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Cohomological Hardy Space for $SU(2,2)$

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Introduction

Let G be a connected real semisimple linear Lie group and let P be a parabolic subgroup. Let G_C and P_C be the complexification of G and P respectively. Our aim is to find a good description of relations between the G-orbits of G_C/P_C and subquotients of degenerate principal series. In this article we treat an example for the group $SU(2, 2)$.

Let $G=SU(2, 2)$ and $K=S(U(2)\times U(2))$. Let P be a parabolic subgroup of G such that G/P is Shilov boundary of G/K. Then G/P is a unique closed G-orbit of G_C/P_C and there exist three open G-orbits of G_C/P_C. Two open orbits are isomorphic to G/K as G-homogeneous space. But in this article we consider the other orbit. This orbit is isomorphic to a semisimple symmetric space $SU(2,2)/S(U(1, 1)\times U(1, 1))$. We call this orbit $\bar{D}$. We consider the homogeneous line bundle L corresponding to the representation in unitary degenerate series with "the most singular parameter". We can get a holomorphic homogeneous line bundle on G_C/P_C whose restriction to G/P is L. We denote this line bundle and the sheaf of its holomorphic sections by the same letter L. We investigate some relation between the Čech cohomology group $H^2(\bar{D}, L)$ and a decomposition of the above degenerate series representation in Kashiwara and Vergne [KV]. Although the K-type of this cohomology group is known by the very general result of Rawnsley, Schmid, and Wolf [RSW], our approach is purely geometric and we construct an injective G-equivariant "boundary map" of the cohomology space to the space of hyperfunction-section of L on G/P using a Mayer-Vietris exact sequence. We remark this construction of the boundary map is applicable in the case of $SO_0(n, 2)$.

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§ 1. The representation in degenerate series of $SU(2, 2)$ with "the most singular parameter"

1.1. Let F_C be the complex Grassmann manifold of all 2-dimensional subspaces in C^4. Let $e_0 \in F_C$ be the subspace of C^4 which is generated by two vectors:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Then $G_C = GL(4, C)$ acts on F_C transitively, and the stabilizer at e_0 is the group:

$$P_C = \left\{ \left(\begin{array}{c|c} * & 0 \\ \hline * & * \end{array} \right) \in G_C \right\}.$$

Here, each $*$ means an arbitrary 2×2 complex matrix. Hence F_C is identified with the homogeneous space G_C / P_C.

Put

$$J = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}.$$

Next we define a real form G of G_C by

$$G_C = \{ \gamma \in G_C \mid \gamma^* J \gamma = J \}.$$

Here, γ^* means the complex conjugate of the transpose of γ.

Next we consider the G-orbit structure of F_C (for example see Wolf [W]). For positive integers p and q such that $0 \leq p + q \leq 2$, we denote by $O^{(p, q)}$ the set of elements x of F_C such that the signature of the restriction to x of the Hermitian form corresponding to J is (p, q).

Then we have the following G-orbital decomposition:

$$F_C = \cup\, O^{(p, q)} \qquad (0 \leq p + q \leq 2; \text{ disjoint union}).$$

The open orbits are $O^{(2,0)}$, $O^{(1,1)}$, and $O^{(0,2)}$. The two orbits $O^{(2,0)}$ and $O^{(0,2)}$ have a structures of Hermitian symmetric spaces. We write O^+, O^-, and $\bar{D}$ for $O^{(2,0)}$, $O^{(0,2)}$, and $O^{(1,1)}$ respectively. Let $e_1 \in F_C$ be the 2-dimensional subspace of C^4 which is generated by two vectors

$$\begin{pmatrix} i \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -i \\ 0 \\ 0 \end{pmatrix}.$$

Then we have $e_1 \in \bar{D}$.

Let E be the 2×2-matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The stabilizer H of G at e_1 is written as follows.

$$\begin{aligned} H &= S(U(1, 1)\times U(1, 1)) \\ &= \left\{ \left(\begin{array}{c|c} AE & -CE \\ \hline EC & EA \end{array} \right) \in G \,\middle|\, AC^* = CA^*,\ AEA^* + CEC^* = 1 \right\}. \end{aligned}$$

Hence $\bar{D}$ has a structure of a semisimple symmetric space.

$O^{(0,0)}$ is a unique closed orbit and we write F for this closed orbit. Then $e_0 \in F$, and the stabilizer of G at e_0 is:

$$P = \left\{ \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array} \right) \in G \right\}.$$

Here, each $*$ means an arbitrary 2×2-matrix. Hence we identify F and G/P.

Next we consider some open cell of F_C and F. Let $H(2)$ be the set of all the 2×2 Hermitian matrices. Put

$$\bar{N} = \left\{ \left(\begin{array}{c|c} 1 & X \\ \hline 0 & 1 \end{array} \right) \,\middle|\, X \in H(2) \right\} \subseteq G,$$

$$\bar{N}_C = \left\{ \left(\begin{array}{c|c} 1 & * \\ \hline 0 & 1 \end{array} \right) \in G_C \right\}.$$

Then $\bar{N}_C \cdot e_0$ is an open dence $\bar{N}_C$-orbit of F_C, and is identified with $M_2(C) = \{2\times 2\text{-matrices}/C\}$ or C^4 via the following correspondence.

$$(1) \quad \begin{pmatrix} 1 & 0 & z_1+z_2 & z_3-iz_4 \\ 0 & 1 & z_3+iz_4 & z_1-z_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longleftrightarrow \begin{pmatrix} z_1+z_2 & z_3-iz_4 \\ z_3+iz_4 & z_1-z_2 \end{pmatrix} \longleftrightarrow (z_1, z_2, z_3, z_4).$$

We can also identify $\bar{N}\cdot e_0$ with $H(2)$ or $\boldsymbol{R}^4$ via the above correspondence (1). $\bar{N}\cdot e_0$ is an open dence subset of F.

We put for $z_q \in \boldsymbol{C}$ $(q=1, \cdots, 4)$

$$z_q = x_q + iy_q \qquad (x_q, y_q \in \boldsymbol{R}^4).$$

Then we have $O^\pm \subseteq \bar{N}_{\boldsymbol{C}}\cdot e_0 = \boldsymbol{C}^4$, and

$$O^+ = \{(z_1, \cdots, z_4) \in \boldsymbol{C}^4 \,|\, y_1^2 - y_2^2 - y_3^2 - y_4^2 > 0, y_1 > 0\},$$
$$O^- = \{(z_1, \cdots, z_4) \in \boldsymbol{C}^4 \,|\, y_1^2 - y_2^2 - y_3^2 - y_4^2 > 0, y_1 < 0\}.$$

These are the realizations of Hermitian symmetric spaces as a tube domains. $\boldsymbol{R}^4 = H(2)$ is the Shilov boundary of $O^\pm$. Next we put $D = \bar{D} \cap \bar{N}_{C\cdot e_0}$. Then D is an open dence subset of $\bar{D}$, and we have

$$D = \{(z_1, \cdots, z_4) \in \boldsymbol{C}^4 \,|\, y_1^2 - y_2^3 - y_3^2 - y_4^2 < 0\}.$$

1.2. According to Kashiwara and Vergne [KV], we describe a representation of G which is realized on a function space on Shilov boundary $H(2)$. Let $L^2(H(2))$ be the L^2-space with respect to the Euclidean measure on $H(2) = \boldsymbol{R}^4$. For $f \in L^2(H(2))$, $X \in H(2)$ and $g \in G$ such that $g^{-1} = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right)$ (a, b, c, d, are 2×2-matrices), we define

$$(T(g)f)(X) = (\det(cX+d))^{-2} f((aX+b)(cX+d)^{-1}).$$

Here, the above formula is well-defined for almost all $X \in H(2)$, and $(T, L^2(H(2))$ is a unitary representation of G.

In fact this representation belongs to the unitary degenerate series and is realized on the space of sections of a homogeneous line bundle L on F defined as follows. First for $Y = \left(\begin{array}{c|c} a & 0 \\ \hline c & d \end{array}\right) \in P_C$ we put

$$\rho'(Y') = (\det d)^2.$$

Then ρ' is a 1-dimensional holomorphic representation of P_C. Let L be the holomorphic homogeneous line bundle on $F_C = G_C/P_C$ associated with ρ'. We also denote the restriction of L to F by the same letter L. Then the space of hyperfunction-sections of L on F is identified with the following space.

$$\mathscr{B}(F; L) = \{f \in \mathscr{B}(G) \,|\, f(gp) = \rho'(p)^{-1} f(g) \qquad \text{for } g \in G, p \in P\}.$$

Here, $\mathscr{B}(G)$ is the space of hyperfunctions on G. The representation corresponding to L belongs to unitary degenerate series and its restriction to the open cell $\bar{N}_C \cdot e_0$ is $(T, L^2(H(2)))$. (See Jakobsen and Vergne [JV].)

Next we consider the Fourier transformation of $(T, L^2(H(2)))$. Let $H(2)^*$ be the dual vector space (over $\boldsymbol{C}$) of $H(2)$. We identify $H(2)$ and $H(2)^*$ via a bilinear form $\operatorname{Tr} XY$ $(X, Y \in H(2))$. Here we have

$$\text{(2)} \qquad \operatorname{Tr}\begin{pmatrix} z_1+z_2 & z_3-iz_4 \\ z_3+iz_4 & z_1-z_2 \end{pmatrix}\begin{pmatrix} v_1+v_2 & v_3-iv_4 \\ v_3+iv_4 & v_1-v_2 \end{pmatrix} = 2(z_1v_1+z_2v_2+z_3v_3+z_4v_4).$$

For $f \in L^2(H(2))$ and $\Xi \in H(2)^*$, we define the Fourier transformation as follows.

$$(\mathscr{F}f)(\Xi) = \hat{f}(\Xi) = \int e^{-i \operatorname{Tr} X\Xi} f(X) dX.$$

Here dX is the Euclidean measure on $\boldsymbol{R}^4 = H(2)$. Let $\mathscr{F}^{-1}$ be the inverse Fourier transformation. For $g \in G$ and $f \in L^2(H(2)^*)$ we put

$$\hat{T}(g)f = \mathscr{F}(T(g)(\mathscr{F}^{-1}f)).$$

Then $(\hat{T}, L^2(H(2)^*))$ is a unitary representation of G which is isomorphic to $(T, L^2(H(2)))$.

