aligned} \tag{4.1.13}$$

Here we have used the formula ([9], Theorem 3.3)

$$\sum_{i \in \mathbf{Z}} (-z)^i q^{i(i-1)/2} \begin{bmatrix} M \\ i \end{bmatrix} = (z)_M. \tag{4.1.14}$$

In general the product $(q^\rho)_{m-1}$ vanishes for $\rho = 0, -1, \cdots, -(m-2)$. In view of this and (4.1.8, 13) it turns out that $K_1^{(+)} + K_2^{(-)} = 0$. Finally we combine $K_2^{(+)}$ and $K_3^{(+)}$. Simplifying the sum of Gaussian polynomials by the formula

(4.1.15) $$\begin{bmatrix} m-1 \\ j \end{bmatrix} + \begin{bmatrix} m-1 \\ j+1 \end{bmatrix} q^{j+1} = \begin{bmatrix} m \\ j+1 \end{bmatrix},$$

we have

(4.1.16) $$\begin{aligned} &K_2^{(+)} + K_3^{(+)} \\ &= \Big(\sum_{\substack{j \geq (m+\mu-1)/2 \\ k \leq (m+\mu-1)/2}} - \sum_{\substack{j \leq (m+\mu-3)/2 \\ k \geq (m+\mu+1)/2}} \Big) (-1)^{j-k} q^{Q_{j,k}^{(m-1)}(0,b) + Q_2^{(+)}(j)} \\ &\quad \times \begin{bmatrix} m-1 \\ k \end{bmatrix} \begin{bmatrix} m \\ j+1 \end{bmatrix} \frac{1}{(q)_{m-1}}. \end{aligned}$$

Replace (j, k) in (4.1.16) by $(k-1, j)$ and recall the assumption $\nu = \mu + 1$. Then the resulting expression coincides with $f_m^{(N)}(b, c)$ (4.1.7) because of the identity

$$Q_{k-1,j}^{(m-1)}(0, b) + Q_2^{(+)}(k-1) = Q_{j,k}^{(m)}(b, c). \qquad \square$$

We call $f_m^{(N)}(b-a, c-a)$ the *fundamental solution* of the equation (4.1.1). It follows from (4.1.1) (though not obvious in (4.1.7)) that the function $f_m^{(N)}(b, c)$ or $\sqrt{q}\, f_m^{(N)}(b, c)$ is a *polynomial* in q with *positive* coefficients.

The fundamental solution satisfies an extra set of linear difference equations *at equal m* (Lemma 4.1.2 below). This is crucial when we consider the linear difference equation in the bounded domain of (b, c) with the restriction (4.0.2) (see section 4.4.)

Lemma 4.1.2 ((3.6) of Part I). *For* $1 \leq b \leq N$ *and* $m \geq 1$,

(4.1.17) $$\sum_{d=b-N, b-N+2, \cdots, N-b} q^{(a+md)/4} f_{m-1}^{(N)}(a+d, a-b) = (a \to -a).$$

Proof. There are two cases to consider:

(4.1.18a) (i) $(a+b-N, a-b+N) \subset R_\rho$ $(\rho \equiv m \bmod 2)$,

(4.1.18b) (ii) $(a+b-N, a-b+N) \subset R_{\rho-1} \cup R_{\rho+1}$ $(\rho \pm 1 \equiv m \bmod 2)$.

We prove here the case (i). The case (ii) is similar. From (4.1.18a) we see $(a+d, a-b) \in R_{\rho,\rho-1}$, $(a+d, a+b) \in R_{\rho,\rho+1}$. Substitute (4.1.7) into (4.1.17). After performing the d-summations therein the formula (4.1.10) leads us to the following expressions.

(4.1.19) $$\begin{aligned} &\text{the l.h.s. of (4.1.17)} = (A(+,+) + A(+,-))/(q)_{m-1}, \\ &\text{the r.h.s. of (4.1.17)} = (A(-,+) + A(-,-))/(q)_{m-1}, \end{aligned}$$

where $A(\varepsilon_1, \varepsilon_2)$ is given by ($\varepsilon_1, \varepsilon_2=\pm 1$)

$$A(\varepsilon_1, \varepsilon_2)=\varepsilon_2\Big(\sum_{\substack{k\leq(m+\rho-2)/2\\ j\geq(m+\rho-1-\varepsilon_1)/2}} - \sum_{\substack{k\geq(m+\rho)/2\\ j\leq(m+\rho-3-\varepsilon_1)/2}}\Big)$$

$$\times\begin{bmatrix} m-1 \\ k \end{bmatrix}\begin{bmatrix} m-1 \\ j+(1+\varepsilon_1)/2 \end{bmatrix}(-1)^{j-k} \tag{4.1.20}$$

$$\times q^{Q^{(m-1)}_{j,k}(a,\, a-\varepsilon_1 b)+\varepsilon_1 a/4+\varepsilon_1\varepsilon_2(b-N-1+\varepsilon_2)(j+1+(\varepsilon_1-1)m/2)/2}.$$

It turns out that $A(\pm, -)=0$. This can be seen from the invariance of the summand in (4.1.20) under the transformation $(j, k)\to(k-(1\pm 1)/2, j+(1\pm 1)/2)$. Now we calculate the difference of the remaining terms in (4.1.19).

$$\begin{aligned}
&A(+, +)-A(-, +)\\
&=-\Big(\sum_{\substack{j\geq(m+\rho)/2\\ k\leq(m+\rho-2)/2}} - \sum_{\substack{j\leq(m+\rho-2)/2\\ k\geq(m+\rho)/2}}\Big)(-1)^{j-k}\begin{bmatrix} m-1 \\ j \end{bmatrix}\begin{bmatrix} m-1 \\ k \end{bmatrix}\\
&\quad\times\Big(q^{Q^{(m-1)}_{j-1,k}(a,\,a-b)+a/4+(1/2)(b-N)j}+q^{Q^{(m-1)}_{j,k}(a,\,a+b)-a/4-(1/2)(N-b)(m-j-1)}\Big).
\end{aligned}$$

Again, this vanishes identically thanks to the invariance of the summand under the change $(j, k)\to(k, j)$. □

4.2. Various representations for $f^{(N)}_m(b, b+N)$

Here we establish various representations for the function

$$\hat{f}^{(N)}_m(v)=q^{m(m+1)N/4-v/2}f^{(N)}_m(b, b+N;\, q^{-1}), \tag{4.2.1a}$$

$$v=\frac{mN-b}{2}, \tag{4.2.1b}$$

other than the one obtained directly from (4.1.7). They are utilized in section 5 in order to examine the $m\to\infty$ behavior of the 1D configuration sum for regime II (see section 3.4 of Part I). Note from (4.1.3) that $\hat{f}^{(N)}_m(v)=0$ unless $0\leq v\leq mN$.

First we rewrite the double sum (4.1.7) into a single sum.

Theorem 4.2.1 ((3.27) of Part I). *For $m\geq 1$, $N\geq 1$ and $0\leq v\leq mN$,*

$$\begin{aligned}
\hat{f}^{(N)}_m(v)=&\sum_{j\in Z}(-1)^j\begin{bmatrix} v+m-(N+1)j-1 \\ m-1 \end{bmatrix}\begin{bmatrix} m-1 \\ j \end{bmatrix}q^{\mathscr{P}}\\
&+\sum_{j\in Z}(-1)^j\begin{bmatrix} v+m-(N+1)j-1 \\ m-1 \end{bmatrix}\begin{bmatrix} m-1 \\ j-1 \end{bmatrix}q^{\mathscr{P}-(N+2)j+v+m},
\end{aligned} \tag{4,2.2a}$$

$$\mathscr{P}=j(j-1)/2+j(mN-v+1). \tag{4.2.2b}$$

Proof. Set $[v/N]=\lambda$. We have

$$0\leq\lambda\leq m, \tag{4.2.3a}$$

$$\begin{aligned}&(b, b+N)=(mN-2v, (m+1)N-2v)\in R_{\mu,\nu},\\ &\mu=m-2\lambda-1, \qquad \nu=m-2\lambda.\end{aligned} \tag{4.2.3b}$$

Thus the formula (4.1.7) with (j, k, q) replaced by $(k-1, m-j, q^{-1})$ yields

$$\begin{aligned}&(q)_{m-1}f_m^{(N)}(b, b+N; q^{-1})\\ &=\Big(\sum_{\substack{j\geq\lambda+1\\ k\geq m-\lambda+1}}-\sum_{\substack{j\leq\lambda\\ k\leq m-\lambda}}\Big)(-1)^{j+k}\begin{bmatrix}m\\ j\end{bmatrix}\begin{bmatrix}m-1\\ k-1\end{bmatrix}\\ &\times q^{(j-k)(j-k+1)/2-(j-m/2)(k-(m+1)/2)N+((b+N)/2)(j-m/2)-(b/2)(k-(m+1)/2)}.\end{aligned} \tag{4.2.4}$$

We rewrite the sum

$$\Big(\sum_{\substack{j\geq\lambda+1\\ k\geq m-\lambda+1}}-\sum_{\substack{j\leq\lambda\\ k\leq m-\lambda}}\Big)$$

in (4.2.4) as

$$\Big(\sum_{\substack{j\in Z\\ k\geq m-\lambda+1}}-\sum_{\substack{j\leq\lambda\\ k\in Z}}\Big).$$

Applying (4.1.14) for the sum over $j\in Z$ or $k\in Z$, we get

$$\begin{aligned}\hat{f}_m^{(N)}(v)=&\sum_{k\geq m-\lambda+1}(-1)^k q^{(k/2)(k-1)+(k-1)v}\begin{bmatrix}m-1\\ k-1\end{bmatrix}\frac{(q^{1-v-(N+1)k+(m+1)N})_m}{(q)_{m-1}}\\ &+\sum_{j\leq\lambda}(-1)^j q^{(j/2)(j-1)+j(mN-v)}\begin{bmatrix}m\\ j\end{bmatrix}\frac{(q^{1+v-(N+1)j})_{m-1}}{(q)_{m-1}}\end{aligned} \tag{4.2.5}$$

$$\begin{aligned}=&-\sum_{j\leq\lambda}(-1)^j q^{(j/2)(j-1)+j(mN-v)}\begin{bmatrix}m-1\\ j-1\end{bmatrix}\frac{(q^{1+v-(N+1)j})_{m-1}}{(q)_{m-1}}\\ &\times(1-q^{v+m-(N+1)j})+\sum_{j\leq\lambda}(-1)^j q^{(j/2)(j-1)+j(mN-v)}\\ &\times\Big(q^j\begin{bmatrix}m-1\\ j\end{bmatrix}+\begin{bmatrix}m-1\\ j-1\end{bmatrix}\Big)\frac{(q^{1+v-(N+1)j})_{m-1}}{(q)_{m-1}}.\end{aligned} \tag{4.2.6}$$

We replaced k by $m+1-j$ in (4.2.5) and used the relation

$$(z)_M=(-z)^M q^{M(M-1)/2}(z^{-1}q^{1-M})_M, \qquad (z)_M=(z)_{M-1}(1-q^{M-1}z).$$

Notice that among the four terms in (4.2.6) the first and the last cancel each other. By virtue of (4.2.3) we can extend the sum $\sum_{j\leq\lambda}$ to $\sum_{j\in Z}$ in the remaining terms and rewrite

$$\frac{(q^{1+v-(N+1)j})_{m-1}}{(q)_{m-1}} \quad \text{as} \quad \begin{bmatrix} v+m-(N+1)j-1 \\ m-1 \end{bmatrix}.$$

Thus we arrive at (4.2.2). □

Theorem 4.2.2. *For* $m \geq 0$ *and* $N \geq 0$,

$$\hat{f}_m^{(N)}(v) = \hat{f}_m^{(N)}(mN-v). \tag{4.2.7}$$

Proof. Since the case $mN=0$ is trivial we assume $mN \geq 1$. Consider the first term in (4.2.2a)

$$\sum_{j \in Z} (-1)^j q^{(j/2)(j-1)+j(mN-v+1)} \begin{bmatrix} m-1 \\ j \end{bmatrix} \frac{(q^{1+v-(N+1)j})_{m-1}}{(q)_{m-1}}. \tag{4.2.8}$$

Expand the product $(q^{1+v-(N+1)j})_{m-1}$ by (4.1.14) and then take the j-sum by the same formula. The expression (4.2.8) is cast into the form with v replaced by $mN-v$. The second term can be handled similarly. □

The following is a slight modification of (4.2.2).

Lemma 4.2.3. *For* $m \geq 1$ *and* $N \geq 1$,

$$\begin{aligned} \hat{f}_m^{(N)}(v) = &\sum_{j \in Z} (-1)^j \begin{bmatrix} v+m-(N+1)j \\ v-(N+1)j \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} q^{\mathscr{P}} \\ &-\sum_{j \in Z} (-1)^j \begin{bmatrix} v+m-(N+1)j-1 \\ v-(N+1)j-1 \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} q^{\mathscr{P}+m-j}, \end{aligned} \tag{4.2.9}$$

where $\mathscr{P}$ *has been defined in* (4.2.2b).

Proof. We write down (4.2.2a) slightly modifying the second term

$$\begin{aligned} \hat{f}_m^{(N)}(v) = &\sum_{j \in Z} (-1)^j \begin{bmatrix} v+m-(N+1)j-1 \\ v-(N+1)j \end{bmatrix} \begin{bmatrix} m-1 \\ j \end{bmatrix} q^{\mathscr{P}} \\ &+\sum_{j \in Z} (-1)^j \begin{bmatrix} v+m-(N+1)j-1 \\ v-(N+1)j \end{bmatrix} \begin{bmatrix} m-1 \\ j-1 \end{bmatrix} q^{\mathscr{P}+m-j} \\ &-\sum_{j \in Z} (-1)^j \begin{bmatrix} v+m-(N+1)j-1 \\ v-(N+1)j \end{bmatrix} \begin{bmatrix} m-1 \\ j-1 \end{bmatrix} q^{\mathscr{P}+m-j} \\ &\times (1-q^{v-(N+1)j}). \end{aligned} \tag{4.2.10}$$

In (4.2.10) simplify the first two terms via the identity

$$\begin{bmatrix} m-1 \\ j \end{bmatrix} + \begin{bmatrix} m-1 \\ j-1 \end{bmatrix} q^{m-j} = \begin{bmatrix} m \\ j \end{bmatrix} \tag{4.2.11}$$

and rewrite the last term using

$$(4.2.12)\qquad \begin{aligned}&\begin{bmatrix} v+m-(N+1)j-1 \\ v-(N+1)j \end{bmatrix}\begin{bmatrix} m-1 \\ j-1 \end{bmatrix}(1-q^{v-(N+1)j}) \\ &=\begin{bmatrix} v+m-(N+1)j-1 \\ v-(N+1)j-1 \end{bmatrix}\begin{bmatrix} m \\ j \end{bmatrix}(1-q^{j}).\end{aligned}$$

We have thus

$$(4.2.13)\qquad \begin{aligned}\hat{f}_m^{(N)}(v)=&\sum(-1)^j\begin{bmatrix} v+m-(N+1)j-1 \\ v-(N+1)j \end{bmatrix}\begin{bmatrix} m \\ j \end{bmatrix}q^{\mathscr{P}} \\ &-\sum_{j\in Z}(-1)^j\begin{bmatrix} v+m-(N+1)j-1 \\ v-(N+1)j-1 \end{bmatrix}\begin{bmatrix} m \\ j \end{bmatrix}(q^{\mathscr{P}+m-j}-q^{\mathscr{P}+m}).\end{aligned}$$

Applying the formula (4.2.11) (with (m, j) replaced by $(v+m-(N+1)j, v-(N+1)j)$) for the first and the third terms in (4.2.13) we obtain (4.2.9). □

The $\hat{f}_m^{(N)}(v)$ is also characterized by a recurrence relation as given below. It is necessary to change both m and N therein.

Lemma 4.2.4. *For $m\geq 0$ and $N\geq 1$,*

$$(4.2.14)\qquad \hat{f}_m^{(N)}(v)=\sum_{0\leq i\leq m}\begin{bmatrix} m \\ i \end{bmatrix}q^{i(i+(N-1)m-v)}\hat{f}_{m-i}^{(N-1)}(v-Ni).$$

Proof. We show that each term in (4.2.9) satisfies the equation (4.2.14). Explicitly, we are to show

$$(4.2.15)\qquad \begin{aligned}&\sum_{j\in Z}(-1)^j\begin{bmatrix} v+m-(N+1)j-\delta \\ v-(N+1)j-\delta \end{bmatrix}\begin{bmatrix} m \\ j \end{bmatrix}q^{(j/2)(j-1)+j(mN-v+1)+\delta m} \\ &=\sum_{\substack{0\leq i\leq m \\ k\in Z}}(-1)^k\begin{bmatrix} m \\ i \end{bmatrix}\begin{bmatrix} v+m-i-(i+k)N-\delta \\ v-(i+k)N-\delta \end{bmatrix}\begin{bmatrix} m-i \\ k \end{bmatrix} \\ &\quad\times q^{(k/2)(k-1)+(i+k)(m(N-1)+i-v)+k+\delta(m-i-k)},\end{aligned}$$

where $\delta=0$ or 1 corresponding to the first and the second term in (4.2.9), respectively. In fact, each j-summand in the l.h.s. is equal to the sum in the r.h.s. with the restriction $i+k=j$:

$$(4.2.16)\qquad \begin{aligned}&\begin{bmatrix} v+m-(N+1)j \\ v-(N+1)j \end{bmatrix}\begin{bmatrix} m \\ j \end{bmatrix} \\ &=\sum_{\substack{0\leq i\leq j \\ k\in Z}}(-1)^i\begin{bmatrix} m \\ i \end{bmatrix}\begin{bmatrix} v+m-i-jN \\ v-jN \end{bmatrix}\begin{bmatrix} m-i \\ j-i \end{bmatrix}q^{(i/2)(i-1)-jm}.\end{aligned}$$

Let us prove (4.2.16). Noting the identity $\begin{bmatrix} m \\ i \end{bmatrix}\begin{bmatrix} m-i \\ j-i \end{bmatrix}=\begin{bmatrix} m \\ j \end{bmatrix}\begin{bmatrix} j \\ i \end{bmatrix}$, we cancel the factor $\begin{bmatrix} m \\ j \end{bmatrix}$. Replacing v by $v+(N+1)j$, we get ($m\geq 0$, $v\geq 0$, $j\geq 0$)

$$\begin{bmatrix} v+m \\ m \end{bmatrix}=\sum_{0\leq i\leq j}(-1)^i\begin{bmatrix} j \\ i \end{bmatrix}\begin{bmatrix} v+m+j-i \\ m-i \end{bmatrix}q^{(i/2)(i-1)-jm}. \tag{4.2.17}$$

Multiply (4.2.17) by z^m and sum it over $m\geq 0$. On account of the formula (4.1.14) and

$$\frac{1}{(z)_M}=\sum_{n\geq 0}\begin{bmatrix} M+n-1 \\ n \end{bmatrix}z^n, \tag{4.2.18}$$

([9], Theorem 3.3) (4.2.17) is equivalent to the obvious identity

$$\frac{1}{(z)_{v+1}}=\frac{(zq^{-j})_j}{(zq^{-j})_{v+j+1}}. \tag{4.2.19}$$

□

The recurrence relation leads us to an expression of $\hat{f}_m^{(N)}(v)$ in terms of the q-multinomial coefficient $\begin{bmatrix} m \\ x_0, \cdots, x_N \end{bmatrix}$. It is defined by [9]

$$\begin{bmatrix} m \\ x_0, \cdots, x_N \end{bmatrix}=(q)_m\Big/\prod_{j=0}^{m}(q)_{x_j}$$

$\qquad$ if $x_0+\cdots+x_N=m$ and $x_j\geq 0$ for all j,

$\qquad =0$ otherwise.

Theorem 4.2.5 ((3.29) of Part I). *For $m\geq 0$ and $N\geq 1$,*

$$\hat{f}_m^{(N)}(v)=\sum\begin{bmatrix} m \\ x_0, \cdots, x_N \end{bmatrix}q^{\sum_{j<k}(k-j-1)x_jx_k}, \tag{4.2.20}$$

where the outer sum is taken over all non-negative integers $x_0, \cdots, x_N$ such that

$$\sum_{j=0}^{N}x_j=m, \tag{4.2.21a}$$

$$\sum_{j=0}^{N}jx_j=v. \tag{4.2.21b}$$

Proof. Repeated use of Lemma 4.2.4 yields

(4.2.22a) $$\hat{f}_m^{(N)}(v)=\sum \begin{bmatrix} m \\ x_N \end{bmatrix}\begin{bmatrix} m-x_N \\ x_{N-1} \end{bmatrix}\cdots\begin{bmatrix} m-x_2-\cdots-x_N \\ x_1 \end{bmatrix} q^{\mathscr{X}} \times \hat{f}_{\tilde{m}}^{(0)}\Big(v-\sum_{j=1}^N jx_j\Big),$$

(4.2.22b) $$\tilde{m}=m-\sum_{j=2}^N x_j,$$

(4.2.22c) $$\mathscr{X}=\sum_{j=1}^N x_j\Big(x_j+(j-1)\Big(m-\sum_{k=j+1}^N x_k\Big)-v+\sum_{k=j+1}^N kx_k\Big)$$

where the sum in (4.2.22a) extends over non-negative integers $x_1, \cdots, x_N$ satisfying $x_1+\cdots+x_N\leq m$. Because of (4.1.4) the sum can be restricted to (4.2.21b). Eliminating m in (4.2.22c) by introducing an extra variable x_0 as in (4.2.21a), we have $\mathscr{X}=\sum_{j<k}(k-j-1)x_jx_k$. The product of the Gaussian polynomials in (4.2.22a) is nothing but the q-multinomial coefficient $\begin{bmatrix} m \\ x_0, \cdots, x_N \end{bmatrix}$. □

Remark. For $N=1$, Theorem 4.2.5 together with (4.1.3) and (4.2.1) provides us with a simple expression (cf. (2.3.6) of [5])

(4.2.23) $$f_m^{(1)}(b, c; q)=q^{bc/4}\begin{bmatrix} m \\ (m+b)/2 \end{bmatrix} \qquad (c=b\pm 1).$$

Although the multinomial expression (4.2.20) looks neat, it is not quite adequate for examining the large m behavior of $H_m^{(N)}(\beta)$ defined in section 5 (see (5.2.6)). The rest of this section is devoted to introducing $h_m^{(N)}(w)$ and its limit $h_\infty^{(N)}(w)=\lim_{m\to\infty} h_m^{(N)}(w)$ and deriving various representations for them (see (4.2.28) and (4.2.41)).

