Advanced Studies in Pure Mathematics 14, 1988
Representations of Lie Groups, Kyoto, Hiroshima, 1986
pp. 531–540

A Description of Discrete Series for Semisimple Symmetric Spaces II

Toshihiko Matsuki

§ 1. Introduction

In [F], Flensted-Jensen constructed countably many discrete series for a semisimple symmetric space G/H when

$$\text{(1.1)} \qquad \operatorname{rank}(G/H)=\operatorname{rank}(K/K\cap H).$$

Conversely, [OM1] proved that (1.1) holds if there exist discrete series for G/H. Moreover [OM1] constructed Harish-Chandra modules B_λ^j which parametrize all the discrete series for G/H, where j runs through finite indices and λ runs through lattice points contained in a positive Weyl chamber. In this paper, we give a necessary condition for j and λ so that the module B_λ^j is nontrivial. In the subsequent paper [OM2], we will prove that the condition also assures $B_\lambda^j\neq\{0\}$. We remark that our results also covers "limits of discrete series" for G/H. In the appendix, we give a certain simplification of the proof of a main result in [OM1]. To state the precise result in this paper, we prepare some notations.

Let $\mathfrak{g}$ be a semisimple Lie algebra and σ an involution (automorphism of order 2) of $\mathfrak{g}$. Fix a Cartan involution θ of $\mathfrak{g}$ such that $\sigma\theta=\theta\sigma$. Let $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ (resp. $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$) be the decomposition of $\mathfrak{g}$ into the $+1$ and -1 eigenspaces for σ (resp. θ). Let $\mathfrak{g}_c$ denote the complexification of $\mathfrak{g}$ and put

$$\mathfrak{k}^d=\mathfrak{k}\cap\mathfrak{h}+\sqrt{-1}(\mathfrak{p}\cap\mathfrak{h}), \qquad \mathfrak{p}^d=\sqrt{-1}(\mathfrak{k}\cap\mathfrak{q})+\mathfrak{p}\cap\mathfrak{q},$$
$$\mathfrak{h}^d=\mathfrak{k}\cap\mathfrak{h}+\sqrt{-1}(\mathfrak{k}\cap\mathfrak{q}), \qquad \mathfrak{q}^d=\sqrt{-1}(\mathfrak{p}\cap\mathfrak{h})+\mathfrak{p}\cap\mathfrak{q},$$
$$\mathfrak{g}^d=\mathfrak{k}^d+\mathfrak{p}^d=\mathfrak{h}^d+\mathfrak{q}^d.$$

Let G_c be a connected complex Lie group with Lie algebra $\mathfrak{g}_c$, and let G, K, H, G^d, K^d, H^d, H_c and K_c be the analytic subgroups of G_c corresponding to $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{h}$, $\mathfrak{g}^d$, $\mathfrak{k}^d$, $\mathfrak{h}^d$, $\mathfrak{h}_c$ and $\mathfrak{k}_c$, respectively.

In [OM1], we studied the discrete series for G/H and proved that

Received April 6, 1987.

rank $(G/H)=$ rank $(K/K\cap H)$ if there exist discrete series for G/H. So we may choose a maximal abelian subspace $\mathfrak{a}_{\mathfrak{p}}$ of $\mathfrak{p}^d$ contained in $\mathfrak{p}^d\cap\mathfrak{h}^d$ $(=\sqrt{-1}(\mathfrak{k}\cap\mathfrak{q}))$. Let Σ denote the root system of the pair $(\mathfrak{g}^d, \mathfrak{a}_{\mathfrak{p}})$ and fix a positive system Σ^+ of Σ. Let M be the centrailzer of $\mathfrak{a}_{\mathfrak{p}}$ in G^d and put $A_{\mathfrak{p}}=\exp\mathfrak{a}_{\mathfrak{p}}$, $\mathfrak{n}^+=\sum_{\alpha\in\Sigma^+}\mathfrak{g}^d(\mathfrak{a}_{\mathfrak{p}};\alpha)$, $N^+=\exp\mathfrak{n}^+$, $\rho=\frac{1}{2}\sum_{\alpha\in\Sigma^+}m_\alpha\alpha$, $\rho_t=\frac{1}{2}\sum_{\alpha\in\Sigma^+}m_\alpha^+\alpha$ where $\mathfrak{g}^d(\mathfrak{a}_{\mathfrak{p}};\alpha)=\{X\in\mathfrak{g}^d\,|\,[Y,X]=\alpha(Y)X \text{ for all } Y\in\mathfrak{a}_{\mathfrak{p}}\}$, $m_\alpha=\dim\mathfrak{g}^d(\alpha_{\mathfrak{p}};\mathfrak{a})$ and $m_\alpha^+=\dim(\mathfrak{g}^d(\mathfrak{a}_{\mathfrak{p}};\alpha)\cap\mathfrak{h}^d)$ for $\alpha\in\Sigma$. Then $P=MA_{\mathfrak{p}}N^+$ is a minimal parabolic subgroup of G^d. For $\lambda\in(\mathfrak{a}_{\mathfrak{p}})_c^*$, we define the space of hyperfunction sections of class 1 principal series for G^d:

$$\mathscr{B}(G^d/P;L_\lambda)=\{f\in\mathscr{B}(G^d)\,|\,f(xman)=a^{\lambda-\rho}f(x) \quad \text{for } x\in G^d,\ m\in M,\ a\in A_{\mathfrak{p}} \text{ and } n\in N^+\}$$

where $a^{\lambda-\rho}=e^{\langle\lambda-\rho,\log a\rangle}$.