Put

$$\bar{P} = \left\{ \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array} \right) \in G \right\},$$

and

$$L = \left\{ \left(\begin{array}{c|c} a & 0 \\ \hline 0 & (a^*)^{-1} \end{array} \right) \middle| \, a \in GL(2, \boldsymbol{C}) \det(a) \in \boldsymbol{R} \right\}.$$

Then $\bar{P} = L\bar{N}$ is a Levi decomposition of a maximal parabolic subgroup. For $a \in GL(2, \boldsymbol{C})$ such that $\det(a) \in \boldsymbol{R}$ and $X \in H(2)$ we have the followings.

$$\left(\hat{T}\left(\left(\begin{array}{c|c} a & 0 \\ \hline 0 & (a^*)^{-1} \end{array} \right) \right) f \right)(\Xi) = (\det(a))^2 f(a\Xi a^*),$$

$$\left(\hat{T}\left(\left(\begin{array}{c|c} 1 & X \\ \hline 0 & 1 \end{array} \right) \right) f \right)(\Xi) = e^{i \operatorname{Tr} X\Xi} f(\Xi).$$

Let V_+, V, and V_- be the spaces of the elements of $H(2)^*$ whose signature as Hermitian forms are (2, 0), (1, 1), and (0, 2) respectively. From the above formulas we have the following decomposition of $\bar{P}$-representations.

$$(3) \qquad L^2(H(2)^*)=L^2(V_+)\oplus L^2(V)\oplus L^2(V_-).$$

Then we easily have:

Lemma 1.2.1. *The decomposition* (3) *is a decomposition into irreducible $\bar{P}$-representations.*

The following theorem is a special case of the result of Kashiwara and Vergne [KV].

Theorem 1.2.2. *The decomposition* (3) *is a decomposition into irreducible G-representations.*

The representations $L^2(V_\pm)$ are realized as Hardy spaces on the Hermitian symmetric space with respect to G. So, we consider the representation $L^2(V)$ hereafter.

§ 2. Factorization of the inverse Fourier transformation

2.1. We identify $H(2)^*$ and $\boldsymbol{R}^4$ via the following correspondence.

$$(v_1, \cdots, v_4)\longleftrightarrow\begin{pmatrix} v_1+v_2 & v_3-iv_4 \\ v_3+iv_4 & v_1-v_2 \end{pmatrix}.$$

Then we have

$$V=\{(v_1, \cdots, v_4)\in \boldsymbol{R}^4 \mid v_1^2-v_2^2-v_3^2-v_4^2<0\}.$$

Here we consider the following 2-sphere

$$S^2=\{(v_2, v_3, v_4)\mid v_2^2+v_3^2+v_4^2=1\}.$$

For $x=(x_2, x_3, x_4)\in \boldsymbol{R}^3-\{(0, 0, 0)\}$ we define $p(x)\in S^2$ by

$$p(x)=(x_2/|x|, x_3/|x|, x_4/x|),$$

where $|x|=(x_2^2+x_3^2+x_4^2)^{1/2}$.

We have:

Lemma 2.1.1. *There exists a family of triangulations of S^2 $\{\Theta_n|n\in N\}$ which satisfies the following conditions.*

(A) *Each edge of Θ_n $(n\in N)$ is a geodesic arc with respect to the Riemannian metric induced from the Euclidean metric of $\boldsymbol{R}^3$.*

(B) *Θ_{n+1} is a subdivision of Θ_n for all $n \in \boldsymbol{N}$.*

(C) *If n tends to ∞ then the maximum of diameters of faces of Θ_n tends to zero.*

(D) *Each triangle of Θ_n does not have an obtuse angle.*

Proof. We consider an octahedron H whose vertices are $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$. We define a triangulation Ψ_n of H whose edges are all straight line segments as follows. First let Ψ_1 be the triangulation whose vertices, edges, and faces are vertices, edges, and faces of the octahedorn H respectively.

Next, using the following subdivisional triangulations of each face of H consisting of only regular triangles (Fig. 1), we can define subdivisions of Ψ_n for all $n \geq 2$ which have the above mentioned property (B), (C), and (D).

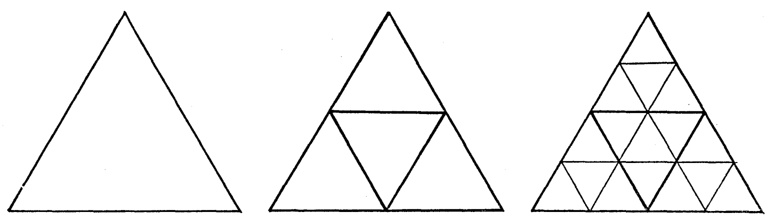


Fig. 1

Finally we define Θ_n by the image of Ψ_n under p. Then we can easily see $\{\Theta_n \mid n \in \boldsymbol{N}\}$ has desired properties. □

Hereafter we fix some Θ_n which is sufficiently fine. We write Θ for Θ_n for simplicity. Let Δ be a triangle of Θ whose vertices are ξ^1, ξ^2, ξ^3. Here, $\xi^i = (\xi_2^i, \xi_3^i, \xi_4^i) \in \boldsymbol{R}^3$, and $\sum_{j=2}^4 (\xi_j^i)^2 = 1$. We, if necessary, change the numeration and we assume

$$\det \begin{pmatrix} \xi_1 \\ \xi^2 \\ \xi^3 \end{pmatrix} > 0.$$

We define

$$V_\Delta = \left\{ (v_1, \cdots, v_4) \in V \;\middle|\; \det \begin{pmatrix} v' \\ \xi^2 \\ \xi^3 \end{pmatrix} \geq 0,\ \det \begin{pmatrix} \xi^1 \\ v' \\ \xi^3 \end{pmatrix} \geq 0,\ \det \begin{pmatrix} \xi^1 \\ \xi^2 \\ v' \end{pmatrix} \geq 0 \right\},$$

where $v' = (v_2, v_3, v_4)$. Then the convex hull of V_Δ is a proper convex cone.

We denote the set of triangles of Θ or the set of vertices of Θ by the same letter Θ. Then we easily get:

Lemma 2.1.2. *$V=\bigcup_{\Delta\in\Theta}V_\Delta$. This union is disjoint except for a set of measure zero.*

We define a function χ_Δ on V by

$$\chi_\Delta=\begin{cases}1 & \text{if } x\in V_\Delta\\ 0 & \text{otherwise.}\end{cases}$$

For $f\in L^2(V)$ we put $f_\Delta=f\cdot\chi_\Delta$. Then we have

$$f=\sum_{\Delta\in\Theta}f_\Delta \qquad \text{(as } L^2\text{-functions).}$$

2.2. Put

$$S_\Delta=\bigcup_{\substack{i=1,2,3\\ \varepsilon=\pm1}}\{(\varepsilon t,\, th\xi_2^i,\, th\xi_3^i,\, th\xi_4^i)\in \boldsymbol{R}^4\,|\,t\geq 0,\, h\geq 1\}.$$

We denote the closure of the face of $\Delta\in\Theta$ by the same letter Δ. Then we have:

Lemma 2.2.1. *V_Δ is contained in the convex hull of S_Δ.*

Proof. Let $v=(v_1, v_2, v_3, v_4)\in V_\Delta$. Put $|v'|=(v_2^2+v_3^2+v_4^2)^{1/2}$. If $v_1\neq 0$, v is contained in the convex hull of

$$\bigcup_{i=1,2,3}\{(v_1,\, h|v_1|\xi_2^i,\, h|v_1|\xi_3^i,\, h|v_1|\xi_4^i)\in\boldsymbol{R}^4\,|\,h\geq 1\},$$

since $|v_1|<|v'|$ and $(v_2/|v'|,\, v_3/|v'|,\, v_4/|v'|)$ is contained in Δ whose vertices are ξ^1, ξ^2, ξ^3.

If $v_1=0$, then we have $(\pm\delta, v_2, v_3, v_4)\in V_\Delta$ for sufficiently small δ. □

For a vertex $\xi=(\xi_2, \xi_3, \xi_4)$ in Θ, we put

$$W'_\xi=\{(z_1,\cdots,z_4)\in\boldsymbol{C}^4|\ \ |y_1|<\xi_2y_2+\xi_3y_3+\xi_4y_4\}.$$

Here, $y_i=\Im z_i$ for $i=2, 3, 4$. We define

$$D_\Theta=\bigcup_{\xi\in\Theta}W'_\xi.$$

Lemma 2.2.2. *Let Δ be a triangle of Θ and ξ^i $(i=1, 2, 3)$ the, vertices of Δ. Then $\mathscr{F}^{-1}f_\Delta$ is holomorphic on $W'_{\xi^1}\cap W'_{\xi^2}\cap W'_{\xi^3}$.*

Proof. For $z_i\in\boldsymbol{C}$ $(i=1,\cdots,4)$ we put $y_i=\Im z_i$. Put

$$U_\Delta^* = \{(z_1, \cdots, z_4) \in C^4 \mid \forall (v_1, \cdots, v_4) \in S_\Delta\ y_1v_1 + \cdots + y_4v_4 > 0\}.$$

From Lemma 2.2.1, $\mathscr{F}^{-1}f_\Delta$ is holomorphic on U_Δ^*. On the other hand we have

$$\begin{aligned} U_\Delta^* &= \bigcap_{i=1,2,3} \{(z_1, \cdots, z_4) \in C^4 \mid \forall h \geq 1 \forall t > 0 \forall \varepsilon = \pm \\ &\quad 1\ \varepsilon t y_1 + thy_2\xi_2^i + thy_3\xi_3^i + thy_4\xi_4^i > 0\} \\ &= \bigcap_{i=1,2,3} \{(z_1, \cdots, z_4) \in C^4 \mid y_2\xi_2^i + y_3\xi_3^i + y_4\xi_4^i > |y_1|\} \\ &= \bigcap W'_{\xi^i} \end{aligned}$$

□

For each vertex ξ we denote by St (ξ) the open kernel of

$$\bigcup_{\substack{\Delta \in \Theta \\ \xi \in \Delta}} \Delta.$$

Here, we identify each triangle Δ and the closure of its face as above. We put

$$W_\xi = \{(z_1, \cdots, z_4) \in W'_\xi \mid p(y_2, y_3, y_4) \in \mathrm{St}\,(\xi)\}.$$

Here, $y_i = \Im z_i$ $(i=2, 3, 4)$. We can immediately see W_ξ is convex. Especially each W_ξ is Stein. From (D) of Lemma 2.1.1, we easily have:

Lemma 2.2.3.

$$D_\Theta = \bigcup_{\xi \in \Theta} W_\xi.$$

Let $\mathscr{O}$ be the sheaf of germs of holomorphic functions. For a triangle $\Delta \in \Theta$ let ξ_Δ^i $(i=1, 2, 3)$ be the vertices of Δ. We can assume

$$\det \begin{pmatrix} \xi_\Delta^1 \\ \xi_\Delta^2 \\ \xi_\Delta^3 \end{pmatrix} > 0.$$

Let $f \in L^2(V)$. Since $\mathscr{F}^{-1}f_\Delta \in \mathscr{O}(W_{\xi_\Delta^1} \cap W_{\xi_\Delta^2} \cap W_{\xi_\Delta^3})$ we can define

$$\varphi_\Theta(f) = \sum_{\Delta \in \Theta} \mathscr{F}^{-1}f_\Delta W_{\xi_\Delta^1} \wedge W_{\xi_\Delta^2} \wedge W_{\xi_\Delta^3}.$$

Here, we use the notation of V.P. Palamodov for cochains. See Palamodov [P] Chapter 3, Section 3 p. 105–110.

