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Multi-Tensors of Differential Forms on the Hilbert Modular Variety and on Its Subvarieties, II

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Dedicated to Prof. Ichiro Satake and Prof. Friedrich Hirzebruch on their sixtieth birthdays

Let Γ_K denote the Hilbert modular group associated with a totally real algebraic number field K of degree $n>1$. Let X_K be the Hilbert modular variety H^n/Γ_K. The present paper is the continuation of a study [8], and our purpose is to extend the known range of K for which an assertion (☆) holds where

(☆) *any subvariety in X_K of codimension one is of general type.*

We show that *if $n\geq 3$, then* (☆) *holds only with finite exceptions.* It was shown in our previous paper [8] that if the dimension $n\geqq 3$ is fixed, then (☆) holds with finite exceptions. The main theorem of the present paper is as follows:

Theorem. (☆) *holds if $n>26$, or if $n>14$ and the ideal in the maximal order of K generated by 2 is unramified at any prime of degree one.*

As stated in [8], (☆) has the consequent on the property of X_K which we restate here for reader's convenience.

(I) *Let X_K° denote the smooth locus of X_K, and let $\tilde{X}_K$*[1] *be any smooth variety having X_K° as an open subset. Then for any birational morphism φ of $\tilde{X}_K$ to a smooth variety, $\varphi|_{X_K^\circ}$ gives rise to an open embedding.*

(II) *The birational automorphism group of X_K (or equivalently, the automorphism group of the Hilbert modular function field over $\boldsymbol{C}$) is equal to the automorphism group of X_K, which is canonically isomorphic to a semi-direct product $H_K^{(2)}\rtimes \mathrm{Aut}\,(K/\boldsymbol{Q})$ where $H_K^{(2)}=\{x\in H_K\,|\,x^2=1\}$, H_K denoting the ideal class group of K in the narrow sense.*

As we see in §2, in order to prove Theorem we need to show

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[1] ~ is missing in [8], Cor. 1, p. 660.

existence of a Hilbert modular form g for every irreducible divisor D of X_K such that (i) $g \not\equiv 0$ on D and that (ii) the quotient by weight (g), of the vanishing order of g at the cusps is at least $n/2(n-1)$. By the dimension formula by Shimizu it follows that there exist modular forms f satisfying the condition (ii), or better one as well, except for a finite number of K. In [8], under a certain condition we have got a modular form g satisfying (i) as well as (ii) by differentiating an "irreducible" factor of f vanishing on D, where non-existence of automorphy factors of very low weight plays an important role, which has been shown by Gundlach [3]. The basic idea of the present paper is to consider a "shifted" modular form $f(\alpha z)$ for α totally positive. Namely, if α will be taken suitably, then $f(\alpha z)$ will still have zeros of large order at cusps and it will not vanish identically on D. $f(\alpha z)$ is generally a modular form for a congruence subgroup, not for Γ_K, and so we must construct desired modular forms from those of congruence subgroups. Combining this method with our previous method, we can find a modular form g satisfying both (i) and (ii) in the case of dim $K \geq 3$ with finite exceptions, where the proof is served by some refinement [9] of [3].

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§ 1. Preliminaries

Let K be a totally real algebraic number field of degree $n>1$. $SL_2(K)$ acts on the product H^n of n copies of the upper half plane $H=\{z_1 \in C \mid \operatorname{Im} z_1>0\}$ by the usual modular substitution

$$z=(z_1, \cdots, z_n) \longrightarrow Mz=\left(\frac{\alpha^{(1)}z_1+\beta^{(1)}}{\gamma^{(1)}z_1+\delta^{(1)}}, \cdots, \frac{\alpha^{(n)}z_n+\beta^{(n)}}{\gamma^{(n)}z_n+\delta^{(n)}}\right)$$

for $M=\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in SL_2(K)$, where $\alpha^{(1)}, \cdots, \alpha^{(n)}$ denote the conjugates of $\alpha \in K$. Let O_K be the maximal order of K. We put $\Gamma_K=SL_2(O_K)$, which is called the *Hilbert modular group* associated with K and which acts properly discontinuously on H^n. Let $\hat{K}=K \cup \{\infty\}$. We define an equivalence relation in $\hat{K}$ in terms of Γ_K; $\lambda_1, \lambda_2 \in \hat{K}$ are equivalent if $\lambda_1=M\lambda_2=(\alpha\lambda_2+\beta)/(\gamma\lambda_2+\delta)$ for some $M=\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in \Gamma_K$ where $\zeta/0=\infty$ for $\zeta \in K, \neq 0$, and $\zeta/\infty=0$, $\infty+\zeta=\infty$ for any $\zeta \in K$. Let $\mathfrak{a}=(\rho, \sigma)$, $\rho, \sigma \in K$, be a non-zero fractional ideal. We associate with $\mathfrak{a}$, an element λ of $\hat{K}$ given by $\lambda=\rho/\sigma$. Then if we denote by $C(K)$ the (fractional) ideal class group of

K, then we have a well-defined bijective map of $C(K)$ onto $\hat{K}/\Gamma_K$ by sending $\mathfrak{a}$ to λ (see for instance, Siegel [6, Chap. III, Sect. 2]). $\hat{K}/\Gamma_K$ can be regarded as the set of inequivalent cusps of Γ_K, and so Γ_K has h inequivalent cusps where h denotes the class number of K.