Lemma 4.2.6. *For $m\geq 1$ and $N\geq 1$,*

(4.2.24a) $$\hat{f}_m^{(N)}(v)=\sum(-1)^{y_0}\Big(\begin{bmatrix} v+m-2s_0-s_1+y_N-1 \\ v-s_1-y_0 \end{bmatrix}\frac{1}{(q)_{y_0}\cdots(q)_{y_N}} -\begin{bmatrix} v+m-2s_0-s_1+y_N \\ v-s_1-y_0 \end{bmatrix}\frac{1}{(q)_{y_0}\cdots(q)_{y_{N-1}}}\Big)q^{\mathscr{Z}},$$

(4.2.24b) $$s_0=\sum_{j=0}^N y_j,$$

(4.2.24c) $$s_1=\sum_{j=0}^N jy_j$$

(4.2.24d)
$$\mathcal{Q}=\sum_{0\le j<k\le N}(j-k)y_j y_k+\frac{y_0(y_0-1)}{2}-y_0(v-s_1)-y_N+(mN+1-s_1)s_0.$$

The sum in (4.2.24a) *extends over all non-negative integers* $y_0, \cdots, y_N$.

Proof. Using (4.2.11) we rewrite the r.h.s. of (4.2.24a) as

(4.2.25)
$$\sum_{s_0\ge 0}\Bigg(\sum_{y_0+\cdots+y_N=s_0}(-1)^{y_0}\begin{bmatrix} v+m-2s_0-s_1+y_N-1 \\ v-s_1-y_0\end{bmatrix}\frac{q^{\mathcal{Q}+y_N}}{(q)_{y_0}\cdots(q)_{y_N}} - \sum_{y_0+\cdots+y_N=s_0}(-1)^{y_0}\begin{bmatrix} v+m-2s_0-s_1+y_N-1 \\ v-s_1-y_0-1\end{bmatrix} \times\frac{q^{\mathcal{Q}+m-2s_0+y_0+y_N}}{(q)_{y_0}\cdots(q)_{y_N-1}}\Bigg).$$

We compare the first and the second terms in the sum with those in (4.2.2) through the identification of s_0 with j. A little calculation shows that it is sufficient to show

(4.2.26a)
$$\sum_{y_0+\cdots+y_K=s_0}(-1)^{y_1+\cdots+y_N}\begin{bmatrix} v+m-2s_0-s_1+y_N-1 \\ v-s_1-y_0\end{bmatrix}\times\frac{q^{\tilde{\mathcal{Q}}}}{(q)_{y_0}\cdots(q)_{y_N}} =\begin{bmatrix} v+m-(N+1)s_0-1 \\ v-(N+1)s_0,\ m-s_0-1,\ s_0\end{bmatrix},$$

(4.2.26b)
$$\tilde{\mathcal{Q}}=\frac{1}{2}\sum_{j=1}^{N}y_j^2+\sum_{1\le j<k\le N}(j-k+1)y_j y_k+\left(v+\frac{1}{2}\right)\sum_{j=1}^{N}y_j-s_0\left(\sum_{j=1}^{N}(j+1)y_j\right).$$

Here we have eliminated y_0 by (4.2.24b) and written the product of Gaussian polynomials in (4.2.2a) as the q-trinomial coefficient. We are going to show (4.2.26) by an induction with respect to v and $m(\ge 1)$. Let us denote the identity (4.2.26) (to be proved) by $[v, m, s_0]$. Clearly $[0, 1, s_0]$ holds as an equality $\delta_{0s_0}=\delta_{0s_0}$; so is $[v, m, s_0]$ for $v<0$ as $0=0$. Assume $[\hat{v}, \hat{m}, s_0]$ for $-\infty<\hat{v}\le v$, $1\le\hat{m}\le m$, $\hat{v}+\hat{m}<v+m$, $-\infty<s_0<\infty$. We split the both sides of $[v, m, s_0]$ into three terms using the standard formulas for the q-multinomial:

(4.2.27a)
$$\begin{bmatrix} C \\ C_1,\ C_2,\ C_3 \end{bmatrix} = \begin{bmatrix} C-1 \\ C_1,\ C_2-1,\ C_3 \end{bmatrix} + \begin{bmatrix} C-1 \\ C_1-1,\ C_2,\ C_3 \end{bmatrix} q^{C_2} + \begin{bmatrix} C-1 \\ C_1,\ C_2,\ C_3-1 \end{bmatrix} q^{C_1+C_2},$$

(4.2.27b)
$$\begin{bmatrix} D \\ D_1 \end{bmatrix} = \begin{bmatrix} D-1 \\ D_1 \end{bmatrix} + \begin{bmatrix} D-1 \\ D_1-1 \end{bmatrix} q^{D-D_1-y_N} - \begin{bmatrix} D-1 \\ D_1-1 \end{bmatrix} q^{D-D_1-y_N}(1-q^{y_N})$$

where $C=v+m-(N+1)s_0-1$, $C_1=v-(N+1)s_0$, $C_2=m-s_0-1$, $C_3=s_0$, $D=v+m-2s_0-s_1+y_N-1$ and $D_1=v-s_1-y_0$. The contributions to (4.2.26a) from the first, the second and the third terms in (4.2.27a) are respectively equal to those from (4.2.27b) due to $[v, m-1, s_0]$, $[v-1, m, s_0]$ and $[v-N-1, m-1, s_0-1]$. Thus we have proved $[v, m, s_0]$. □

Theorem 4.2.7. *For $m\geq 1$ and $N\geq 1$,*

(4.2.28a)
$$\hat{f}_m^{(N)}(v) = \sum_{w\in \mathbf{Z}} (-1)^w q^{-w(w-1)/2+wm} \begin{bmatrix} Nm-2w \\ v-w \end{bmatrix} h_m^{(N)}(w),$$

(4.2.28b)
$$h_m^{(N)}(w) = {\sum}' (-1)^{\bar{s}_1} \frac{q^{\mathscr{R}}}{(q)_{y_1}\cdots(q)_{y_{N-1}}} h_\infty^{(N)}(w-\bar{s}_1),$$

(4.2.28c)
$$\bar{s}_0 = \sum_{j=1}^{N-1} y_j, \qquad \bar{s}_1 = \sum_{j=1}^{N-1} j y_j,$$

(4.2.28d)
$$\mathscr{R} = -\sum_{j=1}^{N-1} j y_j^2 - 2 \sum_{1\leq j<k\leq N-1} k y_j y_k + (1+mN-2(w-\bar{s}_1))\bar{s}_0 + (w-m-(\bar{s}_1+1)/2)\bar{s}_1,$$

(4.2.28e)
$$h_\infty^{(N)}(w) = \delta_{w0} \qquad \text{if } N=1,$$
$$= q^{w(w-1)} \sum_j (-1)^{Nj} \frac{q^{N^2j^2/2-(N+1)wj}}{(q)_{w-Nj}} \left(\frac{q^{Nj/2}}{(q)_j} - \frac{q^{-Nj/2+w}}{(q)_{j-1}} \right),$$
otherwise.

Here the sum ${\sum}'$ in (4.2.28b) *is taken over non-negative integers $y_1, \cdots, y_{N-1}$ under the restriction $0\leq \bar{s}_1 \leq w$. The j-sum in* (4.2.28e) *extends over $0\leq j\leq [w/N]$ or $1\leq j\leq [w/N]$ for the first or second term, respectively.*

Proof. The case $N=1$ is clear from (4.2.1) and (4.2.23) (See also (4.2.33) below.) For $N\geq 2$ we reduce (4.2.28) to (4.2.24). Expanding $\begin{bmatrix} Nm-2w \\ v-w \end{bmatrix}$ using the formula (4.1.14), we get

(4.2.29)
$$\begin{bmatrix} Nm-2w \\ v-w \end{bmatrix} = (q^{Nm-v-w+1})_{v-w}/(q)_{v-w}$$
$$= \sum_{y_0 \ge 0} (-1)^{y_0} q^{y_0(y_0-1)/2+(Nm-v-w+1)y_0} \frac{1}{(q)_{y_0}(q)_{v-w-y_0}}.$$

Combine the factor $1/(q)_{v-w-y_0}$ in (4.2.29) with $1/(q)_{w-\bar{s}_1-Nj}$ coming from $h_\infty^{(N)}(w-\bar{s}_1)$ as

(4.2.30)
$$\frac{1}{(q)_{v-w-y_0}(q)_{w-\bar{s}_1-Nj}} = \frac{1}{(q)_{v-\bar{s}_1-y_0-Nj}} \begin{bmatrix} v-\bar{s}_1-y_0-Nj \\ w-\bar{s}_1-Nj \end{bmatrix}.$$

After replacing w by $w+\bar{s}_1+Nj$, we perform the w-summation by applying (4.1.14). Then we have

(4.2.31a)
$$\hat{f}_m^{(N)}(v) = \sum (-1)^{y_0} \frac{q^{\mathscr{R}'}}{(q)_{y_0}(q)_{y_1}\cdots(q)_{y_{N-1}}(q)_{v-\bar{s}_1-y_0-Nj}}$$
$$\times \left(\frac{(q^{m-2\bar{s}_0-y_0-j})_{v-\bar{s}_1-y_0-Nj}}{(q)_j} - \frac{(q^{m-2\bar{s}_0-y_0-j+1})_{v-\bar{s}_1-y_0-Nj}}{(q)_{j-1}} \right).$$

(4.2.31b)
$$\mathscr{R}' = \mathscr{R}|_{w=0} + y_0(y_0-1)/2 + (Nm-v+1)y_0$$
$$+\bar{s}_1(\bar{s}_1/2+m-2\bar{s}_0-y_0+1/2) - Nj(j-m+2\bar{s}_0+y_0).$$

Here the sum $\sum$ in (4.2.31a) extends over all non-negative integers $j, y_0, y_1, \cdots, y_{N-1}$. Setting

(4.2.32)
$$j = y_N, \quad \bar{s}_0 = s_0 - y_0 - y_N, \quad \bar{s}_1 = s_1 - Ny_N,$$

we obtain the expression identical with (4.2.24). □

Remark. From (4.2.28b–e) the following formula holds:

(4.2.33)
$$h_m^{(1)}(w) = \delta_{w0}, \qquad h_m^{(N)}(0) = 1.$$

The $h_\infty^{(N)}(w)$ is characterized by the following recurrence relation along with the initial condition $h_m^{(1)}(w) = \delta_{w0}$.

Lemma 4.2.8. *For $N \geq 2$,*

(4.2.32)
$$h_\infty^{(N)}(w) = \sum_{k \ge 0} \frac{(-1)^{Nk}}{(q)_k} q^{-(N^2/2-2N+1)k^2+((N-3)w-N/2+1)k}$$
$$\times h_\infty^{(N-1)}(w-(N-1)k).$$

Proof. Substitute (4.2.28e) into the r.h.s. of (4.2.34). After replacing j by $j-k$, we combine the products as

(4.2.35)
$$\frac{1}{(q)_k(q)_{j-k-\delta}}=\frac{1}{(q)_{j-\delta}}\begin{bmatrix} j-\delta \\ k \end{bmatrix},$$

where $\delta=0$ or 1 corresponding to the first and the second terms in (4.2.28e). Performing the k-summations in the resulting expression by applying the formula (4.1.14), we get

(4.2.36)
$$\begin{aligned}
&(\text{the r.h.s. fo (4.2.34)})\\
&=q^{w(w-1)}\sum_{j\geq 0}(-1)^{(N-1)j}\frac{q^{(N-1)^2j^2/2-Nwj}}{(q)_{w-(N-1)j}}\\
&\quad\times\left(\frac{(q^{(N-1)j-w+1})_j q^{(N-1)j/2}}{(q)_j}-\frac{(q^{(N-1)j-w+1})_{j-1}q^{-(N-1)j/2+w}}{(q)_{j-1}}\right)\\
&=q^{w(w-1)}\sum_{j\geq 0}(-1)^{Nj}\frac{q^{N^2j^2/2-(N+1)wj}}{(q)_{w-Nj}}\\
&\quad\times\left(\frac{(1-q^{w-Nj})q^{Nj/2}}{(q)_j(1-q^{w-(N-1)j})}+\frac{q^{-(3N/2-1)j+2w}}{(q)_{j-1}(1-q^{w-(N-1)j})}\right).
\end{aligned}$$

where we used the formulas such as

$$(q)_{w-(N-1)j}=(q)_{w-Nj}(1-q^{w-Nj+1})\cdots(1-q^{w-(N-1)j}),$$
$$(q^{(N-1)j-w+1})_j=(-1)^j q^{(N-1/2)j^2-(w-1/2)j}(q^{w-Nj})_j, \quad etc.$$

The first and the second terms in (4.2.36) decompose into two terms

(4.2.37a) $$(1-q^{w-Nj})q^{Nj/2}=(1-q^{w-(N-1)j})q^{Nj/2}-(1-q^j)q^{-Nj/2+w},$$

(4.2.37b) $$q^{-(3N/2-1)j+2w}=-(1-q^{w-(N-1)j})q^{-Nj/2+w}+q^{-Nj/2+w},$$

where the first terms give rise to $h_\infty^{(N)}(w)$ while the second terms cancel each other. □

Similarly as Lemma 4.2.4 leads to Theorem 4.2.5, Lemma 4.2.8 leads to the following.

Lemma 4.2.9. *For* $N\geq 1$,

(4.2.38a) $$h_\infty^{(N)}(w)=\sum{}'(-1)^{w+\sum_{1\leq j\leq N-1}x_j}\frac{q^{\mathscr{T}_N(w)}}{(q)_{x_1}\cdots(q)_{x_{N-1}}},$$

(4.2.38b)
$$\begin{aligned}
\mathscr{T}_N(w)=&-\sum_{1\leq j\leq N-1}\left(j-\frac{1}{2}\right)x_j^2-2\sum_{1\leq j<k\leq N-1}jx_jx_k\\
&+\frac{1}{2}\sum_{1\leq j\leq N-1}x_j+\frac{1}{2}w(w-1),
\end{aligned}$$

where the sum $\sum'$ *in* (4.2.38a) *is taken over non-negative integers* $x_1, \cdots, x_{N-1}$ *under the restriction*

(4.2.38c)
$$\sum_{1\le j\le N-1} jx_j = w.$$

Proof. The case $N=1$ is clearly true. (See (4.2.28e).) For $N\ge 2$, we are going to show that the r.h.s. of (4.2.38a) satisfies the recurrence relation (4.2.34). Let us denote x_{N-1} by i. Direct calculation shows that

(4.2.39a)
$$\mathcal{T}_N(w) = \mathcal{T}_{N-1}(w-(N-1)i) - (N^2/2-2N+1)i^2 + ((N-3)w - N/2+1)i,$$

(4.2.39b)
$$w + \sum_{1\le j\le N-1} x_j = Ni + w - (N-1)i + \sum_{1\le j\le N-2} x_j.$$

Thus the r.h.s. of (4.2.38a) can be written as

(4.2.40a)
$$\sum_{i\ge 0} \frac{(-1)^{Ni}}{(q)_i} q^{-(N^2/2-2N+1)i^2+((N-3)w-N/2+1)i} \times \sum{}'' (-1)^{w-(N-1)i+\sum_{1\le j\le N-2} x_j} \frac{q^{\mathcal{T}_{N-1}(w-(N-1)i)}}{(q)_{x_1}\cdots(q)_{x_{N-2}}}.$$

Here the sum $\sum''$ is taken over non-negative integers $x_1, \cdots, x_{N-2}$ such that

(4.2.40b)
$$\sum_{1\le j\le N-2} jx_j = w-(N-1)i.$$

This is nothing but the r.h.s. of (4.2.38a) with N and w replaced by $N-1$ and $w-(N-1)i$, respectively. Thus we conclude that the r.h.s. of (4.2.38a) satisfies (4.2.34). □

Lemma 4.2.9 gives another expression for $h_m^{(N)}(w)$ as given below. It is remarkable that as a consequence we have so many different expressions for one and the same quantity $\hat{f}_m^{(N)}(v)$.

Theorem 4.2.10. *For* $N\ge 1$,

(4.2.41a)
$$h_m^{(N)}(w) = \sum{}' (-1)^{w+\sum_{1\le j\le N-1} x_j} q^{\mathcal{T}_N(w)} \times \begin{bmatrix} m \\ x_{N-1} \end{bmatrix} \begin{bmatrix} 2m-2x_{N-1} \\ x_{N-2} \end{bmatrix} \cdots \times \begin{bmatrix} (N-1)m - 2(x_2+2x_3+\cdots+(N-2)x_{N-1}) \\ x_1 \end{bmatrix}.$$

Here $\mathcal{T}_N(w)$ *is given by* (4.2.38b). *The sum* $\sum'$ *is taken over non-negative integers* $x_1, \cdots, x_{N-1}$ *subject to*

$$\sum_{1\le j\le N-1} jx_j = w. \tag{4.2.41b}$$

Proof. Expand all the Gaussian polynomials in a way like (see (4.1.14))

$$\begin{aligned}\begin{bmatrix} m \\ x_{N-1}\end{bmatrix} &= \frac{(q^{m-x_{N-1}+1})_{x_{N-1}}}{(q)_{x_{N-1}}} \\ &= \sum_{y_{N-1}} (-1)^{y_{N-1}} \frac{q^{y_{N-1}(y_{N-1}-1)/2+y_{N-1}(m-x_{N-1}+1)}}{(q)_{y_{N-1}}(q)_{x_{N-1}-y_{N-1}}}, \quad etc.\end{aligned} \tag{4.2.42}$$

Then the r.h.s. of (4.2.41a) is cast into the form

$$\begin{aligned}&\sum_{0\le \bar{s}_1\le w} \sum{}' \frac{q^{\sum_{1\le j\le N-1} y_j((y_j+1)/2+m(N-j))}}{(q)_{y_1}\cdots(q)_{y_{N-1}}} \\ &\quad\times \sum{}'' (-1)^{w+\sum_{1\le j\le N-1}(x_j-y_j)} \\ &\quad\times \frac{q^{\mathcal{T}_N(w)-\sum_{1\le j\le N-1} x_j y_j - 2\sum_{1\le j<k\le N-1}(k-j)x_k y_j}}{(q)_{x_1-y_1}\cdots(q)_{x_{N-1}-y_{N-1}}},\end{aligned} \tag{4.2.43}$$

where the sum $\sum'$ (resp. $\sum''$) is taken over non-negative integers $y_1, \cdots, y_{N-1}$ (resp. $x_1, \cdots, x_{N-1}$) satisfying $\sum_{1\le j\le N-1} jy_j = \bar{s}_1$ (resp. $\sum_{1\le j\le N-1} jx_j = w$). If we replace x_j by x_j+y_j, the condition $\sum_{1\le j\le N-1} jx_j = w$ changes to

$$\sum_{1\le j\le N-1} jx_j = w-\bar{s}_1. \tag{4.2.44}$$

After a little calculation (4.2.43) turns out to be written as follows

$$\begin{aligned}&\sum_{0\le \bar{s}_1\le w} (-1)^{\bar{s}_1} \sum{}' \frac{q^{\mathcal{R}}}{(q)_{y_1}\cdots(q)_{y_{N-1}}} \\ &\quad\times \sum{}''' (-1)^{w-\bar{s}_1+\sum_{1\le j\le N-1} x_j} \frac{q^{\mathcal{T}_N(w-\bar{s}_1)}}{(q)_{x_1}\cdots(q)_{x_{N-1}}}.\end{aligned} \tag{4.2.45}$$

Here the sum $\sum'$ is taken in the same manner as (4.2.43) and the sum $\sum''$ extends over all non-negative integers $x_1, \cdots, x_{N-1}$ with the condition (4.2.44). The power $\mathcal{R}$ is given by (4.2.28c-d). Now we apply Lemma 4.2.9 to identify the sum $\sum'''$ in (4.2.45) with $h_\infty^{(N)}(w-\bar{s}_1)$. Comparing the resulting expression with (4.2.28b-d) we arrive at (4.2.41). □

Remark. For $N=2$, (4.2.41) yields the simple formula

(4.2.46) $$h_m^{(2)}(w)=\begin{bmatrix} m \\ w \end{bmatrix}.$$

4.3. Fundamental solution for the modified linear difference equation

In this paragraph we assume that $L \geq 2N+3$. Let $H(n, a, b, c)$ denote the function $H(a, b, c)$ (3.1.12) with n regarded as an integer parameter (not necessarily $[L/2]$). Note that $H(n, a, b, c)$ now acquires the translational invariance

(4.3.1) $$H(n, a, b, c)=H(n+1, a+1, b+1, c+1).$$

In this paragraph we study the function $g_m^{(N)}(n; b, c)$ ((b, c): weakly admissible, $m \geq 0$, $N \geq 1$) characterized by the following linear difference equation.

(4.3.2a) $$g_m^{(N)}(n; b, c)=\sum_d{}' g_{m-1}^{(N)}(n; d, b)q^{mH(n,d,b,c)},$$

(4.3.2b) $$g_0^{(N)}(n; b, c)=\delta_{b0}.$$

Here the sum $\sum'$ is taken over d such that the pair (d, b) is weakly admissible. We set $g_m^{(N)}(n; b, c)=0$ if (b, c) is not weakly admissible.

We recall that the $H(a, b, c)$ is akin to the negative of $|a-c|/4$ employed in the linear difference equation (4.1.1). The effect of this modification is fully absorbed by patchwork construction of $g(n; b, c)$ (see (4.3.12)). By the definition (4.3.2) together with the symmetries of $H(n, a, b, c)$ (3.1.13), (4.3.1) the following formulas are valid.

(4.3.3) $$g_m^{(N)}(n; b, c)= g_m^{(N)}(-n-1; -b, -c) \qquad \text{for } m \geq 0,$$

(4.3.4) $$g_m^{(N)}(n; b, c)=0 \quad \text{unless } |b| \leq mN, \quad |c| \leq (m+1)N, \quad \text{and } b \equiv mN \bmod 2.$$

In order to describe $g_m^{(N)}(n; b, c)$ we prepare some notations. For a weakly admissible pair of integers (b, c) define the four regions $S_1(n), \cdots, S_4(n)$ as follows.

