

On Dimension Formula for Siegel Modular Forms

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§ 0. Introduction

Let \mathfrak{S}_g be the Siegel upper half plane of degree g , and let $Sp(g, \mathbf{Z})$ be the Siegel modular group of degree g . Let Γ be a subgroup of $Sp(g, \mathbf{Z})$ of finite index, and let μ be a holomorphic representation of $GL(g, \mathbf{C})$ into $GL(r, \mathbf{C})$. By an *automorphic form of type μ* with respect to Γ , we mean a holomorphic mapping f of \mathfrak{S}_g to the r -dimensional complex vector space \mathbf{C}^r which satisfies the following equalities:

$$f(M\langle Z \rangle) = \mu(CZ + D)f(Z),$$

for any $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$ and $Z \in \mathfrak{S}_g$, where $M\langle Z \rangle$ is defined to be $(AZ + B)(CZ + D)^{-1}$. (We need to assume the holomorphy of f at “cusps” if $g=1$.) We denote by $A_\mu(\Gamma)$ the complex vector space of automorphic forms of type μ with respect to Γ . It is known that $A_\mu(\Gamma)$ is finite dimensional ([8]). In case $\mu(CZ + D) = \det(CZ + D)^k$, an automorphic form of type μ is also called an automorphic form of *weight k* , and $A_\mu(\Gamma)$ is also denoted by $A_k(\Gamma)$. In case the degree of μ is greater than one, an automorphic form of type μ is called a *vector-valued* automorphic form.

Our main problem is to find a formula for $\dim A_\mu(\Gamma)$ as a function in the signature of μ . The first result to our main problem was obtained by J.-I. Igusa in case $g=2$. (The case $g=1$ is classical.) Let $\Gamma_g(N)$ be the principal congruence subgroup of $Sp(g, \mathbf{Z})$ of level N , i.e.,

$$\Gamma_g(N) = \{M \in Sp(g, \mathbf{Z}) \mid M \equiv \mathbf{1}_{2g} \pmod{N}\}.$$

$A(\Gamma) := \bigoplus_{k \geq 0} A_k(\Gamma)$ has a structure of a graded ring. By using the theory of theta series, he explicitly determined the generators of $A(\Gamma)$ and represented these generators by theta constants for some cases such as $\Gamma = \Gamma_2(1)$ or $\Gamma_2(2)$ ([29], [30]). Especially $\dim A_k(\Gamma_2(1))$ and $\dim A_k(\Gamma_2(2))$ were known in this work. In [31], he constructed a graded ring homomorphism ρ_g from a subring R_g of $A(\Gamma_g(1))$ to the graded ring $S(2, 2g+2)$ of binary

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$(2g+2)$ -forms. R_g contains $A(\Gamma_g(1))^{(2)} := \bigoplus_{k \geq 0} A_{2k}(\Gamma_g(1))$ and coincides with $A(\Gamma_g(1))$ if g is odd or $g=2, 4$. ρ_1 is bijective and ρ_2 is injective. He reproved the structure theorem of $A(\Gamma_2(1))$ by using this homomorphism ρ_2 and the structure theorem of $S(2, 6)$. But these methods were restricted to special cases of Γ and the case of weight k .

For general group Γ and representation μ , two main approaches are known. The first is a geometric one which uses Riemann-Roch-Hirzebruch's formula and the holomorphic Lefschetz fixed points formula (when Γ has fixed points). The second is a group-theoretical one which uses Selberg's trace formula.

If $N \geq 3$, then $\Gamma_g(N)$ acts on \mathfrak{S}_g without fixed points. So $X_g := \Gamma_g(N) \backslash \mathfrak{S}_g$ is a manifold in this case. But X_g is not compact. I. Satake constructed a compactification \bar{X}_g of X_g which is a projective variety ([43]). But if $g \geq 2$, then \bar{X}_g has bad singularities along its cusps: $\bar{X}_g - X_g$. J.-I. Igusa constructed a desingularization \tilde{X}_2, \tilde{X}_3 of \bar{X}_2, \bar{X}_3 which is a blowing up of \bar{X}_2, \bar{X}_3 along its cusps, respectively ([32]). For general g , a smooth compactification \tilde{X}_g of X_g was constructed in [3] and in [41]. When we need to specify the level N , we denote X_g, \bar{X}_g and \tilde{X}_g by $X_g(N), \bar{X}_g(N)$ and $\tilde{X}_g(N)$, respectively.

The first result to our main problem from the first approach was obtained by T. Yamazaki ([57]). He applied Riemann-Roch-Hirzebruch's formula to \tilde{X}_2 and calculated $\dim A_k(\Gamma_2(N))$ with $N \geq 3$. The first result to our main problem from the second approach was obtained by Y. Morita and U. Christian independently ([39], [9] and [10]). They calculated $\dim A_k(\Gamma_2(N))$ with $N \geq 3$ by Selberg's trace formula. The next result would be the author's one. In 1979, he calculated $\dim A_k(\Gamma_3(N))$ with $N \geq 3$ by applying Riemann-Roch-Hirzebruch's formula to \tilde{X}_3 ([50]). Next in 1980, the author applied the holomorphic Lefschetz fixed points formula to the action on $\tilde{X}_2(N)$ of $\Gamma_2(1)/\Gamma_2(N)$ with $N \geq 3$ and calculated $\dim A_k(\Gamma_2(1))$, and then $\dim A_k(\Gamma_2(2))$ similarly ([51]). About the same time K. Hashimoto calculated $\dim A_k(\Gamma_2(1))$ and $\dim A_k(\Gamma_2(2))$ by Selberg's trace formula ([19] I). These results can be regarded as the third proof of the structure theorem of $A(\Gamma_2(1))$. He (and T. Ibukiyama) also explicitly calculated $\dim A_k(\Gamma)$ for more general congruence subgroups such as $\Gamma = \Gamma_0(p)$, where p is a prime number and $\Gamma_0(p) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid C \equiv 0 \pmod{p} \right\}$ ([19] I, [22] and [28]). Next in 1982, the author explicitly calculated the dimension of the spaces of vector-valued automorphic forms $A_\mu(\Gamma_2(N))$ with $N \geq 1$ by geometric method ([52]).

There are discrete subgroups of $Sp(g, \mathbf{R})$ which have quite different properties. For discrete subgroups Γ of $Sp(g, \mathbf{R})$ such that $\Gamma \backslash \mathfrak{S}_g$ is compact, $\dim A_\mu(\Gamma)$ was calculated in [26], [33] and [27] by geometric method

and in [37] by Selberg's trace formula. They expressed the dimension of $A_\mu(\Gamma)$ as a finite sum of invariants attached to each elliptic fixed point of Γ , or what is the same, elliptic conjugacy classes of Γ . So the dimension can be calculated once a complete list of such data is known. (They solved this problem for general bounded symmetric domains.) $Sp(2, \mathbf{R})$ has discrete arithmetic subgroups which are related to a quaternion unitary groups. The quotient space of \mathfrak{S}_2 by a group of this type is not compact but it is compactified by adding a finite number of points to it. As to such a group Γ , $\dim A_k(\Gamma)$ was calculated in [56], [46] by geometric method and in [1], [19] II by Selberg's trace formula. We refer the reader to a note by I. Satake [45] and Satake-Ogata's article in this volume.

So our next problem should be to calculate $\dim A_k(\Gamma_3(1))$. The author was studying this problem by geometric method and K. Hashimoto was studying by Selberg's trace formula. But there were some difficulties unsolved on both sides (see § 2). So recently we were studying this problem jointly ([23]). But this work has not been completed now. While we were studying this problem, S. Tsuyumine succeeded to solve it ([54]). His method is similar (but far more complicated) to the Igusa's second proof for the structure theorem of $A(\Gamma_2(1))$. The kernel of Igusa's graded ring homomorphism $\rho_3: A(\Gamma_3(1)) \rightarrow S(2, 8)$ was known to be an ideal which is generated by a cusp form χ_{18} of weight eighteen ([31]), and the structure of $S(2, 8)$ was known by T. Shioda ([48]). Tsuyumine explicitly determined the structure and the generators of the graded ring $A(\Gamma_3(1))$ by using these results. So especially $\dim A_k(\Gamma_3(1))$ was known.