Let $\mathfrak{a}=(\rho, \sigma)$ be a non-zero fractional ideal, and let $\lambda=\rho/\sigma$. We can take the generators ξ, η of an ideal $\mathfrak{a}^{-1}$ for which $\rho\eta-\sigma\xi=1$. Let us put

$$M_\lambda=\begin{pmatrix}\rho & \xi\\ \sigma & \eta\end{pmatrix} \in SL_2(K). \tag{1}$$

The modular substitution corresponding to M_λ maps a cusp λ to ∞, and $M_\lambda^{-1}\Gamma_K M_\lambda$ equals $\left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in SL_2(K) \,|\, \alpha, \delta \in O_K, \gamma \in \mathfrak{a}^2, \beta \in \mathfrak{a}^{-2}\right\}$.

Let us fix a subgroup Γ in $SL_2(K)$ commensurable with Γ_K. Let J be the automorphy factor for Γ which is of the form

$$J(M, z)=v(M)\prod_{i=1}^{n}(\gamma^{(i)}z_i+\delta^{(i)})^{k_i}, \qquad M=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \Gamma, \tag{2}$$

where $k_1, \cdots, k_n \in \boldsymbol{Q}$, and v is the multiplier whose value for $M \in \Gamma$ is a root of unity. v is, of course, depending on the choice of the branches of $(\gamma^{(i)}z_i+\delta^{(i)})^{k_i}$ if $k_i \in \boldsymbol{Q}-\boldsymbol{Z}$ $(i=1, \cdots, n)$. A holomorphic function f on H^n is called a *(Hilbert) modular form* for Γ associated with J if it satisfies

$$f(Mz)=J(M, z)f(z) \qquad \text{for any } M \in \Gamma.$$

f is said to be of *vector weight* $(k_1, \cdots, k_n)$, and conversely $(k_1, \cdots, k_n)$ is called the *weight vector* of f. If all the k_i's are equal to k, then f is said to be of *scalar weight* k, and in notation $k=\text{weight}(f)$. A map $\beta \to v\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}$ gives a *finite* character of an additive group $\left\{\beta \in K \,|\, \begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix} \in \Gamma\right\}$, and moreover it is true also for the unipotent subgroup of the stabilizer subgroup in Γ at every cusp $\lambda \in K$ (for the references, see [8, p. 665]). So a modular form f for Γ has a Fourier expansion centered at λ. Let $\lambda=\rho/\sigma$, $\mathfrak{a}=(\rho, \sigma)$, M_λ be as above, and let

$$w=M_\lambda^{-1}z, \qquad w_i=(M_\lambda^{-1})^{(i)}z_i \qquad (1\leq i\leq n). \tag{3}$$

Then

$$f_\lambda(w)=\prod_{i=1}^{n}(-\sigma^{(i)}z_i+\rho^{(i)})^{k_i}f(z)$$

is a modular form in w for $M_\lambda^{-1}\Gamma M_\lambda$, and it has a Fourier expansion

$$f_\lambda(w)=\sum_\nu c_\nu \exp(2\pi\sqrt{-1}\,\mathrm{tr}(\nu w)) \tag{4}$$

where $\mathrm{tr}(\nu w)=\nu^{(1)}w_1+\cdots+\nu^{(n)}w_n$, and ν runs over the set composed of 0 and totally positive numbers contained in some lattice in K. f_λ is defined independently of $\rho, \sigma \in K$ with $\lambda=\rho/\sigma$, up to a constant multiple. Now we define the vanishing order $\mathrm{ord}_\lambda(f)$ of f at λ to be the minimum of the set of non-negative rational numbers

$$\{\mathrm{tr}(\nu\zeta) \mid \text{totally positive } \zeta \in \mathfrak{a}^{-2},\ \nu \text{ with } c_\nu \neq 0\}.$$

It is easy to see that the above definition is independent of $\rho, \sigma \in K$ for which $\lambda=\rho/\sigma$, and further that it is independent of the choice of λ in one equivalence class of cusps of Γ.

Remark. We give a comment about the vanishing order. For simplicity we suppose that a multiplier v is trivial and $k_1=\cdots=k_n \in 2\mathbf{Z}$ in (2). Let U be a neighborhood at a cusp λ in the analytic space given by compactifying H^n/Γ. If U is small enough, then $f_\lambda(w)$ can be regarded as a function on U. Let $\varphi: \tilde{U} \to U$ be a desinguralization, and $\{E_i\}$, the exceptional divisors. Then $\mathrm{ord}_\lambda(f)$ has a geometric meaning, namely, it equals the minimum of the vanishing orders of $\varphi^* f$ at the E_i's, provided that $\Gamma=\Gamma_K$. But this is not necessarily true for general Γ. To make general definition, the "width" of cusps of Γ must be taken into account. However as far as we focus only on modular forms for Γ_K, the above definition works well.

Let f, g be modular forms for Γ. The inequalities $\mathrm{ord}_\lambda(fg) \geqq \mathrm{ord}_\lambda(f) + \mathrm{ord}_\lambda(g)$, $\mathrm{ord}_\lambda(f+g) \geqq \min\{\mathrm{ord}_\lambda(f), \mathrm{ord}_\lambda(g)\}$ holds[2]. In the Fourier expansion (4), we call ν *minimal* if $c_\nu \neq 0$ and if ν cannot be written as $\nu=\nu'+\nu''$ with $c_{\nu'} \neq 0$, $c_{\nu''} \neq 0$. We write

$$g \preccurlyeq f$$

if at every cusp the following holds; for any minimal ν in the Fourier expansion of f, there is a minimal ν' in that of g for which $\nu'^{(i)} \leq \nu^{(i)}$ $(i=1, \cdots, n)$. In such a case $\mathrm{ord}_\lambda(fh) \geq \mathrm{ord}_\lambda(gh)$ holds for any modular form h[3]. Finally in this section we define the vanishing order $\mathrm{ord}(f)$ of f at cusps by

$$\mathrm{ord}(f) = \min_{\lambda \in \hat{K}} \{\mathrm{ord}_\lambda(f)\}$$

where λ may actually run over a finite set of representatives of cusps with respect to the equivalence under Γ.