(4.3.5)
$$\begin{aligned} S_1(n)&=\{(b, c) \mid b+c<2n+1-N\}, \\ S_2(n)&=\{(b, c) \mid b+c>2n+1-N,\ b<n+1/2\}, \\ S_3(n)&=\{(b, c) \mid b+c<2n+1+N,\ b>n+1/2\}, \\ S_4(n)&=\{(b, c) \mid b+c>2n+1+N\}. \end{aligned}$$

We set

(4.3.6) $$\kappa=[(2n+1)/2N],$$

and further subdivide the regions $S_{2,3}(n)$.

(4.3.7)
$$\begin{aligned} S_2^{(\pm)}(n)&=\{(b,c)\,|\,(b,c)\in S_2(n),\ b\in R_{\kappa\pm1}\}, &&\text{if } m+\kappa \text{ is even,}\\ &=\{(b,c)\,|\,(b,c)\in S_2(n),\ b\gtrless 2n+1-(\kappa+1)N\}, &&\text{otherwise.}\\ S_3^{(\pm)}(n)&=\{(b,c)\,|\,(b,c)\in S_3(n),\ b\lessgtr 2n+1-\kappa N\}, &&\text{if } m+\kappa \text{ is even,}\\ &=\{(b,c)\,|\,(b,c)\in S_3(n),\ b\in R_{\kappa+1\mp1}\}. &&\text{otherwise.}\end{aligned}$$

These regions are schematically shown in Fig. 4.1.

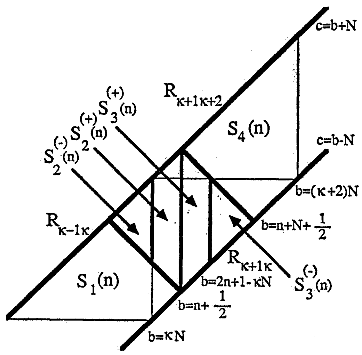


Fig. 4. 1 a Regions of (b, c) when $m+\kappa$ is even. $S_1(n)$ and $S_3(n)$ are the infinite domains bounded by the bold lines.

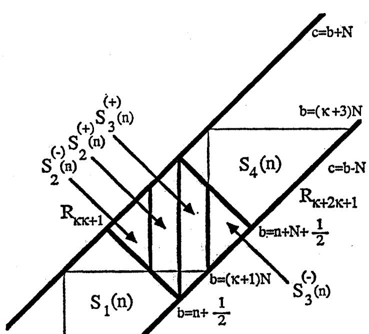


Fig. 4. 1 b The case $m+\kappa$ is odd.

The following relations are readily derivable from (4.3.5–7)

(4.3.8a) $$(b,c)\in S_1(n)\longleftrightarrow(-b,-c)\in S_4(-n-1),$$

(4.3.8b) $$(b,c)\in S_2^{(\pm)}(n)\longleftrightarrow(-b,-c)\in S_3^{(\pm)}(-n-1).$$

For a weakly admissible pair $(b, c) \in R_{\mu,\nu}$ with μ, ν satisfying (4.1.6) we define a function $\bar{f}_m^{(N)}(b, c; \mu, \nu)$ by

$$(q)_{m-1}\bar{f}_m^{(N)}(b, c; \mu, \nu) \tag{4.3.9a}$$
$$=\Big(\sum_{\substack{j\geq 1+(m+\nu)/2 \\ k\geq (m-\mu+1)/2}} - \sum_{\substack{j\leq (m+\nu)/2 \\ k\leq (m-\mu-1)/2}}\Big)(-1)^{j+k}q^{P_{j,k}^{(m)}(b,c)}\begin{bmatrix} m-1 \\ j-1 \end{bmatrix}\begin{bmatrix} m \\ k \end{bmatrix},$$

$$P_{j,k}^{(m)}(b, c) = -\frac{j}{2}b + \frac{k}{2}c + \Big(\frac{m}{2}j + \frac{m+1}{2}k - jk\Big)N + \frac{1}{2}(j-k)(j-k-1), \qquad \text{if } m\geq 1, \tag{4.3.9b}$$

$$\bar{f}_0^{(N)}(b, c; \mu, \nu) = \delta_{b0}. \tag{4.3.9c}$$

The $\bar{f}_m^{(N)}$ satisfies the following linear difference equation.

Lemma 4.3.1. *For $m\geq 1$,*

$$\bar{f}_m^{(N)}(b, c; \mu, \nu) = \sum_d{}' \bar{f}_{m-1}^{(N)}(d, b\colon \lambda, \mu)q^{m(\min(d,c)-b+N)/2}, \tag{4.3.10}$$

where the sum $\sum'$ is taken over d such that the pair (d, b) is weakly admissible and λ is specified by $d \in R_\lambda$, $\lambda\equiv m$ mod 2 (see (4. 1.5–8)).

Proof. Comparing (4.1.7) and (4.3.9) it is straightforward to see $(m\geq 0)$

$$\bar{f}_m^{(N)}(b, c; \mu, \nu) = f_m^{(N)}(b, c; q^{-1})q^{Nm^2/4-(m/4)(b-c-N)-b/4}. \tag{4.3.11}$$

We substitute this into (4.3.10). The resulting equation for $f_m^{(N)}(b, c; q^{-1})$ turns out to be equal to (4.1.1a) with q replaced by q^{-1} (note that $\min(d, c) = (d+c-|d-c|)/2$). Thus we have (4.3.10) by Theorem 4.1.1. □

Now we construct $g_m^{(N)}(n; b, c)$ from $\bar{f}_m^{(N)}$ by patchwork.

Theorem 4.3.2 ((3.14) of Part I). *For all weakly admissible pairs (b, c) $\in R_{\mu,\nu}$ and integers m, $N\geq 1$,*

$g_m^{(N)}(n; b, c)$

(4.3.12a) $\quad = \bar{f}_m^{(N)}(b, c; \mu, \nu), \qquad$ *if* $(b, c) \in S_1(n) \cap R_{\mu,\nu}$,

(4.3.12b) $\quad = \bar{f}_m^{(N)}(-b, -c; -\mu, -\nu), \qquad$ *if* $(b, c) \in S_4(n) \cap R_{\mu,\nu}$,

(4.3.12c) $\quad = \bar{f}_m^{(N)}(b, 2n-b-N; \kappa\pm(1-\rho), \kappa\mp\rho), \qquad$ *if* $(b, c) \in S_2^{(\pm)}(n)$, (4.3.12d) $$=\bar{f}_m^{(N)}(-b,\ -2n-2+b-N;\ -\kappa-1\pm\rho,\ -\kappa-1\pm(\rho-1)),\quad \text{if } (b,c)\in S_3^{(\pm)}(n),$$

where $\rho=0$ *or* 1 *according as* $m+\kappa$ *is even or odd, respectively.*

Proof. The properties (4.3.3) and (4.3.4) can be directly checked by using (4.1.3) and (4.3.8, 11, 12). In view of (4.3.3, 8) we restrict ourselves to the case $(b, c)\in S_1(n)$ or $S_2^{(\pm)}(n)$. If $(b, c)\in S_1(n)$ and $b<n-N+1/2$, then for any d such that (d, b) is weakly admissible,

$$H(n, d, b, c)=(\min(d, c)-b+N)/2 \quad \text{and} \quad (d, b)\in S_1(n).$$

By virtue of (4.3.12a) equation (4.3.2a) in this case reduces to (4.3.10), which has been already proved. We list up the non-trivial cases (see Fig. 4.1).

If $m+\kappa$ is odd:

(4.3.13a) $(b, c)\in S_1(n)\cap R_{\kappa,\kappa\pm1}$,

(4.3.13b) $\in S_2^{(\pm)}(n)$.

If $m+\kappa$ is even:

(4.3.13c) $(b, c)\in S_1(n)\cap R_{\kappa-1,\kappa-1\pm1}$,

(4.3.13d) $\in S_1(n)\cap R_{\kappa+1,\kappa}$,

(4.3.12e) $\in S_2^{(\pm)}(n)$.

We prove here the case (4.3.13a). The other cases are similar. Let us write down the r.h.s. of (4.3.2a) taking (4.3.12) and (4.3.13a) into account.

(4.3.14)
$$\begin{aligned}&\sum_{b-N\le d\le\kappa N}{}' \bar{f}_{m-1}^{(N)}(d, b;\ \kappa-1,\ \kappa)q^{m((\min(d,c)-b+N)/2)}\\ &+\sum_{\kappa N\le d<2n+1-N-b}{}' \bar{f}_{m-1}^{(N)}(d, b;\ \kappa+1,\ \kappa)q^{m((\min(d,c)-b+N)/2)}\\ &+\sum_{2n+1-N-b<d<n+1/2}{}' \bar{f}_{m-1}^{(N)}(d,\ 2n-d-N;\ \kappa+1,\ \kappa)q^{m((c-b+N)/2)}\\ &+\sum_{n+1/2<d\le b+N}{}' \bar{f}_{m-1}^{(N)}(-d,\ -2n-2+d-N;\ -\kappa-1,\ -\kappa-2)\\ &\qquad\times q^{m((c-b+N)/2)}.\end{aligned}$$

Here d in the sums $\sum'$ runs over the series $d=b-N, b-N+2, \cdots, b+N$ divided into the four regions as above. The term $d=\kappa N$ is contained in the first (resp. second) sum if $(d, b)\in R_{\kappa-1,\kappa}$ (resp. $R_{\kappa+1,\kappa}$). On the other hand by using Lemma 4.3.1 the l.h.s. of (4.3.2a) is expressed as

$$
\begin{aligned}
&\sum_{b-N\le d\le \kappa N}' \bar{f}^{(N)}_{m-1}(d, b; \kappa-1, \kappa) q^{m((\min(d,c)-b+N)/2)} \\
(4.3.15)\qquad &+ \sum_{\kappa N\le d<2n+1-N-b}' \bar{f}^{(N)}_{m-1}(d, b; \kappa+1, \kappa) q^{m((\min(d,c)-b+N)/2)} \\
&+ \sum_{2n+1-N-b<d\le b+N}' \bar{f}^{(N)}_{m-1}(d, b; \kappa+1, \kappa) q^{m((c-b+N)2)},
\end{aligned}
$$

where the d-sum is taken in the same manner as (4.3.14). Comparing (4.3.14) with (4.3.15) we are led to the following equality to show

$$(4.3.16)\qquad T_1 = T_2 + T_3,$$

$$(4.3.17a)\qquad T_1 = \sum_{2n+2-N-b\le d\le b+N}' \bar{f}^{(N)}_{m-1}(d, b; \kappa+1, \kappa),$$

$$(4.3.17b)\qquad T_2 = \sum_{2n+2-N-b\le d\le \tilde{n}}' \bar{f}^{(N)}_{m-1}(d, 2n-d-N; \kappa+1, \kappa),$$

$$(4.3.17c)\qquad T_3 = \sum_{\tilde{n}+2\le d\le b+N}' \bar{f}^{(N)}_{m-1}(-d, -2n-2+d-N; -\kappa-1, -\kappa-2),$$

with d taking the values in $\{b-N, b-N+2, \cdots, b+N\}$ and $\tilde{n}=n$ or $n-1$ according as $b+N\equiv n$ or $n-1 \bmod 2$. Substituting (4.3.9) into (4.3.17) and performing the d-summations, we get

$$(4.3.18)\qquad T_i = T_i^{(+)} + T_i^{(-)}, \qquad i=1, 2, 3,$$

$$
\begin{aligned}
&T_i^{(\pm)} = \pm\Big(\sum_{\substack{j\ge(m+\kappa+1)/2\\ k\ge(m-\kappa-1)/2}} - \sum_{\substack{j\le(m+\kappa-1)/2\\ k\le(m-\kappa-3)/2}}\Big)(-1)^{j+k} \\
(4.3.19a)\qquad &\times \begin{bmatrix} m-2 \\ \alpha_i-1 \end{bmatrix} \begin{bmatrix} m-1 \\ \beta_i \end{bmatrix} \frac{q^{P_i(\pm)}}{1-q^{\gamma_i}},
\end{aligned}
$$

$$
\begin{aligned}
(4.3.19b)\qquad &(\alpha_1, \alpha_2, \alpha_3) = (j, j, k) \quad (\beta_1, \beta_2, \beta_3) = (k, k, j), \\
&(\gamma_1, \gamma_2, \gamma_3) = (j, j+k, j+k),
\end{aligned}
$$

$$
\begin{aligned}
&P_1(+) = P^{(m-1)}_{j,k}(b+N, b), \\
&P_1(-) = P_2(-) = P^{(m-1)}_{j,k}(2n-N-b, b), \\
(4.3.19c)\qquad &P_2(+) = P^{(m-1)}_{j,k}(\tilde{n}, 2n-\tilde{n}-N), \\
&P_3(+) = P^{(m-1)}_{k,j}(-\tilde{n}-2, \tilde{n}-2n-N), \\
&P_3(-) = P^{(m-1)}_{k,j}(-b-N-2, b-2n).
\end{aligned}
$$

First consider $T_1^{(+)}$ and rewrite the factor $\dfrac{1}{1-q^j}\begin{bmatrix} m-1 \\ j-1 \end{bmatrix}$ therein by the formula (4.1.10b). Under the replacement (j, k) by $(m-1-k, m-1-j)$ the summand in (4.3.19a) for $T_1^{(+)}$ is invariant because of

$$P^{(m-1)}_{m-1-k, m-1-j}(b+N, b) = P^{(m-1)}_{j,k}(b+N, b),$$

while the sum $(\sum - \sum)$ changes the sign. Hence we have $T_1^{(+)}=0$. Next we combine $T_2^{(+)}$ and $T_3^{(+)}$. Using the formula

$$\text{(4.3.20)}\quad \begin{aligned}&\frac{1}{1-q^{j+k}}\left(\begin{bmatrix} m-2 \\ j-1 \end{bmatrix}\begin{bmatrix} m-1 \\ k \end{bmatrix}+\begin{bmatrix} m-2 \\ k-1 \end{bmatrix}\begin{bmatrix} m-1 \\ j \end{bmatrix}q^j\right)\\ &=(j\leftrightarrow k)=\frac{1}{1-q^{m-1}}\begin{bmatrix} m-1 \\ j \end{bmatrix}\begin{bmatrix} m-1 \\ k \end{bmatrix},\end{aligned}$$

we have

$$\text{(4.3.21)}\quad \begin{aligned}T_2^{(+)}+T_3^{(+)}=&\Big(\sum_{\substack{j\geq(m+\kappa+1)/2\\ k\geq(m-\kappa-1)/2}}-\sum_{\substack{j\leq(m+\kappa-1)/2\\ k\leq(m-\kappa-3)/2}}\Big)(-1)^{j+k}\\ &\times\begin{bmatrix} m-1 \\ j \end{bmatrix}\begin{bmatrix} m-1 \\ k \end{bmatrix}\frac{q^P}{1-q^{m-1}},\end{aligned}$$

where $P=P_{j,k}^{(m-1)}(n, n-N)$ or $P_{k,j}^{(m-1)}(-n-1, -n-1-N)$ according as $\tilde{n}=n$ or $n-1$. The same argument exploiting the transformation $(j, k)\rightarrow(m-1-k, m-1-j)$ leads us to $T_2^{(+)}+T_3^{(+)}=0$. Finally we verify the equality that holds among the remaining terms in (4.3.16):

$$\text{(4.3.22)}\quad T_1^{(-)}-T_2^{(-)}=T_3^{(-)}.$$

Using the identity

$$\left(\frac{1}{1-q^j}-\frac{1}{1-q^{j+k}}\right)\begin{bmatrix} m-2 \\ j-1 \end{bmatrix}\begin{bmatrix} m-1 \\ k \end{bmatrix}=\frac{1}{1-q^{j+k}}\begin{bmatrix} m-1 \\ j \end{bmatrix}\begin{bmatrix} m-2 \\ k-1 \end{bmatrix}q^j,$$

the l.h.s. of (4.3.22) becomes

$$\text{(4,3.23)}\quad \begin{aligned}-&\Big(\sum_{\substack{j\geq(m+\kappa+1)/2\\ k\geq(m-\kappa-1)/2}}-\sum_{\substack{j\leq(m+\kappa-1)/2\\ k\leq(m-\kappa-3)/2}}\Big)(-1)^{j+k}\\ &\times\begin{bmatrix} m-1 \\ j \end{bmatrix}\begin{bmatrix} m-2 \\ k-1 \end{bmatrix}\frac{q^{P_{j,k}^{(m-1)}(2n-2N-b,b)}}{1-q^{j+k}}.\end{aligned}$$

This is identical with $T_3^{(-)}$ since we have

$$P_{j,k}^{(m-1)}(2n-2-N-b, b)=P_{k,j}^{(m-1)}(-b-N-2, b-2n).$$

This establishes (4.3.16). □

4.4. 1D configuration sums as superpositions of the fundamental solutions

Here we prove that the 1D configuration sums defined in section 3 (see (3.1.1), (3.1.10–12)) are expressed as linear superpositions of the fundamental solutions discussed in section 4.1–3. We begin by redefining them in terms of linear difference equations and initial conditions.

Regime III

(4.4.1a) $$X_m(a, b, c)=\sum_d{}'' X_{m-1}(a, d, b)q^{m|d-c|/4},$$

(4.4.1b) $$X_0(a, b, c)=\delta_{a,b}.$$

Regime I ($L\geq 2N+3$)

(4.4.2a) $$Y_m(a, b, c)=\sum_d{}'' Y_{m-1}(a, d, b)q^{mH(d,b,c)},$$

(4.4.2b) $$Y_0(a, b, c)=\delta_{a,b}.$$

Here the sum $\sum''$ is taken over d such that the pair (d, b) is admissible and the function $H(a, b, c)$ has been given in (3.1.12). Note that we retain the original definition of n: $n=[L/2]$. We set $X_m(a, b, c)=Y_m(a, b, c)=0$ unless (b, c) is admissible and $0<a<L$.

Remark. By the definition (4.4.1, 2) the 1D configuration sums have the following properties.

(4.4.3a) $X_m(a, b, c)=0$ if $a\not\equiv b+mN \bmod 2$,

(4.4.3b) $X_m(a, b, c)=X_m(L-a, L-b, L-c)$,

(4.4.4a) $Y_m(a, b, c)=0$ if $a\not\equiv b+mN \bmod 2$,

(4.4.4b) $Y_m(a, b, c)=Y_m(L-a, L-b, L-c)$ if L is odd.

Theorem 4.4.1 ((3.5) of Part I). *For* $m\geq 0$,

(4.4.5a) $X_m(a, b, c)=q^{-a/4}(F_m(a, b, c)-F_m(-a, b, c))$,

(4.4.5b) $F_m(a, b, c)=\sum_{\lambda\in Z} q^{-L\lambda^2+(L/2-a)\lambda+a/4}f_m^{(N)}(b-a-2L\lambda, c-a-2L\lambda)$.

Proof. The property (4.4.3a) is clear from (4.4.5) and (4.1.3). So is (4.4.3b) by the invariance of the summand in $q^{-a/4}F_m(\pm a, b, c)$ under the change

$$(a, b, c, \lambda)\longrightarrow\left(L-a, L-b, L-c, -\lambda+\frac{1\mp 1}{2}\right).$$

Since the function $X_m(a, b, c)$ in (4.4.5) is a linear superposition of $f_m^{(N)}(b, c)$, it also satisfies the same equation (4.1.1). In order to prove

(4.4.1a) it is enough to show the cancellation of non-admissible summands corresponding to the following values of d (see (4.0.1, 2)).

$$d=b-N,\ b-N+2,\cdots,N-b,\qquad \text{if } b\leq N,$$
$$d=2L-N-b,\ 2L-N-b+2,\cdots,b+N,\qquad \text{if } L-b\leq N. \tag{4.4.6}$$

The latter case can be reduced to the former thanks to the symmetry (4.4.3b). Thus we are to show that the following is equal to zero for $1\leq b\leq N$.

$$\begin{aligned}\sum_{d=b-N,b-N+2,\cdots,N-b}\ \sum_{\lambda\in Z} &\times(q^{-L\lambda^2+(L/2-a)\lambda+a/4}f^{(N)}_{m-1}(d-a-2L\lambda,\ b-a-2L\lambda)\\ &-q^{-L\lambda^2+(L/2+a)\lambda-a/4}f^{(N)}_{m-1}(d+a-2L\lambda,\ b+a-2L\lambda))q^{m|c-d|/4}\\ =\sum_{\lambda\in Z}q^{-L\lambda^2-a\lambda+mc/4}\sum_{d} &\times(q^{(a+2L\lambda-md)/4}f^{(N)}_{m-1}(d-a-2L\lambda,\ b-a-2L\lambda)\\ &-q^{(-a-2L\lambda-md)/4}f^{(N)}_{m-1}(d+a+2L\lambda,\ b+a+2L\lambda)).\end{aligned}\tag{4.4.7}$$

Here we have used the fact $c>N-b\geq d$ (see (4.0.2)) in reducing $|c-d|$. The vanishing of (4.4.7) follows directly from Lemma 4.1.2. □

Theorem 4.4.2 ((3.15 of Part I). *For* $m\geq 0$,

$$Y_m(a,b,c)=q^{a/2}(G_m(a,b,c)-G_m(-a,b,c)),\tag{4.4.8a}$$

$$G_m(a,b,c)=\sum_{\lambda\in Z}q^{2L\lambda^2+(2a-L)\lambda-a/2}\times g^{(N)}_m(n-a-2L\lambda;\ b-a-2L\lambda,\ c-a-2L\lambda).\tag{4.4.8b}$$

Proof. We assume that

$$(b,c)\in S_1(n)\quad\text{or}\quad S_2^{(\pm)}(n).\tag{4.4.9}$$

The other cases are similar. As in the proof of Theorem 4.4.1, we show the cancellation of the non-admissible summands. From (4.4.9) together with $n=[L/2]$ and $L\geq 2N+3$ we have

$$L-b>N,\tag{4.4.10a}$$

$$(d,b)\in S_1(n),\ H(n,d,b,c)=\frac{d-b+N}{2},\quad \text{if } b\leq N \text{ and } d\in\{b-N,\ b-N+2,\cdots,N-b\}.\tag{4.4.10b}$$

In view of (4.4.6) and (4.4.10a) we assume $b\leq N$ without loss of generality. Then the cancellation identity reads as

$$0=\sum_{d=b-N,b-N+2,\cdots,N-b}\sum_{\lambda\in Z}$$
$$\times(q^{2L\lambda^2+(2a-L)\lambda-a/2}g^{(N)}_{m-1}(n-a-2L\lambda;\, d-a-2L\lambda,\, b-a-2L\lambda)$$
$$-q^{2L\lambda^2+(2a+L)\lambda+a/2}g^{(N)}_{m-1}(n+a+2L\lambda;\, d+a+2L\lambda,\, b+a+2L\lambda))$$
$$\text{(4.4.11)}\qquad \times q^{mH(n,d,b,c)}$$
$$=\sum_{\lambda\in Z}q^{2L\lambda^2+2a\lambda+m(N-b)/2}\sum_{d}$$
$$\times(q^{(-a-2L\lambda+md)/2}\bar{f}^{(N)}_{m-1}(d-a-2L\lambda,\, b-a-2L\lambda;\, \mu_-,\nu_-)$$
$$-q^{(a+2L\lambda+md)/2}\bar{f}^{(N)}_{m-1}(d+a+2L\lambda,\, b+a+2L\lambda;\, \mu_+,\nu_+)),$$

where $\mu_\pm, \nu_\pm$ are integers defined by $(d\pm(a+2L\lambda), b\pm(a+2L\lambda))\in R_{\mu_\pm,\nu_\pm}$. We used (4.4.10b) and (4.3.12a) in deriving (4.4.11). Substitute (4.3.11) into (4.4.11) and replace $(a+2L\lambda, d)$ by $(a, -d)$. The resulting sum $\sum_d$ is equal to (the l.h.s.—the r.h.s.) of (4.1.17) with q replaced by q^{-1}. Thus (4.4.11) follows from Lemma 4.1.2. □

Remark. From (4.4.8) and (4.3.12) $Y_m(a, b, c)$ enjoys the following symmetries

$$\text{(4.4.12a)}\qquad Y_m(a,b,c)=Y_m(L-a,L-b,L-c),\qquad \text{if } (b,c)\in S_4(n),$$
$$\text{(4.4.12b)}\qquad =Y_m(a,b,2n-b-N),\qquad \text{if } (b,c)\in S_2(n),$$
$$\text{(4.4.12c)}\qquad =Y_m(a,b,2n+2-b+N),\qquad \text{if } (b,c)\in S_3(n).$$

In this way the evaluation of $Y_m(a, b, c)$ for $(b, c)\in S_4(n)$, $S_2(n)$ and $S_3(n)$ reduces to that for $S_1(n)$, $S_1(n)$ and $S_4(n)$, respectively.

