Advanced Studies in Pure Mathematics 15, 1989
Automorphic Forms and Geometry of Arithmetic Varieties
pp. 29–40

Cusps on Hilbert Modular Varieties and Values of L-Functions

Robert Sczech

§ 1.

Let s be a cusp, and $D=\sum S_\tau$ the corresponding cusp divisor on a Hilbert modular variety X. Every such a cusp belongs to a pair (M, V) where M is a lattice (isomorphic to $\boldsymbol{Z}^n$), and V a group of units (isomorphic to $\boldsymbol{Z}^{n-1}$) in a totally real number field F of degree n over $\boldsymbol{Q}$, subject to the restriction that all elements in V are totally positive, and that V acts on M by multiplication, $VM=M$. However, the cusp divisor D is not unique for a given pair (M, V).

The divisor D is a normal crossing divisor, i.e. the irreducible components S_τ (hypersurfaces on X) intersect only in simple normal crossings. The complicated intersection behavior of the S_τ can be described in terms of a triangulation of the $(n-1)$-torus $\boldsymbol{R}^{n-1}/V$. Every hypersurface S_τ corresponds to a vertex τ of this triangulation, and k different hypersurfaces S_{τ_j} $(1\leq j\leq k)$ intersect either in a $(n-k)$-dimensional submanifold S_σ, or the intersection set is empty. In the first case, σ is the unique simplex of the triangulation having the τ_j as vertices.

This description of the cusp divisor D was given for the first time by Hirzebruch [4] in the case of a real quadratic field F $(n=2)$. He showed in particular that the corresponding triangulation of the torus $S^1=\boldsymbol{R}/V$ is given by the continued fraction expansion of a quadratic irrationality associated with M. In the same paper, Hirzebruch defined a rational number $\varphi(s)=\varphi(M, V)$ called the signature defect of s, in the following way: let Y be a small closed neighbourhood in X of the cusp s. Then Y is a manifold with boundary ∂Y which is a $T^n=\boldsymbol{R}^n/M$ bundle over the torus $T^{n-1}=\boldsymbol{R}^{n-1}/V$ completely determined by the pair (M, V). Let $L(Y)$ be the L-polynomial in the relative Chern classes of Y, and sign (Y) the signature of Y. From the signature theorem [1], it follows that

$$\varphi(M, V) := L(Y) - \operatorname{sign}(Y)$$

Received March 28, 1987.
Received April 19, 1988.

depends only on the boundary ∂Y of Y, but not on Y itself (which depends on the particular divisor D). This number is called the signature defect of s because in the hypothetical case $\partial Y=\phi$ we would have $\varphi(M, V)=0$. Although easy to define, this number is not easy to compute. But in the case $n=2$, using the explicit description of D, Hirzebruch found the beautiful formula

$$\varphi(M, V)=-\frac{1}{3}\sum_{k}(b_k-3), \tag{1}$$

where the b_k are the integers arising from the continued fraction expansion of a quadratic irrationality associated with M (the number of terms depends on V).

On the other hand, the pair (M, V) determines a Hecke L-function

$$L(M, V, s)=\sum_{m\in M/V}{}' \frac{\operatorname{sign} N(m)}{|N(m)|^s}, \qquad \operatorname{Re}(s)>1.$$

Here $Y(m)$ is the norm of m, and $m\neq 0$ runs through a set of representatives for the equivalence classes of M modulo V. The special value $L(M, V, 1)$ is well defined because $L(M, V, s)$ has an analytic continuation to the whole complex s-plane. In the quadratic case $n=2$, the calculation of this number was begun by Hecke and completed by C. Meyer and Siegel. Using their results, Hirzebruch proved the relation

$$\varphi(M, V)=\frac{d(M)}{(\pi i)^n}L(M, V, 1) \tag{2}$$

for $n=2$, and then conjectured it for all $n>2$. Here $d(M)$ denotes the volume of a fundamental domain for the action of M on $\boldsymbol{R}^n$. The attraction of this conjecture stems from the fact that the left-hand side reflects only pure topological properties of the cusp whereas the right-hand side is a pure arithmetic quantity. However, at the time of Hirzebruch's paper, it was not even known that the right hand side is always a rational number.

