Advanced Studies in Pure Mathematics 15, 1989
Automorphic Forms and Geometry of Arithmetic Varieties
pp. 281–286

Any Irreducible Smooth GL_2-Module is Multiplicity Free for any Anisotropic Torus

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Dedicated to Prof. Ichiro Satake on his sixtieth birthday

§ 1.

Let k be a non-archimedean local field, B be a quaternion algebra, i.e. a central simple algebra of rank 4 over k. Let L be a separable quadratic subfield of B. The group $G=B^\times$, of the regular elements of B, is a T.D.L.C. (= totally disconnected locally compact) group by the induced topology from B, and $H=L^\times$ is a closed subgroup of G. In other words, G is a k-form of GL_2, and H is a maximal torus anisotropic modulo center. Let (π, E) be a smooth representation of G on the complex vector space E. The purpose of this paper is to prove the following:

Theorem A. *If (π, E) is irreducible as G-module, then it is multiplicity free as H-module. Namely, there is a subset $\hat{H}(\pi)$ of the set $\hat{H}$ of all quasicharacters of H such that*

$$\pi=\bigoplus_{\chi\in\hat{H}(\pi)}\chi \qquad \textit{as H-module.}$$

§ 2.

The irreducible smooth representations of $G=B^\times$ are classified into several series (cf. [J-L], [K] for split G, and [G-G], [Ho] for non-split G). To identify the set $\hat{H}(\pi)$ for all L amounts to get a complete knowledge for the representation π, at least character-theoretically. In this respect, there are no difficulties if k has odd residual characteristic. While, in dyadic case, I have determined $\hat{H}(\pi)$ (for all L) for some series of π's, but not yet for all series.

When G is non-split, i.e. B is a division algebra, there is a close connection between Theorem A and the Basis Problem of modular forms as

Received March 27, 1987.

indicated in Part II Chap. 9 of [H-P-S]. This connection is the motivation of this work.

When G is split, i.e. $B=M_2(k)$, and $G=GL_2(k)$, let K be a maximal compact modulo center subgroup of G. There are two such K's up to conjugacy. The standard one, the normalizer of a maximal compact subgroup of G, contains unramified $L^\times$, while the other one, the normalizer of an Iwahori subgroup of G, contains any ramified $L^\times$. Hence we have the following:

Corollary. *Any irreducible smooth representation π of $GL_2(k)$ is multiplicity free as k-module. In particular, π is admissible.* (The last statement is well known, and it is valid for any reductive group G as shown in [B]).

§ 3.

As for the proof, Theorem A is a formal consequence of the following simple

Proposition B. *For each L, there is a topological antiautomorphism τ of the algebra B satisfying*:

(i) *τ is of order* 2,

(ii) *$\tau(a)=a$ for any $a \in L$,*

(iii) *each coset Hg contains a τ-fixed element.*

Proof. Let $a \mapsto \bar{a}$ denote the Galois action of L over k. By Skolem-Noether theorem, there exists $y \in B^\times$ such that

$$yay^{-1}=\bar{a} \qquad \text{for any } a \in L.$$

Then it follows that $B=L\oplus yL$, $y^2 \in k^\times$ and

$$i\colon a+yb \longmapsto \bar{a}-yb$$

is the canonical involution of B.

By Hilbert theorem 90, there exists $c \in L^\times$ such that

$$\bar{c}c^{-1}=-1.$$

Define τ as the composite $i \circ I(cy)$ of the canonical involution i and the inner automorphism $I(cy)\colon x \mapsto (cy)x(cy)^{-1}$, i.e.

$$\tau\colon a+yb \longmapsto a+y\bar{b}.$$

Clearly, τ is a topological antiautomorphism of order 2, fixing each element of L. Since $G=B^{\times}=L^{\times}((L+y)\cup\{1\})$, τ also satisfies the last condition (iii).

§ 4.

The formal argument to derive Theorem A from Proposition B can be summarized as Proposition C below after introducing some notation.