[2] Although a resulting inequality is true, there is inaccuracy at the last equality in the sequence of inequalities in [8, p668, line 1~2].

[3] This gives a correct proof of the above.

§ 2. Résumé of [8]

We put

$$\omega_i=(-1)^i dz_1\wedge\cdots\wedge d\check{z}_i\wedge\cdots\wedge dz_n \in \Omega_{H^n}^{n-1} \qquad (1\leq i\leq n),$$

and

$$\omega=\omega_1\otimes\cdots\otimes\omega_n \in (\Omega_{H^n}^{n-1})^{\otimes n}.$$

Then

$$M\cdot\omega=\prod_{i=1}^{n}(\gamma^{(i)}z_i+\delta^{(i)})^{-2(n-1)}\omega \qquad \text{for } M=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL_2(K).$$

If f is a Hilbert modular form for Γ_K of scalar weight $2r(n-1)$, then $f\omega^{\otimes r}$ is Γ-invariant, and it may be regarded as a multi-tensor of differential forms on the smooth locus X_K° of X_K. It is extendable over all elliptic fixed points except for the cases listed in [8, Lemma 3], and it is always so in particular for $n>6$. It is extendable to a projective non-singular model of X_K, then its restriction to D gives a multi-tensor of canonical differential forms on D. The restriction is not zero unless f vanishes at D.

Proposition 1. *Let $n>6$. Let D be a subvariety in X_K of codimension one. If there is a modular form f for Γ_K of scalar weight satisfying that* (i) $f|_D\not\equiv 0$ *and that* (ii) ord(f)/weight$(f)>n/2(n-1)$, *then D is of general type.*

Proof. The proposition was essentially proved in [8, Sect. 6]. Here we give only a sketch. We may assume that weight$(f)=2r(n-1)$, $r\in \boldsymbol{Z}$, replacing f by its power if necessary. Let $g_1,\cdots,g_t$ be modular forms for Γ_K of weight $2r'(n-1)$, $r'\in\boldsymbol{Z}$, by which X_K is embedded into a projective space. If $m\in\boldsymbol{Z}$ is large enough, then ord$(f^m g_j)$/weight$(f^m g_j)$, $1\leq j\leq t$, are at least $n/2(n-1)$, and hence $f^m g_j\omega^{\otimes(mr+r')}$, $1\leq j\leq t$, are extendable to a projective non-singular model of X_K. Since $f|_D\not\equiv 0$, D is of general type. q.e.d.

The proof of Theorem is reduced to find the modular form satisfying the condition in Proposition 1 for any fixed D, which in substance, we carry out in the present paper.

§ 3. Vanishing order

Let μ be a totally positive integer in O_K. Let m be a positive rational integer such that

$$(5) \qquad m\mu^{-1}\in O_K.$$

Let $f(z)$ be a modular form of weight $(k_1, \cdots, k_n)$ for some subgroup in $SL_2(K)$ commensurable with Γ_K. Then such is $f(\mu z)=f(\mu^{(1)}z_1, \cdots, \mu^{(n)}z_n)$. Let $M=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \Gamma_K$. Then

$$g(z) := \prod_{i=1}^{n} (\gamma^{(i)}z_i+\delta^{(i)})^{-k_i} f(\mu Mz)$$

is also such a modular form. We estimate ord (g) in terms of ord (f) and m in (5).

Lemma 1. *Let the notation be as above. Then* ord $(g) \geq m^{-1}$ ord (f).

Proof. Let $\lambda=\rho/\sigma$ be a cusp. Let us put $\lambda'=\mu M(\lambda)$ and $M_{\lambda'}=\begin{pmatrix}\sqrt{\mu} & 0\\ 0 & \sqrt{\mu}^{-1}\end{pmatrix} MM_\lambda \begin{pmatrix}\sqrt{\mu}^{-1} & 0\\ 0 & \sqrt{\mu}\end{pmatrix}=\begin{pmatrix}\rho' & *\\ \sigma' & *\end{pmatrix}$, M_λ being as in (1). We have the equalities $\lambda'=M_{\lambda'}(\infty)=\rho'/\sigma'$ and $\rho'=\alpha\rho+\beta\sigma$, $\sigma'=\mu^{-1}(\gamma\rho+\delta\sigma)$. $f(z)$ has a Fourier expansion centered at λ'

$$\prod_{i=1}^{n} (-\sigma'^{(i)}z_i+\rho'^{(i)})^{k_i} f(z)=\sum_\nu c_\nu \exp(2\pi\sqrt{-1}\,\mathrm{tr}(\nu M_{\lambda'}^{-1}z)).$$