Although our problem was already solved, it is still not meaningless to solve this problem by our method, because Tsuyumine's method is restricted to the case of the full modular group $\Gamma_3(1)$ and of weight k . If $\dim A_k(\Gamma_3(1))$ is calculated by our method, it is easy to extend this result to the case of general congruence subgroups or the case of vector-valued automorphic forms, since the process to reach there has been already reduced to a routine work (modulo the vanishing theorem in the vector-valued case).

To solve our problem by geometric method, we need to classify the fixed points sets of the action of the finite group $H := \Gamma_3(1)/\Gamma_3(N)$ with $N \geq 3$ on the smooth compactification $\tilde{X}_3(N)$ and to classify all conjugacy classes of H which have fixed points on $\tilde{X}_3(N)$. We need to calculate the contributions to the dimension formula of these conjugacy classes one by one by the holomorphic Lefschetz fixed points formula. We define that two fixed points sets of H are equivalent to each other if there is an element of H which maps one fixed points set to another. In this classification there are more than one hundred kinds of fixed points sets (in the case of degree two, there were *only* twenty-five) and there are about three hundreds

conjugacy classes which have fixed points on $\tilde{X}_3(N)$.

To solve our problem by Selberg's trace formula, we need to classify all conjugacy classes of $\Gamma_3(1)$ which have non-zero contributions to the dimension formula. Although in the case of geometric method, we study the finite group $\Gamma_3(1)/\Gamma_3(N)$ and in the case of Selberg's trace formula, we study the infinite group $\Gamma_3(1)$, the classifications of these conjugacy classes go almost in parallel with each other. So in both methods, we need to calculate the contributions of about three hundred conjugacy classes. To execute such a hard calculation, it is very effective to compare the results from these two approaches with each other.

Let $f(t)$ be the generating function of $A(\Gamma_3(1))$, i.e., $f(t) = \sum_{k \geq 0} t^k \dim A_k(\Gamma_3(1))$. Then this is a rational function and the degree of the denominator of $f(t)$ is one hundred and ten. (There is a misprint in [54] p. 832, $(1-T^{12})^3$ in the denominator should be $(1-T^{12})^2$.) So the expansion of $f(t)$ to partial fractions has essentially fifty five terms, since $f(t)$ is an even function. We are calculating the coefficients of these fifty five terms one by one and we have calculated these coefficients except the following four terms: $1/(1-t)$, $1/(1-t)^2$, $1/(\rho-t)$ and $1/(\rho^2-t)$, where $\rho = \exp(2\pi i/3)$. Among them, the coefficient of $1/(1-t)$ is the most difficult one. To determine this exactly, we need to calculate the contributions of about one hundred conjugacy classes. Perhaps we need to spend much time and effort such as mathematicians of old times spent to calculate the circular constant.

The joint work of K. Hashimoto and the author was expected to be one of main parts of this volume. But unfortunately it is still incomplete, and therefore instead the author presents here an expository note.

In 1986 a paper of M. Eie and C.-Y. Lin was published in American Journal of Mathematics which states that they found the formula for $\dim A_k(\Gamma_3(1))$ by Selberg's trace formula ([15]). If this were true, our effort will be of no use. But the author should confess that this paper seems very deceptive to him. In the end of this note, we comment on this paper.

§ 1. Dimension formula for $\Gamma_2(N)$ and $\Gamma_3(N)$ with $N \geq 3$

For the sake of simplicity we mainly study the case of weight k in this note. In this section we present the calculation of $\dim A_k(\Gamma_2(N))$ with $N \geq 3$. The method we employ now was developed by the author in [50] and is more systematic than the original method of T. Yamazaki.

In both methods (geometric and Selberg), we do not calculate the dimension of $A_\mu(\Gamma)$ directly, but calculate the dimension of the space of "cusp forms" $S_\mu(\Gamma)$. $S_\mu(\Gamma)$ is a subspace of $A_\mu(\Gamma)$, and $\dim A_\mu(\Gamma)$ is

calculated by a theory of Eisenstein series from $\dim S_\mu(\Gamma)$. So we need to define the space $S_\mu(\Gamma)$.

We recall the construction of the compactification \bar{X}_g of the quotient space X_g . Since \mathfrak{S}_g is not a bounded domain, a part of its boundary is in "infinite place", so we need to consider a bounded domain \mathcal{D}_g which is biholomorphic to \mathfrak{S}_g . We put

$$\mathcal{D}_g := \{Z \in M(g, \mathbf{C}) \mid {}^t Z = Z, Z^* Z < \mathbf{1}_g\}.$$

Then \mathcal{D}_g is a bounded domain and the Cayley transformation $c: \mathfrak{S}_g \rightarrow \mathcal{D}_g$ is defined to be

$$c(Z) = (Z - \sqrt{-1}\mathbf{1}_g)(Z + \sqrt{-1}\mathbf{1}_g)^{-1},$$

and this is a biholomorphic mapping. The action of $Sp(g, \mathbf{R})$ on \mathcal{D}_g extends to its closure $\bar{\mathcal{D}}_g$ in $\mathbf{C}^{g(g+1)/2}$.

We put

$$F_{g'} := \left\{ \begin{pmatrix} Z & 0 \\ 0 & \mathbf{1}_{g-g'} \end{pmatrix} \mid Z \in \mathcal{D}_{g'} \right\},$$

where $0 \leq g' < g$. Then $F_{g'}$ is in $\bar{\mathcal{D}}_g - \mathcal{D}_g$ and biholomorphic to $\mathcal{D}_{g'}$. $F_{g'}$ and its images by the action of $Sp(g, \mathbf{R})$ are called *boundary components* of degree g' of \mathcal{D}_g . $\bar{\mathcal{D}}_g - \mathcal{D}_g$ is divided into a disjoint union of boundary components. $F_{g'}$ and its images by the action of $Sp(g, \mathbf{Z})$ are called *rational boundary components* of degree g' (for the definition of a (rational) boundary component in general, see a text book for example [44]). We define a certain topology on $\mathcal{D}_g^s := \mathcal{D}_g \cup \{\text{rational boundary components}\}$ ([43]), and Satake's compactification $\bar{X}_g(N)$ is the quotient space of \mathcal{D}_g^s by $\Gamma_g(N)$. $\bar{X}_g(N) - X_g(N)$ is a disjoint union of a finite number of copies of $X_{g'}(N)$ ($0 \leq g' < g$) and a copy of $X_g(N)$ in $\bar{X}_g(N) - X_g(N)$ is called a *cusp* of degree g' . Any smooth compactification \tilde{X}_g has a morphism $\pi_g: \tilde{X}_g \rightarrow \bar{X}_g$ which is the identity on the quotient space X_g . Here we take a so-called toroidal compactification due to D. Mumford ([3]). Then $E := \tilde{X}_g - X_g$ is a divisor with normal crossings and each irreducible component in E has a structure of a fiber space over a cusp of degree $g-1$ through π_g whose general fiber is an abelian variety of dimension $g-1$.

Let $\mu: GL(g, \mathbf{C}) \rightarrow GL(r, \mathbf{C})$ be a holomorphic representation. We define an action of $Sp(g, \mathbf{R})$ on the product space $\mathfrak{S}_g \times \mathbf{C}^r$ by

$$M(Z, u) := (M\langle Z \rangle, \mu(CZ + D)u),$$

for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \mathbf{R})$, $Z \in \mathfrak{S}_g$ and $u \in \mathbf{C}^r$. If $N \geq 3$, then the quotient

space V_μ of $\mathfrak{S}_g \times \mathbb{C}^r$ by $\Gamma_g(N)$ has a structure of a vector bundle on X_g . From the construction of the smooth compactification \tilde{X}_g ([3], [41]), it is proved that this vector bundle V_μ has a natural extension to a vector bundle \tilde{V}_μ on \tilde{X}_g . (V_μ is not extended to a vector bundle on \bar{X}_g in general.) In case $\mu(CZ+D)=\det(CZ+D)$, V_μ and \tilde{V}_μ are line bundles and denoted by L_g and \tilde{L}_g , respectively. L_g is extended to a line bundle \bar{L}_g on \bar{X}_g such that the restriction of \bar{L}_g to a cusp of degree g' is isomorphic to $L_{g'}$ and \tilde{L}_g is the pullback of \bar{L}_g by π_g . This line bundle \bar{L}_g is ample and the projectivity of \bar{X}_g follows from this fact ([5]). The space $A_\mu(\Gamma_g(N))$ is naturally identified with $\Gamma(\tilde{X}_g, \mathcal{O}(\tilde{V}_\mu))$.