Let M^* denote the normalizer of $\mathfrak{a}_{\mathfrak{p}}$ in K^d and $W=M^*/M$ the Weyl group of Σ. Then by [M1] § 3 Proposition 2, we can choose elements $w_1=1, w_2,\cdots,w_m$ of M^* such that $\{H^dw_jP\,|\,j=1,\cdots,m\}$ is the set of all the closed H^d-P double cosets in G^d $(H^dw_iP\neq H^dw_jP$ if $i\neq j)$. Put

$B_\lambda^j=\{f\in\mathscr{B}(G^d/P;L_\lambda)\,|\,\operatorname{supp}f\subset H^dw_jP$ and f transforms according to a finite dimensional representation of H^d which can be extended to a holomorphic representation of $K_c\}$.

In [OM1], we proved that all the K-finite functions of all the discrete sreies for G/H are given by $(\eta^{-1}\circ\mathscr{P}_\lambda)B_\lambda^j$ $(j=1,\cdots,m,\ \lambda\in L_{K/K\cap H}-\rho+2\rho_t,\ \langle\lambda,\alpha\rangle>0$ for all $\alpha\in\Sigma^+)$ where η: $\mathscr{A}_K(G/H)\rightrightarrows\mathscr{A}_{H^d}(G^d/K^d)$ is the Flensted-Jensen's isomorphism, $\mathscr{P}_\lambda$: $\mathscr{B}(G^d/P;L_\lambda)\rightrightarrows\mathscr{A}(G^d/K^d;\mathscr{M}_\lambda^d)$ is the Poisson transform and $L_{K/K\cap H}$ is the lattice in $\mathfrak{a}_{\mathfrak{p}}^*$ generated by the highest weights of finite-dimensional representations of K having a $K\cap H$-fixed vector. (See [OM1] for precise notations.) In [OM1] § 1, we announced a proposition which describes the condition for $B_\lambda^j\neq\{0\}$. One of the aim of this paper is to prove a part of the following theorem which is a revised version of the proposition. (There was a mistake in the formulation of the proposition. See the remark following Theorem 1.1.)

Since $\mathscr{B}(G^d/P;L_\lambda)\simeq\mathscr{B}(G^d/w_jPw_j^{-1};L_{w_j^{-1}\lambda})$ by the identification $xw_jP\rightarrow xw_jPw_j^{-1}$ of G^d/P and $G^d/w_jPw_j^{-1}$, we have only to study B_λ^1 for any choice of the positive system Σ^+ of Σ. Put $\mu_\lambda=\lambda+\rho-2\rho_t$, $m_\alpha^-=m_\alpha-m_\alpha^+$ and $m_\alpha^0=m_\alpha^+-m_\alpha^-$ $(\alpha\in\Sigma)$. Let $\boldsymbol{Z}$ denote the ring of integers.

Theorem 1.1. *Suppose that*

$$\mu_\lambda\in L_{K/K\cap H} \tag{1.2}$$

and that

$$(1.3) \qquad \langle \lambda, \alpha \rangle \geq 0 \qquad \text{for all } \alpha \in \Sigma^+.$$

Then $B_\lambda^1 \neq \{0\}$ *if and only if the following condition* (P) *holds.*

(P) *Let* $\{\beta_1, \cdots, \beta_k\}$ *be a sequence of roots in* Σ^+ *satisfying the following conditions* (i) *and* (ii). *Then*

$$\langle \mu_\lambda, \beta_k \rangle \geq 0.$$

(i) β_i *is a simple root in the set* $\{\alpha \in \Sigma^+ \mid \langle \alpha, \beta_1 \rangle = \cdots = \langle \alpha, \beta_{k-1} \rangle = 0\}$.

(ii) *Put* $n_i = \sum_{\alpha \in \Sigma \cap (\beta_i + Z\beta_1 + \cdots + Z\beta_{i-1})} m_\alpha^0$. *Then* $n_i < m_{\beta_i}$ *for* $i = 1, \cdots, k-1$ *and* $n_k = m_{\beta_k}$.

We will prove in this paper that $B_\lambda^1 \neq \{0\}$ implies the condition (P). The converse assertion will be proved in a subsequent paper [OM2].

Remark. (i) The condition (P) for $k=1$ is equivalent to the condition

$$(1.4) \qquad \langle \mu_\lambda, \alpha \rangle \geq 0 \text{ for any simple root } \alpha \text{ in } \Sigma^+ \text{ satisfying } \mathfrak{g}^d(\mathfrak{a}_\mathfrak{p}; \alpha) \subset \mathfrak{h}^d$$

(the condition (a) in [OM1] Theorem (iii)). [OM1] § 1 Proposition is false even for $k=1$. There is a counter example when Σ is of type B_2 and β_1 is the long simple root. The condition (a′) in the proposition should be replaced by the condition (P).