Today, this conjecture is a theorem of Atiyah, Donnelly and Singer who gave a proof in the important recent paper [2]. But in comparison to Hirzebruch's original proof for $n=2$, their proof is not constructive in the sense that it does not answer the question how to calculate both sides of (2) independently. This question makes sense since Ehlers [3] generalized Hirzebruch's construction of the cusp divisor D to all totally real number fields which leads to a closed expression for $\varphi(M, V)$ in terms of the divisor D, and Shintani [8] established a closed formula for $L(M, V, 1)$

which makes it evident that the right hand side of (2) is a rational number. Indeed, a key element in Shintani's formula is also a triangulation of the torus $\boldsymbol{R}^{n-1}/V$. But unfortunately, a proof of (2) along these lines is not easy to accomplish, mostly because of an unexpected relation between $\varphi(M, V)$ and $\varphi(M^*, V)$ which we explain in section 5.

In this paper, we give an explicit formula for $\varphi(M, V)$ in terms of the triangulation of $\boldsymbol{R}^{n-1}/V$ generalizing Hirzebruch's formula (1) in the case $n=2$. In addition, we discuss a new idea for calculating $L(M, V, 1)$ which would lead to the same closed formula for $L(M, V, 1)$ as for $\varphi(M, V)$. The idea is essentially to identify (the conditionally convergent series) $L(M, V, 1)$ as a partial fraction decomposition of $\varphi(M, V)$ using Euler's formula

$$\pi \cot \pi x = \sum_{m \in \boldsymbol{Z}+x}{}' \frac{1}{m},$$

which is in some sense the case $n=1$ of Hirzebruch's conjecture. If successful, this idea would lead to an elementary proof of Hirzebruch's conjecture. However, at the time of this report, not all of the technical difficulties are solved which arise in the case $n>2$.

§ 2.

We explain our notation. Let $C(k)$ be the set of all $(k-1)$-simplices σ $(1 \leq k \leq n)$ in the triangulation of $\boldsymbol{R}^{n-1}/V$, and denote by C the set of all simplices. In this paper, the letter τ will be used exclusively to denote an element of $C(1)$, and the letter t will always stand for an element in $C(n)$. For example, $\tau \in t$ means τ is a vertex of t. The complex C associated with the divisor D has a second characteristic property. For every $\sigma \in C(k)$, there are k linearly independent lattice points $A_\tau \in M$ (determined only up to a unit in V) such that

$$M \cap \sum_{\tau \in \sigma} \boldsymbol{R} A_\tau = \sum_{\tau \in \sigma} \boldsymbol{Z} A_\tau.$$

(In writing this equation, we assume that M is embedded in $\boldsymbol{R}^n$ through the n different embeddings $F \hookrightarrow \boldsymbol{R}$.) In particular, for every $t \in C(n)$ there is a distinguished $\boldsymbol{Z}$-basis $\{A_\tau, \tau \in t\}$ for M,

$$M = \sum_{\tau \in t} \boldsymbol{Z} A_\tau.$$

Let $\{B_\tau^t\}$ be the dual basis of $M^* = \sum_{\tau \in t} \boldsymbol{Z} B_\tau^t$ given by

$$\text{(3)} \qquad \operatorname{tr}(A_{\tau'} B_\tau^t) = \begin{cases} 1, & \text{if } \tau = \tau' \in t \\ 0, & \text{if } \tau, \tau' \in t, \ \tau \neq \tau'. \end{cases}$$

The compact submanifolds S_σ define homology classes in $H_{2n-2k}(Y, \boldsymbol{Z})$ if $\sigma \in C(k)$. We want to compute the intersection product ΠS_σ of these homology classes in the homology ring $H_*(Y, \boldsymbol{Z})$. From the construction of the divisor D, c.f. [3], it is clear that

$$\prod_{\tau \in t} S_\tau = 1$$

because the n different hypersurfaces corresponding to the n vertices of $t \in C(n)$ intersect transversally in one point. More generally, we have

$$\prod_{\tau \in \sigma} S_\tau = S_\sigma.$$

To allow multiplicities, we consider a partition p of n,

$$n = \sum_{\tau \in \sigma} p_\tau, \qquad p_\tau \geq 1,$$

where the indices of p run over the vertices of σ. Then the intersection numbers

$$S_\sigma^p := \prod_{\tau \in \sigma} S_\tau^{p_\tau} \in H_0(Y, \boldsymbol{Z}) = \boldsymbol{Z}$$

are well defined integers. Provided that all hypersurfaces S_τ do not have any self-intersections, we can calculate these intersection numbers using the following