Here $\mathrm{ord}_{\lambda'}(f)=\min\{\mathrm{tr}(\nu\zeta) \mid \text{totally positive } \zeta \in (\rho', \sigma')^{-2}, \nu \text{ with } c_\nu\neq 0\} \geq$ ord (f). We replace z by μMz in the above identity. Then a simple calculation leads to

$$\prod_{i=1}^{n} (-\sigma^{(i)}z_i+\rho^{(i)})^{k_i} g(z)=\sum_\nu c_\nu \exp(2\pi\sqrt{-1}\,\mathrm{tr}(\nu\mu w)),$$

w being as in (3). Then $\mathrm{ord}_\lambda(g)=\min\{\mathrm{tr}(\zeta\mu\nu) \mid \text{totally positive } \zeta \in (\rho, \sigma)^{-2}, \nu \text{ with } c_\nu\neq 0\} = \min\{\mathrm{tr}(\zeta\nu) \mid \text{totally positive } \zeta \in \mu(\rho, \sigma)^{-2}, \nu \text{ with } c_\nu\neq 0\}$. Since $m^{-1}(\rho', \sigma')=m^{-1}\mu^2(\mu(\alpha\rho+\beta\sigma), \gamma\rho+\delta\sigma)^{-2} \supset m^{-1}\mu^2(\alpha\rho+\beta\sigma, \gamma\rho+\delta\sigma)^{-2} \supset m^{-1}\mu^2(\rho, \sigma)^{-2} \supset \mu(\rho, \sigma)^{-2}$, $\mathrm{ord}_\lambda(g)$ is at least $m^{-1}\,\mathrm{ord}_{\lambda'}(f) \geq m^{-1}$ ord (f).
q.e.d.

Let us construct a modular form for Γ_K from f;

$$(6) \qquad f_{\mu,r}(z) := \sum_M \{\prod_i (\gamma^{(i)}z_i+\delta^{(i)})^{-k_i} f(\mu Mz)\}^r$$

where r is a positive rational integer, and $M=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \Gamma_K$ runs through a system of right representatives of Γ_K mod $\Gamma \cap \Gamma_K$, Γ denoting the subgroup in $SL_2(K)$ commensurable with Γ_K for which $f(\mu z)$ is modular form.

Corollary. *Let f be a modular form for a subgroup $SL_2(K)$ commensurable with Γ_K, and let μ be a totally positive integer in O_K. Then $f_{\mu,r}$ is* *a modular form for Γ_K whose weight vector is an r times the weight vector of f, and* ord $(f_{\mu,r})$ *is at least* $m^{-1}r$ ord (f) *where m is as in* (5).

The common zero of $f_{\mu,r}$, $r=1, 2, \cdots$, is equal to the common zero of $f(\mu Mz)$'s, M being as above. In particular $f_{\mu,r}$ does not vanish identically for infinitely many r unless f vanishes identically.

§ 4. Modular groups

First we introduce several congruence subgroups of Γ_K. Let $\mathfrak{b}$ be a non-zero integral ideal of O_K. We define subgroups of Γ_K associated with $\mathfrak{b}$;

$$\Gamma(\mathfrak{b}) := \left\{M \in \Gamma_K \mid M \equiv \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \bmod \mathfrak{b}\right\},$$

$$\Gamma_0(\mathfrak{b}) := \left\{\begin{pmatrix}\alpha & \beta\\\gamma & \delta\end{pmatrix} \in \Gamma_K \mid \gamma \equiv 0 \bmod \mathfrak{b}\right\},$$

$$\Gamma^1(\mathfrak{b}) := \left\{\begin{pmatrix}\alpha & \beta\\\gamma & \delta\end{pmatrix} \in \Gamma_K \mid \alpha \equiv \delta \equiv 1 \bmod \mathfrak{b},\ \beta \in \mathfrak{b}^2\right\}.$$

Let μ be a totally positive integer in O_K. A simple calculation shows that $M(\mu z) = \mu M_{(\mu)} z$ with

$$M_{(\mu)} = \begin{pmatrix}\sqrt{\mu}^{-1} & 0\\0 & \sqrt{\mu}\end{pmatrix} M \begin{pmatrix}\sqrt{\mu} & 0\\0 & \sqrt{\mu}^{-1}\end{pmatrix} = \begin{pmatrix}\alpha & \mu^{-1}\beta\\\mu\gamma & \delta\end{pmatrix} \qquad \text{for } M = \begin{pmatrix}\alpha & \beta\\\gamma & \delta\end{pmatrix}.$$

We note that

$$\Gamma_K \cap \begin{pmatrix}\sqrt{\mu}^{-1} & 0\\0 & \sqrt{\mu}\end{pmatrix} \Gamma_K \begin{pmatrix}\sqrt{\mu} & 0\\0 & \sqrt{\mu}^{-1}\end{pmatrix} = \Gamma_0(\mu),$$

and

$$\begin{pmatrix}\sqrt{\mu}^{-1} & 0\\0 & \sqrt{\mu}\end{pmatrix} \Gamma^1(\mu) \begin{pmatrix}\sqrt{\mu} & 0\\0 & \sqrt{\mu}^{-1}\end{pmatrix} = \Gamma(\mu).$$

Let f be a modular form. We denote by Γ_f, the maximal subgroup of $SL_2(K)$ for which f is a modular form, namely,

$$\Gamma_f = \{M \in SL_2(K) \mid f(Mz)/f(z) \text{ is holomorphic}\}.$$

Then $\Gamma_{f(\mu z)} = \begin{pmatrix}\sqrt{\mu}^{-1} & 0\\0 & \sqrt{\mu}\end{pmatrix} \Gamma_f \begin{pmatrix}\sqrt{\mu} & 0\\0 & \sqrt{\mu}^{-1}\end{pmatrix}$.