We consider a triple (G, Z, ω) consisting of a T.D.L.C. group G, its closed normal subgroup Z, and a locally constant homomorphism $\omega: Z \to C^{\times}$, normalized by G, $\omega(gzg^{-1})=\omega(z)$ for any $z \in Z$, $g \in G$. Let $S(G, \omega)$ denote the vector space of all locally constant complex valued functions f on G, of which supports are compact mod Z, and which are ω-semiinvariant, $f(zg)=\omega(z)f(g)$ for any $z \in Z$. $S(G, \omega)$ is an associative algebra over C by the convolution product,

$$f_1 * f_2(g_0)=\int f_1(g)f_2(g^{-1}g_0)d\bar{g},$$

where $d\bar{g}$ is a left invariant Haar measure of $\bar{G}=G/Z$.

Let H be a closed subgroup of G containing Z and having a compact quotient H/Z. Let $\varepsilon: H \to C^{\times}$ be a locally constant homomorphism which coincides with ω on Z. Let $S(G, H, \varepsilon)$ denote the subalgebra of $S(G, \omega)$ consisting of all ε-bi-semiinvariant functions f, $f(hg)=f(gh)=\varepsilon(h)f(g)$ for any $h \in H$. Let (π, E) be a smooth representation of G, on which Z acts as ω^{-1}, $\pi(z)v=\omega(z)^{-1}v$ for $z \in Z$, $v \in V$. Finally let $E(H, \varepsilon^{-1})$ denote the ε^{-1}-eigen subspace under H,

$$E(H, \varepsilon^{-1})=\{v \in E \mid \pi(h)v=\varepsilon(h)^{-1}v \text{ for } h \in H\}.$$

Proposition C. *There are the implications*: (I) $\Rightarrow$ (II) $\Rightarrow$ (III).

(I) *G has a topological antiautomorphism τ satisfying*:

(1) $\tau(Z)=Z$, $\tau(H)=H$, $\varepsilon \circ \tau=\varepsilon$,

(2) *the automorphism* τ': $g \mapsto \tau(g)^{-1}$ *is of finite order,*

(3) *each double coset HgH contains a τ-fixed element.*

(II) *The algebra $S(G, H, \varepsilon)$ is commutative.*

(III) *If (π, E) is irreducible, then* $\dim E(H, \varepsilon^{-1}) \leq 1$.

§ 5.

In the rest of this paper, we retain all the notation of Section 4. The first implication '(I) $\Rightarrow$ (II)' is rather obvious. The first assumption (1) implies that the map $f \mapsto \tau f := f \circ \tau^{-1}$ is a linear isomorphism of $S(G, \omega)$. It also implies that τ' induces an automorphism $\bar{\tau}'$ of $\bar{G}$, hence $d(\bar{\tau}'(\bar{g}))= c\,d\bar{g}$ by some positive constant c. Then the second assumption (2) implies that $c=1$, hence $\tau(f_1*f_2)=\tau f_2*\tau f_1$ for $f_1, f_2 \in S(G, \omega)$. The third assumption (3) implies $\tau f=f$ if $f \in S(G, H, \varepsilon)$, hence $f_1*f_2=f_2*f_1$ for $f_1, f_2 \in S(G, H, \varepsilon)$.

The next implication '(II) $\Rightarrow$ (III)' is more or less known, at least if H is open in G (cf. [C], [B-Z]). In particular, if Z is a trivial subgroup $\{1\}$, hence ω is also trivial, and moreover if H is open and compact, '(II) $\Rightarrow$ (III)' is a part of Proposition 2.10 of [B-Z]. Although there is no difficulty to modify their method (of embedding $S(G, \omega)$ into the algebra of distributions) to be capable of covering our case of non-trivial ω and not open H, the points to be checked might not be clear without giving the exact statement at each step. Here, we will give a shorter proof relying on a result of [C], under an extra condition,

(4) Z is a closed subgroup of the center of G.

Note that $(G, Z)=(B^\times, k^\times)$ of Section 1 certainly satisfies (4). Note also, as a general theory, the assumption (4) is not essentially restrictive, since we may work on the quotient by the kernel of ω, of G, Z and everything.

§ 6.