Theorem 1. $$S_\sigma^p = \sum_{t \in St(\sigma)} \Big(\prod_{\tau \in t} B_\tau^t\Big)^{-1} \prod_{\tau \in \sigma} (B_\tau^t)^{p_\tau}.$$

Here t runs over all $(n-1)$-simplices in $\mathrm{St}(\sigma) = \{\sigma' \in C \mid \sigma \subseteq \sigma'\}$, the star of σ, and $B_\tau^t \in F \subseteq \boldsymbol{R}$ denotes the algebraic number defined by (3). The stated formula is a special case of the more general formula

$$S_\sigma^p = \sum_{t \in \mathrm{St}(\sigma)} \Big(\prod_{\tau \in t} \mathrm{tr}(xB_\tau^t)\Big)^{-1} \prod_{\tau \in \sigma} \mathrm{tr}(xB_\tau^t)^{p_\tau} \tag{4}$$

valid for all $x \in F$ as long as none of the denominators vanishes. After embedding F in $\boldsymbol{R}^n$, the formula extends (by continuity) to all $x \in \boldsymbol{R}^n$ outside the finite set of hyperplanes defined by $\mathrm{tr}(xB_\tau^t) = 0$, $\tau \in t$, $t \in \mathrm{St}(\sigma)$. In particular, taking a unit vector for x, we get the special case stated in Theorem 1.

All these formulas can be proved by using induction over k if $\sigma \in C(k)$, starting with $k = n$ which is easy since $\mathrm{St}(\sigma) = \{\sigma\}$ and all $p_\tau = 1$ in this case. For the induction step $k \to k-1$, we use the fact that the intersection number $d \circ S$ of

$$S = \prod_{\tau \in \sigma} S_\tau^{q_\tau}, \quad \sigma \in C(k-1), \qquad \sum_{\tau \in \sigma} q_\tau = n-1, \quad q_\tau \geq 1,$$

with the divisor d of a meromorphic function on Y is zero. In particular, for

$$d=\sum_{\tau\in C(1)} c_\tau S_\tau, \qquad c_\tau \in \mathbf{Z}$$

we get

$$\left(\sum\nolimits_1+\sum\nolimits_2\right)c_\tau S_\tau\circ S=0$$

where $\sum_1$, runs over all $\tau\in\sigma$, and $\sum_2$ runs over all $\tau\notin\sigma$. By induction hypothesis, we know all the intersection numbers in $\sum_2$. Varying the divisor d, we get in this way a system of linear equations for the intersection numbers $S_\tau\circ S$, $\tau\in\sigma$, which has a unique solution given by (4).

The formula stated in Theorem 1 looks complicated, but it has the following nice property: assume n is even, and let $\sigma, \sigma'\in C(n/2)$. Defining $B_\tau^t=0$ for τ not contained in t, we can write the intersection number $S_\sigma\circ S_{\sigma'}$ as

$$S_\sigma\circ S_{\sigma'}=\sum_t\Big(\prod_{\tau\in\sigma} B_\tau^t\Big)\Big(\prod_{\tau\in t} B_\tau^t\Big)^{-1}\Big(\prod_{\tau\in\sigma} B_\tau^t\Big),$$

where now t runs over all simplices in $C(n)$. To put this formula into the right perspective, write

$$x_{\sigma,t}:=\prod_{\tau\in\sigma} B_\tau^t$$

for $\sigma\in C$, $t\in C(n)$, and consider the matrices

$$X=(x_{\sigma,t}),\quad T=(x_{t,t'}),\quad t,\ t'\in C(n).$$

In particular, T is a diagonal matrix whose signature sign (T) equals

$$\operatorname{sign}(T)=\sum_t \operatorname{sign}\Big(\prod_{\tau\in t} B_\tau^t\Big).$$