Lemma 2. *Let $f(z)$ be a non-constant modular form for Γ_K. Then $\Gamma_{f(\mu z)} \cap \Gamma_K = \Gamma_0(\mu)$ for a totally positive integer μ in O_K.*

Proof. By the above fact it is enough to show that $\Gamma_f=\Gamma_K$, where $\Gamma_f\supset\Gamma_K$ is our assumption. By Maass [5] it was shown that Hurwitz's extension $\tilde{\Gamma}_K$ of Γ_K is the unique maximal extension acting properly discontinuously on H^n except for an extension by a group acting trivially on H^n. $\tilde{\Gamma}_K$ is consisting of matrices

$$\begin{bmatrix}\alpha/\sqrt{\omega} & \beta/\sqrt{\omega}\\ \gamma/\sqrt{\omega} & \delta/\sqrt{\omega}\end{bmatrix}$$

where $\alpha, \beta, \gamma, \delta \in K$, and $\omega=\alpha\delta-\beta\gamma$ is totally positive, and $\alpha/\sqrt{\omega}$, $\beta/\sqrt{\omega}$, $\gamma/\sqrt{\omega}$, $\delta/\sqrt{\omega}$ are integral over O_K. $\tilde{\Gamma}_K\cap SL_2(K)$ equals Γ_K, which implies $\Gamma_f=\Gamma_K$. q.e.d.

Let $\mathfrak{p}$ be a prime ideal of O_K. It is easy to check that the set $\Gamma_0(\mathfrak{p}^e)\backslash\Gamma_K$ is naturally isomorphic to

$$\left\{\begin{pmatrix}0 & 1\\ -1 & \delta\end{pmatrix} \mid \delta\in\mathfrak{p} \bmod \mathfrak{p}^e\right\}\cup\left\{\begin{pmatrix}1 & 0\\ \gamma & 1\end{pmatrix} \mid \gamma\in O_K \bmod \mathfrak{p}^e\right\}.$$

In particular the index $[\Gamma_K:(\mathfrak{p}^e)]$ equals $\mathrm{Nm}(\mathfrak{p}^e)(1+\mathrm{Nm}(\mathfrak{p})^{-1})$ where Nm denotes the norm of K over $\boldsymbol{Q}$. The general case is reduced to the prime power case by the Chinese remainder theorem. We have the following:

Lemma 3. *Let* $\mathfrak{b}$ *be a non-zero integral ideal of* O_K. *Then the index* $[\Gamma_K:\Gamma_0(\mathfrak{b})]$ *equals* $\mathrm{Nm}(\mathfrak{b})\prod_{\mathfrak{p}}(1+\mathrm{Nm}(\mathfrak{p})^{-1})$ *where* $\mathfrak{p}$ *runs through prime ideals dividing* $\mathfrak{b}$.

§ 5. Irreducibility

We assume $n\geq 3$ in what follows. Let Γ be a subgroup in $SL_2(K)$ commensurable with Γ_K. Then any holomorphic automorphy factor for Γ is, up to trivial automorphy factors, of the form (2) (Freitag [2]). If there is a non-constant modular form associated with J in (2), then the entries of the weight vector are all positive (Freitag [1, Sect. 2]). A modular form f for Γ is said to be *irreducible in* Γ if its divisor corresponds to an irreducible divisor of the modular variety H^n/Γ. Then any modular form f has a unique irreducible decomposition up to a constant factor;

$$f=f_1^{r_1}\cdots f_t^{r_t}$$

where f_i $(1\leq i\leq t)$ are irreducible in Γ.

Let us suppose that $\Gamma\subset\Gamma_K$. Let g be a modular form for Γ, and let div(g) be the divisor of g on H^n. Then div(g) is written as

$$\sum_{j=1}^{t} m_j(\sum_M M \cdot D_j), \quad m_j \in \boldsymbol{Z}, >0,$$

where $D_1, \cdots, D_t$ are a finite number of irreducible divisors on H^n inequivalent under Γ and where M runs over a system of left representatives of Γ mod the stabilizer subgroup in Γ at D_j. We note that D_j's may be equivalent under Γ_K. Suppose that in the equivalence relation under Γ_K, D_j's are divided into several classes $\{D_1, \cdots, D_{j_1}\}, \{D_{j_1+1}, \cdots, D_{j_2}\}, \cdots, \{D_{j_s+1}, \cdots, D_j\}$. We define a new divisor invariant under Γ_K by

$$\sum_{k=0}^{s} \max\{m_{j_k+1}, \cdots, m_{j_{k+1}}\} (\sum_k M \cdot D_{j_k+1}) \tag{7}$$

where $j_0 = 0, j_{s+1} = j$, and where M runs over a system of left representatives of Γ_K mod the stabilizer subgroup in Γ_K at D_{j_k+1}. We denote by $N(g)$ the modular form for Γ_K whose divisor equals (7). $N(g)$ is divisible by g as modular forms for Γ, and moreover it is minimal in the sense that it divides any such modular form for Γ_K. We do not mind the ambiguity in the definition of $N(g)$ up to a constant multiple, which will never cause a trouble in the present paper.

Lemma 4. *Let $n \geq 3$, and let Γ be a normal subgroup of Γ_K. Let g be a modular form for Γ of weight $(k_1, \cdots, k_n)$ which is irreducible in Γ. Then $N(g)$ is given as*

$$\prod_M \prod_i (\gamma^{(i)} z_i + \delta^{(i)})^{-k_i} g(Mz) \tag{8}$$

where $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ runs through a system of right representatives of Γ_K mod *$\Gamma_K \cap \Gamma_g$. In particular the weight vector of $N(g)$ is a $[\Gamma_K : \Gamma_K \cap \Gamma_g]$ times the weight vector of g.*

Proof. Since (8) is a modular form for Γ_K, it is divisible by $N(g)$. We prove the contrary. Since Γ is normal in Γ_K, for each $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_K$ $\prod_i (\gamma^{(i)} z_i + \delta^{(i)})^{-k_i} g(Mz)$ is a modular form for Γ which is also irreducible in Γ, and which divides $N(g)$. If M runs through a system of right representatives of Γ_K mod $\Gamma_K \cap \Gamma_g$, then any two of them have no common divisor of zero. This shows our assertion. q.e.d.