§ 5. One Dimensional Configuration Sums as Modular Functions

In this section we identify the limit $m\to\infty$ of the 1D configuration sums $X_m(a, b, c; q^{\pm 1})$ and $Y_m(a, b, c; q^{\pm 1})$ (up to some power corrections in q) with modular functions appearing in theta function identities. Regime I was fully treated in Part I. In what follows we shall deal with the other regimes III, IV and II.

5.1. Regimes III and IV

The modular functions $c^{(\pm)}_{j_1 j_2 j_3}(\tau)$ (which we called the branching coefficients) have been defined in Appendix A of Part I. For our present purpose let us quote the expression (A.6) therein.

Assume $j_1, m_1\in Z/2$, $j_2, m_2\in Z$, $0<j_i\leq m_i\neq 0$ $(i=1, 2)$ and $m_3=m_1+m_2-2>0$. We choose $j_3\in Z+j_1$ with the restriction $0<j_3\leq m_3$ for

$c^{(-)}_{j_1j_2j_3}(\tau)$ and $0<j_3<m_3$ for $c^{(+)}_{j_1j_2j_3}(\tau)$. The branching coefficient $c^{(\pm)}_{j_1j_2j_3}(\tau)$ is given by

(5.1.1a) $$c^{(\pm)}_{j_1j_2j_3}(\tau)=\varepsilon^{m_3}_{j_3}\frac{q^{\gamma(j_1j_2j_3)}}{\varphi(q)^3}\Big(\sum_{\xi+1\le t,\eta\le\mathscr{L}_1\in Z}-\sum_{\xi\ge t,\eta-1\ge\mathscr{L}_1\in Z}\Big)\text{ (summand)},$$

(5.1.1b) $$\text{(summand)}=(-1)^{t+(\varepsilon_1+\varepsilon_2)/2}(\pm 1)^{n_1}q^{t(t-1)/2+t\mathscr{L}_1+\mathscr{L}_0},$$

(5.1.1c) $$\mathscr{L}_0=\sum_{i=1}^{2}(m_in_i^2+j_in_i),$$

(5.1.1d) $$\mathscr{L}_1=\frac{j_3+1}{2}+\sum_{i=1}^{2}\varepsilon_i(m_in_i+j_i/2),$$

(5.1.1e) $$\gamma(j_1,j_2,j_3)=\frac{j_1^2}{4m_1}+\frac{j_2^2}{4m_2}-\frac{1}{8}-\frac{j_3^2}{4m_3}.$$

Here the sum is over integers t, n_1, n_2 and $\varepsilon_1, \varepsilon_2=\pm 1$ with the restriction written as above. The integers $\xi=\xi(\varepsilon_1, n_1)$ and $\eta=\eta(\varepsilon_1, n_1)$ can be chosen arbitrarily for fixed ε_1 and n_1. As for further properties of these branching coefficients (the modular invariance, the small q behavior etc, $\cdots$) see Appendix A in Part I. The modular functions $c^{(+)}_{j_1j_2j_3}(\tau)$ $(j_1, m_1\in Z)$ appear as the branching coefficinets in the irreducible decompositions of tensor products of $A_1^{(1)}$ modules. The 1D configuration sums in regime III coincide with the $c^{(+)}_{j_1j_2j_3}(\tau)$:

Theorem 5.1.1.

(5.1.2a) $$\lim_{m\text{ even}\to\infty}X_m(a,b,c)=\lim_{m\text{ odd}\to\infty}X_m(a,c,b)=q^{\rho}c^{(+)}_{rsa}(\tau),$$

(5.1.2b) $$r=\frac{b+c-N}{2},\qquad s=\frac{b-c+N}{2}+1,$$

(5.1.2c) $$\rho=\frac{b-a}{4}-\gamma(r,s,a),$$

where $c^{(+)}_{rsa}(\tau)$ *is given by* (5.1.1) *with*

(5.1.2d) $$m_1=L-N,\quad m_2=N+2,\quad m_3=L.$$

Proof. Substitute (4.1.7) into (4.4.5) and replace (j, k) by

$$\Big(j+\frac{m}{2}, k+\frac{m}{2}\Big)\quad\text{or}\quad\Big(k+\frac{m-1}{2}, j+\frac{m+1}{2}\Big)$$

according as m is even or odd, respectively. Using that

$$\lim \begin{bmatrix} M \\ \frac{M+j}{2} \end{bmatrix} = \frac{1}{\varphi(q)}$$

(j fixed, $M \to \infty$, with $M \equiv j \bmod 2$), we obtain

(5.1.3a) $$\varphi(q)^3 \lim_{m \text{ even} \to \infty} X_m(a, b, c) = \varphi(q)^3 \lim_{m \text{ odd} \to \infty} X_m(a, c, b)$$
$$= \varphi(q)^3 q^{-a/4}(F(a, b, c) - F(-a, b, c)),$$

(5.1.3b) $$\varphi(q)^3 q^{-a/4} F(\pm a, b, c)$$
$$= \sum_{\lambda \in Z} \Big(\sum_{\substack{j \geq \nu^\pm/2 \\ k \leq (\mu^\pm - 1)/2}} - \sum_{\substack{j \leq \nu^\pm/2 - 1 \\ k \geq (\mu^\pm + 1)/2}} \Big) (-1)^{j+k} q^{A^{(\pm)}(\lambda, j, k)},$$

(5.1.3c) $$A^{(\pm)}(\lambda, j, k) = -L\lambda^2 + \left(\frac{L}{2} \mp a\right)\lambda - \frac{1 \mp 1}{4} a$$
$$+ Q^{(0)}_{j,k}(b \mp a - 2L\lambda, c \mp a - 2L\lambda),$$

where $Q^{(m)}_{j,k}(b, c)$ is defined in (4.1.7) and the integers $\mu^\pm$, $\nu^\pm$ are determined by

(5.1.3d) $(b \mp a - 2L\lambda, c \mp a - 2L\lambda) \in R_{\mu^\pm, \nu^\pm}$, $\quad \mu^\pm$: odd, $\nu^\pm$: even.

On the other hand we have from (5.1.1) (note that $\varepsilon_a^{L/2} = 1$ for $0 < a < L/2$)

(5.1.4a) $$\varphi(q)^3 q^\rho c^{(+)}_{rsa}(\tau)$$
$$= \Big(\sum_{\xi + 1 \leq t, \eta \leq \mathscr{L}_1} - \sum_{\xi \geq t, \eta - 1 \geq \mathscr{L}_1} \Big) (-1)^{t + (\varepsilon_1 + \varepsilon_2)/2} q^{B(\varepsilon_1, \varepsilon_2, t, n_1, n_2)},$$

(5.1.4b) $$B(\varepsilon_1, \varepsilon_2, t, n_1, n_2) = t(t-1)/2 + t\mathscr{L}_1(\varepsilon_1, \varepsilon_2) + (b-a)/4$$
$$+ (L-N)n_1^2 + rn_1 + (N+2)n_2^2 + sn_2,$$

(5.1.4c) $\mathscr{L}_1(\varepsilon_1, \varepsilon_2) = (a+1)/2 + \varepsilon_1((L-N)n_1 + r/2) + \varepsilon_2((N+2)n_2 + s/2),$

where the sum in (5.1.4a) is taken in the same manner as in (5.1.1a). We have explicitly exhibited the $(\varepsilon_1, \varepsilon_2)$-dependence of the summands. Note that the condition $\mathscr{L}_1 \in Z$ is fulfilled because of the restriction $a - b \equiv mN \equiv 0 \bmod 2$. As the first step to identify (5.1.4) with (5.1.3), we seek a transformation from the variables (t, n_1, n_2) to (λ, j, k) such that the summands in (5.1.4a) are transformed into those in (5.1.3a, b). This is achieved in the following way. We have

(5.1.5) $$B(\mp, \varepsilon_2, t, n_1, n_2) = A^{(\pm)}(\lambda, j, k),$$

if (t, n_1, n_2) is chosen as follows:

(5.1.6)

$(\varepsilon_1, \varepsilon_2)$	$(+,+)$	$(+,-)$	$(-,+)$	$(-,-)$
$t=t(\varepsilon_1, \varepsilon_2)$	$2\lambda+j+k$	$2\lambda+j+k$	$-2\lambda-j-k$	$-2\lambda-j-k$
$n_1=n_1(\varepsilon_1, \varepsilon_2)$	$-\lambda$	$-\lambda$	$-\lambda$	$-\lambda$
$n_2=n_2(\varepsilon_1, \varepsilon_2)$	$-\lambda-k$	$\lambda+j$	$\lambda+j$	$-\lambda-k$

From (5.1.4c) and (5.1.6) we deduce

(5.1.7a) $\quad \mathscr{L}_1(-\varepsilon_1, -\varepsilon_2)=a+1-\mathscr{L}_1(\varepsilon_1, \varepsilon_2),$

(5.1.7b) $\quad \mathscr{L}_1(+,+)=(a+b)/2+1-(L+2)\lambda-(N+2)k,$

(5.1.7c) $\quad \mathscr{L}_1(+,-)=(a+c-N)/2-(L+2)\lambda-(N+2)j.$

Our next step is to partch up the summation domains in (5.1.4a) so as to make them coincide with those in (5.1.3b). To describe the constraints on the summation variables in (5.1.4a) we introduce the domain $D_\lambda(\kappa; \varepsilon_1, \varepsilon_2)$ $(\kappa, \varepsilon_1, \varepsilon_2=\pm 1)$ by

(5.1.8)
$$D_\lambda(\kappa; \varepsilon_1, \varepsilon_2)=\{(j,k)\in \boldsymbol{Z}^2 \mid \kappa t(\varepsilon_1, \varepsilon_2)>\kappa(\xi(\varepsilon_1, n_1(\varepsilon_1, \varepsilon_2))+1/2), \quad \kappa\mathscr{L}_1(\varepsilon_1, \varepsilon_2)>\kappa(\eta(\varepsilon_1, n_1(\varepsilon_1, \varepsilon_2))-1/2)\}.$$

For example $D_\lambda(+:+,+)$ reads as (see (5.1.6, 7))

$$D_\lambda(+;+,+)=\{(j,k)\in \boldsymbol{Z}^2 \mid j+k\geq \xi(+,-\lambda)-2\lambda+1, \quad (N+2)k\leq -\eta(+,-\lambda)-(L+2)\lambda+(a+b)/2+1\}.$$

Using the relation (5.1.5) and the fact $(-1)^{j+k}=(-1)^t$ (see (5.1.6)) we recombine (5.1.4a) as follows.

(5.1.9)
$$\begin{aligned}&\varphi(q)^3 q^\rho c^{(+)}_{rsa}(\tau)\\ &=\sum_{\lambda\in \boldsymbol{Z}}\Big(\sum_{(j,k)\in D_\lambda(+;-,+)\cup D_\lambda(-;-,-)}-\sum_{(j,k)\in D_\lambda(-;-,+)\cup D_\lambda(+;-,-)}\Big)\\ &\quad\times(-1)^{j+k}q^{A^{(+)}(\lambda,j,k)}\\ &\quad-\sum_{\lambda\in \boldsymbol{Z}}\Big(\sum_{(j,k)\in D_\lambda(+;+,+)\cup D_\lambda(-;+,-)}-\sum_{(j,k)\in D_\lambda(-;+,+)\cup D_\lambda(+;+,-)}\Big)\\ &\quad\times(-1)^{j+k}q^{A^{(-)}(\lambda,j,k)}.\end{aligned}$$

Compare this expression with the one given in (5.1.3). They are identical if there exist $\xi(\mp, -\lambda)$ and $\eta(\mp, -\lambda)$ such that

(5.1.10a)
$$D_\lambda(\pm;\mp,\pm)\cup D_\lambda(\mp,\mp,\mp) = \{(j,k)\in Z^2 \mid j\geq \nu^\pm/2,\ k\leq(\mu^\pm-1)/2\},$$

(5.1.10b)
$$D_\lambda(\mp;\mp,\pm)\cup D_\lambda(\pm,\mp,\mp) = \{(j,k)\in Z^2 \mid j\leq \nu^\pm/2-1,\ k\geq(\mu^\pm+1)/2\}.$$

Schematic explanation of the patch-up procedure (5.1.10a) is given in Fig. 5.1.

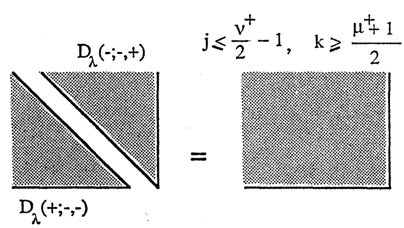


Fig. 5. 1 **An example of the patch up procedure of the summation domains in the (j, k) plane.**

Thus the remaining task is to check (5.1.10). In the following we denote $\xi(-,-\lambda)$, $\eta(-,-\lambda)$, μ^+, ν^+ simply by ξ, η, μ, ν and verify the upper case in (5.1.10a). Now that we have

(5.1.11a)
$$D_\lambda(+;-,+)=\{(j,k)\in Z^2 \mid -2\lambda-j-k\geq\xi+1,\ (a-c+N)/2+1+(L+2)\lambda+(N+2)j\geq\eta\},$$

(5.1.11b)
$$D_\lambda(-;-,-)=\{(j,k)\in Z^2 \mid -2\lambda-j-k\leq\xi,\ (a-b)/2+(L+2)\lambda+(N+2)k\leq\eta-1\},$$

the condition that assures the existence of η is stated as follows.

(5.1.12)
$$E_1\cap E_2\neq\varnothing,$$

(5.1.13a)
$$E_1=\left\{\eta\in Z \,\middle|\, \left\{j\in Z \mid j\geq\frac{1}{N+2}\left(\eta-(L+2)\lambda-\frac{a-c+N}{2}-1\right)\right\} = \{j\in Z \mid j\geq\nu/2\}\right\},$$

(5.1.13b)
$$E_2=\left\{\eta\in Z \,\middle|\, \left\{k\in Z \mid k\leq\frac{1}{N+2}\left(\eta-(L+2)\lambda-\frac{a-b}{2}-1\right)\right\} = \{k\in Z \mid k\leq(\mu-1)/2\}\right\}.$$

It is easy to see

$$E_1=\{\eta \in Z \mid \eta_{\min}<\eta\leq\eta_{\max}\},$$

(5.1.14a)
$$\eta_{\substack{\max\\ \min}}=(L+2)\lambda+(a-c+N)/2+1+(N+2)\times\begin{cases}\nu/2\\ \nu/2-1\end{cases},$$

$$E_2=\{\eta \in Z \mid \eta'_{\min}\leq\eta<\eta'_{\max}\},$$

(5.1.14b)
$$\eta'_{\substack{\max\\ \min}}=(L+2)\lambda+(a-b)/2+1+(N+2)\times\begin{cases}(\mu+1)/2\\ (\mu-1)/2\end{cases}.$$

Now (5.1.12) is clear since we have

$$\eta'_{\max}-\eta_{\min}=(c-b+N)/2+(N+2)(\mu-\nu+1)/2+2>0,$$
$$\eta_{\max}-\eta'_{\min}=(b-c+N)/2+(N+2)(\nu-\mu+1)/2\geq 0. \qquad \square$$

Much the same as in regime III, our result in regime IV is stated as follows.

Theorem 5.1.2. *For $(b, c) \in S_1(n)$,*

(5.1.15a)
$$\begin{aligned}&\lim_{m\text{ even}\to\infty} Y_m(a, b, c; q^{-1})q^{(m/4)(c-b)+m(m+1)N/4}\\ &= \lim_{m\text{ odd}\to\infty} Y_m(a, c, b; q^{-1})q^{((m+1)/4)(b-c)+m(m+1)N/4}\\ &=q^{\sigma}c^{(-)}_{rsa}(\tau), \qquad \textit{if } L \textit{ is odd},\end{aligned}$$

(5.1.15b)
$$\begin{aligned}&\lim_{m\text{ even}\to\infty} \varepsilon_a^{L/2}(Y_m(a, b, c; q^{-1})\\ &\qquad\mp Y_m(L-a, b, c; q^{-1}))q^{(m/4)(c-b)+(m(m+1)N)/4}\\ &= \lim_{m\text{ odd}\to\infty} \varepsilon_a^{L/2}(Y_m(a, c, b; q^{-1})\\ &\qquad\mp Y_m(L-a, c, b; q^{-1}))q^{((m+1)/4)(b-c)+(m(m+1)N)/4}\\ &=q^{\sigma}c^{(\pm)}_{rsa}(\tau), \qquad \textit{if } L \textit{ is even}.\end{aligned}$$

Here

(5.1.15c)
$$\sigma=\frac{b-a}{2}-\gamma(r, s, a),$$

where the integers r, s are defined in (5.1.2b) *and the $c^{(\pm)}_{rsa}(\tau)$ is given by* (5.1.1) *with*

(5.1.15d)
$$m_1=L/2-N, \quad m_2=N+2, \quad m_3=L/2.$$

Proof. As was done for Theorem 5.1.1 the proof consists of two steps:

(i) Write down the l.h.s. (resp. r.h.s.) as a sum over (λ, j, k) (resp. (t, n_1, n_2)) and find the transformations between (λ, j, k) and (t, n_1, n_2) such that the summands are mapped to each other.

(ii) Utilizing the parameters ξ and η, patch up the summation domains in the r.h.s. to make them coincide with those in the l.h.s.

Here we shall only demonstrate (i). The step (ii) is almost the same as the one for Theorem 5.1.1.