The formula for the intersection number $S_\sigma\circ S_{\sigma'}$ given above means that the intersection matrix $I=(S_\sigma\circ S_{\sigma'})$, $\sigma, \sigma'\in C(n/2)$, can be factorized as

$$I=(S_\sigma\circ S_{\sigma'})=XT^{-1}X'.$$

If we would know that the homology classes $[S_\sigma]$, $\sigma\in C(n/2)$, generate the middle homology group $H_n(Y,\mathbf{Z})$, and in addition, that the signatures of I and T are equal, then by the definition of the signature of Y we could conclude that

Conjecture. (5) $$\operatorname{sign}(Y)=\sum_t \operatorname{sign}\Big(\prod_{\tau\in t} B_\tau^t\Big).$$

Unfortunately, at this point we do not know whether the last two assumptions are true, so the last equation is only a conjecture (which is true for $n=2$). The next result was proved in [6]. Modulo the conjecture (5), it solves the problem (posed by Hirzebruch in [4]) to find an arithmetic expression for $\varphi(M, V)$.

Theorem 2. $$\varphi(M, V) = -\operatorname{sign}(Y) + 2^n \sum_{\sigma \in C} \sum_{p_\tau > 1} S_\sigma^p \prod_{\tau \in \sigma} \frac{B_{p_\tau}}{p_\tau !}.$$

For given σ, p runs here over all partitions of n with length $=\#$ of vertices in σ,

$$n = \sum_{\tau \in \sigma} p_\tau, \qquad p_\tau > 1.$$

The condition $p_\tau > 1$ implies that only the $\sigma \in C(k)$ with $1 \leq k \leq n/2$ contribute to the sum. The numbers B_{p_τ} are the classical Bernoulli numbers

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \cdots$$

§ 3.

In this section we want to discuss two further geometric invariants of the cusp (M, V) which are closely related to the signature defect $\varphi(M, V)$. The first one, $\varphi(M, V, x)$, is the equivariant version of $\varphi(M, V)$ arising from the equivariant signature theorem [1, p. 589] (it is usually called α-invariant, although in [1] it is called σ-invariant). The second one, $\psi(M, V, x)$, could be called equivariant Todd genus because it comes from the holomorphic Lefschetz theorem [1, p. 566]. To define these numbers, let $x \in F$ be a solution of the congruence $V(M+x) = M+x$. (For a given group of units V there are only finitely many different residue classes $M+x$ which will satisfy this equation. Conversely, for every x in F there is some power $V^k = \{v^k / v \in V\}$ such that $V^k(M+x) = M+x$.) Then x induces an action on Y as follows: Recall that the boundary ∂Y is a $\boldsymbol{R}^n/M$-bundle over the base space $\boldsymbol{R}^{n-1}/V$. On this boundary x acts by translation in the fiber $\boldsymbol{R}^n/M$, $w \mapsto w+x$, and this action has a natural extension to Y. For $x \notin M$ (not the identity action) the fixed-point set Y^x is concentrated in the divisor D, so we can define

$$\varphi(M, V, x) := L(Y, x)[Y^x] - \operatorname{sign}(Y, x),$$

where $\operatorname{sign}(Y, x)$ is the equivariant signature, and $L(Y, x)$ is the corresponding cohomology class in $H^*(Y^x, \boldsymbol{C})$. Because x acts trivially on the cohomology group $H^n(Y, C)$, we have actually $\operatorname{sign}(Y, x) = \operatorname{sign}(Y)$. Again, $\varphi(M, V, x)$ depends only on the boundary ∂Y and the action of x on it, but not on Y and the particular divisor D. For a proof of the following theorem, see [6].

Theorem 3. $\varphi(M, V, x) = -\operatorname{sign}(Y) + i^n \sum_{\sigma} (-1)^k \sum_{p_\tau > 0} S_\sigma^p \prod_{\tau \in \sigma} c_{p_\tau}(x_\tau)$.