Lemma 5. *Let $n \geq 3$, and let Γ be a subgroup of $SL_2(K)$ commensurable with Γ_K. Let g be a modular form for Γ of weight $(k_1, \cdots, k_n)$ irreducible in Γ. Suppose that g is not irreducible in some subgroup in Γ of finite index. Then for any i within $1 \leq i \leq n$ and for any positive even r,* *there is a modular form h for Γ of weight $r(k_1, \cdots, k_{i-1}, k_i+2, k_{i+1}, \cdots, k_n)$ satisfying that* (i) $g^r \preccurlyeq h$ *and that* (ii) g, h *have no common divisors.*

Proof. By assumption, there is a subgroup Γ' of Γ in which g is not irreducible. Replacing Γ' by a smaller one if necessary, we may assume that Γ' is normal in Γ. Let $g=g_1\cdots g_t$, $t\geq 2$, be an irreducible decomposition in Γ'. Let k, l be distinct integers in $\{1, \cdots, t\}$. Because of the irreducibility of g in Γ, g_k, g_l are transformed into each other for any k, l, more precisely, for some $M=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \Gamma$, $\prod_i (\gamma^{(i)}z_i+\delta^{(i)})^{-e_i}g_k(Mz)$ equals $g_l(z)$ up to a constant factor, $(e_1, \cdots, e_n)$ being the weight vector of g_k. Hence, in particular, g_j's have the same vector weight, namely, $(1/t)(k_1, \cdots, k_n)$. The irreducibility of g in Γ implies also that the multiplicity of each factor is one namely, any two of the g_j's have no common divisors.

$$g_l^2\frac{\partial}{\partial z_i}(g_k/g_l)=g_l\frac{\partial}{\partial z_i}g_k-g_k\frac{\partial}{\partial z_i}g_l \tag{9}$$

is modular form for Γ' of weight $(2k_1/t, \cdots, 2k_{i-1}/t, 2+2k_i/t, 2k_{i+1}/t, \cdots, 2k_n/t)$. By a matrix M in Γ, (9) is transformed into a similar one. So

$$h:=\sum_{k<l}\left(g_l\frac{\partial}{\partial z_i}g_k-g_k\frac{\partial}{\partial z_i}g_l\right)^r\prod_{j\neq k,l}g_j^r \tag{10}$$

is a modular form for Γ of weight $r(k_1, \cdots, k_{i-1}, k_i+2, k_{i+1}, \cdots, k_n)$. A function (9) is $\succcurlyeq g_kg_l$ (see [8, Proof of Proposition 3]), and so each term of (10) is $\succcurlyeq g^r$. This implies that (i) $g^r\preccurlyeq h$. Finally we show that h is not divisible by g, which implies (ii). If otherwise, h is divisible by g_1 in Γ'. $(\partial/\partial z_i)g_1$ does not vanish identically on any component of $\operatorname{div}(g_1)$ on H^n by [8, Lemma 5] where the proof is given only for Γ_K, but it is easy to observe validity in general. Expanding (10), we get only one term $((\partial/\partial z_i)g_1)^r\prod_{j\neq 1}g_j^r$ that does not contain g_1. This implies that h is not divisible by g_1, a contradiction. q.e.d.

Corollary. *Let g be as in Lemma* 5. *Then there is a modular form h for Γ such that* (i) *the weight vector is* $2r(1, \cdots, 1)+nr(k_1, \cdots, k_n)$, r *being any positive integer, and that* (ii) $g^{nr}\preccurlyeq h$, *and that* (iii) g, h *have no common divisors.*

Proof. We take a product for $j=1, \cdots, n$ of modular forms given in the lemma. q.e.d.

§ 6. Key proposition

There is a positive rational number k_0 depending only on K for which

$\sum k_i/k_0$ is integral if $(k_1, \cdots, k_n)$ is the weight vector of any automorphy factor for Γ_K. Gundlach [3] (see also [8, p. 666]) shows that $k_0 \geq 1/2$ if n is even. By [9] it was shown that $k_0 \geq 1/2$ for any $n>1$, and that $k_0 \geq 2$ particularly if the ideal in O_K generated by 2 is unramified at any prime ideal of degree one. We note that if f is a non-constant modular form for Γ_K of scalar weight, then n weight $(f) \geq k_0$.

Now we state the key proposition of the paper, which gives a refinement of [8, Proposition 3].

Proposition 2. *Let $n \geq 3$. Let l be a non-negative real number for which there is a non-constant modular form f for Γ_K of scalar weight such that* ord(f)/weight$(f) > l$. *Let D be any effective divisor on H^n invariant under Γ_K which corresponds to an irreducible divisor of the modular variety H^n/Γ_K. Then there exists a modular form g for Γ_K of scalar weight such that* (i) $g|_D \not\equiv 0$, *and that* (ii) *ord*(g)/weight$(g) > \min\{l/(2+1/2^{n-2}k_0)$, $\max\{3l/4(1+1/k_0), l/(1+2/k_0)\}\}$, *which is at least $l/4$ and which is at least $l/(2+2^{-n+1})$ particularly if the ideal in O_K generated by 2 is unramified at any prime ideal of degree one.*

Let F denote the modular form for Γ_K defining D, which is obviously irreducible in Γ_K. Let $(k_1, \cdots, k_n)$ be the weight vector of F. If F is not a factor of f, then there is nothing to prove. So we assume that F is a factor of f in what follows.