We substitute (4.3.9) and (4.3.12a) into (4.4.8) and replace q by q^{-1} and (j, k) by

$$\left(j+\frac{m}{2}+1, \frac{m}{2}-k\right) \quad \text{or} \quad \left(k+\frac{m+1}{2}, \frac{m-1}{2}-j\right)$$

according as m is even or odd. Then in the limit $m\to\infty$ we get

(5.1.16a)
$$\begin{aligned}&\varphi(q)^3 \lim_{m \text{ even}\to\infty} Y_m(a, b, c; q^{-1})q^{(m/4)(c-b)+(m(m+1)N)/4}\\ &=\varphi(q)^3 \lim_{m \text{ odd}\to\infty} Y_m(a, c, b; q^{-1})q^{((m+1)/4)(b-c)+(m(m+1)N)/4}\\ &=\varphi(q)^3 q^{-a/2}(G(a, b, c)-G(-a, b, c)),\end{aligned}$$

(5.1.16b)
$$\begin{aligned}&\varphi(q)^3 q^{-a/2} G(\pm a, b, c)\\ &=\sum_{\lambda\in Z}\Big(\sum_{\substack{j\geq\nu^\pm/2\\ k\leq(\mu^\pm-1)/2}} - \sum_{\substack{j\leq\nu^\pm/2-1\\ k\geq(\mu^\pm+1)/2}}\Big)(-1)^{j+k} q^{\bar{A}^{(\pm)}(\lambda, j, k; a)},\end{aligned}$$

(5.1.16c)
$$\begin{aligned}\bar{A}^{(\pm)}(\lambda, j, k; a)=&-2L\lambda^2-(\pm 2a-L)\lambda\\ &-\frac{1\mp1}{2}a+\frac{j+1}{2}(b\mp a-2L\lambda)\\ &+\frac{k}{2}(c\mp a-2L\lambda)-\left(j+\frac{1}{2}\right)kN\\ &+\frac{1}{2}(j-k)(j-k+1),\end{aligned}$$

where the integers $\mu^\pm, \nu^\pm$ is given by (5.1.3d). Next we write down the branching coefficients $c_{rsa}^{(\pm)}(\tau)$ in the r.h.s. of (5.1.15a, b)

(5.1.17a)
$$\begin{aligned}\varphi(q)^3 q^\sigma c_{rsa}^{(\pm)}(\tau)=\varepsilon_a^{L/2}\Big(&\sum_{\xi+1\leq t, \eta\leq\bar{\mathscr{L}}_1\in Z} - \sum_{\xi\geq t, \eta-1\geq\bar{\mathscr{L}}_1\in Z}\Big)\\ &\times(-1)^{t+(\varepsilon_1+\varepsilon_2)/2}(\pm1)^{n_1} q^{\bar{B}(\varepsilon_1, \varepsilon_2, t, n_1, n_2)},\end{aligned}$$

(5.1.17b)
$$\begin{aligned}\bar{B}(\varepsilon_1, \varepsilon_2, t, n_1, n_2)=&\frac{t(t-1)}{2}+t\bar{\mathscr{L}}_1(\varepsilon_1, \varepsilon_2)+\frac{b-a}{2}\\ &+(L/2-N)n_1^2+rn_1+(N+2)n_2^2+sn_2,\end{aligned}$$

(5.1.17c) $\bar{\mathscr{L}}_1(\varepsilon_1, \varepsilon_2)=\dfrac{a+1}{2}+\varepsilon_1((L/2-N)n_1+r/2)+\varepsilon_2((N+2)n_2+s/2).$

Consider the case L is odd. The condition $\bar{\mathscr{L}}_1 \in \boldsymbol{Z}$ asserts that $n_1 \in 2\boldsymbol{Z}$. In this case we can transform (5.1.17) into (5.1.16) through the following rule:

(5.1.18) $$\bar{B}(\mp, \varepsilon_2, t, n_1, n_2)=\bar{A}^{(\pm)}(\lambda, j, k; a),$$

where (t, n_1, n_2) is given in terms of (λ, j, k) as

(5.1.19)

$(\varepsilon_1, \varepsilon_2)$	$(+,+)$	$(+,-)$	$(-,+)$	$(-,-)$
t	$4\lambda+j+k$	$4\lambda+j+k$	$-4\lambda-j-k$	$-4\lambda-j-k$
n_1	-2λ	-2λ	-2λ	-2λ
n_2	$-2\lambda-k$	$2\lambda+j$	$2\lambda+j$	$-2\lambda-k$

From (5.1.18) and (5.1.19) together with the patch-up argument we conclude (5.1.15a). (Note that $\varepsilon_a^{L/2}=1$ in this case.) In the case L is even, the condition $\bar{\mathscr{L}}_1 \in \boldsymbol{Z}$ is satisfied irrespective of the parity of n_1. Thus $\varphi(q)^3 q^\sigma c_{rsa}^{(\pm)}(\tau)$ in (5.1.17) consists of two kinds of the summands corresponding to $n_1 \in 2\boldsymbol{Z}$ or $n_1 \in 2\boldsymbol{Z}+1$. The former can be treated in the same way as the case L is odd and is identified with

(5.1.20)
$$\begin{aligned}&\varphi(q)^3 \lim_{m \text{ even}\to\infty} \varepsilon_a^{L/2} Y_m(a, b, c; q^{-1}) q^{(m/4)(c-b)+(m(m+1)N)/4}\\ &=\varphi(q)^3 \lim_{m \text{ odd}\to\infty} \varepsilon_a^{L/2} Y_m(a, c, b; q^{-1}) q^{((m+1)/4)(b-c)+(m(m+1)N)/4}.\end{aligned}$$

Transform the latter summands ($n_1 \in 2\boldsymbol{Z}+1$) via

(5.1.21)

$(\varepsilon_1, \varepsilon_2)$	$(+,+)$	$(+,-)$	$(-,+)$	$(-,-)$
t	$4\lambda+j+k+2$	$4\lambda+j+k+2$	$-4\lambda-j-k-2$	$-4\lambda-j-k-2$
n_1	$-2\lambda-1$	$-2\lambda-1$	$-2\lambda-1$	$-2\lambda-1$
n_2	$-2\lambda-k-1$	$2\lambda+j+1$	$2\lambda+j+1$	$-2\lambda-k-1$

Under the rule (5.1.21) we have

(5.1.22) $$\bar{B}(\pm, \varepsilon_2, t, n_1, n_2)=\bar{A}^{(\pm)}(\lambda, j, k; L-a).$$

Thus their contribution to $\varphi(q)^3 q^\sigma c_{rsa}^{(\pm)}(\tau)$ is equal to

$$\mp\varphi(q)^3 \lim_{m \text{ even}\to\infty} \varepsilon_a^{L/2} Y_m(L-a, b, c; q^{-1}) q^{(m/4)(c-b)+(m(m+1)N/4}$$

(5.1.23)
$$= \mp\varphi(q)^3 \lim_{m \text{ odd}\to\infty} \varepsilon_a^{L/2} Y_m(L-a, c, b; q^{-1}) \times q^{((m+1)/4)(b-c)+(m(m+1)N)/4}.$$

From (5.1.20) and (5.1.23) we arrive at (5.1.15b). □

By virtue of the symmetries (4.4.12) the evaluation of the limit $m\to\infty$ of $Y_m(a, b, c; q^{-1})$ for $(b, c) \in S_2(n) \sim S_4(n)$ reduces to the case $(b, c) \in S_1(n)$ described above.

Remark. It is not easy to find the lowest power in q of the $c^{(\pm)}_{j_1 j_2 j_3}(\tau)$ by using (5.1.1). In Appendix A this is done by manipulating the 1D configuration sums.

5.2. Regime II

In this paragraph we prove

Theorem 5.2.1. *Let* $1\leq a,\ b\leq L-1$. *We set* $\bar{\ell}_j = \langle b+(j-1)N\rangle$. *The* $(\bar{\ell}_j)$ *is a ground state configuration, and we have*

(5.2.1)
$$\lim_{m\to\infty} q^{\phi_m(\bar{\ell}_1,\cdots,\bar{\ell}_{m+2})+\nu(a,b)} X_m(a, \bar{\ell}_{m+1}, \bar{\ell}_{m+2}; q^{-1}) = e^{L-2}_{b-1,a-1}(\tau),$$

where

$$\nu(a, b) = -\frac{1}{4(L-2)}\left(\frac{L}{2}-b\right)^2 + \frac{1}{4L}\left(\frac{L}{2}-a\right)^2 + \frac{1}{24}.$$

As for the symbol $\langle\ \rangle$ see Appendix B. There we prove that $(\bar{\ell}_j)$ is a ground state configuration in the sense of Part I, section 2.4. An explicit form of $\phi_m(\bar{\ell}_1, \cdots, \bar{\ell}_{m+2})$ is given in (B.1). In fact, the following relation holds (see Lemma B.3 and the last remark in Appendix B)

(5.2.2)
$$X_m(a, \bar{\ell}_{m+1}, \bar{\ell}_{m+2}; q^{-1}) = X_{m-1}(a, \bar{\ell}_m, \bar{\ell}_{m+1}; q^{-1}) q^{-m|\bar{\ell}_m - \bar{\ell}_{m+2}|/4}(1+O(q^{\alpha m})),$$

where α is some positive constant. Thus the proof of (5.2.1) is reduced to the case $1\leq \bar{\ell}_{m+1}, \bar{\ell}_{m+2} \leq L-1$, $\bar{\ell}_{m+2} = \bar{\ell}_{m+1}+N$ by the repeated use of (5.2.2).

The modular functions $e^{\ell}_{jk}(\tau)$ have been described in Appendix B of Part I. They are the branching coefficients appearing in the irreducible decompositions of level 1 highest weight modules of the affine Lie algebra $A_{2\ell-1}^{(1)}$ with respect to the embedded $C_{\ell}^{(1)}$. Here we shall exploit the characterization of $(e_{jk}^{\ell}(\tau))_{0\le j,k\le\ell}$ in terms of its inverse matrix.

Lemma 5.2.2 ((B. 11, 12) of Part I). *Set*

(5.2.3a) $\tilde{p}_{kj}^{\ell}(\tau)=\varepsilon_j^{\ell}(p_{kj}^{\ell}(\tau)+p_{k,-j}^{\ell}(\tau)),$

$$p_{kj}^{\ell}(\tau)=\sum_{k'\equiv k \bmod 2(\ell+2)}(-)^{(k'-j)/2}q^{(k'-j)(k'-j+2)/8-(k'+1)^2/4(\ell+2)+j^2/4\ell+1/8},$$

(5.2.3b) *if* $j, k\in \boldsymbol{Z}$ *and* $j\equiv k \bmod 2,$

$=0,$ *otherwise.*

Then we have

(5.2.4) $$\sum_{0\le k\le\ell, k\equiv j \bmod 2} e_{jk}^{\ell}(\tau)\tilde{p}_{kj'}^{\ell}(\tau)=\eta(\tau)\varepsilon_{j'}^{\ell}(\tilde{\delta}_{jj'}^{2\ell}+\tilde{\delta}_{j-j'}^{2\ell}),$$

where $\tilde{\delta}_{jj'}^{2\ell}=1$ *if* $j'\equiv j \bmod 2\ell$ *and* $=0$ *otherwise.*

Remark. By the definition, $p_{kj}^{\ell}(\tau)$ has the following symmetries

(5.2.5) $$p_{kj}^{\ell}(\tau)=-p_{-k-2,-j}^{\ell}(\tau)=p_{k+2(\ell+2),j}^{\ell}(\tau)=p_{k,j+2\ell}^{\ell}(\tau).$$

For $\beta\in \boldsymbol{Z}$, define the function $H_m^{(N)}(\beta)$ by

(5.2.6) $$H_m^{(N)}(\beta)=\sum_{j\in \boldsymbol{Z}}(-1)^j q^{j^2/2} f_m^{(N)}(\beta-2j, \beta+N-2j; q^{-1}),$$

where $f_m^{(N)}(b, c; q)$ is the fundamental solution described in sections 4.1–2. By (4.2.1) $H_m^{(N)}(\beta)$ can also be expressed as

(5.2.7a) $$H_m^{(N)}(\beta)=(-1)^{\omega}q^{-m(m+1)N/4+\omega^2/2}\sum_{v\in \boldsymbol{Z}}(-1)^v q^{v(v-2\omega+1)/2}\hat{f}_m^{(N)}(v),$$

(5.2.7b) $$\omega=\frac{Nm-\beta}{2}.$$

Note that the summand in (5.2.7a) is supported in the interval $0\le v\le mN$. In the case $N=1$ (4.2.23), (5.2.6) together with the formula (4.1.14) give

(5.2.8) $$H_m^{(1)}(\beta)=(-1)^{\omega}q^{-m(m+1)/4+\omega^2/2}(q^{1-\omega})_m.$$

From (5.2.7) and Theorem 4.2.2, we deduce the symmetry:

(5.2.9) $$H_m^{(N)}(\beta)=-q^{(\beta+1)/2}H_m^{(N)}(-\beta-2).$$

In what follows we make use of several representations of $\hat{f}_m^{(N)}(v)$ derived in section 4.2, and determine the large m behavior of $H_m^{(N)}(\beta)$ in various regions of β. The results are given in Lemma 5.2.3–6.

Lemma 5.2.3 ((3.28) of Part I). *For $N \geq 1$, $\beta \geq mN$,*

$$H_m^{(N)}(\beta) = O(q^{-m(m+1)N/4+\omega^2/2}). \tag{5.2.10}$$

Proof. First we show that

$$\hat{f}_m^{(N)}(v) = 1 + \cdots \qquad \text{if } 0 \leq v \leq mN \ (m \geq 1), \tag{5.2.11a}$$

$$= 0 \qquad \text{otherwise.} \tag{5.3.11b}$$

By the definition (4.2.1), (5.2.11b) is clear. Recall the expression (4.2.2) for $\hat{f}_m^{(N)}(v)$. Note that the Gaussian polynomial is of the form $1+O(q)$. Under the assumption $0 \leq v \leq mN$, we see ($\mathscr{P}$ defined in (4.2.2b))

$$\min_{j \geq 0} \mathscr{P} = \mathscr{P}|_{j=0} = 0.$$

Thus the lowest order of the first term in (4.2.2a) is unity. Because of the symmetry (4.2.7) we can assume that $v \leq mN/2$. Then it is shown that the second term in (4.2.2a) does not contain the power lower than q^1. Thus we have (5.2.11a). Since the minimum of $v(v-2\omega+1)/2$ in the interval $0 \leq v \leq mN$ is attained at $v=0$ (note that $\omega \leq 0$), we obtain (5.2.10). □

The expression (4.2.2) yields the following estimate of $\hat{f}_m^{(N)}(v)$ as m tends to ∞ (when v is fixed):

$$\hat{f}_m^{(N)}(v) = \begin{bmatrix} v+m-1 \\ m-1 \end{bmatrix} + O(q^{mN-v+1}), \tag{5.2.12}$$

Applying this to (5.2.7) with $\beta = mN$ ($\omega = 0$) we obtain

$$\lim_{m \to \infty} q^{m(m+1)N/4} H_m^{(N)}(mN) = \sum_{v \in Z} (-1)^v \frac{q^{v(v+1)/2}}{(q)_v} = \varphi(q), \tag{5.2.13}$$

where we have used the formula (4.1.14) in the limit $M \to \infty$.

Lemma 5.2.4 ((3.30) of Part I). *Assume that $N \geq 2$. Then for $mN > \beta > m(N-C)$, $C = \min(2, 4(L-N-2)/(L-4))$,*

$$H_m^{(N)}(\beta) = O(q^{-m(m+1)N/4+\omega(m+1-\omega/2)}). \tag{5.2.14}$$

Proof. Substitute the expression (4.2.20) into (5.2.7). After eliminating x_0 by (4.2.21a), we perform the x_1-summation utilizing the formula (4.1.14). The result takes the form:

(5.2.15a)
$$H_m^{(N)}(\beta)=(-1)^{\omega}q^{-m(m+1)N/4+\omega^2/2}\times\sum(-1)^{\sum_{2\leq j\leq N} jx_j}\begin{bmatrix} m \\ x_2, \cdots, x_N, \tilde{m}\end{bmatrix}q^{\mathscr{S}_1}(q^{\mathscr{S}_0})_{\tilde{m}},$$

(5.2.15b)
$$\tilde{m}=m-\sum_{2\leq j\leq N} x_j,$$

(5.2.15c)
$$\mathscr{S}_0=1-\omega+\sum_{2\leq j\leq N}(j-1)x_j,$$

(5.2.15d)
$$\mathscr{S}_1=\sum_{2\leq j\leq N}(j^2-2j+2)x_j^2/2+\sum_{2\leq k<j\leq N}(jk-2k+1)x_jx_k+\sum_{2\leq j\leq N}((j-1)m-(\omega-1/2)j)x_j,$$

where the sum $\sum$ in (5.2.15a) is taken over all non-negative integers $x_2, \cdots, x_N$. In (5.2.15a) the product $(q^{\mathscr{S}_0})_{\tilde{m}}$ is non-zero if

(5.2.16)
$$\mathscr{S}_0\geq 1 \quad \text{or} \quad \mathscr{S}_0\leq -\tilde{m}.$$

Under the assumption $Nm>\beta>m(N-C)$ $(0<\omega<mC/2\leq m)$, we can discard the latter possibility, for

(5.2.17)
$$\mathscr{S}_0+\tilde{m}=1+m-\omega+\sum_{2\leq j\leq N}(j-2)x_j>0.$$

In the former case in (5.2.16), the lowest order term of the product $(q^{\mathscr{S}_0})_{\tilde{m}}$ as well as the q-multionomial coefficient in (5.2.15a) is unity. Thus apart from the obvious overall factor in (5.2.15a), the lowest power of $H_m^{(N)}(\beta)$ is given by

(5.2.17a)
$$\min_D \mathscr{S}_1,$$

where D is the domain of $(x_2, \cdots, x_N)$ specified by

(5.2.17b)
$$D=\{(x_2, \cdots, x_N)\in \boldsymbol{Z}^{N-1}|x_2, \cdots, x_N\geq 0,\ \tilde{m}\geq 0, \mathscr{S}_0\geq 1\}.$$

Let us evaluate (5.2.17). For $N=2$ it is easily seen by using the condition $m\geq x_2\geq\omega$ that the minimum is attained by $x_2=\omega$. This gives the value $\mathscr{S}_1=\omega(m+1-\omega)$ leading to (5.2.14). In the sequel we prove the case $N\geq 3$. First we seek for the point that attains the minimum in $\{(x_2, \cdots, x_N)\in \boldsymbol{R}^{N-1}|x_2, \cdots, x_N\geq 0, \tilde{m}\geq 0, \mathscr{S}_0\geq 1\}$. Consider the derivative of $\mathscr{S}_1$ with respect to x_i $(2\leq i\leq N)$. Then there appear the terms $(i-1)m-\omega i$ (see (5.2.15d)). If we rewrite this as $(i-1)(m-\omega)-\omega$ and use the condition $\omega\leq\sum_{2\leq j\leq N}(j-1)x_j$, we have for $2\leq i\leq N$

(5.2.18)
$$\frac{\partial\mathscr{S}_1}{\partial x_i}\geq(i^2-3i+3)x_i+\sum_{2\leq k<i}((i-3)k+2)x_k+\sum_{i<k\leq N}(k-2)(i-1)x_k+(i-1)(m-\omega)+i/2>0.$$

This implies that the points in question are located on the hyperplane $\sum_{2\le j\le N}(j-1)x_j=\omega$. If $N\geq 4$ we derive further by using $\omega=\sum_{2\le j\le N}(j-1)x_j$ that

$$(5.2.19)\qquad \begin{aligned}&(N-2)\frac{\partial \mathcal{S}_1}{\partial x_N}-(N-1)\frac{\partial \mathcal{S}_1}{\partial x_{N-1}}\\ &=\frac{N-3}{2}x_N+\frac{N-4}{2}x_{N-1}+\sum_{2\le k\le N-2}(k-1)x_k+\frac{\omega-1}{2}\geq 0.\end{aligned}$$

Thus we must set $x_N=0$ to attain the minimum. Repeated use of this argument reduces the problem to the case $x_N=\cdots=x_4=0$. Then, $x_2+2x_3=\omega$ and (5.2.19) reads as

$$(5.2.20)\qquad \frac{\partial \mathcal{S}_1}{\partial x_3}-2\frac{\partial \mathcal{S}_1}{\partial x_2}=x_3-1/2.$$

From this we see that the minimum is attained at $x_2=\omega-1$, $x_3=1/2$, $x_4=\cdots=x_N=0$. Actually x_2 and x_3 range within $\boldsymbol{Z}$. Therefore (5.2.20) leads us to the conclusion for all $N(\geq 3)$ that

$$(5.2.21)\qquad \min_D \mathcal{S}_1=\mathcal{S}_1|_{N=3,x_2=\omega,x_3=0}=\mathcal{S}_1|_{N=3,x_2=\omega-2,x_3=1}=\omega(m+1-\omega).$$

From this and (5.2.15a) we establish (5.2.14). □

Remark. In the case $N=1$ it follows from the expression (5.2.8) that

$$(5.2.22)\qquad H_m^{(1)}(\beta)=0,\qquad \text{for } m>\beta\geq -m.$$

Lemma 5.2.5. *Assume that* $N\geq 2$. *For* $0\leq w\leq mN/2$, *set* $u=w/m$ $(0\leq u\leq N/2)$. *Then we have as* $m\to\infty$

$$(5.2.23)\qquad h_m^{(N)}(um)=O(q^{(N-2)u^2m^2/2N+\text{linear terms in } m}),$$

where $h_m^{(N)}(w)$ *has been defined in* (4.2.28b-d).

Proof. The case $N=2$ is obvious from (4.2.46) and so is the case $u=0$ from $h_m^{(N)}(0)=1$ (see (4.2.33)). In the following we assume $N\geq 3$ and $u\neq 0$. We employ the expression (4.2.28b-e) for the function $h_m^{(N)}(w)$. Our proof here consists of three steps:
(i) Under the conditions $\sum_{1\le j\le N-1}jy_j=\bar{s}_1$ and $y_j\geq 0$, extract the minimum of the power $\mathcal{R}$ (4.2.28d) as a function of $\bar{s}_1$.
(ii) Supply (i) with the contributions from the j-summand in $h_\infty^{(N)}(w-\bar{s}_1)$.
(iii) Under the conditions $0\leq \bar{s}_1\leq w$ and $0\leq j\leq (w-\bar{s}_1)/N$ (or $1\leq j\leq (w-\bar{s}_1)/N$, see (4.2.28e)), minimize the total power obtained in (ii).

In the working (i) $\sim$ (iii) we shall regard $y_j \in \boldsymbol{R}$ rather than $y_j \in \boldsymbol{Z}$. This does not effect the estimations to the order of m^2.

(i) Let $\bar{s}_1$ be a non-negative constant. Eliminate y_1 by using the latter of (4.2.28c). Then the power $\mathscr{R}$ in (4.2.28d) is written as

$$\begin{aligned}\mathscr{R}=&\sum_{2\leq j,k\leq N-1}\mathscr{R}_{j,k}y_j y_k-(mN-2(w-\bar{s}_1)+1)\sum_{2\leq j\leq N-1}(j-1)y_j\\&+((N-1)m-w+(\bar{s}_1+1)/2)\bar{s}_1,\end{aligned}\tag{5.2.24a}$$

$$\mathscr{R}_{j,k}=\mathscr{R}_{k,j}=(\min(j,k)-1)\max(j,k).\tag{5.2.24b}$$

Our aim here is to evaluate the quantity

$$\min_{\bar{D}}\mathscr{R},\tag{5.2.25a}$$

$$\bar{D}=\{(y_2,\cdots,y_{N-1})\in \boldsymbol{R}^{N-1}\mid \sum_{2\leq j\leq N-1} jy_j\leq \bar{s}_1,\ y_j\geq 0\}.\tag{5.2.25b}$$

The quadratic form $\sum_{2\leq j,k\leq N-1}\mathscr{R}_{j,k}y_j y_k$ is positive definite. This follows from the fact that the inverse of $(\mathscr{R}_{jk})_{j,k=2,\cdots,N-1}$ is given by

$$\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & \cdot & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & \cdot & 2 & -1 \\ & & & & -1 & \dfrac{N-2}{N-1} \end{pmatrix}\tag{5.2.26}$$

Thus the minimum of $\mathscr{R}$ is attained by the point $\bar{M}$ where its inward derivatives with respect to $\bar{D}$ are non-negative. It is given by

$$\begin{aligned}\bar{M}:\ & y_2=\cdots=y_{N-2}=0,\quad y_{N-1}=\frac{\bar{s}_1}{N-1} && \text{for } N\geq 4,\\ :\ & y_2=\bar{s}_1/2 && \text{for } N=3.\end{aligned}\tag{5.2.27}$$

The condition is checked as follows (recall the assumption $0\leq w\leq mN/2$):

$$-\frac{\partial\mathscr{R}}{\partial y_{N-1}}\bigg|_{\bar{M}}=(N-2)(mN-2w+1)>0 \qquad \text{for } N\geq 3,$$

$$\left((N-1)\frac{\partial\mathscr{R}}{\partial y_j}-j\frac{\partial\mathscr{R}}{\partial y_{N-1}}\right)\bigg|_{\bar{M}}=(N-1-j)(mN-2w+1)>0$$

for $N\geq 4$ and $2\leq j\leq N-2$.