Here the first sum runs only over those $\sigma \in C(k)$, $1 \leq k \leq n$, which admit a representation of x as

$$x = \sum_{\tau \in \sigma} x_\tau A_\tau$$

with rational coordinates x_τ (which are determined only up to an integer because the A_τ are determined only up to a unit in V). The corresponding submanifolds S_σ belong then to the fixed-point set Y^x. The product is taken over the values of the trigonometric functions

$$c_k(u) = \pi^{-k} \sum_{m \in Z + u} m^{-k}, \qquad k = 1, 2, 3, \cdots$$

(if $k=1$, the series has to be ordered according to increasing values of $|m|$). Note that the c_k are essentially the derivatives of the cotangent function

$$c_1(u) = \begin{cases} 0, & u \in Z \\ \cot \pi u, & u \in C \backslash Z \end{cases}, \quad \frac{d}{du} c_k(u) = -\pi k c_{k+1}(u), \quad u \notin Z.$$

By Euler, we have for integral values of u,

$$c_k(u) = -(-1)^{k/2} 2^k \frac{B_k}{k!}, \qquad k \geq 2$$

with the Bernoulli numbers B_k. In particular, if $x \in M$, then formally $\varphi(M, V, x) = \varphi(M, V)$. But the definition given for $\varphi(M, V, x)$ makes sense only for $x \notin M$.

The second invariant, the equivariant Todd genus $\psi(M, V, x)$, is the value

$$\psi(M, V, x) := \operatorname{td}(Y, x)[Y^x]$$

of the cohomology class $\operatorname{td}(Y, x) \in H^*(Y^x, C)$ contributing to the Lefschetz number of x (acting on the trivial line bundle over Y) in the holomorphic Lefschetz formula [1, p. 566, 6, p. 57]. Again, this makes sense only for $x \notin M$. In the case $x \in M$ we define $\psi(M, V, x) = \psi(M, V)$ as the coefficient of z^n in the formal power series

$$\prod_{\tau} \frac{S_\tau}{1-\exp(-S_\tau z)}.$$

This number was studied, among others by Ehlers [3] and Satake [5]. In [6], we established the following explicit expression for $\psi(M, V, x)$.

Theorem 4. $$\psi(M, V, x)=\left(\frac{i}{2}\right)^n \sum_{\sigma}(-1)^k \sum_{p_\tau>0} S_\sigma^p \prod_{\tau\in\sigma} C_{p_\tau}(x_\tau).$$

Aside from the missing signature term and the additional factor 2^{-n}, the only difference between this expression and the corresponding expression in Theorem 3 is the slightly different definition of the trigonometric functions $C_k(u)$:

$$C_1(u)=c_1(u)+i, \quad C_k(u)=c_k(u) \quad \text{for } k>1.$$

Despite these differences, we conjecture that

Conjecture. (6) $$\varphi(M, V, x)=2^n\psi(M, V, x).$$

This equality, if true, would reflect a very special geometric property of Y. It is equivalent to our conjecture (5) about the signature of Y. This follows from the following result which we found by a very complicated calculation.

Theorem 5. $$\begin{aligned}\varphi(M, V, x)-2^n\psi(M, V, x) &=\operatorname{sign}(Y)-\sum_{\sigma\in C}(-2)^{n-k}, \quad k=\sharp \textit{ of vertices in } \sigma\\ &=\operatorname{sign}(Y)-\sum_{t\in C(n)} \operatorname{sign}\big(\prod_{\tau\in\sigma} B_\tau^t\big).\end{aligned}$$

If n is odd, then $\operatorname{sign}(Y)=0$ by definition. The expressions

$$\sum_{\sigma\in C}(-2)^{n-k}=\sum_{t\in C(n)} \operatorname{sign}\big(\prod_{\tau\in t} B_\tau^t\big)$$

also vanish in this case (although this is not obvious). Therefore, for n odd, the equality (6) is indeed true.

§ 4.