Suppose that $F(2z)$ does not divide $f(z)$ (in $\Gamma(2)$), or equivalently that $F(z)$ does not divide $f(\frac{1}{2}z)$ (in $\Gamma(2)$). Then $F(z)$ does not divide $f(2z)$ since $f(2z)$ (resp. $F(z)$) is equal to $\Pi\,(2z_i)^{-\text{weight}(f)} f(\frac{1}{2}(-z^{-1}))$ (resp. $\Pi\,(z_i)^{-k_i} \times F(-z^{-1})$)) up to a constant multiple and since $F(-z^{-1})$ does not divide $f(\frac{1}{2}(-z^{-1}))$. Then by the comment below Corollary to Lemma 1, $F(z)$ is not a factor of $f_{2,r}$ in (6) for some positive integer r. Since

$$\text{ord}\,(f_{2,r})/\text{weight}\,(f_{2,r}) \geq \frac{1}{2}\,(\text{ord}\,(f)/\text{weight}\,(f)) > \frac{l}{2}$$

by Corollary to Lemma 1, our assertion follows. So the problem is reduced to the case that

(i) F is a factor of f, and

(ii) $F(2z)$ is a factor of $f(z)$, or equivalently $N(F(2z))$ divides f, N being as in the preceding section. We consider two cases;

I: F is not irreducible in $\Gamma^1(2)$,

II: F remains irreducible in $\Gamma^1(2)$.

The proof in the case I is given in the next section, and the case II, in the section after next.

§ 7. Proof in case I

Let $f=F^sG$ be the decomposition in Γ_K where G is not divisible by F. At first we assume that $\sum k_i \geq 2k_0$ for the weight vector $(k_1, \cdots, k_n)$ of F. Obviously n weight $(f) \geq s\sum k_i \geq 2sk_0$. By Corollary to Lemma 5 there is a modular form h for Γ_K such that (i) the weight vector is $4(1, \cdots, 1)+2n(k_1, \cdots, k_n)$, and (ii) $F^{2n} \preccurlyeq h$, and that (iii) $h|_D \not\equiv 0$. Then we take as g, h^sG^{2n} which is a modular form for Γ_K of scalar weight $4s+2n$ weight (f), and which does not vanish identically on D. ord (g)/weight $(g) \geq 2n$ ord $(f)/(4s+2n$ weight $(f))>l/(1+2s/n$ weight $(f)) \geq l(1+1/k_0)$. So in this case our assertion follows, because $l/(1+1/k_0)>\max\{3l/4(1+1/k_0), l/(1+2/k_0)\}$.

Lemma 6. *Let F be a non-constant modular form for Γ_K of weight $(k_1, \cdots, k_n)$ irreducible in Γ_K, and let μ be a totally positive integer in O_K other than units. Then $N(F(\mu z))$ has as a factor, at least one modular form for Γ_K different from $F(z)$. If $\sum k_i=k_0$, then it is not irreducible in $\Gamma(\mu)$.*

We note that in the above lemma F is not assumed to be irreducible in a subgroup of Γ_K.

Proof. Let $F(\mu z)=F_1(z)\cdots F_t(z)$ be an irreducible decomposition in $\Gamma(\mu)$. By the irreducibility of $F(z)$ in Γ_K, any two of the $F_j(z)$'s have no common divisor. If $N(F(\mu z))$ has only $F(z)$ as a factor, then $F(z)$ is divisible by each of the $F_j(z)$, and hence by $F(\mu z)$. So $\Gamma_{F(z)}$ is properly larger than Γ_K, which contradicts to Maass [5] (see the proof of Lemma 2). This shows our first assertion. Now suppose that $\sum k_i=k_0$. If $t \geq 2$, then any of $F_1, \cdots, F_t$ is not a modular form for Γ_K by the definition of k_0. The irreducible factor of $N(F(\mu z))$ equals one of $N(F_j)$, up to a constant multiple, which is not irreducible in $\Gamma(\mu)$. If $t=1$, i.e., $F(\mu z)$ is irreducible in $\Gamma(\mu)$, then Lemma 4 together with Lemma 2 shows that $N(F(\mu z))$ is not irreducible in $\Gamma(\mu)$.

We continue to prove Proposition 2. We assume that $k_0=\sum k_i$. Let $f=F^sF'^{s'}G$ be a decomposition in Γ_K, where F' is an irreducible factor of $N(F(2z))$ in Γ_K different from F, and G is divisible by neither F nor F'. Let $(k_1', \cdots, k_n')$ be the weight vector of F', and let $k'=\sum k_i' \geq k_0$. Then obviously n weight $(f) \geq sk_0+sk' \geq (s+s')k_0$. Suppose that $s'(k'+4) \geq s(2-k_0)$. Let h be as in the beginning of this section. We take as g, $h^s(F'^{s'}G)^{2n}$ which is a modular form for Γ_K of scalar weight $4s+2n$ weight (f) with order at least $2n$ ord (f) and which does not vanish identically on D. Then ord (g)/weight $(g) \geq 2n$ ord $(f)/(4s+2n$ weight $(f))>l/(1+2s/n$ weight $(f)) \geq \max\{3l/4(1+1/k_0), l/(1+2/k_0)\}$. Then our assertion follows. Now let us suppose that $s'(k'+4) \leq s(2-k_0)$, which implies