Put

$$\mathscr{U}_\Theta = \{W_\xi \mid \xi \in \Theta\}.$$

Let $Z^2(\mathscr{U}_\Theta, \mathscr{O})$ be the space of 2-cocycles of the Čech complex of $\mathscr{O}$-coefficient with respect to the Leray (Stein) covering $\mathscr{U}_\Theta$. Since the intersection of any four distinct W_ξ's is empty, $Z^2(\mathscr{U}_\Theta, \mathscr{O})$ conincides with the space of 2-cochains. Hence we have the following map.

$$\varphi_\Theta : L^2(V) \longrightarrow Z^2(\mathscr{U}_\Theta, \mathscr{O}).$$

2.3. Now we review some fundamental facts about the theory of hyperfunctions. For details, see [KKK].

For a sheaf $\mathscr{S}$ on $\boldsymbol{C}^n$, we define a sheaf $\Gamma_{\boldsymbol{R}^n}(\mathscr{S})$ as follows.

$$\Gamma_{\boldsymbol{R}^n}(\mathscr{S})(U) = \{s \in \mathscr{S}(U) | \operatorname{supp}(s) \subseteq U \cap \boldsymbol{R}^n\},$$

for all open subset U of $\boldsymbol{C}^n$. Here $\operatorname{supp}(s)$ means the support of s. Let ι be the natural embedding $\boldsymbol{R}^n \hookrightarrow \boldsymbol{C}^n$.

If we fix an orientation of $\boldsymbol{R}^n$, then we can define the sheaf of germs of hyperfunctions $\mathscr{B}$ by $\iota^{-1} R^n \Gamma_{\boldsymbol{R}^n}(\mathscr{O}_{\boldsymbol{C}^n})$, where $R^n \Gamma_{\boldsymbol{R}^n}$ means the n-th derived functor of $\Gamma_{\boldsymbol{R}^n}$ and $\mathscr{O}_{\boldsymbol{C}^n}$ is the sheaf of germs of holomorphic functions on $\boldsymbol{C}^n$.

We can represent the space of global sections $\mathscr{B}(U)$ on any open subset U of $\boldsymbol{R}^n$ by relative cohomologies as follows.

$$\mathscr{B}(U) = H_U^n(U', \mathscr{O}),$$

where U' is any complex neighbourhood of U in $\boldsymbol{C}^n$. We also have

$$H_U^q(U', \mathscr{O}) = 0. \qquad (q \neq n)$$

Next we consider the (abstract) boundary values of holomorphic functions. Let W be an open subset of $\boldsymbol{C}^n$. We call W a proper convex conic tube domain, if there exists some proper open convex cone Q in $\boldsymbol{R}^n$ whose vertex is the origin such that $W = \boldsymbol{R}^n + iQ$. Let W be a proper convex conic tube domain. Then for each holomorphic function f on W we can define a boundary value $b_W(f) \in \mathscr{B}(\boldsymbol{R}^n)$ (or sometimes we write $b(W; f)$ or simply $b(f)$). The boundary values have the following properties.

(A) *Let W and W' be proper convex conic tube domains such that $W' \subseteq W$. Let f be a holomorphic function on W. Then we have*

$$b_W(f) = b_{W'}(f_{W'}).$$

(B) *Let f be a holomorphic function on W such that $\lim_{t \to 0} f(x+ity)$ $(x, y \in \boldsymbol{R}^n, x+iy \in W)$ exists as a distribution. Then $\lim_{t \to 0} f(x+ity) = (b_W(f))(x)$. Here we can regard the space of distributions as a subspace of the space of hyperfunctions.*

Next we consider the relation between relative cohomologies and boundary values. Let $\{W_1, \cdots, W_m\}$ be a open covering of $\boldsymbol{C}^n-\boldsymbol{R}^n$ such that each W_i is an open proper convex conic tube domain and the intersection of any $n+1$ distinct W_i's is empty. Then we can immediately see $\{W_1, \cdots, W_m\}$ is a Leray covering with respect to not only the sheaf of germs of holomorphic function $\mathcal{O}$ but also constant sheaf of $\boldsymbol{Z}$-coefficient. If we assume $n>1$, then we have

$$H^{n-1}(\boldsymbol{C}^n-\boldsymbol{R}^n, \boldsymbol{Z})\cong \boldsymbol{Z}.$$

The above isomorphism is not canonical. Fixing an orientation of $\boldsymbol{R}^n$ is equivalent to fixing an isomorphism

$$\varepsilon\colon H^{n-1}(\boldsymbol{C}^n-\boldsymbol{R}^n, \boldsymbol{Z})\longrightarrow \boldsymbol{Z}.$$

Let $Z^{n-1}(\boldsymbol{C}^n-\boldsymbol{R}^n, \boldsymbol{Z})$ be the space of $(n-1)$-cocycles of the Čech complex of $\boldsymbol{Z}$-coefficient with respect to the Leray covering $\{W_1, \cdots, W_m\}$. Let $p_1\colon Z^{n-1}(\boldsymbol{C}^n-\boldsymbol{R}^n, \boldsymbol{Z})\longrightarrow H^{n-1}(\boldsymbol{C}^n-\boldsymbol{R}^n, \boldsymbol{Z})$ be a natural projection. $Z^{n-1}(\boldsymbol{C}^n-\boldsymbol{R}^n, \boldsymbol{Z})$ is generated over $\boldsymbol{Z}$ by the following elements.

$$W_{i_1}\wedge\cdots\wedge W_{i_n} \qquad (1\leq i_i<\cdots<i_n\leq m).$$

We put

$$\eta_{i_1,\cdots,i_n}=\varepsilon\circ p_1(W_{i_1}\wedge\cdots\wedge W_{i_n}).$$

Then we have

$$\eta_{i_1,\cdots,i_n}=\pm 1.$$

Next we consider the $\mathcal{O}$-coefficient cohomology. Let $Z^{n-1}(\boldsymbol{C}^n-\boldsymbol{R}^n, \mathcal{O})$ be the space of $(n-1)$-cocycles of the Čech complex of $\mathcal{O}$-coefficient with respect to the Stein covering $\{W_1, \cdots, W_m\}$ and $P_2\colon Z^{n-1}(\boldsymbol{C}^n-\boldsymbol{R}^n, \mathcal{O})\rightarrow H^{n-1}(\boldsymbol{C}^n-\boldsymbol{R}^n, \mathcal{O})$ the natural projection. Any element X of $Z^{n-1}(\boldsymbol{C}^n-\boldsymbol{R}^n, \mathcal{O})$ is written as follows.

$$X=\sum_{1\leq i_1<\cdots<i_n\leq m} g_{i_1,\cdots,i_n}W_{i_1}\wedge\cdots\wedge W_{i_n}.$$

Here, $g_{i_1,\cdots,i_n}\in\mathcal{O}(W_{i_1}\cap\cdots\cap W_{i_n})$. If we identify $\mathscr{B}(\boldsymbol{R}^n)$ and $H^n_{\boldsymbol{R}^n}(\boldsymbol{C}^n, \mathcal{O})=H^{n-1}(\boldsymbol{C}^n-\boldsymbol{R}^n, \mathcal{O})$, then we have

$$p_2(X)=\sum_{1\leq i_1<\cdots<i_n\leq m} b(W_{i_1}\cap\cdots\cap W_{i_n}; g_{i_1,\cdots,i_n})\eta_{i_1,\cdots,i_n}.$$

Next we return to the original situation. For each element $c\in Z^2(\mathscr{U}_\Theta, \mathcal{O})$ has the following expression.

$$c=\sum_{\Delta\in\Theta} g_\Delta W_{\xi_\Delta^1}\wedge W_{\xi_\Delta^2}\wedge W_{\xi_\Delta^3}.$$

Here $g_\Delta\in\mathcal{O}(W_{\xi_\Delta^1}\cap W_{\xi_\Delta^2}\cap W_{\xi_\Delta^3})$ and ξ_Δ^i ($i=1, 2, 3$) are the vertices of Δ such that

$$\det\begin{pmatrix}\xi_\Delta^1\\ \xi_\Delta^2\\ \xi_\Delta^3\end{pmatrix}>0.$$

Then we can define a boundary value map

$$b_\Theta\colon Z^2(\mathcal{U}_\Theta, \mathcal{O})\longrightarrow\mathcal{B}(\boldsymbol{R}^4)$$

as follows.

$$b_\Theta(c)=\sum_{\Delta\in\Theta} b(W_{\xi_\Delta^1}\cap W_{\xi_\Delta^2}\cap W_{\xi_\Delta^3}; g_\Delta).$$

Immediately, we have:

Lemma 2.3.1. $\mathcal{F}^{-1}=b_\Theta\circ\varphi_\Theta.$

2.4. Put

$$\Gamma_\Theta^+=\{(z_1, \cdots, z_4)\in\boldsymbol{C}^4\,|\,(z_1, \cdots, z_4)\notin D_\Theta,\ \Im z_1\geq 0\},$$
$$\Gamma_\Theta^-=\{(z_1, \cdots, z_4)\in\boldsymbol{C}^4\,|\,(z_1, \cdots, z_4)\notin D_\Theta,\ \Im z_1\leq 0\}.$$

We have

$$(\boldsymbol{C}^4-\Gamma_\Theta^+)\cup(\boldsymbol{C}^4-\Gamma_\Theta^-)=\boldsymbol{C}^4-\boldsymbol{R}^4,$$
$$(\boldsymbol{C}^4-\Gamma_\Theta^+)\cap(\boldsymbol{C}^4-\Gamma_\Theta^-)=D_\Theta.$$

Hence we have the following Mayer-Vietris exact sequence

$$(4)\qquad \cdots\longleftarrow H^{q+1}(\boldsymbol{C}^4-\boldsymbol{R}^4, \mathcal{O})\overset{\delta_\Theta}{\longleftarrow} H^q(D_\Theta, \mathcal{O})\longleftarrow H^q(\boldsymbol{C}^4-\Gamma_\Theta^+, \mathcal{O})$$
$$\oplus H^q(\boldsymbol{C}^4-\Gamma_\Theta^-, \mathcal{O})\longleftarrow H^q(\boldsymbol{C}^4-\boldsymbol{R}^4, \mathcal{O})\longleftarrow\cdots.$$

Since $\boldsymbol{C}^4$ is Stein, we have

$$H^q(\boldsymbol{C}^4-\Gamma_\Theta^\pm, \mathcal{O})=H^{q+1}_{\Gamma_\Theta^\pm}(\boldsymbol{C}^4, \mathcal{O})\qquad(q\geq 1),$$
$$H^q(\boldsymbol{C}^4-\boldsymbol{R}^4, \mathcal{O})=H^{q+1}_{\boldsymbol{R}^4}(\boldsymbol{C}^4, \mathcal{O})\qquad(q\geq 1).$$

Here, the right hands of the above equations are relative cohomologies (cf. [KKK]). From Kashiwara and Laurent [KL] Théorèm 1.1.2, we have

$$H^q_{\Gamma^{\pm}_{\Theta}}(\boldsymbol{C}^4, \mathcal{O})=0 \qquad (q\neq 0).$$

From 2.3, we have

$$H^q_{\boldsymbol{R}^4}(\boldsymbol{C}^4, \mathcal{O})=\begin{cases}\mathscr{B}(\boldsymbol{R}^4) & q=4,\\ 0 & \text{otherwise.}\end{cases}$$

Since the intersection of any four distinct W_ξ's is empty, we have

$$H^3(D_\Theta, \mathcal{O})=0.$$

Hence we get:

Lemma 2.4.1. (A) *For* $q\neq 0, 2,$

$$H^q(D_\Theta, \mathcal{O})=0.$$

(B) *We have the following exact sequence.*

$$0\longleftarrow H^4_{\Gamma^+_\Theta}(\boldsymbol{C}^4, \mathcal{O})\oplus H^4_{\Gamma^-_\Theta}(\boldsymbol{C}^4, \mathcal{O})\longleftarrow \mathscr{B}(\boldsymbol{R}^4)\overset{\delta_\theta}{\longleftarrow} H^2(D_\Theta, \mathcal{O})\longleftarrow 0.$$

2.5. Let $pr_\Theta\colon Z^2(\mathscr{U}_\Theta, \mathcal{O})\to H^2(D_\Theta, \mathcal{O})$ be the natural projection. We are going to show:

Lemma 2.5.1. $b_\Theta\circ pr_\Theta=\delta_\Theta.$

Proof. We fix some $\Delta\in\Theta$ with vertices $\xi^1_\Delta, \xi^2_\Delta, \xi^3_\Delta$ such that

$$\det\begin{pmatrix}\xi^1_\Delta\\ \xi^2_\Delta\\ \xi^3_\Delta\end{pmatrix}>0.$$

We have only to show

$$b_\Theta\circ pr_\Theta(fW_{\xi^1_\Delta}\wedge W_{\xi^2_\Delta}\wedge W_{\xi^3_\Delta})=\delta_\Theta(fW_{\xi^1_\Delta}\wedge W_{\xi^2_\Delta}\wedge W_{\xi^3_\Delta})$$

for all $f\in\mathcal{O}(W_{\xi^1_\Delta}\cap W_{\xi^2_\Delta}\cap W_{\xi^3_\Delta})$.

Hereafter we put $y_i=\Im z_i$ for $i=1, 2, 3, 4$. Let μ be a sufficiently small positive number. Put

$$\tilde{W}_\xi=\{(z_1, \cdots, z_4)\in \boldsymbol{C}^4\,|\,(1-\mu)|y_1|<\xi_2y_2+\xi_3y_3+\xi_4y_4,\ p(y_2, y_3, y_4)\in \mathrm{St}(\xi)\}.$$

Here, $\xi=(\xi_2, \xi_3, \xi_4)$ is a vertex of Θ. Let $\tilde{\Gamma}^{\pm}_\Theta$ be the open kernels of $\Gamma^{\pm}_\Theta$ respectively. Put

$$\tilde{\mathscr{U}}_\Theta=\{\tilde{\Gamma}_\Theta^+, \tilde{\Gamma}_\Theta^-\}\cup\{\tilde{W}_\xi\,|\,\xi\in\Theta\}.$$

Then $\tilde{\mathscr{U}}_\Theta$ is a Stein covering of $\boldsymbol{C}^4-\boldsymbol{R}^4$ and we can easily see that any five distinct elements of $\tilde{\mathscr{U}}_\Theta$ do not intersect. Put

$$\tilde{W}_\xi^+=\{(z_1, \cdots, z_4)\in\tilde{W}_\xi\,|\,y_1<\xi_2y_2+\xi_3y_3+\xi_4y_4\},$$
$$\tilde{W}_\xi^-=\{(z_1, \cdots, z_4)\in\tilde{W}_\xi\,|\,-y_1<\xi_2y_2+\xi_3y_3+\xi_4y_4\}.$$

We put

$$\mathscr{U}_\Theta^+=\{\tilde{\Gamma}_\Theta^-\}\cup\{\tilde{W}_\xi^+\,|\,\xi\in\Theta\},$$
$$\mathscr{U}_\Theta^-=\{\tilde{\Gamma}_\Theta^+\}\cup\{\tilde{W}_\xi^-\,|\,\xi\in\Theta\}.$$

Then $\mathscr{U}_\Theta^+$ (resp. $\mathscr{U}_\Theta^-$) is a Leray (Stein) covering of $\boldsymbol{C}^4-\Gamma_\Theta^+$ (resp. $\boldsymbol{C}^4-\Gamma_\Theta^-$).

Let $C^*(\mathscr{U}_\Theta, \mathscr{O})$, $C^*(\tilde{\mathscr{U}}_\Theta, \mathscr{O})$, $C^*(\mathscr{U}_\Theta^+, \mathscr{O})$, and $C^*(\mathscr{U}_\Theta^+, \mathscr{O})$ be the cochain complexes for Čech cohomologies with respect to the Leray coverings $\mathscr{U}_\Theta$, $\tilde{\mathscr{U}}_\Theta$, $\mathscr{U}_\Theta^+$, and $\mathscr{U}_\Theta^-$ respectively.

For ξ, ξ', $\xi''\in\Theta$ we easily have

$$\tilde{W}_\xi^+\cap\tilde{W}_\xi^-=W_\xi,$$
$$\tilde{W}_\xi^+\cup\tilde{W}_\xi^-=\tilde{W}_\xi,$$
$$(\tilde{W}_\xi^+\cap\tilde{W}_{\xi'}^+)\cup(\tilde{W}_\xi^-\cap\tilde{W}_{\xi'}^-)=\tilde{W}_\xi\cap\tilde{W}_{\xi'},$$
$$(\tilde{W}_\xi^+\cap\tilde{W}_{\xi'}^+\cap\tilde{W}_{\xi''}^+)\cup(\tilde{W}_\xi^-\cap\tilde{W}_{\xi'}^+\cap\tilde{W}_{\xi''}^-)=\tilde{W}_\xi\cap\tilde{W}_{\xi'}\cap\tilde{W}_{\xi''},\qquad\text{etc.}$$

Hence we can easily see there exists the following exact sequence of complex.

$$(5)\quad 0\longrightarrow C^*(\tilde{\mathscr{U}}_\Theta, \mathscr{O})\xrightarrow{\alpha^*}C^*(\mathscr{U}_\Theta^+, \mathscr{O})\oplus C^*(\mathscr{U}_\Theta^-, \mathscr{O})\xrightarrow{\beta^*}C^*(\mathscr{U}_\Theta, \mathscr{O})\longrightarrow 0.$$

Since the Mayer-Vietris exact sequence (4) is induced from the exact sequence (5), considering the snake lemma, we can describe

$$\delta_\Theta\circ pr_\Theta(fW_{\xi_\Delta^2}\wedge W_{\xi_\Delta^2}\wedge W_{\xi_\Delta^3})$$

as follows. Here $f\in\mathscr{O}(W_{\xi_\Delta^1}\cap W_{\xi_\Delta^2}\cap W_{\xi_\Delta^3})$ and Δ is a triangle in Θ with vertices ξ_Δ^1, ξ_Δ^2, ξ_Δ^3 such that

$$\det\begin{pmatrix}\xi_\Delta^1\\ \xi_\Delta^2\\ \xi_\Delta^3\end{pmatrix}>0.$$

Since we have

$$(\tilde{W}^+_{\xi^1_\Delta}\cap\tilde{W}^+_{\xi^2_\Delta}\cap\tilde{W}^+_{\xi^3_\Delta})\cup(\tilde{W}^-_{\xi^1_\Delta}\cap\tilde{W}^-_{\xi^2_\Delta}\cap\tilde{W}^-_{\xi^3_\Delta})=\tilde{W}_{\xi^1_\Delta}\cap\tilde{W}_{\xi^2_\Delta}\cap\tilde{W}_{\xi^3_\Delta},$$

$$(\tilde{W}^+_{\xi^1_\Delta}\cap\tilde{W}^+_{\xi^2_\Delta}\cap\tilde{W}^+_{\xi^3_\Delta})\cap(\tilde{W}^-_{\xi^1_\Delta}\cap\tilde{W}^-_{\xi^2_\Delta}\cap\tilde{W}^-_{\xi^3_\Delta})=W_{\xi^1_\Delta}\cap W_{\xi^2_\Delta}\cap W_{\xi^3_\Delta},$$

and $\tilde{W}_{\xi^1_\Delta}\cap\tilde{W}_{\xi^2_\Delta}\cap\tilde{W}_{\xi^3_\Delta}$ is Stein, we can write

$$f=h_+-h_-,$$

where $h_\pm\in\mathcal{O}(\tilde{W}^\pm_{\xi^1_\Delta}\cap\tilde{W}^\pm_{\xi^2_\Delta}\cap\tilde{W}^\pm_{\xi^3_\Delta})$.