(ii) Suppose the conditions (1.2), (1.3) and (1.4). If $\langle \mu_\lambda, \alpha \rangle \geq 0$ holds for all $\alpha \in \Sigma(\mathfrak{h}^d; \mathfrak{a}_\mathfrak{p})^+$ $(= \Sigma(\mathfrak{h}^d; \mathfrak{a}_\mathfrak{p}) \cap \Sigma^+)$, for example when Σ is irreducible and is of type A_l $(l \geq 2)$, D_l, E_l or G_2 (cf. [OM1] Lemma 10), then $B_\lambda^1 \neq \{0\}$ by [OM1] § 1 Remark 2 (i) (Flensted-Jensen's construction of discrete series for G/H in [F]). Hence the conditions (1.2), (1.3) and (1.4) imply the condition (P) in this case.

(iii) When Σ is of type C_l, then we will show in [OM2] that the conditions (1.2), (1.3) and (1.4) imply the condition (P) and $B_\lambda^1 \neq \{0\}$.

(iv) Suppose that Σ is of type B_l, BC_l or F_4. Then we will prove in [OM2] that $B_\lambda^1 \neq \{0\}$ if the conditions (1.2), (1.3), (1.4) and the following (1.5) holds.

(1.5) The condition (P) holds for the sequence $\{\beta_1, \cdots, \beta_k\}$ consisting only of short roots.

(v) If Σ is of type B_l or BC_l and $\{\beta_1, \cdots, \beta_k\}$ consists only of short roots, then the condition (ii) in [OM1] Proposition is equivalent to the condition (ii) in Theorem 1.1. (This is the reason why the authors were not aware of the miswriting in [OM1] Proposition.)

After writing [OM1], Oshima [O] found a theorem describing the precise asymptotic behavior of spherical functions on G/H. By this theorem of Oshima, we have only to prove the following lemma to prove Theorem 1 in [OM1] § 4 instead of Lemma 3 in [OM1], which we announced in [OM1] p. 389 as "(iii) we have obtained a simpler proof of Theorem 1 which does not require case-by-case checking, which will appear in another paper." (We spent 16 pages to prove Lemma 3 in [OM1] by case-by-case checking.)

Let $\mathfrak{a}_\mathfrak{p}^d$ be a maximal abelian subspace of $\mathfrak{p}^d$ such that $\mathfrak{a}=\mathfrak{a}_\mathfrak{p}^d \cap \mathfrak{q}^d$ is maximal abelian in $\mathfrak{p}^d \cap \mathfrak{q}^d$. Let $\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$ be a $\sigma\theta$-compatible positive system of $\Sigma(\mathfrak{a}_\mathfrak{p}^d)$ and P^d the minimal parabolic subgroup of G^d defined by the pair $(\mathfrak{a}_\mathfrak{p}^d, \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+)$. Let y be an element of K^d such that $\mathrm{Ad}(y)\mathfrak{a}_\mathfrak{p}^d=\mathfrak{a}_\mathfrak{p}$ and that $\Sigma^+=\{\alpha \circ \mathrm{Ad}(y)^{-1} \mid \alpha \in \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+\}$. Then $P=yP^dy^{-1}$. Let $\Sigma(\mathfrak{a})^+$ denote the positive system of the root system $\Sigma(\mathfrak{a})$ consisting of the nonzero restrictions of roots in $\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$. Let $\{\alpha_1, \cdots, \alpha_{l_0}\}$ denote the set of simple roots in $\Sigma(\mathfrak{a})^+$ and $\{\omega_1, \cdots, \omega_{l_0}\}$ the dual basis of $\{\alpha_1, \cdots, \alpha_{l_0}\}$.

Lemma 1.2. *Let λ be an element of $(\mathfrak{a}_\mathfrak{p}^d)^*$ such that $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$ and x be an element of G^d. Suppose that one of the following three conditions is satisfied.*

(i) $\mathrm{rank}(G/H) \neq \mathrm{rank}(K/K \cap H)$.

(ii) *$H^d x P^d$ is not closed in G^d.*

(iii) $\mathrm{rank}(G/K)=\mathrm{rank}(K/K \cap H)$ *and there is a j $(1 \leq j \leq m)$ such that $H^d x P^d = H^d w_j y P^d$ and that $\langle \lambda, \alpha \rangle = 0$ for a simple root α of $\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$ satisfying* $\mathrm{Ad}(w_j y)\mathfrak{g}^d(\mathfrak{a}_\mathfrak{p}^d; \alpha) \cap \mathfrak{q}^d \neq \{0\}$.

Then there exists a $w \in W(\mathfrak{a}_\mathfrak{p}^d)$ (the Weyl group of $\Sigma(\mathfrak{a}_\mathfrak{p}^d)$) such that (a) *$H^d x (P^d w P^d)^{\mathrm{cl}}$ contains inner points in G^d and that* (b) $(\langle w^{-1}\lambda, \omega_1 \rangle, \cdots, \langle w^{-1}\lambda, \omega_{l_0} \rangle) \notin (-\infty, 0)^{l_0}$.

We will give a simple proof of this lemma in Appendix.