In this way we obtain

$$(5.2.28)\qquad \min_{\bar{D}} \mathscr{R}=\mathscr{R}|_{\bar{M}}=\bar{s}_1((N-3)(w-(\bar{s}_1+1)/2)+m)/(N-1).$$

(ii) Define the functions $\mathscr{U}_1(\bar{s}_1, j; w)$, $\mathscr{U}_2(\bar{s}_1, j; w)$ by

$$(5.2.29a)\qquad \begin{aligned}\mathscr{U}_1(\bar{s}_1, j; w)=&\bar{s}_1((N-3)(w-(\bar{s}_1+1)/2)+m)/(N-1)\\ &+(w-\bar{s}_1)(w-\bar{s}_1-1)+N^2j^2/2\\ &-(N+1)(w-\bar{s}_1)j+Nj/2,\end{aligned}$$

$$(5.2.29b)\qquad \begin{aligned}\mathscr{U}_2(\bar{s}_1, j; w)=&\bar{s}_1((N-3)(w-(\bar{s}_1+1)/2)+m)/(N-1)\\ &+(w-\bar{s}_1)^2+N^2j^2/2\\ &-(N+1)(w-\bar{s}_1)j-Nj/2.\end{aligned}$$

In addition to (5.2.28), $\mathscr{U}_1$ and $\mathscr{U}_2$ respectively count the power of q coming from the first and the second terms in (4.2.28e) with w replaced by $w-\bar{s}_1$. These are positive definite quadratic forms of $\bar{s}_1$ and j.

(iii) From (i), (ii) and (4.2.28b-e) we now have the estimation

$$(5.2.30a)\qquad h_m^{(N)}(w)=O(q^{\mathscr{U}}),$$

$$(5.2.30b)\qquad \mathscr{U}=\min(\mathscr{U}_1^*, \mathscr{U}_2^*),$$

$$(5.2.30c)\qquad \mathscr{U}_i^*=\min_{D_i^*} \mathscr{U}_i(\bar{s}_1, j; w), \qquad (i=1, 2),$$

$$(5.2.30d)\qquad D_1^*=\{(\bar{s}_1, j)\in \boldsymbol{R}^2 | 0\leq \bar{s}_1\leq w, \quad 0\leq j\leq (w-\bar{s}_1)/N\},$$

$$(5.2.30e)\qquad D_2^*=\{(\bar{s}_1, j)\in \boldsymbol{R}^2 | 0\leq \bar{s}_1\leq w, \quad 1\leq j\leq (w-\bar{s}_1)/N\}.$$

(we put $\mathscr{U}_2^*=\infty$ if $w<N$.) Consider the point $M^*=(0, w/N)$ on the boundaries of D_1^* and D_2^*, and the derivatives of $\mathscr{U}_i$ at M^*

$$(5.2.31a)\qquad \left.\frac{\partial \mathscr{U}_1}{\partial j}\right|_{M^*}=\frac{N}{2}-w, \qquad \left.\frac{\partial \mathscr{U}_2}{\partial j}\right|_{M^*}=-\frac{N}{2}-w,$$

$$(5.2.31b)\qquad \left.\left(N\frac{\partial \mathscr{U}_1}{\partial \bar{s}_1}-\frac{\partial \mathscr{U}_1}{\partial j}\right)\right|_{M^*}=\left.\left(N\frac{\partial \mathscr{U}_2}{\partial \bar{s}_1}-\frac{\partial \mathscr{U}_2}{\partial j}\right)\right|_{M^*}=\frac{mN-2w+N}{N-1}.$$

If we set $w=um$ $(0<u\leq N/2)$ and let $m\to\infty$, (5.2.31a) tends to $-\infty$ while (5.2.31b) remains positive because of the assumption $0\leq w\leq mN$. Thus we can employ the same argument as in (i) to find

$$(5.2.32)\qquad \mathscr{U}_i^* \xrightarrow[m\to\infty]{} \mathscr{U}_i\left(0, \frac{um}{N}; um\right)=\frac{N-2}{2N}m^2+\text{linear terms in } m.$$

From this and (5.2.30) we establish (5.2.23). □

Lemma 5.2.6 ((3.31) of Part I). *Assume that* $N\geq 2$. *For* $m(N-C)\geq|\beta|$ (*C defined in Lemma* 5.2.4), *set* $t=\beta/m$ ($|t|\leq N-C$). *Then we have*

$$H_m^{(N)}(mt)=O(q^{-t^2m^2/4N+\text{linear terms in m}}). \tag{5.2.33}$$

Proof. In view of the symmetry (5.2.9) it is sufficient to verify the case $m(N-C)\geq\beta\geq 0$ ($N-C\geq t\geq 0$). Substitute (4.2.28a) into (5.2.7) and replace v by $v+w$. The v-summation using the formula (4.1.14) yields ω defined in (5.2.7b))

$$\begin{aligned} H_m^{(N)}(\beta)=&(-1)^{\omega}q^{-m(m+1)N/4+\omega^2/2} \\ &\times\sum_{0\leq w\leq mN/2}(q^{w-\omega+1})_{mN-2w}q^{w(m-\omega+1)}h_m^{(N)}(w). \end{aligned} \tag{5.2.34}$$

The product $(q^{w-\omega+1})_{mN-2w}$ is non-zero only if

$$w\geq\omega \quad \text{or} \quad w\geq mN-\omega+1. \tag{5.2.35}$$

By virtue of the assumption $m(N-C)\geq\beta\geq 0$ (i.e. $mC/2\leq\omega\leq mN/2$), the latter can be discarded, which leads to

$$(q^{w-\omega+1})_{mN-2w}=1+O(q). \tag{5.2.36}$$

From (5.2.34–36) and Lemma 5.2.5, we obtain

$$\begin{aligned} &H_m^{(N)}(mt)=O(q^{\sigma m^2+\text{linear terms in } w}), \\ &\sigma=-\frac{N}{4}+\frac{1}{2}\left(\frac{N-t}{2}\right)^2+\min_{(N-t)/2\leq u\leq N/2}\left(u\left(1-\frac{N-t}{2}\right)+\frac{N-2}{2N}u^2\right) \\ &\quad=-t^2/4N. \end{aligned}$$

where we used the assumption $0\leq t(\leq N-C)$. □

Proof of Theorem 5.2.1. It is sufficient to show that

$$\lim_{\substack{m\to\infty \\ mN\equiv\rho \bmod 2(L-2)}} x_m(a,b,b+N)=e^{L-2}_{b-\rho-1,a-1}(\tau), \tag{5.2.37}$$

where $0<b$, $b+N<L$ and

$$x_m(a,b,b+N)=q^{M(m,a,b)}X_m(a,b,b+N;q^{-1}), \tag{5.2.38a}$$

$$\begin{aligned} M(m,a,b)=&\frac{m(m+1)N}{4}-\frac{1}{4(L-2)}\left(mN+\frac{L}{2}-b\right)^2 \\ &+\frac{1}{4L}\left(\frac{L}{2}-a\right)^2+\frac{1}{24}. \end{aligned} \tag{5.2.38b}$$

Let us introduce the function $\tilde{y}_m(b, b')$ by

(5.2.39) $$\tilde{y}_m(b, b') = \sum_{0<a<L} x_m(a, b, b+N)\tilde{p}^{L-2}_{a-1,b'-1}(\tau),$$

where $\tilde{p}^l_{kj}(\tau)$ is given in (5.2.3). Thanks to Lemma 5.2.2, (5.2.37) is equivalent to the following statement

(5.2.40) $$\begin{aligned}\lim_{\substack{m\to\infty \\ mN\equiv\rho \bmod 2(L-2)}} \tilde{y}_m(b, b') &= \eta(\tau) \qquad \text{if } b'-1\equiv\pm(b-\rho-1) \\ &\qquad\qquad\qquad \bmod 2(L-2), \\ &= 0 \qquad\quad \text{otherwise.}\end{aligned}$$

Using Theorem 4.4.1 we express $x_m(a, b, b+N)$ in terms of $f^{(N)}_m(*, *+N; q^{-1})$. Substitute the explicit form (5.2.3) of $\tilde{p}^l_{a-1,b'-1}(\tau)$ into (5.2.39). After a little calculation we find

(5.2.41a) $$\tilde{y}_m(b, b') = q^{1/24}\varepsilon^{L-2}_{b'-1}(y_m(b, b') + y_m(b, 2-b')),$$

(5.2.41b) $$y_m(b, b') = \sum_{\beta\in Z,\ \beta\equiv b-b' \bmod 2(L-2)} z_m(\beta, b),$$

(5.2.41c) $$z_m(\beta, b) = q^{m(m+1)N/4-\omega(mN-\omega+L/2-b)/(L-2)}H^{(N)}_m(\beta),$$

where the function $H^{(N)}_m(\beta)$ has been defined by (5.2.6) and ω by (5.2.7b). First consider the case $N=1$. Then $y_m(b, b')$ is written by using (5.2.8) as

$$\begin{aligned}y_m(b, b') = &\sum_{\substack{\nu\geq 0 \\ 2\nu\equiv b-b'-m \bmod 2(L-2)}} (-1)^\nu q^{\nu(2m+L(\nu+1)-2b)/(2L-4)}(q^{\nu+1})_m \\ + &\sum_{\substack{\nu\geq 1 \\ 2\nu\equiv b-b'-m \bmod 2(L-2)}} (-1)^\nu q^{(L\nu^2+(2m-L+2b)\nu+2m(b-1))/(2L-4)}(q^\nu)_m.\end{aligned}$$

In this form it is straightforward to take the limit $m\to\infty$. We obtain (recall that $b>0$)

(5.2.42) $$\begin{aligned}\lim_{m\to\infty,\ m\equiv\rho \bmod 2(L-2)} y_m(b, b') &= \varphi(q) \qquad \text{if } b'\equiv b-\rho \bmod 2(L-2), \\ &= 0 \qquad\quad \text{otherwise.}\end{aligned}$$

This proves (5.2.40) for $N=1$. Next we treat the case $N\geq 2$. Much the same as in (5.2.42), we find for general N using (5.2.13) and (5.2.41c) that

(5.2.43) $$\lim_{m\to\infty} z_m(mN, b) = \varphi(q).$$

Thus the remaining task is to show that the contributions from all other values of β vanish in the limit $m\to\infty$. By Lemma 5.2.3, 5.2.4 and 5.2.6 we deduce the following estimates:

$$
\begin{aligned}
z_m(\beta, b) &= O(q^{-\omega(2mN-L\omega+L-2b)/(2L-4)}) \\
&\qquad \text{if } \beta > mN \ (\omega < 0), \\
&= O(q^{\omega(2(L-N-2)m-(L-4)\omega+L-4+2b)/(2L-4)}) \\
&\qquad \text{if } mN > \beta > m(N-C) \ (0 < \omega < mC/2), \\
&= O(q^{Am^2+\text{linear terms in } m}), \\
&\qquad \text{if } \beta = mt, \ |t| \leq N-C \ (mC/2 \leq \omega \leq m(N-C/2)),
\end{aligned}
$$

where $A=(L-N-2)(N^2-t^2)/(4N(L-2))>0$ (see (2.2.2)). Now it is clear that $z_m(\beta, b)$ in these regions converges to zero as m tends to ∞. This is also the case for the remaining region $\beta<-m(N-C)$ due to the symmetry (5.2.9). □

Appendix A. Minimum/Maximum Configurations

In section 5 we identified the 1D configuration sums with modular forms. It is important to know the lowest power in the q-expansion of these modular forms, for they are related to the critical exponents of the models. (See section 4 in Part I.) In regime I the modular forms are given in the form of infinite product (Part I (A. 14)), and the lowest power can be easily read off. In regime II the modular forms $e^l_{jk}(\tau)$ have been encountered previously and the lowest power is known (see (5.1.6) in [10] or (2.4) in [7]). In regimes III and IV, where the modular forms are characterized by theta function identities (Part I (A.3)), it is not straightforward to pinpoint the lowest power (the fractional power mod Z follows immediately). Here we do it by singling out the sequence that attains the minimum (regime III)/maximum (regime IV) of

(A.1) $$\phi_m(l_1, \cdots, l_{m+2}) = \sum_{j=1}^{m} jH(l_j, l_{j+1}, l_{j+2}),$$

under the condition that

(A.2) $$l_1=a, \ l_{m+1}=b, \ l_{m+2}=c.$$

A.1. Regime III

$$H(a, b, c) = \frac{|a-c|}{4}$$

We define $2m+1$ integers a_j $(-m \leq j \leq m)$ by

$$
\begin{aligned}
a_j &= b+jN \qquad \text{if } j \equiv m \bmod 2, \\
&= c+jN \qquad \text{otherwise.}
\end{aligned}
$$

Note that if $a>a_m$ or $a<a_{-m}$ there is no sequence satisfying (A.2). Without loss of generality we assume that $a\leq a_0$. We also assume that $a_{-1}\geq 1$. Because of the symmetry (Part I (A. 5)) of the branching coefficient this does not restrict our aim.

Lemma A.1. *Under these assumptions let μ be a positive integer such that $a_{-\mu}\leq a\leq a_{1-\mu}$. The minimum of* (A.1) *among admissible sequence $(l_j)_{j=1,\cdots,m+2}$ satisfying* (A.2) *is attained by the following sequence*:

$$(\text{A.3})\qquad \begin{aligned}\bar{\ell}_j&=a+(j-1)N && \text{if } 1\leq j\leq \mu,\\ &=c && \text{else if } j\equiv m \bmod 2,\\ &=b && \text{otherwise.}\end{aligned}$$

The minimum is

$$(\text{A.4})\qquad \begin{aligned}&\phi_m(\bar{\ell}_1,\cdots,\bar{\ell}_{m+2})\\ &\quad=\frac{a-b-\mu(2a-b-c)-\mu(\mu-1)N}{4} && \text{if } \mu\equiv m \bmod 2,\\ &\quad=\frac{a-c-\mu(2a-b-c)-\mu(\mu-1)N}{4} && \text{otherwise.}\end{aligned}$$

Proof. Because of the assumption $a_{-1}\geq 1$ the $(\bar{\ell}_j)$ of (A.3) is admissible. Therefore it is sufficient to show that this attains the minimum of (A.1) among weakly admissible sequences satisfying (A.2). Assume that (ℓ_j) attains the minimum. Take any successive four $\ell_j, \ell_{j+1}, \ell_{j+2}, \ell_{j+3}$ $(1\leq j\leq m-1)$. They are subject to the following: If $\ell_{j+1}>\ell_{j+3}$ then the weight (A.1) strictly decreases if we replace ℓ_{j+1} by $\ell_{j+1}-2$. Therefore this replacement should violate the weak admissibility of the sequence. This implies that $\ell_j=\ell_{j+1}+N$. Similarly, if $\ell_{j+1}<\ell_{j+3}$ then we have $\ell_j=\ell_{j+1}-N$. The $(\bar{\ell}_j)$ is the unique one which satisfies these restrictions as well as (A.2). Therefore the minimum of (A.1) is attained solely by the $(\bar{\ell}_j)$. It is straightforward to compute (A.4). □

A.2. Regime IV

$$(\text{A.5})\qquad \begin{aligned}H(a,b,c)&=\min\left(n-b, \frac{\min(a,c)-b+N}{2}\right) && \text{if } b\leq n,\\ &=\min\left(b-n-1, \frac{b-\max(a,c)+N}{2}\right) && \text{if } b\geq n+1.\end{aligned}$$

We seek for the sequence $(\bar{\ell}_j)$ that attains the maximum of (A.1) where the weight function $H(a, b, c)$ is given by (A.5). As noted in Part I (eqs.

(3.9–11)) this resembles the negative of the weight in regime III. Therefore it may be expected that the maximum is attained again by (A.3). This is partially true as we shall see below.

We assume that

$$\text{(A.6)} \qquad \frac{b+c-N}{2} \leq n-N.$$

This is not restrictive, for the branching coefficients relevant to regime IV are all obtained as the $m \to \infty$ limit of $Y_m(a, b, c; q^{-1})$ with b, c satisfying (A.6).

Because of the restriction (A.6) we cannot assume that $a \leq a_0$ as we did in Lemma A.1. We need two more candidates maximizing (A.1) (see Fig. A.1): Let μ be a positive integer such that $a_{\mu-1} \leq a \leq a_\mu$. We set

$$\text{(A.7a)} \qquad \begin{aligned} \bar{\ell}_j^{(1)} &= a-(j-1)N && \text{if } 1 \leq j \leq \mu, \\ &= c && \text{else if } j \equiv m \bmod 2, \\ &= b && \text{otherwise.} \end{aligned}$$

With $\mu \geq 2$ as above we set

$$\text{(A.8a)} \qquad \begin{aligned} \bar{\ell}_j^{(2)} &= a-(j-1)N && \text{if } 1 \leq j \leq \mu-1, \\ &= c+N && \text{if } j=\mu \equiv m \bmod 2, \\ &= b+N && \text{if } j=\mu \not\equiv m \bmod 2, \\ &= c && \text{else if } j \equiv m \bmod 2, \\ &= b && \text{otherwise.} \end{aligned}$$

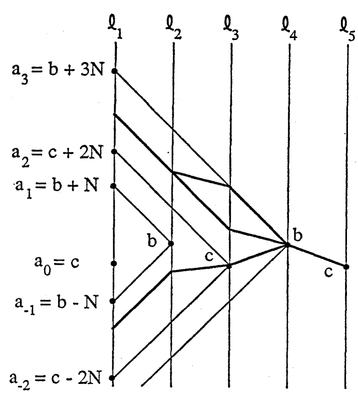


Fig. A. 1 The configurations maximizing the weight $\sum_{j=1}^{m} jH(\ell_j, \ell_{j+1}, \ell_{j+2})$.

The $(\bar{\ell}_j^{(1)})$ differs from the $(\bar{\ell}_j^{(2)})$ only at $j=\mu$. We shall use these sequences under the condition

(A.7b) $$\frac{\bar{\ell}_{m-1}^{(1)}+b}{2}\leq n,$$

(A.8b) $$\frac{\bar{\ell}_{m-1}^{(2)}+b}{2}\geq n+1,$$

respectively. The sequences $(\bar{\ell}_j)$, $(\bar{\ell}_j^{(1)})$ and $(\bar{\ell}_j^{(2)})$ are weakly admissible, but they are not necessarily admissible. For a given a there exists a unique path connecting a to b, c among these three.

Lemma A.2. *The maximum of* (A.1), *among weakly admissible sequences satisfying* (A.2), *is uniquely attained by one of* (A.3), (A.7–8). *Let* $a_{-\mu}\leq a\leq a_{1-\mu}$ $(1-m\leq\mu\leq m)$. *The maximum value* $\phi_{\max}$ *is*

(A.9)
$$\begin{aligned}\phi_{\max}&=\frac{m(c-b)+m(m+1)N+\mu(2a-b-c)+\mu(\mu-1)N}{4}\\&\qquad\qquad\textit{if } \mu\equiv m \bmod 2,\\&=\frac{(m+1)(c-b)+m(m+1)N+\mu(2a-b-c)+\mu(\mu-1)N}{4}\\&\qquad\qquad\textit{otherwise}.\end{aligned}$$

The proof of this lemma will be given in Lemma A.3–8. The sequences used in the proof are weakly admissible unless otherwise stated. For $(\ell_j)_{j=j_0,\cdots,j_1}$ and $(\ell'_j)_{j=j_0,\cdots,j_1}$ such that

(A.10a) $$\ell_j=\ell'_j \qquad \text{if } j=j_0, j_0+1, j_1-1, j_1,$$

or when $j_0=1$

(A.10b) $$\ell_j=\ell'_j \qquad \text{if } j=1, j_1-1, j_1,$$

we denote by $(\ell_j)<(\ell'_j)$ and say the latter *dominates* the former if

$$\phi_m(\ell_1, \cdots, \ell_{m+2})<\phi_m(\ell'_1, \cdots, \ell'_{m+2}),$$

where $\ell_j=\ell'_j$ $(j<j_0$ or $j>j_1)$ are supplemented arbitrarily. We abbreviate $H(\ell_j, \ell_{j+1}, \ell_{j+2})$ (resp. $H(\ell'_j, \ell'_{j+1}, \ell'_{j+2})$) to H_j (resp. H'_j). If we replace ϕ_m of (A.1) with $\sum_{j=1}^m \alpha_j H(\ell_j, \ell_{j+1}, \ell_{j+2})$ where $\alpha_j<\alpha_{j+1}$ $(j=1, \cdots, m)$ the proof goes well without change. In fact we need this generalization later in the proof of Lemma A.9.

Lemma A.3. *Consider an* (ℓ_j) *such that*

$$\ell_j=\ell_m-(m-j)N \qquad \text{if } i+1\leq j\leq m-1$$

where $1\leq i\leq m-1$. *We also assume that* $\ell_m<c$ *and* $\ell_i>\ell_{i+1}-N$. *Then the* (ℓ'_j) *defined by*

$$\begin{aligned}\ell'_j&=\ell_j+2 &&\text{if } i+1\leq j\leq m,\\ &=\ell_j &&\text{otherwise,}\end{aligned}$$

dominates the (ℓ_j).

Proof. We have $H'_{i-1}\geq H_{i-1}$, $H'_i=H_i-1$, $H'_j=H_j=0$ $(i+1\leq j\leq m-1)$, and $H'_m=H_m+1$. This implies $(\ell_j)<(\ell'_j)$. □

Lemma A.3 tells that if (ℓ_j) attains the maximum and $\ell_m<c$, then (ℓ_j) must be of the form (A.3). Now we consider the case $\ell_m>c$.

Lemma A.4. *Assume that* $(\ell_j)_{j=i,\cdots,i+4}$ *and* $(\ell'_j)_{j=i,\cdots,i+4}$ *satisfy* (A.10), *and that*

$$\frac{\ell_{i+3}+\ell_{i+4}-N}{2}\leq n-N, \qquad c\leq\ell_{i+2}=\ell'_{i+2}-2.$$

If $\ell_{i+2}\geq n+1$ *then* $(\ell_j)<(\ell'_j)$, *and if* $\ell'_{i+2}\leq n$ *then* $(\ell_j)>(\ell'_j)$.