We denote by $Z^3(\tilde{\mathscr{U}}_\Theta, \mathcal{O})$ the space of 2-cocycle with respect to the covering $\tilde{\mathscr{U}}_\Theta$ and let $pr_{\tilde{\mathscr{U}}_\Theta}: Z^3(\tilde{\mathscr{U}}_\Theta, \mathcal{O})\to H^3(\boldsymbol{C}^4-\boldsymbol{R}^4, \mathcal{O})$ be the natural projection.

Considering the smake lemma, we can easily deduce

$$\begin{aligned}&\delta_\Theta\circ pr_\Theta(fW_{\xi^1_\Delta}\wedge W_{\xi^2_\Delta}\wedge W_{\xi^3_\Delta})\\&\quad=pr_{\tilde{\mathscr{U}}_\Theta}(h_+\tilde{\Gamma}^+_\Theta\wedge\tilde{W}_{\xi^1_\Delta}\wedge\tilde{W}_{\xi^2_\Delta}\wedge\tilde{W}_{\xi^3_\Delta})-pr_{\tilde{\mathscr{U}}_\Theta}(h_-\tilde{W}_{\xi^1_\Delta}\wedge\tilde{W}_{\xi^2_\Delta}\wedge\tilde{W}_{\xi^3_\Delta}\wedge\tilde{\Gamma}^-_\Theta).\end{aligned}$$

If we consider the orientation, we have

$$\begin{aligned}&\delta_\Theta\circ pr_\Theta(fW_{\xi^1_\Delta}\wedge W_{\xi^2_\Delta}\wedge W_{\xi^3_\Delta})\\&\quad=b(\tilde{\Gamma}^+_\Theta\cap\tilde{W}_{\xi^1_\Delta}\cap\tilde{W}_{\xi^2_\Delta}\cap\tilde{W}_{\xi^3_\Delta};h_+)-b(\tilde{W}_{\xi^1_\Delta}\cap\tilde{W}_{\xi^2_\Delta}\cap\tilde{W}_{\xi^3_\Delta}\cap\tilde{\Gamma}^-_\Theta;h_-)\\&\quad=b(\tilde{W}^+_{\xi^1_\Delta}\cap\tilde{W}^+_{\xi^2_\Delta}\cap\tilde{W}^+_{\xi^3_\Delta};h_+)-b(\tilde{W}^-_{\xi^1_\Delta}\cap\tilde{W}^-_{\xi^2_\Delta}\cap\tilde{W}^-_{\xi^3_\Delta};h_-)\\&\quad=b(W_{\xi^1_\Delta}\cap W_{\xi^2_\Delta}\cap W_{\xi^3_\Delta};f)\\&\quad=b_\Theta(fW_{\xi^1_\Delta}\wedge W_{\xi^2_\Delta}\wedge W_{\xi^3_\Delta}).\end{aligned}$$

Q.E.D.

Put

$$\psi_\Theta=pr_\Theta\circ\varphi_\Theta.$$

From Lemma 2.3.1 and Lemma 2.5.1, we immediately have:

Corollary 2.5.2. $\delta_\Theta\circ\psi_\Theta=\mathscr{F}^{-1}$.

2.6. If $n\leq m$, then we have $D_{\Theta_n}\subseteq D_{\Theta_m}$. Then there exists a restriction map

$$r: H^*(D_{\Theta_m}, \mathcal{O})\longrightarrow H(D_{\Theta_m}, \mathcal{O}).$$

We have:

Lemma 2.6.1. *For $m>n$, we have*

$$r\circ\psi_{\Theta_m}=\psi_{\Theta_n}.$$

Proof. Functoriality of the Mayer-Vietris exact sequence implies

$$\delta_{\Theta_m}=\delta_{\Theta_n}\circ r.$$

Hence we have

$$\begin{aligned}\delta_{\Theta_n}\circ r\circ\psi_{\Theta_m}&=\delta_{\Theta_m}\circ\psi_{\Theta_m}\\&=\mathscr{F}^{-1}\\&=\delta_{\Theta_n}\circ\psi_{\Theta_n}.\end{aligned}$$

Since δ_{Θ_n} is injective (Lemma 2.4.1), we have the desired result. □

From Lemma 2.6.1, we get a canonical map

$$\psi'\colon L^2(V)\longrightarrow\varprojlim_n H^2(D_{\Theta_n},\mathcal{O}).$$

Since $D=\bigcup_n D_{\Theta_n}$ from Lemma 2.1.1, we have a canonical map

$$q\colon H^2(D,\mathcal{O})\longrightarrow\varprojlim_n H^2(D_{\Theta_n},\mathcal{O}).$$

We quote:

Lemma 2.6.2 ([KL] Lemma 1.1.6). *Let X be a topological space, F a sheaf on X, and $k\in N$. Let $\{U_n\mid n\in N\}$ be a family of open sets of X which satisfies the following conditions.*

(A) $U_n\subseteq U_{n+1}$ *for all n,*

(B) $\bigcup_n U_n=X$.

(C) *The restriction map* $H^{k-1}(U_{n+1},\mathcal{O})\to H^{k-1}(U_n,F)$ *is surjective for all n.*

Then the canonical map

$$H^k(X,F)\longrightarrow\varprojlim_n H^k(U_n,F)$$

is an isomorphism.

From this lemma and Lemma 2.4.1, we see that q is an isomorphism. Hence from ψ' and q^{-1}, we can define

$$\psi\colon L^2(V)\longrightarrow H^2(D,\mathcal{O}).$$

Let $\Gamma^{\pm}$ be the closures of $D^{\pm}$ in C^4 respectively. Then we have

$$\begin{aligned}(C^4-\Gamma^+)\cup(C^4-\Gamma^-)&=C^4-R^4,\\(C^4-\Gamma^-)\cup(C^4-\Gamma^-)&=D.\end{aligned}$$

Hence we get the following Mayer-Vietris exact sequence.

$$(6)\qquad \cdots \longleftarrow H^3(\boldsymbol{C}^4-\boldsymbol{R}^4, \mathcal{O}) \overset{\delta}{\longleftarrow} H^2(D, \mathcal{O}) \longleftarrow H^2(\boldsymbol{C}^4-\Gamma^+, \mathcal{O}) \oplus H^2(\boldsymbol{C}^4-\Gamma^-, \mathcal{O}) \longleftarrow H^2(\boldsymbol{C}^4-\boldsymbol{R}^4, \mathcal{O}) \longleftarrow \cdots.$$

The above sequence is the inverse limit of (4). Let $\mathfrak{g}$ be the Lie algebra of G and $U(\mathfrak{g})$ the universal enveloping algebra of complexification of $\mathfrak{g}$. We can immediately see all maps in (6) are $U(\mathfrak{g})$ and $\bar{P}$-homomorphism under twisted action compatible with the actions on $\mathscr{B}(\boldsymbol{R}^4)$.

Taking inverse limit, now we can easily have:

Theorem 2.6.3. (A) *For the inverse Fourier transformation*

$$\mathscr{F}^{-1}: L^2(V) \longrightarrow L^2(H(2)) \subseteq \mathscr{B}(H(2))$$

we have $\mathscr{F}^{-1}=\delta\circ\psi$.

(B) $\mathscr{F}^{-1}$, ψ, *and* δ *are all* $U(\mathfrak{g})$ *and* $\bar{P}$*-homomorphisms.*

(C) δ *is injective.*

§ 3. Some cohomology group of the line bundle L on G/H

3.1. From the generalized Borel-Weil-Bott theorem (Kostant [Ko] Theorem 6.4), we have:

Lemma 3.1.1. *Let L be the line bundle defined in* 1.2. *Then we have*

$$H^q(F_C, L)=0 \qquad (q=0, 1, 2, \cdots).$$

3.2. Let $\Gamma^{\pm}$ be the closure of $D^{\pm}$ in F_C respectively. Then we have

$$(F_C-\bar{\Gamma}^+)\cup(F_C-\bar{\Gamma}^-)=F_C-F,$$
$$(F_C-\bar{\Gamma}^+)\cap(F_C-\bar{\Gamma}^-)=\bar{D}.$$

Hence we get the following Mayer-Vietris exact sequence.

$$(7)\qquad \cdots \longleftarrow H^{q+1}(F_C-F, L) \overset{\bar{\delta}}{\longleftarrow} H^q(\bar{D}, L) \longleftarrow H^q(F_C-\bar{\Gamma}^+, L) \oplus H^q(F_C-\bar{\Gamma}^-, L) \longleftarrow H^q(F_C-F, L) \longleftarrow \cdots.$$

From Lemma 3.1.1, for all $q\in N$ we have

$$H^q(F_C-\bar{\Gamma}^{\pm}, L)=H^{q+1}_{\bar{\Gamma}^{\pm}}(F_C, L),$$
$$H^q(F_C-F, L)=H^{q+1}_F(F_C, L).$$

Since we can regard $\bar{\Gamma}^{\pm}$ as a closed convex set in C^4 (See Wolf [Wo] 3.), from the result of Kashiwara-Morimoto (also see [KL] Théorème 1.1.2) we have

$$H^{q+1}_{\bar{\Gamma}^{\pm}}(F_C, L)=0 \qquad (q\neq 3).$$

Since F_C is a complex neighbourhood of F, we have

$$H^q_F(F_C, L)=0 \qquad (q\neq 4),$$
$$H^4_F(F_C, L)=\mathscr{B}(F, L).$$

Hence we have:

Theorem 3.2.1. (A) $H^q(\bar{D}, L)=0$ $(q\neq 2, 3)$.
(B) *The following is a exact sequence of G-equivariant maps.*

$$0\longleftarrow H^3(\bar{D}, L)\longleftarrow H^4_{\bar{\Gamma}^+}(F_C, L)\oplus H^4_{\bar{\Gamma}^-}(F_C, L)\longleftarrow \mathscr{B}(F_C, L) \xleftarrow{\ \bar{\delta}\ } H^2(\bar{D}, L)\longleftarrow 0$$

3.3. Put

$$K=\left\{\left(\begin{array}{c|c} a & -b \\ \hline b & a \end{array}\right)\middle|\ (a+ib, a-ib)\in S(U(2)\times U(2))\right\}.$$

Then K is a maximal compact subgroup of G. We write $\mathfrak{k}$ for the Lie algebra of of K. For a $\mathfrak{k}$-module M we write $M_{\mathfrak{k}}$ for the space of $\mathfrak{k}$-finite elements in M.