Definition (1.1). The space of cusp forms $S_\mu(\Gamma_g(N))$ is defined to be $\Gamma(\tilde{X}_g, \mathcal{O}(\tilde{V}_\mu - E))$, where $\mathcal{O}(\tilde{V}_\mu - E)$ means the subsheaf of $\mathcal{O}(\tilde{V}_\mu)$ consisting of the germs of sections of \tilde{V}_μ which vanish along E . If $g \geq 2$, then any subgroup Γ of $Sp(g, \mathbb{Z})$ of finite index contains $\Gamma_g(N)$ for some $N \geq 3$ ([6] and [38]). $S_\mu(\Gamma)$ is defined to be $A_\mu(\Gamma) \cap S_\mu(\Gamma_g(N))$. In case $g=1$, a subgroup Γ of $SL(2, \mathbb{Z})$ of finite index does not contain $\Gamma_1(N)$ in general but always contains a normal subgroup Γ' of finite index which acts fixed points free, so $S_\mu(\Gamma)$ is defined by using Γ' instead of $\Gamma_1(N)$. In case $\mu(CZ+D)=\det(CZ+D)^k$, $S_\mu(\Gamma)$ is also denoted by $S_k(\Gamma)$.

Let $\omega = h(Z) \prod_{i \leq j} dZ_{ij}$ be a holomorphic n -form on \mathfrak{S}_g ($n = \dim \mathfrak{S}_g = g(g+1)/2$). Then ω is invariant under $\Gamma_g(N)$, i.e., $f^*(\omega) = \omega$ for any $f \in \Gamma_g(N)$, if and only if h belongs to $A_{g+1}(\Gamma_g(N))$. So the line bundle $L_g^{\otimes(g+1)}$ on X_g is isomorphic to the canonical line bundle K_{X_g} of X_g . We denote by $\tilde{\omega}$ the n -form on X_g which is induced from ω .

Proposition (1.2). $\tilde{\omega}$ may have a single pole along E , so the sheaf $\mathcal{O}(\tilde{L}_g^{\otimes(g+1)})$ is isomorphic to $\mathcal{O}(K_{\bar{X}_g} + E)$.

Outline of the proof. Let E_1 be an irreducible component of E . Then E_1 consists of the limits of the points in X_g , and E_1 is defined by an equation for example $\text{Im}(Z_{11}) = \infty$. In this case the coordinates of the generic point on E_1 consist of $W_{11} := \exp(2\pi i Z_{11}/N)$ and $(Z_{ij})_{1 < i \leq j}$, and E_1 is defined by $W_{11} = 0$. By these coordinates, $\tilde{\omega}$ is represented as

$$h(Z) NdW_{11}/2\pi i W_{11} \prod_{1 < i \leq j} dZ_{ij},$$

since $dZ_{11} = NdW_{11}/2\pi i W_{11}$. So $\tilde{\omega}$ may have a single pole along E .

The reason to calculate $\dim S_k(\Gamma)$ is that we can apply the theorem of Riemann-Roch-Hirzebruch by the following

Theorem (1.3). For $i > 0$ and $k > g+1$, we have

$$H^i(\tilde{X}_g, \mathcal{O}(\tilde{L}_g^{\otimes k} - E)) = 0.$$

Hence $\dim S_k(\Gamma_g(N)) = \chi(\tilde{X}_g(N), \mathcal{O}(\tilde{L}_g^{\otimes k} - E))$ (the Euler-Poincaré characteristic).

Proof. Let $[E]$ be the line bundle associated with the divisor E . Then the sheaf $\mathcal{O}(\tilde{L}_g^{\otimes k} - E)$ is isomorphic to $\mathcal{O}(\tilde{L}_g^{\otimes k} \otimes [E]^{\otimes (-1)})$, and by the above proposition this is isomorphic to $\mathcal{O}(\tilde{L}_g^{\otimes(k-g-1)} \otimes K_{\tilde{X}_g})$. Since \tilde{L}_g is a pull-back of the ample line bundle \bar{L}_g , our assertion is essentially a consequence of the Kodaira vanishing theorem ([36]).

Let $c_i (1 \leq i \leq n)$ be the i -th Chern class of \tilde{X}_g and let Q_n be the Riemann-Roch polynomial of dimension n . By definition the space of cusp form $S_k(\Gamma_g(N))$ is $\Gamma(\tilde{X}_g, \mathcal{O}(\tilde{L}_g^{\otimes k} - E))$, so if $k \geq g + 2$, then from the above theorem the dimension of this space is equal to

$$Q_n(c_1(\tilde{L}_g^{\otimes k} \otimes [E]^{\otimes (-1)}); c_1, \dots, c_n),$$

where $c_1(L)$ means the first Chern class of L .

So our problem is reduced to the calculation of this polynomial. To execute this calculation we introduce the notion of logarithmic Chern class ([50]).

Definition (1.4). Let X be a compact complex manifold of dimension n and E a reduced divisor on X with simple normal crossings. Then for $p \in E$, one may take a coordinate system (z_1, \dots, z_n) around p such that E is defined by $z_1 \cdots z_l = 0$. Let Θ_X be the sheaf of germs of local holomorphic vector fields on X and let $\Theta_X(\log E)$ denote the (locally free) subsheaf of Θ_X consisting of germs of those local holomorphic vector fields which can be expressed in the form

$$\sum_{i=1}^l f_i(z) z_i \frac{\partial}{\partial z_i} + \sum_{i=l+1}^n f_i(z) \frac{\partial}{\partial z_i} \quad (f_i(z) \text{ holomorphic}).$$

$\Theta_X(\log E)$ is the dual sheaf of $\Omega_X^1(\log E)$ which was defined in [11]. We denote by $T_X(\log E)$ the vector bundle which corresponds to $\Theta_X(\log E)$. Then the i -th logarithmic Chern class \bar{c}_i of X relative to E is defined to be $c_i(T_X(\log E))$.

We denote the total Chern class of X by $c(X) = 1 + c_1 + \dots + c_n$ and the total logarithmic Chern class of X relative to E by $\bar{c}(X, E) = 1 + \bar{c}_1 + \dots + \bar{c}_n$. Let $E = \cup_{i \in I} E_i$ be the decomposition of E into a union of irreducible components and let $\varepsilon_i := c_1([E_i])$ be the dual cohomology class of E_i . Then there is the following relation between Chern classes and

logarithmic Chern classes.