§ 2. H^d-P double cosets in G^d

Let $\mathfrak{t}$ be a maximal abelian subspace of $\mathfrak{p}^d$, $\mathfrak{t}^*$ the space of real linear forms on $\mathfrak{t}$ and $\mathfrak{s}$ a subalgebra of $\mathfrak{g}^d$. Then we put $\mathfrak{s}(\mathfrak{t}; \alpha) = \{X \in \mathfrak{s} \mid [Y, X]=\alpha(Y)X$ for all $Y \in t\}$ for $\alpha \in \mathfrak{t}^*$ and $\Sigma(\mathfrak{s}; \mathfrak{t}) = \{\alpha \in \mathfrak{t}^* - \{0\} \mid \mathfrak{s}(\mathfrak{t}; \alpha) \neq \{0\}\}$. Then $\Sigma=\Sigma(\mathfrak{g}^d; \mathfrak{a}_\mathfrak{p})$.

Put $\Sigma_0^+ = \{\alpha \in \Sigma^+ \mid \alpha \notin \Sigma\}$. Let D be an H^d-P double coset in G^d. Then we define a number $N(D)$ by

$$N(D)=\tfrac{1}{2}\,\sharp((\Sigma_y^+)_0 \cap \theta(\Sigma_y^+)_0) + \tfrac{1}{2}\dim(\mathrm{Ad}(y)\mathfrak{a}_\mathfrak{p} \cap \mathfrak{h}^d)$$

where $y \in K^d$ is a representative of D such that $\mathrm{Ad}(y)\mathfrak{a}_\mathfrak{p}$ is σ-stable and $(\Sigma_y^+)_0 = \{\alpha \circ \mathrm{Ad}(y)^{-1} \in \Sigma(\mathfrak{g}^d; \mathrm{Ad}(y)\mathfrak{a}_\mathfrak{p}) \mid \alpha \in \Sigma_0^+\}$. By [M1] Theorem 1, we can see that the definition of $N(D)$ does not depend on the choice of $y \in K^d$.

Lemma 2.1. *Let D and D' be H^d-P double cosets in G^d.*

(i) *If $D' \subset DP_\gamma$ for a simple root γ in Σ^+ $(P_\gamma = P \cup Pw_\gamma P)$. Then*

$$N(D') - N(D) = \mathrm{sgn}(\dim D - \dim D').$$

Here sgn *is the usual signature function with the range* $\{-1, 0, 1\}$.

(ii) *If $w \in W$ satisfies $l(w) < |N(D) - N(D')|$. Then*

$$D(PwP)^{cl} \cap D' = \emptyset.$$

Here $l(w) = \#(\Sigma_0^+ \cap -w\Sigma_0^+)$ is the length of w.

Proof. (i) follows easily from the argument in [M2] Lemma 3. (ii) is clear from (i) because $(PwP)^{\mathrm{cl}} = P_{\gamma_1} P_{\gamma_2} \cdots P_{\gamma_{l(w)}}$ for a minimal expression $w = w_{\gamma_1} w_{\gamma_2} \cdots w_{\gamma_{l(w)}}$. ($\gamma_1, \cdots, \gamma_{l(w)}$ are simple roots in Σ^+.)

Let β be a simple root of Σ^+ such that $m_\beta^- > 0$. Choose an element X_β of $\mathfrak{g}^d(\mathfrak{a}_\mathfrak{p}; \beta) \cap \mathfrak{q}^d$ so that $2\langle\beta, \beta\rangle B(X_\beta, \sigma X_\beta) = -1$ and put $c_\beta = \exp(\pi/2)(X_\beta + \sigma X_\beta)$, $\mathfrak{a}_\mathfrak{p}'' = \mathrm{Ad}(c_\beta)\mathfrak{a}_\mathfrak{p}$ (cf. [M1] § 2). Then $\mathfrak{a}_\mathfrak{p}'' = \mathfrak{a}_\mathfrak{p}'' \cap \mathfrak{h}^d + \mathfrak{a}_\mathfrak{p}'' \cap \mathfrak{q}^d$, $\mathfrak{a}_\mathfrak{p}'' \cap \mathfrak{a}_\mathfrak{p} = \mathfrak{a}_\mathfrak{p}'' \cap \mathfrak{h}^d$ and $\dim(\mathfrak{a}_\mathfrak{p}'' \cap \mathfrak{q}^d) = 1$. Let Y_β be the element of $\mathfrak{a}_\mathfrak{p}$ defined by $B(Y, Y_\beta) = \beta(Y)$ $(Y \in \mathfrak{a}_\mathfrak{p})$. Then $Y_\beta' = \mathrm{Ad}(c_\beta) Y_\beta$ generates the one-dimensional space $\mathfrak{a}_\mathfrak{p}'' \cap \mathfrak{q}^d$.