Proof. Assume that $\ell_{i+2}\geq n+1$. Then we have $H'_i\geq H_i-1$, $H'_{i+1}\geq H_{i+1}+1$ and $H'_{i+2}=H_{i+2}$. From this follows $(\ell_j)<(\ell'_j)$. The other case is proved similarly. □

Lemma A.5. *Assume that* $(\ell_j)_{j=i,\cdots,i+4}$ *and* $(\ell'_j)_{j=i\cdots,i+4}$ *satisfy* (A.10), *and that*

$$\frac{\ell_{i+3}+\ell_{i+4}-N}{2}\leq n-N, \quad \ell_{i+2}=\ell_{i+1}+N, \quad \ell'_{i+2}=\ell_{i+4}.$$

Then we have $(\ell_j)<(\ell'_j)$.

Proof. We have $H'_i\geq H_i-n+(\ell_{i+2}+\ell_{i+4})/2$, $H'_{i+1}\geq H_{i+1}+n+1-(\ell_{i+2}+\ell_{i+4})/2$ *and* $H'_{i+2}=H_{i+2}$. Noting that

$$n+1-\frac{\ell_{i+2}+\ell_{i+4}}{2}\geq n+1-\frac{\ell_{i+3}+N+\ell_{i+4}}{2}\geq 1,$$

we have $(\ell_j)<(\ell'_j)$. □

From Lemma A.4–5 we can conclude that if (ℓ_j) satisfying (A.2) attains the maximum of ϕ_m and if $\ell_m > c$ then one of the following is valid:
(i) $\ell_m = b + N$,
(ii) $\ell_m = \ell_{m-1} - N$.
Note that $(\bar{\ell}_j^{(1)})$ and $(\bar{\ell}_j^{(2)})$ satisfies (i) and (ii), respectively. Now we distinguish these two as in (A.7b) and (A.8b).

Lemma A.6. *Assume that* $(\ell_j)_{j=i,\cdots,i+4}$ *and* $(\ell'_j)_{j=i\cdots,i+4}$ *satisfy* (A.10), *and that*

$$\frac{\ell_{i+3}+\ell_{i+4}-N}{2} \leq n-N, \quad \ell_{i+2}=\ell_{i+1}-N, \quad \ell'_{i+2}=\ell_{i+3}+N.$$

We set

$$\omega = \frac{\ell_{i+1}+\ell_{i+3}}{2} - n.$$

Then we have

$$\text{(A.11a)} \qquad (\ell_j) > (\ell'_j) \qquad \textit{if } \omega \leq 0,$$

$$\text{(A.11b)} \qquad (\ell_j) < (\ell'_j) \qquad \textit{if } \omega \geq 1.$$

In either case we have

$$\max(H_{i+1}, H'_{i+1}) = \frac{\ell_{i+3}-\ell_{i+1}}{2} + N.$$

Proof. We consider (A.11a) first. We have $H_i \geq H'_i + \omega$, $H_{i+1} \geq H'_i + 1 - \omega$ and $H_{i+2} = H'_{i+2}$. Therefore (A.11a) is valid. If $\omega \geq 1$, we have $H'_j \geq H_j - \omega + 1$, $H'_{i+1} \geq H_{i+1} + \omega$ and $H_{i+2} = H'_{i+2}$. This implies (A.11b). □

Thus we have proved the last four $\ell_{m-1}, \ell_m, \ell_{m+1}, \ell_{m+2}$ are as expected in (A.7–8) if (ℓ_j) attains the maximum. (Remember that we are assuming that $\ell_m > c$). We now prove

$$\text{(A.12)} \qquad \ell_j = \ell_{j+1} + N \qquad \text{if } j=1, \cdots, m-2.$$

Assume that

$$\text{(A.13a)} \qquad \ell_j - \ell_{j+1} = N \qquad \text{if } j=i+1, \cdots, m-1,$$

$$\text{(A.13b)} \qquad \qquad < N \qquad \text{if } j=i.$$

If $(\ell_i+\ell_{i+2})/2\le n$ then $(\ell_{i+2}-\ell_{i+3}-N)/2\le n-N$ and $\ell_{i+1}>\ell_{i+3}$. This is a contradiction, for ℓ_i must be equal to $\ell_{i+1}+N$ from Lemmas A. 4–6. Therefore it is sufficient to show (A.12) when $(\ell_i+\ell_{i+2})/2\ge n+1$.

Lemma A.7. *Let* (ℓ_j) *satisfy* (A.2), (A.13). *We also assume*

$$\frac{\ell_i+\ell_{i+2}}{2}\ge n+2.$$

Define (ℓ'_j) *by* $\ell'_j=\ell_j$ $(j=1,\cdots,i,m+1,m+2)$ *and* $\ell'_j=\ell_j-2$ $(j=i+1,\cdots,m)$, *then we have* $(\ell'_j)>(\ell_j)$.

Proof. We have $H'_j=H_j$ $(j=1,\cdots,i-2)$, $H'_{i-1}\ge H_{i-1}$, $H'_i\ge H_i-1$, $H'_j=H_j=0$ $(j=i+1,\cdots,m-2)$ and $H'_{m-1}\ge H_{m-1}+1$. Therefore (ℓ'_j) dominates (ℓ_j). □

Lemma A.8. *Let* (ℓ_j) *satisfy* (A.2), (A.13). *We also assume*

$$\frac{\ell_i+\ell_{i+2}}{2}=n+1. \tag{A.15}$$

There are three cases to consider:

(*Case* 1) $\ell_i-(m-i-1)N\le b$

Define (ℓ'_j) *by*

$$\begin{aligned}
\ell'_j&=\ell_i-(j-i)N && \text{if } j=1,\cdots,m-3,\\
&=b+N && \text{if } j=m-2 \text{ and } \frac{\ell'_{m-3}+b}{2}\ge n+1,\\
&=\ell'_{m-3}-N && \text{if } j=m-2 \text{ and } \frac{\ell'_{m-3}+b}{2}\le n,\\
&=b && \text{if } j=m-1,\ m+1,\\
&=c && \text{if } j=m,\ m+2.
\end{aligned}$$

(*Case* 2) $b<l_i-(m-i-1)N\le c+N$

Define (ℓ'_j) *by*

$$\begin{aligned}
\ell'_j&=\ell_i-(j-i)N && \text{if } j=1,\cdots,m-2,\\
&=b+N && \text{if } j=m-1 \text{ and } \frac{\ell'_{m-2}+b}{2}\ge n+1,\\
&=\ell'_{m-3}-N && \text{if } j=m-1 \text{ and } \frac{\ell'_{m-2}+b}{2}\le n,\\
&=b && \text{if } j=m+1,\\
&=c && \text{if } j=m,\ m+2.
\end{aligned}$$

(*Case* 3) $c+N<\ell_i-(m-i-1)N\leq b+2N$
Define (ℓ'_j) *by*

$$\begin{aligned} \ell'_j &= \ell_i-(j-i)N && \text{if } j=1,\cdots,m, \\ &= b && \text{if } j=m+1, \\ &= c && \text{if } j=m+2. \end{aligned}$$

In each case the (ℓ'_j) *dominates the* (ℓ_j).

Proof. From (A.14) and $\ell_{i+1}=\ell_{i+2}+N$, we have

$$\ell_{i+1}-n-1=\frac{\ell_{i+1}-\ell_i+N}{2}>0.$$

We denote this quantity by ρ. Relevant values of H_j and H'_j are:

j	i	$i+1$	$\cdots$	$m-4$	$m-3$	$m-2$	$m-1$
H_j	ρ	0	$\cdots$	0	0	0	$\frac{b-\ell_{m-1}}{2}+N$
H'_j							
(*Case* 1)	0	0	$\cdots$	0	$\frac{b-\ell_{m-1}}{2}+\rho$	$\frac{c-b+N}{2}$	$\frac{b-c+N}{2}$
(*Case* 2)	0	0	$\cdots$	0	0	$\frac{c-\ell_{m-1}+N}{2}+\rho$	$\frac{b-c+N}{2}$
(*Case* 3)	0	0	$\cdots$	0	0	0	$\frac{b-\ell_{m-1}}{2}+N+\rho$

Noting that $(\ell_{m-1}-c-N)/2\geq 1$ we have $(\ell'_j)>(\ell_j)$. □

We have proved that the maximum of ϕ_m, among weakly admissible sequences satisfying (A.2), is attained by one of $(\bar{\ell}_j)$, $(\bar{\ell}_j^{(1)})$, $(\bar{\ell}_j^{(2)})$.

The maximum value (A.9) is obtained by a straightforward computation.

Lemma A.9. *The maximum of* ϕ_m *among admissible sequences is attained by one of* $(\bar{\ell}_j)$, $(\bar{\ell}_j^{(1)})$, $(\bar{\ell}_j^{(2)})$, *if it is admissible. Otherwise we need the following modification*: (*We assume that m is even. If m is odd, we must interchange b and c.*)

(i) $\mu=1$ *and* $a+c<N+2$

(A.15a) $$\bar{\ell}_2=N+2-a,$$

(ii) $\mu=-1$ *and* $a+b>2(L-N-1)$

(A.15b) $$\bar{\ell}_2^{(2)}=2L-N-2-a$$

The maximum value is modified to

(A.16) $$\text{the r.h.s. of (A.9)}+\min\left(0,\ \frac{a+c-N-2}{2},\ L-N-1-\frac{a+b}{2}\right).$$

Proof. Using the assumption that $L\geq 2N+3$, we can verify that the sequences $(\bar{\ell}_j)$, $(\bar{\ell}_j^{(1)})$, $(\bar{\ell}_j^{(2)})$ are admissible except for (i) and (ii). In these cases we have $H(a, \ell_2, \ell_3)\leq H(a, \ell_2, b)$. Therefore ℓ_3 must be equal to b. Since $H(\ell_2\, b, c)$ is independent of ℓ_2 for the admissible values of ℓ_2, our task is to pick up ℓ_2 that maximizes $H(a, \ell_2, b)$. Thus we obtain (A.15), and then (A.16) follows immediately. □

Appendix B. Ground State Configuration in Regime II

Let (b, c) be any admissible pair. In this appendix we shall determine the admissible sequences (ℓ_j) that maximize the weight

$$\phi_m(\ell_1, \cdots, \ell_{m+2})=\sum_{j=1}^{m} j\frac{|\ell_j-\ell_{j+2}|}{4},$$

under the restriction

$$\ell_{m+1}=b, \qquad \ell_{m+2}=c.$$

For $\ell\in \boldsymbol{Z}$ we denote by $\langle\ell\rangle$ the unique integer satisfying

(B.1) $$1\leq\langle\ell\rangle\leq L-1, \qquad \langle\ell\rangle-1\equiv\pm(\ell-1) \bmod 2(L-2).$$

Lemma B.1. *Define* $(\bar{\ell}_j^{(\pm)})$ *by*

$$\begin{aligned} \bar{\ell}_j^{(\pm)}&=\langle b\pm(m-j+1)N\rangle && \text{if } 1\leq j\leq m,\\ &=b && \text{if } j=m+1,\\ &=c && \text{if } j=m+2. \end{aligned}$$

We denote by $\mu_m(b)$ *a positive integer determined by*

$$1-\mu_m(b)(L-2)\leq b-mN<L-1-\mu_m(b)(L-2).$$

Then we have

$$\phi_m(\bar{\ell}_1^{(-)}, \cdots, \bar{\ell}_{m+2}^{(-)})=\mathcal{G}_m(b, c), \qquad \phi_m(\bar{\ell}_1^{(+)}, \cdots, \bar{\ell}_{m+2}^{(+)})=\mathcal{G}_m(L-b, L-c),$$

where

(B.2) $\mathscr{G}_m(b, c)$

$$=\frac{m^2N+m(c-b)+\mu_m(b)(\mu_m(b)-1)(L-2)+2\mu_m(b)(b-mN-1)}{4}.$$

Proof. The integer $\mu_m(b)$ gives the number of reflections along the $(\bar{\ell}_j^{(-)})$ (see Fig. B.1). If $\mu_m(b)=0$ the weight is given by $(m^2N+m(c-b))/4$. The deficiency in the weight at the i-th reflection (counting from the right) is $(1+mN-(i-1)(L-2)-b)/2$. Therefore the exact value of the weight is given by (B.2). □

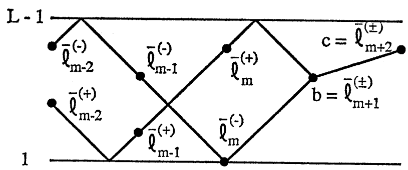


Fig. B.1 The configuration $(\bar{\ell}_j^{(\pm)})$.

Lemma B.2. $m>\mu_m(b)+(\mu_m(b)-1)/N$.

The proof is easy. From this we know, in particular, that $m-\mu_m(b)$ tends to ∞ when $m\to\infty$. Therefore if m is sufficiently large the sequence $\bar{\ell}_j=\langle b+Nj\rangle$ must contain a pair $(\bar{\ell}_i, \bar{\ell}_{i+1})$ such that $\bar{\ell}_{i+1}=\bar{\ell}_i+N$.

Lemma B.3. *Let x be an integer such that $\langle b-N\rangle<x<\langle b+N\rangle$, and set*

$$\mathscr{D}_m(x, b, c)=\max(\mathscr{G}_m(b, c), \mathscr{G}_m(L-b, L-c))-\left(\mathscr{G}_{m-1}(x, b)+\frac{m|x-c|}{4}\right).$$

Then $\mathscr{D}_m(x, b, c)>0$ *and* $\liminf_{m\to\infty}\mathscr{D}_m(x, b, c)/m>0$.

Proof. We set $\mu^{(-)}=\mu_m(b)$, $\mu^{(+)}=\mu_m(L-b)$ and $\mu=\mu_{m-1}(x)$. There are five cases:

	(i)	(ii)	(iii)	(iv)	(v)
$\mu^{(+)}$	μ	$\mu+1$	$\mu+1$	$\mu-1$	μ
$\mu^{(-)}$	μ	$\mu+1$	μ	μ	$\mu+1$

We set $\mathscr{D}^{(-)}=4(\mathscr{G}_m(b, c)-\mathscr{G}_{m-1}(x, b)-m|x-c|/4)$ and $\mathscr{D}^{(+)}=4(\mathscr{G}_m(L-b, L-c)-\mathscr{G}_{m-1}(x, b)-m|x-c|/4)$. The following prove the claim of this lemma:

Case (i), (v), $x\geq c$.

$$\mathscr{D}^{(+)}=(2m-1)N+b-x \qquad \text{if } \mu=0,$$
$$=(2\mu-1)(2L-N-2-b-x) \qquad \text{if } \mu\geq 1.$$

Case (i), (iii), (iv), $x \leq c$.

$$\mathscr{D}^{(-)} = (2m-2\mu-1)(x+N-b).$$

Case (ii), (iii), $x \geq c$.

$$\mathscr{D}^{(+)} = (2\mu+1)(2L-2-N-b-x).$$

Case (ii), (v), $x \leq c$.

$$\begin{aligned} \mathscr{D}^{(-)} &= 2\mu(L-N-2)+x+b-N-2 && \text{if } m=\mu+1, \\ &= 2(x-b+N)(m-\mu-3/2) && \text{if } m \geq \mu+2. \end{aligned}$$

Case (iv), $x \geq c$.

$$\mathscr{D}^{(+)} > (2\mu-3)(2L-N-2-b-x)+2(L-2). \qquad \square$$

From Lemma B.3 follows that the sequence $\bar{\ell}_j = \langle b+jN \rangle$ is a ground state configuration in the sense of section 2 of Part I. In fact, we have $\mathscr{G}_m(\bar{\ell}_{m+1}, \bar{\ell}_{m+2}) > \mathscr{G}_m(L-\bar{\ell}_{m+1}, L-\bar{\ell}_{m+2})$. Therefore $(\bar{\ell}_j)$ maximizes the weight $\phi_m(\ell_1, \cdots, \ell_{m+2})$.

Appendix C. Branching Coefficients and String Functions

Here we relate the branching coefficients $c^{(\varepsilon)}_{j_1 j_2 j_3}(\tau)$ and the string functions of $A_1^{(1)}$ (see Part I, Appendix B). This observation is due to V. G. Kac. Our LHP results in regime III, IV for $N=2$ are shown to coincide with the particular cases studied in [11] by using the expressions in C.3.

C.1. Products of theta functions

For a positive integer m and a real number μ, define [10]

$$\text{(C.1)} \qquad \Theta^{(\pm)}_{\mu,m}(u, \tau) = \sum_{\gamma=n+\mu/m, n\in Z} (\pm)^n q^{m\gamma^2/2} z^{-m\gamma}, \quad q=e^{2\pi i\tau}, \quad z=e^{2\pi i u},$$

The *Theta Null Werte* have the product representation (see (1.5.3))

$$\text{(C.2)} \qquad \Theta^{(\varepsilon)}_{\mu,m}(0, \tau) = q^{\mu^2/2m} E(-\varepsilon q^{\mu+m/2}, q^m), \qquad \varepsilon = \pm.$$

The theta function (C.1) obey the standard multiplication formula

$$\text{(C.3)} \qquad \begin{aligned} &\Theta^{(\varepsilon_1)}_{\mu_1,m_1}(u, \tau)\Theta^{(\varepsilon_2)}_{\mu_2,m_2}(u, \tau) \\ &= \sum_{\nu\in Z/(m_1+m_2)Z} \varepsilon_1^\nu \Theta^{(\varepsilon)}_{\alpha,\beta}(0, \tau)\Theta^{(\varepsilon_1\varepsilon_2)}_{m_1\nu+\mu_1+\mu_2, m_1+m_2}(u, \tau), \end{aligned}$$

where $\varepsilon = \varepsilon_1^{m_2}\varepsilon_2^{m_1}$, $\alpha = m_1 m_2(\nu+\mu_1/m_1-\mu_2/m_2)$, $\beta = m_1 m_2(m_1+m_2)$. The theta functions introduced in Part I, Appendix A and those here are connected by

(C.4) $$\Theta^{(\varepsilon_1,\varepsilon_2)}_{j,m}(z,q)=\Theta^{(\varepsilon_2)}_{j,2m}(u/2,\tau)+\varepsilon_1\Theta^{(\varepsilon_2)}_{-j,2m}(u/2,\tau).$$

C.2. Branching coefficients and string functions

Recall the definition of the branching coefficients $c^{(\varepsilon)}_{j_1j_2j_3}(\tau)$:

(C.5) $$\Theta^{(-,\varepsilon)}_{j_1,m_1}(z,q)\Theta^{(-,+)}_{j_2,m_2}(z,q)/\Theta^{(-,+)}_{1,2}(z,q)=\sum_{j_3} c^{(\varepsilon)}_{j_1j_2j_3}(q)\Theta^{(-,\varepsilon)}_{j_3,m_3}(z,q).$$

Here the sum ranges over $j_3\in Z+j_1$ such that $0<j_3\le m_3$ (if $\varepsilon=+$, then $j_3<m_3$). On the other hand, the level m string functions $c^m_{jk}(\tau)$ $(=c^{m-k\,k}_{m-j\,j}(\tau)$ in the notation of [10]) for $A^{(1)}_1$ are characterized by the identity

(C.6) $$\Theta^{(-,+)}_{k+1,m+2}(z,q)/\Theta^{(-,+)}_{1,2}(z,q)=\sum_{0\le j\le m,\, j\equiv k \bmod 2} c^m_{jk}(\tau)\varepsilon^m_j\Theta^{(+,+)}_{j,m}(z,q).$$

In the l.h.s. of (C.5), replace the part $\Theta^{(-,+)}_{j_2,m_2}(z,q)/\Theta^{(-,+)}_{1,2}(z,q)$ by the r.h.s. of (C.6), use (C.4) and apply (C.3). The result reads

$$\sum c^{m_2-2}_{j,j_2-1}(\tau)\Theta^{(+)}_{2m_1j_3-2m_3j_1,8m_1(m_2-2)m_3}(0,\tau)\Theta^{(-,\varepsilon)}_{j_3,m_3}(z,q)$$

where the sum is taken over $j\in Z/2(m_2-2)Z$ and $j_3\in Z/4(m_2-2)m_3Z$ under the conditions $j\equiv j_2-1 \bmod 2$, $j_3\equiv j+j_1 \bmod 2(m_2-2)$. Equating the coefficients of linearly independent $\Theta^{(-,\varepsilon)}_{j_3,m_3}(z,q)$'s, we get an expression of $c^{(\varepsilon)}_{j_1j_2j_3}(\tau)$ in terms of string functions $c^m_{jk}(\tau)$ and the Theta Null Werte (C.2). Explicit formula for $c^m_{jk}(\tau)$ for small m can be found in [10], pp. 219–220.

C.3. Branching coefficients for $m_2=3, 4$

We give below the resulting formulas for $m_2=3,4$. In the case $m_1\in Z+1/2$, we find it convenient to replace the j_3 sum in the r.h.s. of (C.5) by $0<j_3<2m_3$ with the restriction $j_3+1\equiv j_1+j_2 \bmod 2$ (recall that by the definition in Appendix A.1 of Part I $c^{(\varepsilon)}_{j_1j_2j_3}(\tau)=-\varepsilon c^{(\varepsilon)}_{j_1j_2\,2m_3-j_3}(\tau)$). Recall also that if $j_1, m_1\in Z$ and $j_3+1\equiv j_1+j_2 \bmod 2$, then $c^{(\varepsilon)}_{j_1j_2j_3}(\tau)=0$. In the sequel we set

$$k=m_1j_3-m_3j_1,\quad \ell=m_1j_3+m_3j_1,\quad n=m_1m_3,$$

and assume that $j_1\in Z$, $j_3+1\equiv j_1+j_2 \bmod 2$.

The case $m_2=3$.

Here we use the formula $c^1_{00}(\tau)=\eta(\tau)^{-1}$.