To the triple (M, V, x) there is associated the Hecke L-series

$$L(M, V, x, s)=\sum_{m\in M+x/V}{}' \frac{\operatorname{sign} N(m)}{|N(m)|^s}, \qquad \operatorname{Re}(s)>1$$

where m runs through all $m\neq 0$ in $M\neq x$ which are not equivalent under the action of V. As a natural generalization of Hirzebruch's conjecture we propose

Conjecture. (7) $$d(M)L(M, V, x, 1)=(\pi i)^n\varphi(M, V, x) \\ =(2\pi i)^n\psi(M, V, x).$$

Hirzebruch's original conjecture (now a theorem of Atiyah, Donnelly and Singer) is the case $x=0$. Using the functional equation of the L-function at $s=0$,

$$d(M)L(M, V, 0, 1)=(\pi i)^n L(M^*, V, 0, 0),$$

his conjecture can be reformulated as

$$L(M^*, V, 0, 0)=\varphi(M, V, 0). \tag{8}$$

This is exactly the relation Atiyah, Donnelly and Singer have proved in [2]. At $s=1$, the L-series converges conditionally,

$$L(M, V, x, 1)=\lim_{r\to\infty}\sum_{\substack{m\in M+x/V\\ |N(m)|<r}}{}' \frac{1}{N(m)}.$$

It is this conditionally convergent series which is at the center of our considerations. We want to transform the series into another one where m runs over all elements in $M+x$ instead over the classes in $M+x/V$ only. To this end we apply an old trick of Hecke, and consider an absolutely convergent series

$$\sum_{v\in V} f(vm)=\frac{d(M)}{N(m)} \tag{9}$$

with some (at the moment unspecified) terms $f(vm)$. Then

$$\sum_{m\in M+x/V}{}' \frac{d(M)}{N(m)}=\sum_{m\in M+x/V}{}' \sum_{v\in V} f(vm)=\sum_{m\in M+x}{}' f(m),$$

and therefore,

$$d(M)L(M, V, x, 1)=\lim_{r\to\infty}\sum_{\substack{m\in M+x\\ |N(m)|<r}}{}' f(m).$$

Clearly, $f(m)$ is not determined by this equation. Although we do not have a complete proof that the following choice of $f(m)$ satisfies (9), it would lead (as we will see in a moment) to an elementary proof of the conjecture (7). For $\sigma\in C$ let

$$M(\sigma)=\{m \in M+x \mid m=\sum_{\tau\in\sigma} m_\tau A_\tau,\ m_\tau\neq 0\}.$$

Then

$$f(m)=\sum_{\sigma\in C} f(\sigma, m)$$

with $f(\sigma, m)=0$ unless $m \in M(\sigma)$ where

$$f(\sigma, m)=(-1)^{n-k} \sum_{p_\tau>0} S_\sigma^p \prod_{\tau\in\sigma} m_\tau^{-p_\tau} \qquad \text{for } \sigma \in C(k).$$

Notice that the coordinates m_τ are just the numbers $m_\tau=\operatorname{tr}(mB_\tau^t)$ for any $t \in \mathrm{St}(\sigma)$. In the special case where $\operatorname{tr}(mB_\tau^t)$ does not vanish for all t and $\tau \in t$, $f(m)$ simplifies to

$$f(m)=\sum_{t\in C} \prod_{\tau\in t} \operatorname{tr}(mB_\tau^t)^{-1}.$$

Every term in this sum has a simple geometric meaning: it is the projective volume of the simplex $t \in C$ (with respect to a certain differential form depending on m), and the identity (9) says then basically that the projective volume of all simplices t in $V\cdot C$ equals the projective volume of the totally positive chamber of the $\boldsymbol{R}^n$. The following theorem follows from a new limit formula, c.f. [7].

Theorem 6. $\lim\limits_{r\to\infty} \sum\limits_{\substack{m\in M+x\\ |N(m)|<r}} f(m)=(2\pi i)^n \psi(M, V, x).$

This identity can be viewed as a partial fraction decomposition of the right hand sice, generalizing Euler's formula

$$\lim_{r\to\infty} \sum_{\substack{m\in Z+x\\ |m|<r}} \frac{1}{m}=\pi \cot \pi x. \tag{10}$$

(In fact, the theorem remains true in the case $n=1$ if we add πi to the right hand side; it is then equivalent to (8)). By definition of $f(m)$, we can write the left hand side as

$$\sum_{\sigma\in C} (-1)^{n-k} \sum_{p_\tau>0} S_\sigma^p R(p, \sigma)$$

with

$$R(p, \sigma)=\lim_{r\to\infty} \sum_{\substack{m\in M(\sigma)\\ |N(m)|<r}} \prod_{\tau\in\sigma} m_\tau^{-p_\tau}.$$