Theorem (1.5) ([50]). *We have*

$$c(X) = \bar{c}(X, E) \cdot \prod_{i \in I} (1 + \varepsilon_i).$$

Now we calculate the dimension of $S_k(\Gamma_2(N))$. In the three dimensional case, the Riemann-Roch polynomial $Q_3(K; c_1, c_2, c_3)$ is equal to

$$K^3/6 + c_1 K^2/4 + (c_1^2 + c_2)K/12 + c_1 c_2/24.$$

So we need to calculate this polynomial replacing K by $c_1(\tilde{L}_2^{\otimes k} \otimes [E]^{\otimes (-1)})$ and c_1, c_2 by the first, second Chern class of \tilde{X}_2 , respectively. By Proposition (1.2), $c_1(\tilde{L}_2^{\otimes k} \otimes [E]^{\otimes (-1)})$ is equal to

$$-k\bar{c}_1/3 - \sum_i \varepsilon_i,$$

and by Theorem (1.5), we have

$$\begin{aligned} c_1 &= \bar{c}_1 + \sum_i \varepsilon_i, \\ c_2 &= \bar{c}_2 + \bar{c}_1(\sum_i \varepsilon_i) + \sum_{i < j} \varepsilon_i \varepsilon_j. \end{aligned}$$

Therefore what we need to calculate is (the value at the fundamental class of) the following polynomial:

$$\begin{aligned} & 2^{-3}3^{-4}(-4k^3\bar{c}_1^3 + 18k^2\bar{c}_1^3 - 18k\bar{c}_1^3 - 18k\bar{c}_1\bar{c}_2 + 27\bar{c}_1\bar{c}_2) \\ (*) \quad & + 2^{-3}3^{-2}(-2k^2\bar{c}_1^2 + 6k\bar{c}_1^2 - 3\bar{c}_1^2 - 3\bar{c}_2)(\sum_i \varepsilon_i) \\ & + 2^{-3}3^{-2}(-2k + 3)\bar{c}_1((\sum_i \varepsilon_i)^2 + \sum_{i < j} \varepsilon_i \varepsilon_j) \\ & - 2^{-3}3^{-1}(\sum_i \varepsilon_i)(\sum_{i < j} \varepsilon_i \varepsilon_j) + (0)(\sum_i \varepsilon_i)^3. \end{aligned}$$

These intersection numbers in this polynomial are calculated by the following five methods.

a) The intersection numbers in the first line are proportional to the volume of the fundamental domain of $\Gamma_2(N)$ and calculated by the Hirzebruch's proportionality principle ([40]).

b) The intersection numbers in the second line vanish.

c) The first term in the fourth line is calculated by using the theory of torus embedding ([35]).

d) An irreducible component E_1 of E has a structure of an elliptic surface over a cusp of degree one in \bar{X}_2 by the restriction of the morphism $\pi_2: \tilde{X}_2 \rightarrow \bar{X}_2$ to E_1 :

$$\pi: E_1 \longrightarrow \bar{X}_1.$$

The intersection numbers in the third line are calculated by using the structure of this fiber space.

e) We need not calculate the second term in the fourth line because its coefficient is 0. But in the original work of T. Yamazaki, he did not use the logarithmic Chern classes and in his method the coefficient of this term did not vanish. So he needed to calculate this term. He calculated this term by using Igusa's theory on theta constants. On the other hand in the case of degree three, the coefficient of the term $(\sum_i \varepsilon_i)^6$ does not vanish. So we need to use his method in the case of degree three in fact.

a) Now we return to the general case of degree g . Let $Z \in \mathfrak{S}_g$ and put $Z = (Z_{ij})$. Then for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \mathbf{R})$ we have

$$(d(M\langle Z \rangle))_{i,j} = {}^t(CZ + D)^{-1}(dZ_{i,j})(CZ + D)^{-1}.$$

So if s_2 is the symmetric tensor representation of $GL(g, \mathbf{C})$ of degree two, then the vector bundle V_{s_2} is isomorphic to the cotangent bundle $T_{X_g}^*$ on X_g . We can prove that the vector bundle \tilde{V}_{s_2} is isomorphic to $T_{X_g}(\log E)^*$ similarly as in the proof of Proposition (1.2). For the values of the products of Chern classes of \tilde{V}_μ , the extended Hirzebruch proportionality principle holds ([40]). In particular we have the following

Theorem (1.6). *Let $\check{\mathcal{D}}$ be the compact dual of \mathfrak{S}_g and let \check{c}_i be the i -th Chern class of $\check{\mathcal{D}}$. Let $\nu_i (1 \leq i \leq n)$ be non-negative integers with $\sum_i i\nu_i = n$. Then we have*

$$(\prod_i \bar{c}_i^{\nu_i})[\tilde{X}_g(N)] = c(\prod_i \check{c}_i^{\nu_i})[\check{\mathcal{D}}],$$

where $[\check{\mathcal{D}}]$ (resp. $[\tilde{X}_g(N)]$) is the fundamental class in $H_{2n}(\check{\mathcal{D}}, \mathbf{Z})$ (resp. $H_{2n}(\tilde{X}_g(N), \mathbf{Z})$), This constant c is rational and expressed as

$$c = (-1)^n \frac{\text{vol}(\Gamma_g(N) \backslash \mathfrak{S}_g)}{\text{vol}(\check{\mathcal{D}})}.$$

We do not present here the definition of the compact dual and the relation between the invariant measures on \mathfrak{S}_g and on $\check{\mathcal{D}}$. It is essential that the intersection number $(\prod_i \bar{c}_i^{\nu_i})[\tilde{X}_g(N)]$ is proportional to the volume of the fundamental domain. Therefore the value of the first line in (*) is expressed by a polynomial in k and the volume. This polynomial is determined uniquely by the symmetric domain. In the case of Siegel upper half plane of degree g , this is equal to

$$\prod_{0 < i \leq j \leq g} (2k + i + j - 2g - 2),$$

especially this is equal to $(2k-2)(2k-3)(2k-4)$ in the case of degree two ([26]). So the value of the first line is expressed as

$$d(2k-2)(2k-3)(2k-4),$$

where d is a constant. Comparing the coefficient of k^3 , we have

$$2^3 d = -2^{-1} 3^{-4} \bar{c}_1^3[\tilde{X}_2(N)].$$

Let $n = \dim \mathfrak{S}_g = g(g+1)/2$ and let X_{ij} (resp. Y_{ij}) be the real (resp. imaginary) part of Z_{ij} . Then the invariant measure vol on \mathfrak{S}_g is defined by $(\det Y)^{-(g+1)} \prod_{i \leq j} dX_{ij} \prod_{i \leq j} dY_{ij}$. By [7] and [25], it is known that

$$\bar{c}_1^n[\tilde{X}_g(N)] = (-1)^n \pi^{-n} (g+1)^n n! 2^{-g(g+3)/2} \text{vol}(\Gamma_g(N) \backslash \mathfrak{S}_g).$$

By [8] and by [49], it is known that

$$\begin{aligned} [\Gamma_g(1): \Gamma_g(N)] &= N^{g(2g+1)} \prod_{p|N} \prod_{1 \leq h \leq g} (1 - p^{-2h}), \\ \text{vol}(\Gamma_2(1) \backslash \mathfrak{S}_2) &= 2^{-1} 3^{-3} 5^{-1} \pi^3. \end{aligned}$$

By using the above results, we derive that

$$d = 2^{-10} 3^{-3} 5^{-1} N^{10} \prod_{p|N} (1 - p^{-2})(1 - p^{-4}).$$

So the first line in (*) is calculated.

b) Let E_1 be an irreducible component of $E = \tilde{X}_g - X_g$. Then the morphism $\pi_g: \tilde{X}_g \rightarrow \bar{X}_g$ maps E_1 to a cusp of degree $g-1$. We assume that the restriction of π_g to E_1 :

$$\pi_g|_{E_1}: E_1 \longrightarrow \bar{X}_{g-1}$$

factors through $\pi_{g-1}: \tilde{X}_{g-1} \rightarrow \bar{X}_{g-1}$. If $g \leq 4$, then Namikawa's smooth compactification satisfies this assumption ([41]). We denote the morphism of E_1 to \tilde{X}_{g-1} by π and the inclusion of E_1 to \tilde{X}_g by i . In case the representation μ is the standard representation of $GL(g, \mathbf{C})$, we denote V_μ and \tilde{V}_μ by V_g and \tilde{V}_g , respectively. Then from the construction of \tilde{V}_{s_2} ($\simeq T_{\tilde{X}_g}(\log E)^*$), it is easily seen that there exist the following short exact sequences of vector bundles:

$$\begin{aligned} 0 \longrightarrow \pi^*(T_{\tilde{X}_{g-1}}(\log E)^*) \longrightarrow i^*(T_{\tilde{X}_g}(\log E)^*) \longrightarrow i^*(\tilde{V}_g) \longrightarrow 0, \\ 0 \longrightarrow \pi^*(\tilde{V}_{g-1}) \longrightarrow i^*(\tilde{V}_g) \longrightarrow \mathcal{C}_{E_1} \longrightarrow 0, \end{aligned}$$

where C_{E_1} means the trivial line bundle with fiber C on E_1 . We denote by $c(V)$ the total Chern class of a vector bundle V .