Recall that G is a T.D.L.C. group iff it has a fundamental system of neighbourhoods $\mathcal{U}$ of 1, consisting of open compact subgroups U. Since ε is locally constant, it is trivial on $H \cap U$ for some $U \in \mathcal{U}$. By (4), ZU is an open subgroup normalizing U, and $[H: H \cap ZU]$ is finite, hence the intersection $\bigcap hUh^{-1}$ for $h \in H/(ZU \cap H)$ is an open compact subgroup normalized by H. Thus we may and shall assume that $\mathcal{U}$ consists of open compact subgroups U satisfying

(5) $hUh^{-1}=U$ for $h \in H$, and $U \cap H \subset \ker \omega$.

Hence there is a unique homomorphism u: $HU \to \boldsymbol{C}^\times$ satisfying

(6) $u=\varepsilon$ on H, $u=1$ on U.

Let $\mu(HU)$ denote the volume of HU/Z by the Haar measure $d\bar{g}$ of $\bar{G}$ and let $\dot{u}$ denote the function on G which coincides with $\mu(HU)^{-1}u$ on HU, and zero outside. Since HU is open and compact mod Z, $\dot{u}$ is a member of $S(G, \omega)$, and by the definition of convolution, we have:

$$\dot{u}*f=f \text{ iff } f(xg)=u(x)f(g) \qquad \text{for any } x \in HU,$$
$$f*\dot{u}=f \text{ iff } f(gx)=u(x)f(g) \qquad \text{for any } x \in HU.$$

and

$$S(G, HU, u)=\dot{u}*S(G, \omega)*\dot{u}. \tag{7}$$

Since $S(G, H, \varepsilon)$ is the union of $S(G, HU, u)$, it is commutative iff each $S(G, HU, u)$ is commutative.

By definition, a representation (π, E) is smooth iff E is the union of the U-fixed subspace $E(U, 1)$. Since $E(H, \varepsilon^{-1}) \cap E(U, 1) = E(HU, \varepsilon^{-1})$, $E(H, \varepsilon^{-1})$ is the union of $E(HU, u^{-1})$, and $E(HU, u^{-1}) \subset E(HU', (u')^{-1})$ if $U \supset U'$. Therefore if one knows that $\dim E(HU, u^{-1}) \leq d$ for any $U \in \mathscr{U}$, and $\dim E(hU_0, u_0^{-1}) = d$ for some $U_0 \in \mathscr{U}$, then one can conclude that $E(H, \varepsilon^{-1}) = E(HU_0, u_0^{-1})$.

Since Z acts on E as ω^{-1}, $S(G, \omega)$ acts on E by

$$\pi(f)v = \int f(g)\pi(g)v\, d\bar{g}. \tag{8}$$

In particular, $\pi(\dot{u})$ is the projection operator of E to $E(HU, u^{-1})$, and by (7), $S(G, HU, u)$ acts on $E(HU, u^{-1})$. Also observe

$$\pi(g_0) \circ \pi(f) = \pi(L(g_0)f), \tag{9}$$

where $L(g_0)f = (g \mapsto f(g_0^{-1}g)) \in S(G, \omega)$.

Now '(II) $\Rightarrow$ (III)' is a consequence of the following:

(10) If E is G-irreducible and $E(HU, u^{-1}) \neq 0$, then $E(HU, u^{-1})$ is $S(G, HU, u)$-irreducible. (Hence if $S(G, HU, u)$ is commutative, $\dim E(HU, u^{-1}) = 1$.)

The claim (10) is in [C]. We reproduce its proof. Let v_0 be a non-zero vector in $E(HU, u^{-1})$ and v be an arbitrary vector in $E(HU, u^{-1})$. Since E is G-irreducible, we can find $g_i \in G$, $c_i \in \boldsymbol{C}$ $(i = 1, \cdots, n)$ such that $v = \sum c_i \pi(g_i)v_0$. Since $v_0 = \pi(\dot{u})v_0$, by (9), $\pi(g_i)v_0 = \pi(g_i)\pi(\dot{u})v_0 = \pi(L(g_i)\dot{u})v_0 = \pi(L(g_i)\dot{u})\pi(\dot{u})v_0$. Since $v = \pi(\dot{u})v$, we have $v = \pi(f)v_0$ with

$$f = \sum c_i \dot{u} * L(g_i)\dot{u} * \dot{u}$$

which lies in $S(G, HU, u)$ by (7).

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