If all p_τ are >1, then this last series converges absolutely, and we have by definition of the $C_k(u)$,

$$R(p, \sigma)=\pi^n \prod_{\tau\in\sigma} C_{p_\tau}(\operatorname{tr}(xB_\tau^t)).$$

But if some $p_\tau=1$, then the series converges only conditionally, and special considerations are necessary to determine its value. For example, if all $p_\tau=1$, we prove in [7] that

$$R(p, \sigma)=\text{real part of } \pi^n \prod_{\tau\in\sigma} C_1(\operatorname{tr}(xB_\tau^t)).$$

It is remarkable that all the contributions coming from the special ordering of $R(p, \sigma)$ according to the increasing values of $|N(m)|$ add up to the expression (5) which we conjecture to be the signature of Y.

§ 5.

If we combine the theorem of Atiyah, Donnelly and Singer with our formula for $\varphi(M, V)$, we get the equality

$$L(M^*, V, 0, 0)=-\operatorname{sign}(Y)+2^n \sum_{\sigma\in C}\sum_{p_\tau>1} S_\sigma^p \prod_{\tau\in\sigma} \frac{B_{p_\tau}}{p_\tau!}$$

where for given σ, the inner sum runs over all partitions p of n and length $=\#$ of vertices of σ. Assuming our conjecture (5), we can rewrite this as

$$L(M^*, V, 0, 0)=2^n \sum_{\sigma\in C}\sum_{p_\tau>0} S_\sigma^p \prod_{\tau\in\sigma} \frac{B_{p_\tau}}{p_\tau!}.$$

On the other hand, by a result of Shintani [8], we have

$$L(M, V, 0, 0)=2^n \sum_{\sigma\in C}\sum_{q_\tau\geq 0} \operatorname{tr}\left(\frac{1}{n} \prod_{\tau\in\sigma} \frac{B_{q_\tau}}{q_\tau!} A_\tau^{q_\tau-1}\right)$$

where for given σ, the inner sum runs now over all partition q of k and length $k=\#$ of vertices in σ.

The striking similarity between these two formulas raises the possibility that there is a relation between the rational numbers $L(M, V, 0, 0)$ and $L(M^*, V, 0, 0)$. If $n=2$, then it follows from these formulas that

Conjecture. (11) $L(M, V, 0, 0)=i^n L(M^*, V, 0, 0)$.

Notice that this relation is trivial if the multiplier ring of M in F contains units of negative norm (for example -1 if n is odd) because both sides vanish then. Moreover, if M is fractional ideal in a field of class number 1 whose units have only positive norms, then this relation follows from Hecke's result that

$$\operatorname{sign} N(d) = i^n$$

for every generator d of the different of the field. But beyond this evidence, we do not know whether (11), or equivalently

$$\varphi(M, V) = i^n \varphi(M^*, V),$$

is always true.

References

[1] Atiyah, M. F. and Singer, I. M., The index of elliptic operators III, Ann. of Math., **87** (1968), 546–604.
[2] Atiyah, M. F., Donnelly, H. and Singer, I. M., Eta invariants, signature defects of cusps, and values of L-functions, Ann. of Math., **118** (1983), 131–177.
[3] Ehlers, F., Eine Klasse komplexer Mannigfaltigkeiten und die Auflösung einiger isolierter Singularitäten, Math. Ann., **28** (1975), 127–156.
[4] Hirzebruch, F., Hilbert modular surfaces, Enseign. Math., **19** (1973), 183–281.
[5] Satake, I., On numerical invariants of arithmetic varieties of Q-rank one, in: Automorphic forms of several variables, Kataka 1983, I. Satake and Y. Morita (eds.), Progress in Math. 46, Birkhäuser, Basel, Boston, Stuttgart 1984.
[6] Sczech, R., Zur Summation von L-Reihen, Bonner Mathematische Schriften 141, Bonn 1982.
[7] ——, A new limit formula, in preparation.
[8] Shintani, T., On evaluation of zeta functions of totally real algebraic number fields at non-positive integers, J. Fac. Sci. Univ. Tokyo, **23** (1976), 393–417.

Department of Mathematics
Rutgers University
Newark, NJ 07102
U.S.A.