Proposition (1.7). *From the above exact sequences, we have*

$$i^*(c(T_{\tilde{X}_g}(\log E)^*)) = \pi^*(c(T_{\tilde{X}_{g-1}}(\log E)^*) \cdot c(\tilde{V}_{g-1})).$$

Thus we proved that the pullback by i of the logarithmic Chern classes of \tilde{X}_g relative to E is represented as a pullback by π of Chern classes of vector bundles on \tilde{X}_{g-1} . This property holds for the vector bundle \tilde{V}_μ for general representation μ .

Let us return to the case of degree two. From the above proposition it is clear that the terms in the second line in (*) vanish, since it holds that

$$\bar{c}_1^2 \varepsilon_1[\tilde{X}_2] = (i^*(\bar{c}_1))^2[E_1].$$

$(i^*(\bar{c}_1))^2$ is a pullback by π of a cohomology class on \bar{X}_1 . But \bar{X}_1 is one-dimensional, so this is zero. Similarly $\bar{c}_2 \varepsilon_1[\tilde{X}_2]$ vanishes.

The validity of Hirzebruch's proportionality is one of the reasons to use logarithmic Chern classes instead of usual Chern classes. To hold the above property is another reason. In the case of degree three, the author proved a similar proposition as above by using the structure of a group scheme of $E_1 - W$ over \tilde{X}_2 , where W is the set of points where π is not smooth and used it to prove the vanishing of many intersection numbers. If one uses the usual Chern classes, then the calculation would be much harder because such intersection numbers do not vanish.

c) Now we return to the case of degree two. Let E_1 be an irreducible component of E . The fiber space $\pi: E_1 \rightarrow \bar{X}_1(N)$ has a structure of a compactification of the universal family of elliptic curves with level N . If $p \in X_1(N)$, then $\pi^{-1}(p)$ is an elliptic curve and if q is a cusp, then $\pi^{-1}(q)$ consists of N rational curves l_1, l_2, \dots, l_N such that l_i and l_{i+1} meet at a point. (We put $l_{N+1} = l_1$.) Since $\pi^{-1}(p)$ and $\pi^{-1}(q)$ are algebraically equivalent and do not intersect, there exists the following relation among intersection numbers on E_1 :

$$l_1 \cdot \pi^{-1}(q) = l_1 \cdot \pi^{-1}(p) = 0.$$

Moreover we have

$$l_1 \cdot \pi^{-1}(q) = l_1^2 + l_1 l_2 + l_1 l_N,$$

hence

$$l_1^2 = -2.$$

l_1 is an intersection of E_1 and another irreducible component in E which we denote by E_2 . So we have

$$i^*(\varepsilon_2^2)[E_1] = \varepsilon_1 \varepsilon_2^2 [\tilde{X}_2(N)] = -2.$$

Since $\bar{X}_2(N)$ has $(1/2)N^4 \prod_{p|N} (1-p^{-4})$ cusps of degree one and $\bar{X}_1(N)$ has $(1/2)N^2 \prod_{p|N} (1-p^{-2})$ cusps, there are $(1/12)N^7 \prod_{p|N} (1-p^{-2})(1-p^{-4})$ points in $\tilde{X}_2(N)$ where three irreducible components of E intersect and there are $(1/8)N^7 \prod_{p|N} (1-p^{-2})(1-p^{-4})$ rational curves where two irreducible components of E intersect. Therefore the first term in the fourth line in (*) is calculated as

$$\begin{aligned} (\sum_i \varepsilon_i) (\sum_{i < j} \varepsilon_i \varepsilon_j) &= \sum_{i < j} (\varepsilon_i \varepsilon_j^2 + \varepsilon_i^2 \varepsilon_j) + 3 \sum_{i < j < k} \varepsilon_i \varepsilon_j \varepsilon_k \\ &= (-4/8 + 3/12)N^7 \prod_{p|N} (1-p^{-2})(1-p^{-4}). \end{aligned}$$

$\varepsilon_1 \varepsilon_2^2 [\tilde{X}_2(N)]$ was easily calculated from the structure of E_1 as above. But in the case of degree three, the structure of E_1 is very complicated so we use the theory of torus embedding ([35]) to calculate such intersection numbers (see [50] §2).

d) From Proposition (1.7) and Proposition (1.2), we have

$$i^*(\bar{c}_1) = -3\pi^*(c_1(\bar{L}_1)).$$

Since $\bar{c}_1 \varepsilon_1 \varepsilon_2 [X_2(N)] = -3\pi^*(c_1(\bar{L}_1))[l_1]$ and $l_1 = E_1 \cap E_2$ is mapped to a point by π , this number vanishes. So to calculate the third line in (*), it suffices to calculate $\bar{c}_1 \varepsilon_1^2 [\tilde{X}_2(N)]$.

We have

$$c_1([l_1]) i^*(\varepsilon_1)[E_1] = \varepsilon_2 \varepsilon_1^2 [\tilde{X}_2(N)] = -2.$$

Let q be as before. Then $[q]$ is a line bundle on $\bar{X}_1(N)$ of degree one and we have

$$\pi^*(c_1([q])) i^*(\varepsilon_1)[E_1] = \sum_{i=1}^N c_1([l_i]) i^*(\varepsilon_1)[E_1] = -2N.$$

So the term $\bar{c}_1 \varepsilon_1^2 [\tilde{X}_2(N)] = -3\pi^*(c_1(\bar{L}_1)) i^*(\varepsilon_1)[E_1]$ is equal to $6N \cdot \deg(\bar{L}_1)$. Since $\text{vol}(G_1(1) \backslash \mathcal{S}_1)$ is equal to $\pi/3$, it is similarly proved as before that the degree of \bar{L}_1 is equal to $(1/24)N^8 \prod_{p|N} (1-p^{-2})$. So we have

$$\begin{aligned} \bar{c}_1 (\sum_i \varepsilon_i)^2 [\tilde{X}_2] &= \bar{c}_1 (\sum_i \varepsilon_i^2) [\tilde{X}_2] \\ &= (1/8)N^8 \prod_{p|N} (1-p^{-2})(1-p^{-4}). \end{aligned}$$

Thus we calculated the third line and proved the following

Theorem (1.8). *If $N \geq 3$ and $k \geq 4$, then the dimension of $S_k(\Gamma_2(N))$ is equal to*

$$(2^{-10}3^{-3}5^{-1}(2k-2)(2k-3)(2k-4) - 2^{-6}3^{-2}(2k-3)N^{-2} + 2^{-5}3^{-1}N^{-3})[\Gamma_2(1): \Gamma_2(N)].$$

e) Now we calculate the second term of the fourth line in (*). The following method is due to T. Yamazaki. To calculate this term, it suffices to calculate $\varepsilon_1^3[\tilde{X}_2(N)]$ and to calculate such a self-intersection number we need to replace the cohomology class ε_1 by another cohomology class. We recall the definition of theta constants with half-integral characteristics.

Definition (1.9). Let $m = (m', m'')$ be in Z^{2g} and let m' and m'' be the first and the last g components of m , respectively. For (τ, z) in $\mathfrak{S}_g \times C^g$, we put

$$\theta_m(\tau, z) = \sum_{p \in Z^{2g}} e[(p+m'/2)\tau'(p+m'/2)/2 + (p+m'/2)'(z+m''/2)],$$

where $e[x]$ is defined to be $e^{2\pi i x}$. This series converges absolutely and uniformly on every compact subset of $\mathfrak{S}_g \times C^g$. We put $\theta_m(\tau) = \theta_m(\tau, 0)$ and call this *theta constant of characteristic m* .