Put

$$U=\bar{N}_C\cdot e_0=C^4,$$
$$S=F_C-U.$$

Then we have $\bar{\Gamma}^{\pm}\cap U=\bar{\Gamma}^{\pm}$. Hence we get

$$H^q_{\bar{\Gamma}^{\pm}\cap U}(U, L)=H^q_{\bar{\Gamma}^{\pm}}(C^4, \mathcal{O})=0 \qquad (q\neq 4).$$

Here, we identify $\mathcal{O}$ and the sheaf of germs of holomorphic sections of the restriction of L to U.

Therefore we easily get the following commutative diagram, from the flabbiness of the hyperfunction, Lemma 2.4.1 (B), and Theorem 3.2.1 (B).

$$\begin{array}{ccccccc}
0 & & & & & & \\
\downarrow & & & & & & \\
H^4_{\Gamma^+\cap S}(F_C, L)\oplus H^4_{\Gamma^-\cap S}(F_C, L) & & & & & & \\
\downarrow p & & & & & & \\
H^4_{\Gamma^+}(F_C, L)\oplus H^4_{\Gamma^-}(F_C, L) & \xleftarrow{i^*} & \mathscr{B}(F, L) & \xleftarrow{\bar{\delta}} & H^2(\bar{D}, L) & \longleftarrow & 0 \\
\downarrow r' & & \downarrow r & & \downarrow r'' & & \\
H^4_{\Gamma^+}(C^4, \mathcal{O})\oplus H^4_{\Gamma^-}(C^4, \mathcal{O}) & \xleftarrow{j^*} & \mathscr{B}(H(2)) & \xleftarrow{\delta} & H^2(D, \mathcal{O}) & \longleftarrow & 0 \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

Here, r', r, r'' are restriction maps and all rows and columns are exact.

The following lemma will be proved in the next section.

Lemma 3.3.1. $H^4_{\Gamma^\pm\cap S}(F_C, L)_{\mathfrak{k}}=0.$

Using this lemma, we have:

Lemma 3.3.2. *If $f\in\mathscr{B}(F, L)_{\mathfrak{k}}$ satisfies $r(f)\in\mathrm{Im}(\delta)$, then $f\in\mathrm{Im}(\bar{\delta})$.*

Proof. Since

$$r'\circ i^*(f)=j^*\circ r(f)=0,$$

there exists some element g of $H^4_{\Gamma^+\cap S}(F_C, L)\oplus H^4_{\Gamma^-\cap S}(F_C, L)$ such that $p(g)=i^*(f)$. Since p is injective, g is $\mathfrak{k}$-finite. Hence we have $g=0$. Therefore $i^*(f)=0$. From the exactness, we have the desired conclusion. □

Now we have the main result of this section.

Theorem 3.3.3. *The restriction map*

$$r''\colon H^2(\bar{D}, L)_{\mathfrak{k}}\longrightarrow H^2(D, \mathcal{O})_{\mathfrak{k}}$$

is an $U(\mathfrak{g})$-isomorphism.

Proof. Surjectivity of r'' is immediately deduced from Lemma 3.3.2. Injectivity is deduced from the injectivity of

$$r\colon \mathscr{B}(F, L)_{\mathfrak{k}}\longrightarrow\mathscr{B}(H(2)).$$

Hence r'' gives an isomorphism of $H^2(\bar{D}, L)_{\mathfrak{k}}$ to $H^2(D, \mathcal{O})_{\mathfrak{k}}$. Q.E.D.

From Theorem 2.6.3, we have:

Corollary 3.3.4. *We get an embedding of a* $U(\mathfrak{g})$*-module*:

$$L^2(V)_{\mathfrak{k}} \hookrightarrow H^2(\bar{D}, L).$$

Remark. In the general result of [RSW] 4.28, $H^2(\bar{D}, L)_{\mathfrak{k}}$ is calculated.

§ 4. Proof of Lemma 3.3.1.

4.1. We fix the following Levi part of P_C.

$$L_C = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right) \middle| A, B \in GL(2, \boldsymbol{C}), \det(A) = \det(B)^{-1} \right\}.$$

We fix the following Cartan subalgebra $\mathfrak{h}_C$ of L as well as G.

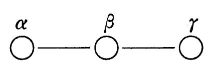
$$\mathfrak{h}_C = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \middle| a+b+c+d=0 \right\}.$$

The Killing form of $\mathfrak{g}_C = \mathfrak{sl}(4, \boldsymbol{C})$ coincides with $\operatorname{Tr} XY$ up to scalar factor. Using this bilinear form, we will identify $\mathfrak{h}_C$ and its dual $\mathfrak{h}_C^*$.

Let Σ be the root system of $\mathfrak{sl}(4, \boldsymbol{C})$ with respect to the Cartan subalgebra $\mathfrak{h}_C$. Let α, β, and γ be roots corresponding to

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

respectively. Then $\{\alpha, \beta, \gamma\}$ forms a fundamental system of roots.



Let Σ^+ be the positive system of Σ with respect to the above fundamental system.

Let W (resp. W_s) be the Weyl group of $G_C = SL(4, \boldsymbol{C})$ (resp. L_C) with respect to the Cartan subalgebra $\mathfrak{h}_C$. Put

$$W_u = \{ w \in W \mid (-w\Sigma^+) \cap \Sigma^+ \subseteq \{\beta, \beta+\gamma, \alpha+\beta+\gamma, \alpha+\beta\}\}.$$

Then we have

$$W_u = \{e, s_\beta, s_\beta s_\alpha, s_\beta s_\gamma, s_\alpha s_\alpha s_\gamma, s_\beta s_\alpha s_\gamma s_\beta\}.$$

Here, e means the identity element of W and s_* means the simple reflection with respect to the simple root $*$ ($*=\alpha, \beta, \gamma$).

Put

$$U_C^+=\left\{\begin{pmatrix}1 & * & 0 & 0\\ 0 & 1 & 0 & 0\\ * & * & 1 & 0\\ * & * & * & 1\end{pmatrix} \in SL(4, C)\right\}.$$

The following is a special case of Borel-Kostant's generalized Bruhat decomposition (Warner [Wa] Proposition 1.2.4.9).

Lemma 4.1.1.

$$G_C=\coprod_{w^{-1}\in W_u} U_C^+\tilde{w}P_C \qquad (\textit{disjoint union}).$$

Here $\tilde{w}$ is some representative of w in G_C.

Put

$$w_0=s_\beta s_\alpha s_\gamma s_\beta s_\gamma s_\alpha=s_\alpha s_\gamma s_\beta s_\gamma s_\alpha s_\gamma.$$

Then we have

$$w_0\Sigma^+=-\Sigma^+,$$

namely w_0 is the longest element of W.

Put

$$U_C^-=w_0U_C^+w_0.$$

Therefore

$$G_C=\coprod_{w^{-1}\in W_u} U_C^-\tilde{w}_0\tilde{w}P_C \qquad (\text{disjoint union}),$$

$$\{w_0w \mid w^{-1}\in W_u\}=\{w_0,\ s_\alpha s_\gamma s_\beta s_\gamma s_\alpha,\ s_\beta s_\alpha s_\beta s_\gamma,\ s_\beta s_\gamma s_\beta s_\alpha,\ s_\beta s_\alpha s_\gamma,\ s_\alpha s_\gamma\}.$$

We can choose

$$\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},$$

which are contained in P_C, as representatives in G_C of s_α and s_γ respectively. Hence, if we put

$$W^*=\{w_0, s_\alpha s_\gamma s_\beta s_\gamma s_\alpha, s_\beta s_\alpha s_\beta, s_\beta s_\gamma s_\beta, s_\beta, e\},$$

then we have

$$G_C=\coprod_{w\in W^*} U_C^- \tilde{w} P_C \qquad \text{(disjoint union).}$$

We choose representatives of the elements of W^* as follows.

$$w_0 \rightsquigarrow \tilde{w}_5=\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$s_\alpha s_\gamma s_\beta s_\gamma s_\alpha \rightsquigarrow \tilde{w}_4=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$s_\beta s_\gamma s_\beta \rightsquigarrow \tilde{w}_3=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$s_\beta s_\alpha s_\beta \rightsquigarrow \tilde{w}_2=\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$s_\beta \rightsquigarrow \tilde{w}_1=\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$e \rightsquigarrow \tilde{w}_0=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hereafter we assume $i=1, 2, 3, 4$. Put

$$e_i=\tilde{w}_i \cdot e_0 \in F_C.$$

Then we have

$$F_C=\coprod_{i=0}^{5} (U_C^- \cdot e_i) \qquad \text{(disjoint union).}$$

Next we consider the following local coordinate system of F_C. Put

$$U_i=\tilde{w}_i\bar{N}_C\cdot e_0=\tilde{w}_i\bar{N}_C\tilde{w}_i^{-1}\cdot e_i.$$

We can introduce a coordinate on U_i as follows.

$$U_i \ni \tilde{w}_i\begin{pmatrix}1 & 0 & z_1+z_2 & z_3-iz_4\\ 0 & 1 & z_3+iz_4 & z_1-z_2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\cdot e_0 \longleftrightarrow (z_1, \cdots, z_4)\in C^4.$$

Then we immediately have

$$U_C^-\cdot e_i\subseteq U_i.$$

The following result follows from some direct calculations.