(i) $m_1\in Z+1/2$.

$$c^{(\varepsilon)}_{j_1j_2j_3}(\tau)=\eta(\tau)^{-1}(\Theta^{(+)}_{2k,8n}(0,\tau)-\Theta^{(+)}_{2\ell,8n}(0,\tau)).$$

(ii) $m_1 \in \boldsymbol{Z}$.

$$c^{(\varepsilon)}_{j_1 j_2 j_3}(\tau)=\eta(\tau)^{-1}\varepsilon^{m_3}_{j_3}(\Theta^{(\varepsilon)}_{k,2n}(0, \tau)-\Theta^{(\varepsilon)}_{\ell,2n}(0, \tau)).$$

The case $m_2=4$.

Put $\gamma_{\pm}(\tau)=c^2_{00}(\tau)\pm c^2_{20}(\tau)$, $\gamma_0(\tau)=c^2_{11}(\tau)$. Then we have

$$\gamma_+(\tau)=e^{-\pi i/24}\eta\left(\frac{\tau+1}{2}\right)\eta(\tau)^{-2},\ \gamma_-(\tau)=\eta\left(\frac{\tau}{2}\right)\eta(\tau)^{-2},$$

$$\gamma_0(\tau)=\eta(2\tau)\eta(\tau)^{-2}.$$

We give the results in the case $j_3+1\equiv j_1+j_2$ mod 4. The other case $j_3+1\equiv j_1+j_2+2$ mod 4 can be obtained by negating $\gamma_-(\tau)$.

(i) $m_1 \in \boldsymbol{Z}+1/2$, $j_2=1, 3$.

$$c^{(\varepsilon)}_{j_1 j_2 j_3}(\tau)=\frac{1}{2}\gamma_+(\tau)(\Theta^{(+)}_{k,4n}(0, \tau)-\Theta^{(+)}_{\ell,4n}(0, \tau))$$
$$+\frac{1}{2}\gamma_-(\tau)(\Theta^{(-)}_{k,4n}(0, \tau)-(-)^{j_1}\Theta^{(-)}_{\ell,4n}(0, \tau)).$$

(ii) $m_1 \in \boldsymbol{Z}+1/2$, $j_2=2$.

$$c^{(\varepsilon)}_{j_1 j_2 j_3}(\tau)=\gamma_0(\tau)(\Theta^{(+)}_{k,4n}(0, \tau)-\Theta^{(+)}_{\ell,4n}(0, \tau)).$$

(iii) $m_1 \in \boldsymbol{Z}$, $j_2=1, 3$.

$$c^{(\varepsilon)}_{j_1 j_2 j_3}(\tau)=\frac{1}{2}\gamma_+(\tau)\varepsilon^{m_3}_{j_3}(\Theta^{(\varepsilon)}_{k/2,n}(0, \tau)-\Theta^{(\varepsilon)}_{\ell/2,n}(0, \tau))$$
$$+\frac{1}{2}\gamma_-(\tau)\varepsilon^{m_3}_{j_3}(\Theta^{((-)^{m_1}\varepsilon)}_{k/2,n}(0, \tau)-(-)^{j_1}\Theta^{((-)^{m_1}\varepsilon)}_{\ell/2,n}(0, \tau)).$$

(iv) $m_1 \in \boldsymbol{Z}$, $j_2=2$.

$$c^{(\varepsilon)}_{j_1 j_2 j_3}(\tau)=\gamma_0(\tau)\varepsilon^{m_3}_{j_3}(\Theta^{(\varepsilon)}_{k/2,n}(0, \tau)-\Theta^{(\varepsilon)}_{\ell/2,n}(0, \tau)).$$

Appendix D· Free Energy

In this appendix we shall give the free energy for the fusion vertex models (section 2.3) and the restricted SOS models (section 2.2), and discuss its critical behavior. The calculation is based on the inversion relation method [8, 12]. As it turns out, the inversion relations for the restricted SOS models are formally identical with those for the vertex models (with the parameter λ replaced by $2K/L$); consequently the free energy itself has the same form under this correspondence.

D.1. Unitarity and crossing symmetry

First let us recall the unitarity relation for vertex models. As in section 2.3 we denote by $C \in \mathrm{End}\,(V\otimes V)$ the transposition operator.

Lemma D.1. *Let $R(u)$ be a solution to the Yang-Baxter equation* (2.3.2) (*with $V_1 = V_2 = V_3$*) *satisfying the initial condition* (2.3.3a). *Then we have*

(D.1) $$R^{12}(u)R^{21}(-u) = \rho(u)I,$$

where $R^{12}(u) = R(u)$, $R^{21}(u) = CR(u)C$ and $\rho(u)$ is a scalar function (*Fig.* D.1).

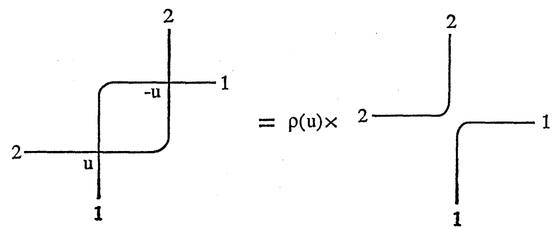


Fig. D. 1 The unitarity relation for vertex models. The r.h.s. is proportional to the scalar operator.

Proof. Setting $v = -u$ in (2.3.2) and using (2.3.3a) we find

$$R^{12}(u)R^{21}(-u) = R^{23}(-u)R^{32}(u).$$

This implies that the l.h.s. of (D.1) commutes with matrixes of the form $X\otimes I$, $I\otimes X$ ($X \in \mathrm{End}\,(V)$). Hence it must be a scalar. □

The function $\rho(u)$ in (D.1) can be determined by comparing a particular matrix element. For the original eight vertex weight (2.3.1) we have

(D.2) $$R(u)R(-u) = \begin{bmatrix} 1+u \\ 1 \end{bmatrix}\begin{bmatrix} 1-u \\ 1 \end{bmatrix} I.$$

(Note that in this case $R(u) = CR(u)C$.) The unitarity relation for the (M, N)-weight $R_{MN}(u)$ in section 2.3 reads as follows. We set $R^{12}_{MN}(u) = R_{MN}(u)$, $R^{21}_{NM}(u) = C_{NM}R_{MN}(u-M+N)C_{NM}$, and regard both as acting on the same space $\underbrace{V\otimes\cdots\otimes V}_{M+N}$ (see section 2.3 for C_{MN}).

Lemma D.2. *Assuming $M \geq N$ we have*

(D.3) $$R^{12}_{MN}(u)R^{21}_{NM}(-u) = \begin{bmatrix} M+u \\ N \end{bmatrix}\begin{bmatrix} N-u \\ N \end{bmatrix} I.$$

Proof. For definiteness, let $M=3$ and $N=2$. Then by the definition

$$R^{1\bar{2}}_{32}(u)=\sigma(u)P_{123}P_{\bar{1}\bar{2}}R^{1\bar{2}}(u+2)R^{2\bar{2}}(u+1)R^{3\bar{2}}(u)R^{1\bar{1}}(u+1)R^{2\bar{1}}(u)R^{3\bar{1}}(u-1),$$

$$R^{21}_{23}(-u)=\sigma(-u-1)P_{123}P_{\bar{1}\bar{2}}$$
$$\times R^{\bar{1}3}(-u+1)R^{\bar{1}2}(-u)R^{\bar{1}1}(-u-1)R^{\bar{2}3}(-u)R^{\bar{2}2}(-u-1)R^{\bar{2}1}(-u-2),$$

where $\sigma(u)=[1]^6/([2]_2[u+2][u+1]^2[u])$. In the second line we have reshaffled the superfixes $(\bar{1}, \bar{2})$ and $(1, 2, 3)$ under the symmetrizers (cf. (2.3.6)). Using (2.3.6) and (D.2) repeatedly we obtain (D.3) (Fig. D.2). The general case is similar. □

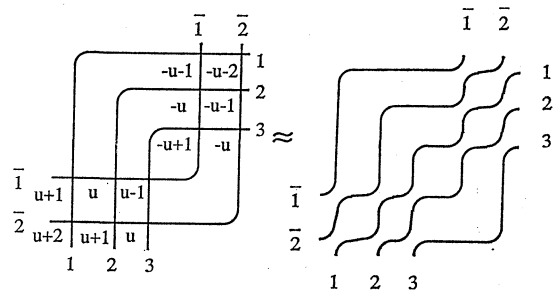


Fig. D. 2 The unitarity relation for fusion vertex models.

Hereafter we shall be concerned with the case $M=N$. Eq. (D.3), to be also called the "first inversion relation", then reads

(D.4) $$R_{NN}(u)CR_{NN}(-u)C=\begin{bmatrix}N+u\\N\end{bmatrix}\begin{bmatrix}N-u\\N\end{bmatrix}I,$$

where $C=C_{NN}$.

Let $\tilde{R}(u)$ denote the matrix of the eight vertex weight obtained by rotating the lattice through 90°, *i.e.*

(D.5) $$\tilde{R}^{\alpha\beta}_{\gamma\delta}(u)=R^{\beta\gamma}_{\delta\alpha}(u).$$

The following crossing symmetry holds:

$$R(-1-u)=-(\sigma^y\otimes I)\tilde{R}(u)(\sigma^y\otimes I),\qquad \sigma^y=\begin{pmatrix} & -i\\ i & \end{pmatrix}.$$

This implies together with (D.2) and the symmetry $R(u)=CR(u)C$ (Fig. D.3)

$$\tilde{R}(u)\tilde{R}(-2-u)=\begin{bmatrix}2+u\\1\end{bmatrix}\begin{bmatrix}-u\\1\end{bmatrix}I.$$

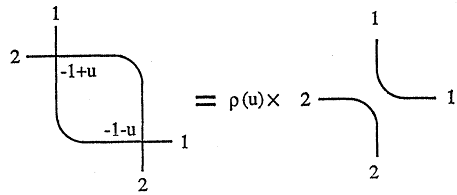


Fig. D. 3 The second inversion relation.

More generally, for the (N, N)-weight $R_{NN}(u)$ define $\tilde{R}_{NN}(u)$ by (D.5). In the same way as Lemma D.2 we have the "second inversion relation" for $R_{NN}(u)$:

Lemma D.3.

(D.6) $$\tilde{R}_{NN}(u)C\tilde{R}_{NN}(-2-u)C=\begin{bmatrix} N+1+u \\ N \end{bmatrix}\begin{bmatrix} N-1-u \\ N \end{bmatrix}I.$$

D.2. The free energy of the fusion vertex models and the restricted SOS models

In order to discuss the free energy for the fusion vertex models, we must specify the regimes to consider. Following the case of the restricted SOS models, we deal with the four cases below.

(D.7)
$$\begin{array}{lll} \text{Regime I:} & -1<p<0, & 0<u<K/\lambda-1, \\ \text{Regime II:} & 0<p<1, & 0<u<K/\lambda-1, \\ \text{Regime III:} & 0<p<1, & -1<u<0, \\ \text{Regime IV:} & -1<p<0, & -1<u<0. \end{array}$$

In regimes II and III, K' is real and positive, while in regimes I and IV so is $\tilde{K}'=K'-iK$ rather than K'. The end points $u=-1, 0$ are "inversion points" and $K/\lambda-1$ is a "virtual inversion point" in the terminology of [12]. We define w by

(D.8)
$$\begin{array}{ll} w=e^{-2\pi\lambda u/K'} & \text{in regime II, III } (p=e^{-\pi K'/K}), \\ =e^{-\pi\lambda u/\tilde{K}'} & \text{in regime I, IV } (p=-e^{-\pi\tilde{K}'/K}), \end{array}$$

and set

(D.9) $$\Lambda=2K/\lambda.$$

Now let κ denote the partition function per site

$$\kappa=\lim_{\mathscr{M}\to\infty} Z^{1/\mathscr{M}},$$

where Z is the partition function and $\mathcal{M}$ the number of sites of the lattice. The free energy per site is given by $f=-k_B T\log\kappa$. From (D.4) and (D.6) we obtain the following inversion relations.

Regime III, IV:

(D.10)
$$\kappa(u)\kappa(-u)=\begin{bmatrix}N+u\\N\end{bmatrix}\begin{bmatrix}N-u\\N\end{bmatrix},$$
$$\kappa(u)\kappa(-2-u)=\begin{bmatrix}N+1+u\\N\end{bmatrix}\begin{bmatrix}N-1-u\\N\end{bmatrix}.$$

Regime I, II:

(D.11)
$$\kappa(u)\kappa(-u)=\begin{bmatrix}N+u\\N\end{bmatrix}\begin{bmatrix}N-u\\N\end{bmatrix},$$
$$\kappa(u)\kappa(\Lambda-2-u)=\begin{bmatrix}N+1+u\\N\end{bmatrix}\begin{bmatrix}N-1-u\\N\end{bmatrix}.$$

The inversion relations for the restricted SOS models follow from (2.1.25) and (2.2.19). Identifying λ with $2K/L$ (and Λ with L) we find that they have the same form as (D.10–11). The definitions of regimes (D.7) and of the parameter w (D.8) agree with those for the SOS models.

It is straightforward to compute the free energy. We apply the conjugate modulus transformation (3.1.5–6) (wherein L is to be read as $2K/\lambda$) and solve (D.10–11) for $\kappa(u)$. The result is expressed as follows.

Regime I: $(\nu=\pi\lambda/\tilde{K}',\ \Lambda>2N)$

(D.12)
$$\log\kappa(u)=\sum_{j\in 2Z}F(j)+\sum_{j\in 2Z+1}G(j),$$
$$F(j)=F(-j)=\frac{1}{j}\times\frac{\operatorname{sh}(N\nu j/2)\operatorname{ch}((\Lambda-2N-2)\nu j/4)}{\operatorname{sh}(\nu j/2)\operatorname{ch}((\Lambda-2)\nu j/4)\operatorname{sh}(\Lambda\nu j/4)}$$
$$\times\operatorname{sh}(u\nu j/2)\operatorname{sh}((\Lambda-2-2u)\nu j/4),$$
$$G(j)=G(-j)=\frac{1}{j}\times\frac{\operatorname{sh}(N\nu j/2)\operatorname{sh}((\Lambda-2N-2)\nu j/4)}{\operatorname{sh}(\nu j/2)\operatorname{sh}((\Lambda-2)\nu j/4)\operatorname{ch}(\Lambda\nu j/4)}$$
$$\times\operatorname{sh}(u\nu j/2)\operatorname{ch}((\Lambda-2-2u)\nu j/4).$$

Regime II: $(\nu=2\pi\lambda/K',\ \Lambda>2N)$

$$\log\kappa(u)=\sum_{j\in Z}F(j),$$

(D.13)
$$F(j)=F(-j)=\frac{1}{j}\times\frac{\mathrm{sh}\,(N\nu j/2)\,\mathrm{sh}\,(u\nu\, j/2)}{\mathrm{sh}\,(\nu j/2)\,\mathrm{sh}\,((\Lambda-2)\nu j/2)\,\mathrm{sh}\,(\Lambda\nu j/2)}$$
$$\times(\mathrm{sh}((\Lambda-N-2)\nu j/2)\,\mathrm{sh}\,((\Lambda-1-u)\nu j/2)$$
$$-\mathrm{sh}\,(N\nu j/2)\,\mathrm{sh}\,((u+1)\nu j/2)).$$

Regime III: $(\nu=2\pi\lambda/K',\ \Lambda>N)$

$$\log\kappa(u)=\sum_{j\in Z}F(j),$$

(D.14)
$$F(j)=F(-j)=-\frac{2}{j}\times\frac{\mathrm{sh}\,(N\nu j/2)\,\mathrm{ch}\,((\Lambda-N-1)\nu j/2)}{\mathrm{sh}\,(\nu j)\,\mathrm{sh}\,(\Lambda\nu j/2)}$$
$$\times\,\mathrm{sh}\,(u\nu j/2)\,\mathrm{sh}\,((u+1)\nu j/2).$$

Regime IV: $(\nu=\pi\lambda/\tilde{K}',\ \Lambda>2N)$

$$\log\kappa(u)=\sum_{j\in 2Z}F(j)+\sum_{j\in 2Z+1}G(j),$$

(D.15)
$$F(j)=F(-j)=-\frac{2}{j}\times\frac{\mathrm{sh}\,(N\nu j/2)\,\mathrm{ch}\,((\Lambda-2N-2)\nu j/4)}{\mathrm{sh}\,(\nu j)\,\mathrm{sh}\,(\Lambda\nu j/4)}$$
$$\times\,\mathrm{sh}\,(u\nu j/2)\,\mathrm{sh}\,((u+1)\nu j/2),$$
$$G(j)=G(-j)=-\frac{2}{j}\times\frac{\mathrm{sh}\,(N\nu j/2)\,\mathrm{sh}\,((\Lambda-2N-2)\nu j/4)}{\mathrm{sh}\,(\nu j)\,\mathrm{ch}\,(\Lambda\nu j/4)}$$
$$\times\,\mathrm{sh}\,(u\nu j/2)\,\mathrm{sh}\,((u+1)\nu j/2).$$

Here we have assumed that $\Lambda>N$ or $2N$ for simplicity. Otherwise the expression should be modified. (This comes from the difference of the behavior of the factors $E(x^u, x^L)$ or $E(x^u, -x^{L/2})$ appearing in the r.h.s. of the inversion relations.)

As we remarked before, the free energy for the restricted SOS models are obtained simply by replacing λ by $2K/L$ (Λ by L).

D.3. Critical behavior

The critical behavior of the free energy can be studied in the following way. For a function $f(x)$, let

$$\hat{f}(\xi)=\int_{-\infty}^{\infty}f(x)\exp(2\pi ix\xi)dx$$

denote it Fourier transform. Poisson's summation formula asserts that in the case $f(-x)=f(x)$

$$\sum_{j\in Z}f(j)=\hat{f}(0)+2\sum_{\xi\geq 1}\hat{f}(\xi).$$

The free energy results (D.12–15) are rewritten as

Regime II, III:

$$\sum_{j\in \boldsymbol{Z}} F(j)=\hat{F}(0)+2\sum_{\xi\geq 1}\hat{F}(\xi),$$

Regime I, IV:

$$\sum_{j\in 2\boldsymbol{Z}} F(j)+\sum_{j\in 2\boldsymbol{Z}+1} G(j)=\frac{1}{2}(\hat{F}(0)+\hat{G}(0))+\sum_{\xi\geq 1}(\hat{F}(\xi/2)+(-)^{\xi}\hat{G}(\xi/2)).$$

The 0-th term represents the critical value $\log\kappa^{(c)}(u)$. It is the same for regime I/II or regime III/IV, and is given by

Regime I, II:

$$\log\kappa^{(c)}(u)=\int_{-\infty}^{\infty}\frac{dt}{t}\frac{\operatorname{sh}(Nt/2)\operatorname{sh}(ut/2)}{\operatorname{sh}(t/2)\operatorname{sh}((\varLambda-2)t/2)\operatorname{sh}(\varLambda t/2)}$$
$$\times(\operatorname{sh}((\varLambda-N-2)t/2)\operatorname{sh}((\varLambda-1-u)t/2)$$
$$-\operatorname{sh}(Nt/2)\operatorname{sh}((u+1)t/2)).$$

Regime III, IV:

$$\log\kappa^{(c)}(u)=-2\int_{-\infty}^{\infty}\frac{dt}{t}\frac{\operatorname{sh}(Nt/2)\operatorname{ch}((\varLambda-N-1)t/2)}{\operatorname{sh}(t)\operatorname{sh}(\varLambda t/2)}$$
$$\times\operatorname{sh}(ut/2)\operatorname{sh}((u+1)t/2).$$

The above formula for regime III–IV agrees with the result of [13] (eq. (2.23) there).

To study the behavior of $\hat{F}(\xi)$ (or $\hat{F}(\xi/2)$, $\hat{G}(\xi/2)$) for $\xi>0$, we deform the contour of integration to surround the upper half plane and pick up the residues. Thus we get the following results for general $\varLambda$ (the case of the fusion vertex models). Here $(\log\kappa(u))_{\text{sing}}$ denotes the non-analytic part of $\log\kappa(u)$ in $|p|$.

Regime I, II:

$$(\log\kappa(u))_{\text{sing}}=\frac{4\sin^2(N\pi/(\varLambda-2))}{\sin(2\pi/(\varLambda-2))}\sin(2\pi u/(\varLambda-2))\times|p|^{\varLambda/(\varLambda-2)}$$
$$+O(|p|^{2\varLambda/(\varLambda-2)}).$$

Regime III:

$$(\log\kappa(u))_{\text{sing}}=4\frac{\cos(\varLambda\pi/2)}{\sin(\varLambda\pi/2)}\sin\pi u\times|p|^{\varLambda/2}+O(|p|^{\varLambda})\qquad\text{if } N \text{ is odd,}$$

$\log\kappa(u)$ is regular if N is even.

Regime IV:

$$(\log \kappa(u))_{\text{sing}} = 4\frac{\sin \pi u}{\sin(\Lambda\pi/2)} \times |p|^{\Lambda/2} + O(|p|^{\Lambda}) \qquad \text{if } N \text{ is odd,}$$

$$\log \kappa(u) \quad \text{is regular} \qquad \text{if } N \text{ is even.}$$

Complications occur when Λ is an integer because of the double poles in $F(j)$ and $G(j)$. This is the case of the restricted SOS models. Careful examination shows that the results above should then be modified as follows.

Regime I, II:

$$(\log \kappa(u))_{\text{sing}} = \frac{4\sin^2(N\pi/(L-2))}{\sin(2\pi/(L-2))} \sin(2\pi u/(L-2)) \times |p|^{L/(L-2)} + O(|p|^{2L/(L-2)}).$$

The only exception occurs when $L=4$ and $N=1$ (Ising model). In this case we have

$$(\log \kappa(u))_{\text{sing}} = -\frac{4}{\pi} \sin \pi u \times |p|^2 \log|p| + O(|p|^4 \log|p|).$$

Regime III:

$$(\log \kappa(u))_{\text{sing}} = \frac{4}{\pi} \sin \pi u \times |p|^{L/2} \log|p| + O(|p|^{L} \log|p|)$$

if L is even and N is odd,

$\log \kappa(u)$ is regular otherwise.

Regime IV:

$$(\log \kappa(u))_{\text{sing}} = (-)^{L/2}\frac{4}{\pi} \sin \pi u \times |p|^{L/2} \log|p| + O(|p|^{L} \log|p|)$$

if L is even and N is odd,

$$(\log \kappa(u))_{\text{sing}} = (-)^{(L-1)/2} 4 \sin \pi u \times |p|^{L/2} + O(|p|^{3L/2})$$

if L is odd and N is odd,

$\log \kappa(u)$ is regular otherwise.

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