$\theta_m(\tau)$ depends only on $m \bmod 2$ and this is identically zero if $m''m' = \text{odd}$. Therefore there are $2^{g-1}(2^g+1)$ theta constants which are not identically zero. $\tau \in \mathfrak{S}_g$ is said to be a *reducible* point if τ is equivalent, with respect to $\Gamma_g(1)$, to a point in the form

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix},$$

where $\tau_1 \in \mathfrak{S}_{g'}$ and $\tau_2 \in \mathfrak{S}_{g-g'}$ ($0 < g' < g$). Then we have the following

Theorem (1.10) ([29] and [18]). *In case $g=2$, there are ten non-zero theta constants. Let χ_{10} be the product of the squares of them. Then $\chi_{10} \in S_{10}(\Gamma_2(1))$ and $p \in \mathfrak{S}_2$ is reducible if and only if $\chi_{10}(p) = 0$.*

Let R be the zeros of χ_{10} in X_2 and \bar{R} its closure in \tilde{X}_2 . From the above theorem we have

$$\tilde{L}_2^{\otimes 10} \simeq [\bar{R}]^{\otimes 2} \otimes [E]^{\otimes N}.$$

So it follows that

$$10c_1(\tilde{L}_2) = 2c_1([\bar{R}]) + Nc_1([E]).$$

We multiply this equation by ε_1^2 and calculate (the values of) both sides. The left hand side is calculated as $10c_1(\tilde{L}_2)\varepsilon_1^2[\tilde{X}_2] = (-10/3)\bar{c}_1\varepsilon_1^2[\tilde{X}_3] = (-5/6)N^4 \prod_{p|N}(1-p^2)$. Let R_1 be an irreducible component of \bar{R} which intersects E_1 . Then R_1 is biholomorphic to $\bar{X}_1 \times \bar{X}_1$ and $E_1 \cap R_1$ has a form: $\{p\} \times \bar{X}_1$ (or $\bar{X}_1 \times \{p\}$), where p is a cusp of \bar{X}_1 . $\{p\} \times \bar{X}_1$ and $\{q\} \times \bar{X}_1$ are algebraically equivalent on R_1 and they do not intersect if $p \neq q$. Therefore we have

$$c_1([R_1])\varepsilon_1^2[\tilde{X}_2] = j^*(\varepsilon_1^2[R_1]) = 0,$$

where $j: R_1 \rightarrow \tilde{X}_2$ is the inclusion. So it follows that

$$c_1([\bar{R}])\varepsilon_1^2[\tilde{X}_2] = 0.$$

Next $c_1([E])\varepsilon_1^2 = \varepsilon_1^3 + \sum_{1 < i} \varepsilon_1^2 \varepsilon_i$. So the value of this is equal to $\varepsilon_1^3[\tilde{X}_2] - N^3 \prod_{p|N}(1-p^{-2})$. Thus we conclude that

$$\varepsilon_1^3[\tilde{X}_2] = (1/6)N^3 \prod_{p|N}(1-p^{-2}).$$

In the above five methods, we have used only the structure of \tilde{X}_2 and did not use special results except in the last method. Since the intersection number $\varepsilon_1^3[\tilde{X}_2]$ is of topological nature, it is desirable to calculate this without using Theorem (1.10). So we present here the following

Problem (1.11). *Calculate the intersection number $\varepsilon_1^3([\tilde{X}_2])$ without using J.-I. Igusa's results on theta constants.*

In the case of degree three, $\dim S_k(\Gamma_3(N))$ with $N \geq 3$ was calculated by the combinations of the above five methods. The results corresponding to Theorem (1.10) is the following

Theorem (1.12) ([31]). *In case $g=3$, there are thirty six non-zero theta constants. Let χ_{18} be the product of them and Σ_{140} the thirty fifth fundamental symmetric polynomial of the eighth powers of them. The χ_{18} and Σ_{140} belong to $S_{18}(\Gamma_3(1))$ and $S_{140}(\Gamma_3(1))$, respectively and $p \in \mathfrak{S}_3$ is reducible if and only if $\chi_{18}(p) = \Sigma_{140}(p) = 0$.*

By using this theorem we can replace the cohomology class $c_1([E])^2$ by another cohomology class and calculate such a self-intersection number as $c_1([E])^2[\tilde{X}_3]$. And the result is as follows.

Theorem (1.13). *If $N \geq 3$ and $k \geq 5$, then the dimension of $S_k(\Gamma_3(N))$ is equal to*

$$(2^{-16}3^{-6}5^{-2}7^{-1}(2k-2)(2k-3)(2k-4)^2(2k-5)(2k-6) - 2^{-10}3^{-2}5^{-1}(2k-6)N^{-5} + 2^{-8}3^{-3}N^{-6})[\Gamma_3(1): \Gamma_3(N)].$$

§ 2. Dimension formula for $\Gamma_2(1)$ and $\Gamma_3(1)$

Since $\Gamma_g(1)$ has fixed points on \mathfrak{S}_g , $\Gamma_g(1) \backslash \mathfrak{S}_g$ has singularities. So we need to use the holomorphic Lefschetz fixed points formula to calculate the dimension formula for $\Gamma_g(1)$. First we recall the holomorphic Lefschetz formula. Let X be a compact complex manifold and V a holomorphic vector bundle on X , and let H be a finite group of automorphisms of the pair (X, V) . For any $h \in H$, let X^h be the fixed points set of h and let

$$N^h = \sum_{\theta} N^h(\theta)$$

denote the normal bundle of X^h decomposed according to the eigenvalues $e^{i\theta}$ of h . We put

$$\mathcal{Q}^{\theta}(N^h(\theta)) = \prod_{\beta} \left(\frac{1 - e^{-x_{\beta} - i\theta}}{1 - e^{-i\theta}} \right)^{-1},$$

where the (total) Chern class of $N^h(\theta)$ is

$$c(N^h(\theta)) = \prod_{\beta} (1 + x_{\beta}).$$

Let $\mathcal{T}(X^h)$ be the Todd class of X^h and $\text{ch}(V|X^h)(h)$ the Chern character of $V|X^h$ with h -action ([4]). Put

$$\mu(h) = \left(\frac{\text{ch}(V|X^h)(h) \cdot \prod_{\theta} \mathcal{Q}^{\theta}(N^h(\theta)) \cdot \mathcal{T}(X^h)}{\det(1-h|(N^h)^*)} \right) [X^h].$$

Then we have

Theorem (2.1) ([4]).

$$\sum_{p \geq 0} (-1)^p \text{Trace}(h|H^p(X, \mathcal{O}(V))) = \mu(h).$$

Let $H^p(X, \mathcal{O}(V))^H$ be the H -invariant subspace of $H^p(X, \mathcal{O}(V))$. Then by this theorem we have

$$\sum_{p \geq 0} (-1)^p \dim H^p(X, \mathcal{O}(V))^H = \frac{1}{|H|} \sum_{h \in H} \mu(h).$$

Let $N \geq 3$. The group $H := \Gamma_g(1)/\Gamma_g(N)$ acts on the pair $(\tilde{X}_g(N), \tilde{\mathcal{L}}_g^{\otimes k})$, so it acts on $\Gamma(\tilde{X}_g(N), \mathcal{O}(\tilde{\mathcal{L}}_g^{\otimes k}(-E)))$. $S_k(\Gamma_g(1))$ is identified with the

invariant subspace $\Gamma(\tilde{X}_g(N), \mathcal{O}(\tilde{L}_g^{\otimes k} - (E)))^H$. Therefore $\dim S_k(\Gamma_g(1))$ is calculated as

$$\frac{1}{|H|} \sum_{h \in H} \mu(h),$$

under the assumption that the vanishing theorem holds.

Now we consider the case of degree two. To execute this calculation, we need to classify the fixed points sets of H in \tilde{X}_2 in the sense we defined in the Introduction. The fixed points sets of $\Gamma_2(1)$ on \mathfrak{S}_2 were classified by E. Gottschling in [16] and [17]. By this result the fixed points sets of $H := \Gamma_2(1)/\Gamma_2(N)$ in the quotient space X_2 are classified. Let E_1 be an irreducible component of $E = \tilde{X}_2 - X_2$. Then E_1 has a structure of an elliptic surface over X_1 . The action of an element h of H which maps E_1 to itself is decomposed into an action of horizontal direction and an action of vertical direction. If h fixes a point in E_1 , then the action of horizontal direction of h fixes a point in \tilde{X}_1 . The fixed points in \tilde{X}_1 come from the fixed points of $\Gamma_1(1)$ on \mathfrak{S}_1 which are well known. Since the action of vertical direction of h is decomposed into a translation and the involution with respect to the origin of the elliptic curve, its fixed points are easily classified. Thus we can classify the fixed points over cusps of degree one. Let p be a cusp of degree zero in \tilde{X}_2 and $\pi_2: \tilde{X}_2 \rightarrow \tilde{X}_2$ as before. Then $\pi_2^{-1}(p)$ is a reducible rational variety composed of $(1/4)N^3 \prod_{p|N} (1-p^{-2})$ projective lines meeting three at each one of the $(1/6)N^3 \prod_{p|N} (1-p^{-2})$ points ([32]) and fixed points in $\pi_2^{-1}(p)$ are easily classified.