Let w_1 be the element of W satisfying

$$w_1 \Sigma^+ = \{\gamma \in \Sigma \mid \langle\gamma, \beta\rangle > 0\} \cup \{\gamma \in \Sigma^+ \mid \langle\gamma, \beta\rangle = 0\}.$$

Put $P'' = c_\beta w_1 P w_1^{-1} c_\beta^{-1}$. Let L_1 (resp. $\mathfrak{l}_1$) be the centralizer of Y_β' in G^d (resp. $\mathfrak{g}^d$). Put $\mathfrak{l} = [\mathfrak{l}_1, \mathfrak{l}_1]$ and let L be the analytic subgroup of G^d for $\mathfrak{l}$. Let P_1 (resp. $\mathfrak{P}_1$) be the parabolic subgroup of G^d (resp. subalgebra of $\mathfrak{g}^d$) defined by the element Y_β' (i.e. $\mathfrak{P}_1 = \sum_{c \geq 0} \{X \in \mathfrak{g}^d \mid [Y_\beta', X] = cX\}$). Then we have the following natural maps ([M2] § 4)

$$\begin{aligned} &L \cap H^d \backslash L / L \cap P'' \xrightarrow[q]{} L_1 \cap H^d \backslash L_1 / L_1 \cap P'' \xleftarrow[p]{\sim} P_1 \cap H^d \backslash P_1 / P'' \\ &\xrightarrow{\sim} H^d \backslash H^d P_1 / P'' \end{aligned} \tag{2.1}$$

given by the inclusions $L \to L_1$, $P_1 \to H^d P_1$ and the projection $p \colon P_1 \to L_1$ with respect to the Langlands decomposition $P_1 = L_1 \exp \mathfrak{n}_1$. Here $\mathfrak{n}_1$ is the nilpotent radical of $\mathfrak{P}_1$ and q is surjective.

Lemma 2.2. (i) *H^dP_1 is open in G^d.*

(ii) *$(L\cap H^d)(L\cap P'')$ is closed in L.*

(iii) *$N(H^dc_\beta w_1P)=N(H^dP)-l(w_\beta w_1)$.*

(iv) *$N(D)\leq N(H^dP)-l(w_\beta w_1)$ for any H^d-P double coset D contained in $H^dP_1c_\beta w_1$.*

(v) *$N(D)<N(H^dP)-l(w_\beta w_1)$ for any H^d-P double coset D contained in $H^dP_1c_\beta w_1$ such that* $\dim D>\dim H^dc_\beta w_1P$.

Proof. (i) $\mathfrak{h}^d+\mathfrak{P}_1=\mathfrak{h}^d+\mathfrak{P}_1+\theta\mathfrak{P}_1=\mathfrak{g}^d$ since $\theta Y'_\beta=-Y'_\beta$. Hence H^dP_1 is open in G^d.

(ii) Since $\mathfrak{a}''_\mathfrak{p}\cap\mathfrak{h}^d\subset\mathfrak{l}\cap\mathfrak{P}''$ and since $\mathfrak{a}''_\mathfrak{p}\cap\mathfrak{h}^d$ is a maximal abelian subspace of $\mathfrak{l}\cap\mathfrak{p}^d$, $(L\cap H^d)(L\cap P'')$ is closed in L by [M1] § 3.

(iii) Since $\#((\Sigma^+_{c_\beta w_1})_0\cap\theta(\Sigma^+_{c_\beta w_1})_0)=\#((\Sigma^+_{c_\beta w_1})_0\cap\Sigma(\mathfrak{l}_1;\mathfrak{a}''_\mathfrak{p}))=\#\Sigma^+_0-2l(w_1)-1$ by the definition of w_1, we have $N(H^dc_\beta w_1P)=N(H^dP)-l(w_1)-1=N(H^dP)-l(w_\beta w_1)$

(iv) By (2.1), we can choose a representative y of D such that $y\in(L\cap K^d)c_\beta w_1$ and that $\mathfrak{a}^y_\mathfrak{p}=\mathrm{Ad}(y)\mathfrak{a}_\mathfrak{p}$ is σ-stable. Since $P_y=yPy^{-1}\subset P_1$, we have $P_y\cap\theta P_y\subset P_1\cap\theta P_1=L_1$ and therefore

$$\begin{aligned}\#((\Sigma^+_y)_0\cap\theta(\Sigma^+_y)_0)&\leq\#((\Sigma^+_y)_0\cap\Sigma(\mathfrak{l}_1;\mathfrak{a}^y_\mathfrak{p}))\\&=\#((\Sigma^+_{c_\beta w_1})_0\cap\Sigma(\mathfrak{l}_1;\mathfrak{a}'_\mathfrak{p}))\\&=\#\Sigma^+_0-2l(w_1)-1.\end{aligned}\tag{2.2}$$

Hence $N(D)\leq N(H^dP)-l(w_\beta w_1)$.

(v) Let y be as above. Then $\dim D-\dim H^dc_\beta w_1P=\dim(L\cap H^d)(L\cap P_y)-\dim(L\cap H^d)(L\cap P'')$ by (2.1). Thus the assertion follows from (2.2) since the equality holds in (2.2) only when $\mathfrak{a}^y_\mathfrak{p}\cap\mathfrak{l}\subset\mathfrak{h}^d$ ($\Leftrightarrow(L\cap H^d)(L\cap P_y)$ is closed in L by [M1] § 3 Proposition 2) and since all the closed $(L\cap H^d)$-$(L\cap P_y)$ double cosets in L have the same dimension (cf. [M2] § 5 Lemma 7). Q.E.D.