Lemma 4.1.2. *Under the above coordinates of* U_i $(i=1, \cdots, 4)$, *we have the following description of* $U_C^-\cdot e_i$.

$$\begin{aligned}
&U_C^-\cdot e_0=U_0=\bar{N}_C\cdot e_0,\\
&U_C^-\cdot e_1=\{(z_1, \cdots, z_4)\in C^4\,|\,z_1+z_4=0\}\subseteq U_1,\\
&U_C^-\cdot e_2=\{(z_1, \cdots, z_4)\in C^4\,|\,z_1+z_2=0,\ z_3-iz_4=0\}\subseteq U_2,\\
&U_C^-\cdot e_3=\{(z_1, \cdots, z_4)\in C^4\,|\,z_1+z_2=0,\ z_3+iz_4=0\}\subseteq U_3,\\
&U_C^-\cdot e_4=\{(z_1, \cdots, z_4)\in C^4\,|\,z_1-z_2=0,\ z_3=z_4=0\}\subseteq U_4,\\
&U_C^-\cdot e_5=\{(0, \cdots, 0)\}\subseteq U_5.
\end{aligned}$$

Next we try to represent $D\cap U_i$ by the coordinate on U_i.

Lemma 4.1.3. *If* $i=0$, 1, 4, 5, *then we have*

$$\tilde{D}\cap U_i=\{(z_1, \cdots, z_4)\in C^4\,|\,y_1^2-y_2^2-y_3^2-y_4^2<0,\ y_i=\Im z_i\ i=1, \cdots, 4\}.$$

Especially, if $i==4$, 5, *then*

$$(U_C^-\cdot e_i)\cap\tilde{D}=\emptyset.$$

Proof. If $i=0$, then the above statement means $D=\tilde{D}\cap U_i$. If $i=$ 1, 4, 5, then we have $\tilde{w}_i\in G$. Hence we have the statement of the lemma from the case of $i=0$, □

After direct calculations, we have:

Lemma 4.1.4. *If* $i=2,3$, *then*

$$U_{\bar{C}}^{-}\cdot e_i\subseteq\tilde{D}.$$

4.2. Now we prove Lemma 3.3.1. We will show

$$H^4_{\bar{\Gamma}^+\cap S}(F_C, L)_{\mathfrak{k}}=0.$$

The case of $\bar{\Gamma}^-$ is similar.

First from Lemma 4.1.3, and Lemma 4.1.4, we have

$$\bar{\Gamma}^+\cap S=\{e_5\}\cup(U_{\bar{C}}^{-}\cdot e_4\cap\bar{\Gamma}^+)\cup(U_{\bar{C}}^{-}\cdot e_1\cap\bar{\Gamma}^+).$$

Since $\{e_5\}\cup(U_{\bar{C}}^{-}\cdot e_4\cap\bar{\Gamma}^+)$ is closed in $\bar{\Gamma}^+\cap S$, we have the following exact sequence.

$$H^3_{U_{\bar{C}}^{-}\cdot e_1\cap\bar{\Gamma}^+}(F_C, L)\longrightarrow H^4_{\{e_5\}\cup(U_{\bar{C}}^{-}\cdot e_4\cap\bar{\Gamma}^+)}(F_C, L)\longrightarrow H^4_{\bar{\Gamma}^+\cap S}(F_C, L)$$
$$\longrightarrow H^4_{U_{\bar{C}}^{-}\cdot e_1\cap\bar{\Gamma}^+}(F_C, L).$$

On U_1, we introduce the following new coordinate.

$$\zeta_1=z_1+z_2,\ \zeta_2=z_1-z_2,\ \zeta_3=z_3,\ \zeta_4=z_4.$$

Put

$$C^+=\{\zeta_2\in C\,|\,\Im\zeta_2\geq 0\}.$$

Under the coordinate $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ on U_1, we have

$$H^q_{U_{\bar{C}}^{-}\cdot e_1\cap\bar{\Gamma}^+}(F_C, L)\cong H^q_{U_{\bar{C}}^{-}\cdot e_1\cap\bar{\Gamma}^+}(U_1, \mathcal{O})$$
$$\cong H^q_{\{0\}\times(C^+)\times R^2}(C^4, \mathcal{O}).$$

From [KL] Théorème 1.1.2, we have

$$H^3_{\{0\}\times(C^+)\times R^2}(C^4, \mathcal{O})=0.$$

Hence we have the following exact sequence.

(8)
$$0\longrightarrow H^4_{\{e_5\}\cup(U_{\bar{C}}^{-}\cdot e_4\cap\bar{\Gamma}^+)}(F_C, L)\xrightarrow{\kappa} H^4_{\bar{\Gamma}^+\cap S}(F_C, L)$$
$$\xrightarrow{\tau} H^4_{\{0\}\times(C^+)\times R^2}(C^4, \mathcal{O}).$$

Let f be any element of $H^4_{\bar{\Gamma}^+\cap S}(F_C, L)_{\mathfrak{k}}$, then $\tau(f)\in H^4_{\{0\}\times(C^+)\times R^2}(C^4, \mathcal{O})$ is also κ-finite.

Now we prove:

Lemma 4.2.1. $H^4_{\{0\}\times(C^+)\times R^2}(C^4, \mathcal{O})_{\mathfrak{k}}=0.$

Proof. Put

$$H=\boldsymbol{C}-\{is \mid s\in\boldsymbol{R},\ |s|\geq 1\}.$$

Then H is simply-connected and an analytic function $\sqrt{1+z^2}$ is well-defined on H. From the excision property, we have

$$H^4_{\{0\}\times(\boldsymbol{C}^+)\times\boldsymbol{R}^2}(\boldsymbol{C}^4,\ \mathcal{O})=H^4_{\{0\}\times(\boldsymbol{C}^+)\times\boldsymbol{R}^2}(H\times\boldsymbol{C}^3,\ \mathcal{O}).$$

We introduce new (global) coordinate $(t_1, \cdots, t_4)$ on $H\times\boldsymbol{C}^3$ as follows.

$$\begin{aligned}\zeta_1&=t_1,\\ \zeta_2&=t_2+t_1(t_3^2+t_4^2),\\ \zeta_3&=t_3\sqrt{1+t_1^2},\\ \zeta_4&=t_4\sqrt{1+t_1^2}.\end{aligned}$$

Then we can easily see

$$(t_1, \cdots, t_4)\rightsquigarrow(\zeta_1, \cdots, \zeta_4)$$

gives a holomorphic automorphism of $H\times\boldsymbol{C}^3$.

Next we consider the following 1-parameter subgroup of K.

$$k(\theta)=\tilde{w}_1\begin{pmatrix}\cos\theta & 0 & -\sin\theta & 0\\ 0 & 1 & 0 & 0\\ \sin\theta & 0 & \cos\theta & 0\\ 0 & 0 & 1 & 1\end{pmatrix}(\tilde{w}_1)^{-1}.$$

After direct calculation we have

$$\left(\begin{pmatrix}\cos\theta & 0\\ 0 & 1\end{pmatrix}X+\begin{pmatrix}\sin\theta & 0\\ 0 & 0\end{pmatrix}\right)\left(\begin{pmatrix}-\sin\theta & 0\\ 0 & 0\end{pmatrix}X+\begin{pmatrix}\cos\theta & 0\\ 0 & 1\end{pmatrix}\right)^{-1}$$
$$=\begin{pmatrix}t_1' & \sqrt{1+(t_1')^2}\,(t_3-it_4)\\ \sqrt{1+(t_1')^2}\,(t_3+it_4) & t_2+t_1'(t_3^2+t_4^2)\end{pmatrix}.$$

Here

$$X=\begin{pmatrix}t_1 & \sqrt{1+t_1^2}(t_3-it_4)\\ \sqrt{1+t_1^2}(t_3+it_4) & t_2+t_1(t_3^2+t_4^2)\end{pmatrix},$$
$$t_1'=\frac{t_1+\tan\theta}{1-t_1\tan\theta}.$$

We also have

$$\left(\det\left(\begin{pmatrix}-\sin\theta & 0\\ 0 & 0\end{pmatrix}X+\begin{pmatrix}\cos\theta & 0\\ 0 & 1\end{pmatrix}\right)\right)^{-2}=\frac{1}{(t_1\sin\theta-\cos\theta)^2}.$$

Hence in the coordinate $(t_1, \cdots, t_4)$, the action of the infinitesimal generator of $\{k(\theta)\}$ was represented as follows.

$$\frac{\partial}{\partial\theta}\left(\frac{1}{(t_1\sin\theta-\cos\theta)^2}f(t_1', t_2, t_3, t_4)\right)\Big|_{\theta=0}=\left((1+t_1^2)\frac{\partial}{\partial t_1}+2t_1\right)f.$$

Put

$$P=(1+t_1^2)\frac{\partial}{\partial t_1}+2t_1.$$

Put

$$\boldsymbol{R}_+=\{x\in\boldsymbol{R}\,|\,x\geq 0\},$$
$$\boldsymbol{R}_-=\{x\in\boldsymbol{R}\,|\,x\leq 0\}.$$

For a closed set A of $H\times\boldsymbol{C}^3$ we put

$$A^c=H\times\boldsymbol{C}^3-A.$$

Then we have the following Mayer-Vietris exact sequence.

$$H^2((\boldsymbol{R}\times\boldsymbol{C}^+\times\boldsymbol{R}^2)^c, \mathcal{O})\longrightarrow H^3((\{0\}\times\boldsymbol{C}^+\times\boldsymbol{R}^2)^c, \mathcal{O})$$
$$\longrightarrow H^3((\boldsymbol{R}_+\times\boldsymbol{C}^+\times\boldsymbol{R}^2)^c, \mathcal{O})\oplus H^3((\boldsymbol{R}_-\times\boldsymbol{C}^+\times\boldsymbol{R}^2)^c, \mathcal{O}).$$

$H\times\boldsymbol{C}^3$ is Stein, since it is a cylinder domain. Hence for all closed subset A of $\boldsymbol{C}^4$ we have

$$H^q(A^c, \mathcal{O})=H_A^{q+1}(H\times\boldsymbol{C}^3, \mathcal{O})\qquad(q\geq 1).$$