What we need to do next is to describe exactly the structure of the fixed points sets and to calculate the intersection numbers which appear in the holomorphic Lefschetz fixed points theorem. We refer the reader to the original paper by the author [51].

Now we consider the case of degree three. First we recall Selberg's trace formula. For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \mathbf{R})$ and $Z \in \mathfrak{S}_g$, we put $J(M, Z) = \det(CZ + D)$ and $Z_{i,j} = X_{i,j} + \sqrt{-1}Y_{i,j}$. Then we have

Theorem (2.2) ([8] Exposé 10). *Let Γ be a subgroup of $Sp(g, \mathbf{Z})$ of finite index. If $k > 2g$, then*

$$\dim S_k(\Gamma) = q \int_{\Gamma \backslash \mathfrak{S}_g} \sum_{M \in \Gamma} J(M, Z)^{-k} \det \left(\frac{M\langle Z \rangle - \bar{Z}}{2\sqrt{-1}} \right)^{-k} (\det Y)^k dZ,$$

where q is a polynomial in k of degree $g(g+1)/2$ and dZ is the invariant measure defined by $(\det Y)^{-(g+1)} \prod_{i \leq j} dX_{i,j} dY_{i,j}$.

If C is a conjugacy class or subset of Γ , then the following integral

$$q \int_{\Gamma \backslash \mathfrak{g}} \sum_{M \in C} J(M, Z)^{-k} \det \left(\frac{M \langle Z \rangle - \bar{Z}}{2\sqrt{-1}} \right)^{-k} (\det Y)^k dZ,$$

is called the contribution of C to $\dim S_k(\Gamma)$.

In the case of degree three, unsolved problems remain on both sides (geometric and Selberg, or Tsushima and Hashimoto) to calculate $\dim S_k(\Gamma_3(1))$. On the side of the geometric method, we need to classify the fixed points sets of H on \tilde{X}_3 . The fixed points sets in the boundary: $\tilde{X}_3 - X_3$ are easily classified as before. To classify the fixed points in the quotient space X_3 , we need to classify the fixed points sets of $\Gamma_3(1)$ on \mathfrak{S}_3 . This problem was solved by K. Hashimoto. In [20], he introduced a principle of Milnor and Springer-Steinberg, by which the classification of elliptic conjugacy classes is reduced to a problem of Hermitian forms. By this method, the elliptic conjugacy classes of $\Gamma_3(1)$ were enumerated completely, and their contributions have been determined explicitly ([23]).

On the side of Selberg's trace formula, unsolved problem is to calculate special values of zeta-functions or L -functions of various kinds. These special values appear as the contributions of parabolic of elliptic/parabolic conjugacy classes of $\Gamma_3(1)$ (see [19]). The conjugacy classes of $H = \Gamma_3(1)/\Gamma_3(N)$ corresponding to these conjugacy classes of $\Gamma_3(1)$ fix points in $E = \tilde{X}_3 - X_3$. As to the contributions of these conjugacy classes, the geometric method is more effective than Selberg's trace formula and these contributions are calculated by the geometric method. So if we use the geometric method and the Selberg's trace formula jointly, there would be no problem unsolved. This was the starting point of the joint work of K. Hashimoto and the author.

Recently T. Arakawa [2] established the method to calculate these special values of zeta-functions except the value at the origin of Shintani's zeta-function of degree three which appears as the contribution of the conjugacy classes of $\begin{pmatrix} \mathbf{1}_3 & S \\ 0 & \mathbf{1}_3 \end{pmatrix}$ with rank $S=3$. So the difficulties on the side of Selberg's trace formula were solved for the most part. But to determine the value of Shintani's zeta-function, we need to use the result of the geometric method.

We recall the definition of Shintani's zeta-function.

Definition (2.3) ([47]). We put

$$\zeta_r^*(s) = \sum \varepsilon(T)^{-1} (\det T)^{-s},$$

where the summation is taken over all the $SL(r, \mathbf{Z})$ -equivalence classes of

positive definite half-integral symmetric matrices and

$$\varepsilon(T) = \#\{S \in SL(r, \mathbf{Z}) \mid {}^tSTS = T\}.$$

It is known that the summation in the above definition converges if $\operatorname{Re}(s) > (r+1)/2$. Furthermore $\zeta_r^*(s)$ has an analytic continuation to the whole complex plane with possible simple poles at $s = j/2$ ($j = 1, \dots, r+1$). Let Π_r be the subset of $\Gamma_g(N)$ consisting of elements which are conjugate, in $\Gamma_g(1)$, to matrices of the form $\begin{pmatrix} \mathbf{1}_g & S \\ 0 & \mathbf{1}_g \end{pmatrix}$, where S is an integral symmetric matrix of size g and of rank r . The importance of Shintani's zeta-function is based on the following theorem and conjecture.

Theorem (2.4) ([47]). *Let $\zeta(s)$ be the Riemannian zeta-function and put*

$$\begin{aligned} P_{g, g'}(k) &= \prod_{i=1}^{g'} (2k - g - i)(2k - g - i + 2) \cdots (2k - g + i - 2), \\ \omega_r &= \zeta(2)\zeta(4) \cdots \zeta(2r) \quad (\omega_0 = 1), \\ u_r &= \prod_{i=1}^r 2\pi^i / i! \quad (u_0 = 1). \end{aligned}$$

Then the contribution $I_g(\Pi_r, N, k)$ of Π_r to $\dim S_k(\Gamma_g(N))$ ($N \geq 3$) is equal to

$$[\Gamma_g(1) : \Gamma_g(N)] N^{-r(2g-r+1)/2} \frac{2^{r(n-r)-1} \omega_{n-r} \zeta_r^*(-g+r)}{u_{n-r} (4\pi)^{(n-r)(n-r+1)/2}} P_{g, g-r}(k).$$

Conjecture (2.5). If $N \geq 3$ and $k > 2g$, then

$$\dim S_k(\Gamma_g(N)) = \sum_{r=0}^g I_g(\Pi_r, N, k).$$

This means that the conjugacy classes of $\Gamma_g(N)$ ($N \geq 3$) other than in Π_r ($0 \leq r \leq g$) have no contributions to $\dim S_k(\Gamma_g(N))$. This conjecture is true if $g=1$ or 2 , and in case $g=3$, it was proved by P. Ploch, K. Hashimoto (and M. Eie), independently. $\zeta_1^*(s)$ is the Riemannian zeta-function $\zeta(s)$, so its special values at negative integers are wellknown. The special values of $\zeta_2^*(s)$ at negative integers were calculated by T. Shintani and others.

The special values of $\zeta_3^*(s)$ at the origin and negative integers have not been calculated. So we determine the value of $\zeta_3^*(0)$ by Theorem (1.13), Theorem (2.4), and Conjecture (2.5). Since Conjecture (2.5) was verified in case $g=3$, we have the following

Theorem (2.6). *We have*

$$\zeta_3^*(0) = 2^{-7} 3^{-8}.$$

In [12] M. Eie claims to have “calculated” all the contributions of the conjugacy classes of $\Gamma_3(2)$ to $\dim S_k(\Gamma_3(2))$ except that of $\Pi_3 \cap \Gamma_3(2)$. By using the fact that the contribution of $\Pi_3 \cap \Gamma_3(2)$ to $\dim S_k(\Gamma_3(2))$ is $2^{-6}[\Gamma_3(1): \Gamma_3(2)]\zeta_3^*(0)$ and $\dim S_k(\Gamma_3(2))$ is an integer, he proved that

$$\zeta_3^*(0) = 2^{-7}3^{-3} + 2^{-3}3^{-4}5^{-17}l,$$

where l is an integer. So Theorem (2.6) is proved modulo integers by a different method. He also used Theorem (1.13) to determine the value of $\zeta_3^*(0)$ exactly and used it to calculate $\dim S_k(\Gamma_3(2))$.