For a root $\alpha\in\Sigma(\mathfrak{a}_\mathfrak{p})$, define a root α'' of $\Sigma(\mathfrak{g}^d;\mathfrak{a}''_\mathfrak{p})$ by $\alpha''=\alpha\circ\mathrm{Ad}(c_\beta)^{-1}$. Then the positive system $\Sigma(\mathfrak{l})^+=\Sigma(\mathfrak{l};\mathfrak{a}''_\mathfrak{p})^+=\{\alpha''\in\Sigma(\mathfrak{l};\mathfrak{a}''_\mathfrak{p})\mid\alpha\in\Sigma^+\}$ of $\Sigma(\mathfrak{l};\mathfrak{a}''_\mathfrak{p})$ corresponds to the minimal parabolic subgroup $L\cap P''$ of L because of the choice of w_1. Put $m_{\alpha''}=\dim\mathfrak{g}^d(\mathfrak{a}''_\mathfrak{p};\alpha)$, $m^+_{\alpha''}=\dim(\mathfrak{g}^d(\mathfrak{a}''_\mathfrak{p};\alpha'')\cap\mathfrak{h}^d)$, $m^-_{\alpha''}=m_{\alpha''}-m^+_{\alpha''}$ and $m^0_{\alpha''}=m^+_{\alpha''}-m^-_{\alpha''}$ for $\alpha''\in\Sigma(\mathfrak{l};\mathfrak{a}''_\mathfrak{p})$. Put $\rho^L=\frac{1}{2}\sum_{\alpha''\in\Sigma(\mathfrak{l})^+}m_{\alpha''}\alpha''$ and $\rho^L_t=\frac{1}{2}\sum_{\alpha''\in\Sigma(\mathfrak{l})^+}m^+_{\alpha''}\alpha''$.

Lemma 2.3. (i) *Let α be an element of $\mathfrak{a}^*_\mathfrak{p}$ such that $\langle\alpha,\beta\rangle=0$. Then $\sum_{\gamma\in\Sigma\cap(\alpha+\boldsymbol{R}\beta)}m^0_\gamma=m^0_{\alpha''}$. (We put $m^0_{\alpha''}=0$ when $\alpha\notin\Sigma$. $\boldsymbol{R}$ is the field of real numbers.)*

(ii) $(2\rho_t-\rho)|_{\mathfrak{a}''_\mathfrak{p}\cap\mathfrak{l}}=2\rho^L_t-\rho^L$.

Proof. (i) Put $V=\sum_{\gamma\in\Sigma\cap(\alpha+\boldsymbol{R}\beta)}\mathfrak{g}^d(\mathfrak{a}_\mathfrak{p};\gamma)$. Then $\sum_{\gamma\in\Sigma\cap(\alpha+\boldsymbol{R}\beta)}m_\gamma^0=\dim(V\cap\mathfrak{h}^d)-\dim(V\cap\mathfrak{q}^d)$. On the other hand, it is clear that $V=\sum_{\gamma''\in\Sigma(\mathfrak{a}_\mathfrak{p}'')\cap(\alpha''+\boldsymbol{R}\beta'')}\mathfrak{g}^d(\mathfrak{a}_\mathfrak{p}'';\gamma'')$. Since $\theta\mathfrak{g}^d(\mathfrak{a}_\mathfrak{p}'';\alpha''+c\beta'')=\mathfrak{g}^d(\mathfrak{a}_\mathfrak{p}'';\alpha''-c\beta'')$, we have

$$\begin{aligned}&\dim((\mathfrak{g}^d(\mathfrak{a}_\mathfrak{p}'';\alpha''+c\beta'')+\mathfrak{g}^d(\mathfrak{a}_\mathfrak{p}'';\alpha''-c\beta''))\cap\mathfrak{h}^d)\\&\quad=\dim((\mathfrak{g}^d(\mathfrak{a}_\mathfrak{p}'';\alpha''+c\beta'')+\mathfrak{g}^d(\mathfrak{a}_\mathfrak{p}'';\alpha''-c\beta''))\cap\mathfrak{q}^d)\end{aligned}$$

when $c\neq 0$. Hence $\dim(V\cap\mathfrak{h}^d)-\dim(V\cap\mathfrak{q}^d)=m_{\alpha''}^0$.

(ii) Since $2\rho_t-\rho=\sum_{\alpha\in\Sigma^+}m_\alpha^0$ and since $\Sigma\cap(\gamma+\boldsymbol{R}\beta)\subset\Sigma^+$ if $\gamma\in\Sigma^+-\{\beta,2\beta\}$, the assertion follows easily from (i). Q.E.D.

§ 3. Proof of $B_\lambda^1\neq\{0\}\Rightarrow$(P)

Suppose that $B_\lambda^1\neq\{0\}$ and let $\{\beta_1,\cdots,\beta_k\}$ be a sequence of roots satisfying the conditions (i) and (ii) in (P). Then we want to prove that $\langle\mu_\lambda,\beta_k\rangle\geq 0$. We will prove this assertion by induction on the rank l of $\mathfrak{g}^d$.

If $k=1$, then the condition (ii) implies that $\mathfrak{g}^d(\mathfrak{a}_\mathfrak{p};\beta_1)\subset\mathfrak{h}^d$. In this case we have already proved in [OM1] Theorem (iii) (a) that $\langle\mu_\lambda,\beta_1\rangle=\langle\lambda-\rho,\beta_1\rangle\geq 0$.

So we may assume that $k>1$. Then we have $\mathfrak{g}^d(\mathfrak{a}_\mathfrak{p};\beta_1)\not\subset\mathfrak{h}^d$. Write $\beta=\beta_1$ and define c_β, $\mathfrak{a}_\mathfrak{p}''$, w_1, P_1, L and P'' as in § 2.