$k_0<2$. By Lemma 6, F' is not irreducible in $\Gamma(2)$. Then by Corollary to Lemma 5, there is a modular form h' for Γ_K such that (i) the weight vector is $4(1, \cdots, 1)+2n(k_1', \cdots, k_n')$ and that (ii) $F'^{2n} \preccurlyeq h'$, and that (iii) h', F' have non common divisor. Let $g'=h'^{s'}(F^s G)^{2n}$, which is a modular form for Γ_K of scalar weight $4s'+2n$ weight(f). ord(g') is at least $2n$ ord(f). g' is not divisible by $F(2z)$, because if otherwise, g' is divisible by F' just as f, which is not the case. Then $g'(2z)$ is not divisible by $F(z)$ (see the argument in the preceding section). We take as g, $(g')_{2,r}$ in (6) where r is taken so that $(g')_{2,r}$ is not divisible by $F(z)$. Then by Corollary to Lemma 1, ord(g)/weight$(g) \geqq \frac{1}{2}$(ord(g')/weight$(g')) > l/(2+4s'/n$ weight $(f)) \geq l/(2+(-2k_0+4)/(k'+2k_0)) \geq 3l/4(1+1/k_0) = \max\{3l/4(1+1/k_0), l/(1+2/k_0)\}$, since $k_0<2$. We are done.

§ 8. Proof in case II

By assumption $F(2z)$ is irreducible in

$$\Gamma(2)=\begin{pmatrix}\sqrt{2}^{-1} & 0\\ 0 & \sqrt{2}\end{pmatrix}\Gamma^1(2)\begin{pmatrix}\sqrt{2} & 0\\ 0 & \sqrt{2}^{-1}\end{pmatrix},$$

and by Lemma 2 $\Gamma_{F(2z)} \cap \Gamma_K = \Gamma_0(2)$. By Lemma 4, $F'(z) := N(F(2z))$ is irreducible in Γ_K and its weight vector is $[\Gamma_K : \Gamma_0(2)]\,(k_1, \cdots, k_n)$. Let $f=F^sF'^{s'}G$ be a decomposition in Γ_K, where G is divisible neither F nor F'. We give a similar proof as in the last part of the preceding section. By Corollary to Lemma 5, there is a modular form h' for Γ_K such that (i) the weight vector is $4(1, \cdots, 1)+2n[\Gamma_K : \Gamma_0(2)](k_1, \cdots, k_n)$ and that (ii) $F'^{2n} \preccurlyeq h'$, and that (iii) h' is not divisible by F'. Let $g'=h'^{s'}(F^sG)^{2n}$, which is a modular form for Γ_K of scalar weight $4s'+2n$ weight(f) and with order $\geq 2n$ ord(f). $g'(z)$ is not divisible by $F(2z)$, in other words, $g'(2z)$ is not divisible by $F(z)$. Then we take as g, $(g')_{2,r}$ in (6) which is not divisible by $F(z)$. Then by Corollary to Lemma 1, ord(g)/weight$(g) \geq \frac{1}{2}$(ord(g')/weight$(g')) > l/(2+4s'/(s+s'[\Gamma_K : \Gamma_0(2)]\sum k_i)) > l/(2+4/[\Gamma_K : \Gamma_0(2)]k_0) > l/(2+1/2^{n-2}k_0)$ since $[\Gamma_K : \Gamma_0(2)] > 2^n$ by Lemma 3. We have proved Proposition 2.

§ 9. Proof of Theorem

Let us recall the asymptotic dimension of the space of modular forms for Γ_K of scalar weight $k \in 2Z$ with a trivial multiplier and with ord$(f)/k > l$ ([8], see also [7]). It is

$$\{2^{-2n+1}\pi^{-2n}d_K^{3/2}\zeta_K(2)-2^{n-1}l^n n^{-n}d_K^{1/2}hR\}k^n+O(k^{n-1})$$

where d_K, h, R denote the discriminant, the class number, the regulator of

K respectively. In particular, if

$$l<2^{-3}\pi^{-2}n\left(\frac{4d_K\zeta_K(2)}{hR}\right)^{1/n},$$

then there are such modular forms for sufficiently large even k.

Theorem 1. *Let $n>6$. If*

$$2^{-4}\pi^{-2}(n-1)\left(\frac{4d_K\zeta_K(2)}{hR}\right)^{1/n}>1, \tag{11}$$

then (☆) holds. If the ideal in O_K generated by 2 is unramified at any prime ideal of degree one and if

$$2^{-3}(1+2^{-n+1})^{-1}\pi^{-2}(n-1)\left(\frac{4d_K\zeta_K(2)}{hR}\right)^{1/n}>1, \tag{12}$$

then (☆) holds.

Proof. By Proposition 2, for any subvariety D in X_K of codimension one, there is a modular form g of scalar weight such that (i) $g|_D\not\equiv 0$, and that (ii) $\mathrm{ord}(g)/\mathrm{weight}(g)>\{2^{-3}\pi^{-2}n(4d_K\zeta_K(2)/hR)^{1/n}\}/4$ which is larger than $n/2(n-1)$ by the condition (11). By Proposition 1, (☆) holds. Also to the second assertion, the similar argument is applicable. q.e.d.

$d_K\zeta_K(2)/hR$ is at least $2^{n-2}\pi^n$ (cf. Lang [4, p. 261]). Hence (11) holds for $n>26$, and (12) holds for $n>14$. This proves our theorem.

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