Thus we solved all the problems to calculate $\dim S_k(\Gamma_3(1))$. No essential problem remains unsolved. But if the calculation is too much, too-muchness may be an essential problem.

In the end we shall make a brief comment on M. Eie and C.-Y. Lin’s paper [15] published in American Journal of Mathematics in 1986 and M. Eie’s [13] in Memoirs AMS in 1987. In the first paper they claim that they calculated $\dim S_k(\Gamma_3(1))$ (Main Theorem I), $\dim S_k(\Gamma_3(2))$ (Main Theorem II), and $\dim S_k(\Gamma_3(N))$, $N \geq 3$ (Main Theorem III). But we can admit none of them because of its incompleteness or lack of originality.

First concerning Theorem I, in the Acknowledgement in [15], they assert that they obtained their results independently of S. Tsuyumine. It is, however, hard to believe as we show below.

As mentioned in Introduction, our problem is to calculate the coefficients of the partial fractions of the generating function of $A(\Gamma_3(1))$. There are two different standpoints to solve this problem. One is to calculate all the contributions of the conjugacy classes of $\Gamma_3(1)$ and this standpoint is the same as ours. The other is, as written in [15] p. 1064, to calculate only contributions of conjugacy classes which are easy to calculate and determine the coefficients of some part of partial fractions, and to determine the coefficients of the remaining partial fractions by the knowledge of modular forms of lower weights instead of the calculation of the contributions of difficult conjugacy classes.

Their “proof” of Main Theorem I in [15] is not complete from either point of view. From the first standpoint they refer in [15] Proposition 10 to their papers [12], [14] and a paper titled “Conjugacy classes of the modular group $Sp(3, \mathbf{Z})$, Manuscript (1984)” instead of detailed calculations. Since the last paper has not been published, we cannot say anything on it. It might be substituted by [13], on which we comment later. On the other hand their papers [12], [14] are not enough complete to be able to serve as references. First [12] contains many elementary errors. For example in Chapter III Lemma 2, the case when $P(X) = (X^2 + X + 1)(X^4 - X^2 + 1)$ is missing. But we shall stop listing up all the errors since this

was written several years ago.

Now we comment on [14] which was published in Transactions of the American Mathematical Society in 1985. In this paper they “classified” the conjugacy classes of elliptic elements of $Sp(3, \mathbf{Z})$ which fix a single point in \mathfrak{S}_3 . But their result is wrong, since some conjugacy classes are missing. We give an example due to K. Hashimoto. Put

$$M = \begin{pmatrix} 0 & -1 & 1 & -2 & -1 & 0 \\ -1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 1 & 1 & -1 \\ 2 & -1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 0 & -1 \end{pmatrix}$$

and

$${}^tP_{=(12)^{-1}} = \begin{pmatrix} -2\sqrt{6} & -3\sqrt{6}-3\sqrt{2} & 3\sqrt{6}-3\sqrt{2} & 2\sqrt{6} & \sqrt{6}+3\sqrt{2} & -\sqrt{6}+3\sqrt{2} \\ 2\sqrt{6} & 3\sqrt{6}-3\sqrt{2} & -3\sqrt{6}-3\sqrt{2} & -2\sqrt{6} & -\sqrt{6}+3\sqrt{2} & \sqrt{6}+3\sqrt{2} \\ 8\sqrt{3} & 0 & 0 & 4\sqrt{3} & -4\sqrt{3} & 4\sqrt{3} \\ 2\sqrt{6} & -\sqrt{6}-3\sqrt{2} & \sqrt{6}-3\sqrt{2} & 2\sqrt{6} & -3\sqrt{6}-3\sqrt{2} & 3\sqrt{6}-3\sqrt{2} \\ -2\sqrt{6} & \sqrt{6}-3\sqrt{2} & -\sqrt{6}-3\sqrt{2} & -2\sqrt{6} & 3\sqrt{6}-3\sqrt{2} & -3\sqrt{6}-3\sqrt{2} \\ -4\sqrt{3} & -4\sqrt{3} & 4\sqrt{3} & 8\sqrt{3} & 0 & 0 \end{pmatrix}.$$

Then $M \in Sp(3, \mathbf{Z})$ and $P \in Sp(3, \mathbf{R})$. The characteristic polynomial of M is $(X^2+1)(X^4-X^2+1)$. And we have

$$P^{-1}MP_{=(2)^{-1}} = \begin{pmatrix} \sqrt{3} & 0 & 0 & 1 & 0 & 0 \\ 0 & -\sqrt{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ -1 & 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & -1 & 0 & 0 & -\sqrt{3} & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \end{pmatrix}.$$

So M is conjugate to $e[-1/6, 5/6, 1/2]$ (in the sense of [14]) in $Sp(3, \mathbf{R})$ and its conjugacy class is missing in [14], since elliptic elements which are conjugate to $e[1/6, 5/6, 1/2]$ or $e[7/6, 11/6, 3/2]$ do not appear in the Table II in [14].

Therefore their paper cannot be complete from the first standpoint since their paper uses a wrong result.

If they wish their paper to be complete from the second standpoint, they need to make clear the references of *their knowledge of modular forms of lower weights*. As far as we know the dimension of $S_k(\Gamma_3(1))$ with $k \geq 12$ was known by Tsuyumine's work for the first time. ($\dim S_k(\Gamma_3(1))$ with

$0 \leq k \leq 10$ was known to be 0 in [31].) In [15] they claim that they have obtained the same result for $\dim S_k(\Gamma_3(1))$ as Tsuyumine's, by using the knowledge of $\dim S_k(\Gamma_3(1))$ with $10 \leq k \leq 44$, which determines the coefficients of eighteen partial fractions that are difficult to calculate, without giving any reference. We do not know, however, any literature on it except for Tsuyumine's paper.

Now we consider their Main Theorems II and III, calculating $\dim S_k(\Gamma_3(2))$ and $\dim S_k(\Gamma_3(N))$, respectively. In both cases the essential point of the proof is to determine the exact value of $\zeta_3^*(0)$, for which they use the author's Theorem (1.13) without giving any reference. Theorem III is then even a tautology, while Theorem II could be said their new "result" aside from gaps in the proof.

Next we give some comments on M. Eie's recent paper [13].

In the second chapter he "classified" all the conjugacy classes of $Sp(3, \mathbf{Z})$ which have non-zero contributions to dimension formula and in the third chapter he "gave" their contributions. They are very detailed results and may give an impression of a completed and elaborate work. But he still uses the wrong result [14] and there exists another conjugacy class which is missing. In Lemma 1 in the second chapter, he states (without proof) that a conjugacy class which has non-zero contribution fixes a single point on \mathfrak{S}_3 or has a representative in Γ_3^2 , Γ_3^1 or Γ_3^0 , where Γ_3^i is the stabilizer in $Sp(3, \mathbf{Z})$ of a rational boundary component of degree i . But this is wrong. In fact, put

$$N = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then N is a torsion element of $Sp(3, \mathbf{Z})$. The fixed points set of N is a one-dimensional submanifold F of \mathfrak{S}_3 and N has non-zero contribution to the dimension formula. Of course, F is biholomorphic to \mathfrak{S}_1 . The subgroup Δ of $\Gamma_3(1)$ consisting of elements which map F into itself is isomorphic to a discrete subgroup Δ' of $SL(2, \mathbf{R})$. We can prove that Δ' is commensurable with the unit group of a maximal order of an indefinite division quaternion algebra over \mathbf{Q} which is generated by

$$\begin{pmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}.$$

Therefore $\Delta \backslash F$ is compact. This means that F does not intersect rational

boundary components of \mathfrak{S}_g , so N does not fix rational boundary components.

Therefore it is impossible that they can reach the same result as Tsuyumine's without adding the contributions for these missing conjugacy classes.

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