Put $\mu=w_1^{-1}w_\beta\lambda$ and $U=G^d-\bigcup_{w'<w_\beta w_1}H^d(Pw'P)^{\text{cl}}$. Then applying [O] Lemma 3.2 to a nonzero function f in B_λ^1, we get a function $g\in\mathscr{B}_{H^d}(U/P;L_\mu)$ such that $\operatorname{supp} g=U\cap H^d(Pw_\beta w_1P)^{\text{cl}}$. Here $\mathscr{B}_{H^d}(U/P;L_\mu)$ is the space of H^d-finite hyperfunctions h on the open subset U of G^d satisfying $h(xman)=a^{\mu-\rho}h(x)$ for any $(x,m,a,n)\in U\times M\times A\times N$ and the hyperfunction g is given as the image of $f\in B_\lambda^1$ by the "local intertwining operator" corresponding to $w^{-1}w_\beta$.

On the other hand, by Lemma 2.1 (ii) and Lemma 2.2 we have

$$H^dP_1c_\beta w_1\subset U,\tag{3.1}$$

$$H^dP_1c_\beta w_1\cap H^d(Pw_\beta w_1P)^{\text{cl}}\supset H^dc_\beta w_1P\tag{3.2}$$

and that

(3.3) every H^d-P double coset contained in $H^dP_1c_\beta w_1\cap H^d(Pw_\beta w_1P)^{\text{cl}}$ has the same dimension as $H^dc_\beta w_1P$.

So by (3.1), (3.2) and (3.3), we can define the restriction g' of g to $H^dP_1c_\beta w_1$ such that $\operatorname{supp} g' \supset H^dc_\beta w_1P$ and that $\dim \operatorname{supp} g' = \dim H^dc_\beta w_1P$.

Put $g''(x)=g'(xc_\beta w_1)$ for $x \in H^dP_1$ and $\mu''=\mu\circ \mathrm{Ad}(c_\beta w_1)^{-1} \in (\mathfrak{a}_\mathfrak{p}'')^*$. Then $g'' \in \mathscr{B}_{H^d}(H^dP_1/P''; L_{\mu''})$ and $\operatorname{supp} g'' \supset H^dP''$. Since $P_1=LP''$ and since g'' is left H^d-finite and right P''-finite, we can define $g''|_L \in \mathscr{B}_{L\cap H^d}(L/L\cap P''; L_{\mu''})$ so that $\operatorname{supp} g''|_L \supset (L\cap H^d)(L\cap P'')$ and that $\dim \operatorname{supp} g''|_L = \dim (L\cap H^d)(L\cap P'')$. (If necessary, we take a left H^d-translation of g'' instead of g'' itself.)

Define the roots $\beta_2'', \cdots, \beta_k''$ of $\Sigma(\mathfrak{l}; \mathfrak{a}_\mathfrak{p}'')$ as in § 2. Then it follows from Lemma 2.3 (i) that

$$\begin{aligned} n_i &= \sum\nolimits_{\alpha\in\Sigma\cap(\beta_i+Z\beta_1+\cdots+Z\beta_{i-1})} m_\alpha^0 \\ &= \sum\nolimits_{\alpha''\in\Sigma(\mathfrak{a}_\mathfrak{p}'')\cap(\beta_i''+Z\beta_2''+\cdots+Z\beta_{i-1}'')} m_{\alpha''}^0. \end{aligned}$$

Thus the sequence $\{\beta_2'', \cdots, \beta_k''\}$ of roots in $\Sigma(\mathfrak{l})^+$ satisfies the conditions (i) and (ii) in (P) for the Lie algebra $\mathfrak{l}$. So the assumption of induction implies

$$\langle \mu''+\rho^L-2\rho_t^L, \beta_k''\rangle \geq 0$$

because of the existence of the hyperfunction $g''|_L$ on L which we considered above. It is clear that $\lambda|_{\mathfrak{a}_\mathfrak{p}''\cap\mathfrak{l}}=\mu''|_{\mathfrak{a}_\mathfrak{p}''\cap\mathfrak{l}}$ from the definition of μ''. Thus we have

$$\langle \mu_\lambda, \beta_k\rangle = \langle \lambda+\rho-2\rho_t, \beta_k\rangle = \langle \mu''+\rho^L-2\rho_t^L, \beta_k''\rangle \geq 0$$

by Lemma 2.3 (ii) and therefore we have proved that the condition (P) is satisfied. Q.E.D.

Appendix.

Proof of Lemma 1.2. By [M1] Theorem 1, there exist an $h \in H^d$ and a $p \in P^d$ such that $x'=hxp \in K^d$ and that $\mathfrak{a}_\mathfrak{p}'=\mathrm{Ad}(x')\mathfrak{a}_\mathfrak{p}^d$ is σ-stable. If either of the conditions (i) or (ii) is satisfied, then

(A.1) $$\mathfrak{a}_\mathfrak{p}'\cap \mathfrak{q}^d \neq \{0\}$$

by [M1] § 3 Proposition 2. The case (iii) is reduced to the case (ii) by a similar argument as in [OM1] § 5. So we may assume (A.1) in the followings.

We may moreover assume that $\mathfrak{a}_\mathfrak{p}'\cap\mathfrak{q}^d\subset\mathfrak{a}$ by [M1] Theorem 2. Put $t=\dim(\mathfrak{a}_\mathfrak{p}'\cap\mathfrak{q}^d)$ and choose an orthogonal basis $\{Y_1, \cdots, Y_l\}$ of $\mathfrak{a}_\mathfrak{p}^d$ so that $\{Y_1, \cdots, Y_t\}$ and $\{Y_1, \cdots, Y_{l_0}\}$ are the basis of $\mathfrak{a}_\mathfrak{p}'\cap\mathfrak{q}^d$ and $\mathfrak{a}$, respectively.

Let w_1 be the element of $W(\mathfrak{a}_\mathfrak{p}^d)$ such that $w_1\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$ is the lexicographic order defined by the sequence $\{Y_1, \cdots, Y_l\}$. (i.e. $\alpha \in w_1\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$ if and only if there exists a u $(1 \leq u \leq l)$ such that $\alpha(Y_1) = \cdots = \alpha(Y_{u-1}) = 0$ and that $\alpha(Y_u) > 0$.) Since $w_1\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$ is $\sigma\theta$-compatible, $H^d w_1 P^d$ is open in G^d by [M1] § 3 Proposition 1. Let w_2 be the element of $W(\mathfrak{a}_\mathfrak{p}^d)$ such that $w_2\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+ = -w_1\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$. Then $H^d w_2 P^d$ is also open in G^d.

By the choice of w_1, there exists an i $(1 \leq i \leq l_0)$ such that $w_1\omega_i \in \mathfrak{a}_\mathfrak{p}' \cap \mathfrak{q}^d$. There also exists a j $(1 \leq j \leq l_0)$ such that $w_2\omega_j = -w_1\omega_i$.

First consider the case $\langle \lambda, \mathrm{Ad}(x'^{-1}w_1)\omega_i \rangle \geq 0$. Choose $w_3 \in W(\mathfrak{a}_\mathfrak{p}^d)$ so that $\mathrm{Ad}(w_3^{-1}x'^{-1}w_1)\omega_i$ is dominant for $\Sigma(\mathfrak{a}_\mathfrak{p}^d)^+$. Then $w_1^{-1}x'w_3$ is contained in the parabolic subgroup P_{ω_i} of G^d defined by the element $\omega_i \in \mathfrak{a} \subset \mathfrak{a}_\mathfrak{p}^d$. Hence

$$\text{(A.2)} \qquad w_1^{-1}x'w_3P^dw_4P^d \ni 1 \quad \text{and} \quad w_4\omega_i = \omega_i$$

for some $w_4 \in W(\mathfrak{a}_\mathfrak{p}^d)$. Since $w_1^{-1}x'w_3 \in K^d$, we have

$$\text{(A.3)} \qquad w_1^{-1}x'w_3 \in P_{\omega_i} \cap \sigma P_{\omega_i} = Z_{G^d}(\omega_i).$$

There exists a $w_3' \in W(\mathfrak{a}_\mathfrak{p}^d)$ such that

$$\text{(A.4)} \qquad w_3' \leq w_3 \text{ and that } w_3P^dw_4P^d \subset (P^dw_3'w_4P^d)^{cl}.$$

Put $w = w_3'w_4$. Then we have

$$H^dx(P^dwP^d)^{cl} = H^dx'(P^dwP^d)^{cl} \supset H^dx'w_3P^dw_4P^d \supset H^dw_1P^d$$

by (A.2) and (A.4). On the other hand,

$$\begin{aligned}\langle w^{-1}\lambda, \omega_i \rangle &= \langle w_4^{-1}w_3'^{-1}\lambda, \omega_i \rangle \\ &= \langle \lambda, w_3'\omega_i \rangle \\ &\geq \langle \lambda, w_3\omega_i \rangle \\ &= \langle \lambda, \mathrm{Ad}(x'^{-1}w_1)\omega_i \rangle \\ &\geq 0\end{aligned}$$

by (A.2) and (A.3).

In the case that $\langle \lambda, \mathrm{Ad}(x'^{-1}w_1)\omega_i \rangle < 0$, we can prove the assertion by the same argument as above because $\langle \lambda, \mathrm{Ad}(x'^{-1}w_2)\omega_j \rangle > 0$. Q.E.D.

References

[F] M. Flensted-Jensen, Discrete series for semisimple symmetric spaces, Ann. of Math., **111** (1980), 253–311.

[M1] T. Matsuki, The orbits of affine symmetric spaces under the action of minimal parabolic subgroups, J. Math. Soc. Japan, **31** (1979), 331–357.

[M2] ——, Closure relations for orbits on affine symmetric spaces under the action of minimal parabolic subgroups, Advanced Studies in Pure Math., this volume, 541–559.

[O] T. Oshima, Asymptotic behavior of spherical functions on semisimple symmetric spaces, Advanced Studies in Pure Math., this volume, 561–601.

[OM1] T. Oshima and T. Matsuki, A description of Discrete series for semisimple symmetric spaces, Advanced Studies in Pure Math., **4** (1984), 331–390.

[OM2] ——, A description of discrete series for semisimple symmetric spaces III, in preparation.

Department of Mathematics
College of General Education
Tottori University
Tottori, 680
Japan