Hence the above exact sequence is rewritten follows.

$$H^3_{\boldsymbol{R}\times(\boldsymbol{C}+)\times\boldsymbol{R}^2}(H\times\boldsymbol{C}^3, \mathcal{O})\longrightarrow H^4_{\{0\}\times(\boldsymbol{C}+)\times\boldsymbol{R}^2}(H\times\boldsymbol{C}^3, \mathcal{O})$$
$$\longrightarrow H^4_{\boldsymbol{R}_+\times(\boldsymbol{C}+)\times\boldsymbol{R}^2}(H\times\boldsymbol{C}^3, \mathcal{O})\oplus H^4_{\boldsymbol{R}_-\times(\boldsymbol{C}+)\times\boldsymbol{R}^2}(H\times\boldsymbol{C}^3, \mathcal{O}).$$

From the excision property, we have

$$H^3_{\boldsymbol{R}\times(\boldsymbol{C}+)\times\boldsymbol{R}^2}(H\times\boldsymbol{C}^3, \mathcal{O})=H^3_{\boldsymbol{R}\times(\boldsymbol{C}+)\times\boldsymbol{R}^2}(\boldsymbol{C}^4, \mathcal{O}).$$

On the other hand, from [KL] Théorème 1.1.2, we have

$$H^3_{\boldsymbol{R}\times(\boldsymbol{C}+)\times\boldsymbol{R}^2}(\boldsymbol{C}^4, \mathcal{O})=0$$

Hence we have only to show:

$$H^4_{\boldsymbol{R}_\pm\times(\boldsymbol{C}^+)\times\boldsymbol{R}^2}(H\times\boldsymbol{C}^3, \mathcal{O})_{\mathfrak{k}}=0.$$

We prove the case of "plus". The other case is similar.

Since $H\times\boldsymbol{C}^3$ is Stein, we have

$$H^4_{\boldsymbol{R}_+\times(\boldsymbol{C}^+)\times\boldsymbol{R}^2}(H\times\boldsymbol{C}^3, \mathcal{O})=H^3(H\times\boldsymbol{C}^3-\boldsymbol{R}_+\times\boldsymbol{C}^+\times\boldsymbol{R}^2, \mathcal{O}).$$

Let h be any $\mathfrak{k}$-finite element of $H^3(H\times\boldsymbol{C}^3-\boldsymbol{R}_+\times\boldsymbol{C}^+\times\boldsymbol{R}^2, \mathcal{O})$. Since h is $\mathfrak{k}$-finite,

$$h, Ph, P^2h, P^3h, \cdots, P^nh, \cdots$$

are not linearly independent. Hence there exists some differential operator

$$Q=(1+t_1^2)^m\frac{\partial^m}{\partial t_1^m}+Q'\left(t_1, \frac{\partial}{\partial t_1}\right)$$

such that $Qh=0$. Here m is some positive integer and Q' is a differential operator of order $<m$. Q is free from t_i and $\partial/\partial t_i$ for $i=2, 3, 4$.

Now we consider the following Leray covering of $H\times\boldsymbol{C}^3-\boldsymbol{R}_+\times\boldsymbol{C}^+\times\boldsymbol{R}^2$.

$$\mathcal{U}=\{(H-\boldsymbol{R}_+)\times\boldsymbol{C}^3, H\times(\boldsymbol{C}-\boldsymbol{C}^+)\times\boldsymbol{C}^2, H\times\boldsymbol{C}\times(\boldsymbol{C}-\boldsymbol{R})\times\boldsymbol{C}, H\times\boldsymbol{C}^2\times(\boldsymbol{C}-\boldsymbol{R})\}.$$

Then we can identify the space of 3-cocycle $Z^3(\mathcal{U}, \mathcal{O})$ and the space

$$\mathcal{O}((H-\boldsymbol{R}_+)\times(\boldsymbol{C}-\boldsymbol{C}^+)\times(\boldsymbol{C}-\boldsymbol{R})^2).$$

The space of 2-cochain $C^2(\mathcal{U}, \mathcal{O})$ is identified with

$$\mathcal{O}(W_1)\oplus\mathcal{O}(W_2)\oplus\mathcal{O}(W_3)\oplus\mathcal{O}(W_4).$$

Here we put

$$\begin{aligned}
W_1&=H\times(\boldsymbol{C}-\boldsymbol{C}^+)\times(\boldsymbol{C}-\boldsymbol{R})^2,\\
W_2&=(H-\boldsymbol{R}_+)\times\boldsymbol{C}\times(\boldsymbol{C}-\boldsymbol{R})^2,\\
W_3&=(H-\boldsymbol{R}_+)\times(\boldsymbol{C}-\boldsymbol{C}^+)\times\boldsymbol{C}\times(\boldsymbol{C}-\boldsymbol{R}),\\
W_4&=(H-\boldsymbol{R}_+)\times(\boldsymbol{C}-\boldsymbol{C}^+)\times(\boldsymbol{C}-\boldsymbol{R})\times\boldsymbol{C}.
\end{aligned}$$

Then we have the following exact sequence

$$C^2(\mathcal{U}, \mathcal{O})\xrightarrow{d}Z^3(\mathcal{U}, \mathcal{O})\xrightarrow{p}H^3(H\times\boldsymbol{C}^3-\boldsymbol{R}_+\times\boldsymbol{C}^+\times\boldsymbol{R}^2, \mathcal{O})\longrightarrow 0.$$

We choose g such that $p(g)=f$. Since $Qh=0$, there exist some $\varphi_i \in \mathcal{O}(W_i)$ $(i=1, 2, 3, 4)$ such that

$$Qg=d(\varphi_1\oplus\varphi_2\oplus\varphi_3\oplus\varphi_4).$$

Since H and $H-\boldsymbol{R}_+$ are simply-connected and Q is free from t_i and $\partial/\partial t_i$ for $i=2, 3, 4$, we can solve the following (ordinary) differential equation in the domain W_i for $i=1, 2, 3, 4$.

$$Qu_i=\varphi_i.$$

Put

$$g'=g-d(u_1\oplus u_2\oplus u_3\oplus u_4).$$

Then we have $p(g')=h$ and

$$(9) \qquad Qg'=0.$$

Since $g' \in \mathcal{O}((H-\boldsymbol{R}_+)\times(\boldsymbol{C}-\boldsymbol{C}^+)\times(\boldsymbol{C}-\boldsymbol{R})^2)$ satisfies the above differential equation and H is simply-connected, we can extend g' to a holomorphic function on $H\times(\boldsymbol{C}-\boldsymbol{C}^+)\times(\boldsymbol{C}-\boldsymbol{R})^2$. This means $h=p(g')=0$. □

From the above Lemma 4.2.1 and exact sequence (8), we have only to show

$$(10) \qquad H^4_{\{e_5\}\cup(U_{\bar{G}}\cdot e_4\cap\bar{\Gamma}^+)}(F_C, L)_{\mathfrak{k}}=0.$$

Under the coordinate $(\zeta_1, \cdots, \zeta_4)$ on U_4, we have

$$U_{\bar{G}}\cap\bar{\Gamma}^+=\{(\zeta_1, \cdots, \zeta_4)\in \boldsymbol{C}^4 \mid \zeta_2=\zeta_3=\zeta_4=0, \Im\zeta_1\geq 0\}.$$

Hence we have

$$H^q_{U_{\bar{G}}\cdot e_4\cap\bar{\Gamma}^+}(F_C, L)=H^q_{U_{\bar{G}}\cdot e_4\cap\bar{\Gamma}^+}(U_4, \mathcal{O})$$
$$=H^q_{\boldsymbol{C}^+\times\{0\}^3}(\boldsymbol{C}^4, \mathcal{O}).$$

From [KL] Théorème 1.1.2, we have

$$(11) \qquad 0\longrightarrow H^4_{\{e_5\}}(F_C, L)\xrightarrow{\kappa'} H^4_{\{e_5\}\cup(U_{\bar{G}}\cdot e_4\cap\bar{\Gamma}^+)}(F_C, L)\xrightarrow{\tau'} H^4_{\boldsymbol{C}^+\times\{0\}^3}(\boldsymbol{C}^4, \mathcal{O}).$$

The same argument as the proof of Lemma 4.2.1 implies:

Lemma 4.2.2.

$$H^4_{\boldsymbol{C}^+\times\{0\}^3}(\boldsymbol{C}^4, \mathcal{O})_{\mathfrak{k}}=0.$$

Hence we have only to prove

$$H^4_{\{e_5\}}(F_C, L)_{\mathfrak{k}} = 0. \tag{12}$$

However the left side of the above equation is "the space of $\mathfrak{k}$-finite hyperfunction on F whose support is contained in the point $\{e_5\} \in F$". Hence we can see (12) holds. Q.E.D.

References

[KV] M. Kashiwara and M. Vergne, Functions on the Shilov boundary of the generalized half plane, Springer-Verlag Lecture Notes in Math., **728** (1979), 136–176.

[KL] M. Kashiwara and Y. Laurent, Théorèmes d'annulation et deuxième microlocalisation, Universitè de Paris-Sud (1983).

[KKK] M. Kashiwara, T. Kawai and T. Kimura, "Foundations of Algebraic Analysis" Princeton University Press, Princeton, New Jersey (1986).

[RSW] J. Rawnsley, W. Schmid and J. Wolf, Singular unitary representations and indefinite harmonic theory, J. Funct. Anal., **51** (1983), 1–114.

[JV] H. P. Jakobsen and M. Vergne, Wave and Dirac operators, and Representations of the conformal group, J. Funct. Anal., **24** (1977), 52–106.

[P] V. P. Paramodov, "Linear Differential Operators with Constant Coefficients" Springer-Verlag Berlin-Heidelberg-New York (1979).

[Ko] B. Kostant, Lie algebra cohomology and the generalized Borel-Weil theorem, Ann. of Math., **74** (1961), 329–387.

[Wa] G. Warner, "Harmonic Analysis on Semi-Simple Lie Groups I" Springer-Verlag Berlin-Heidelberg-New York (1972).

[Wo] J. Wolf, Fine structure of Hermitian symmetric spaces, in "Symmetric Spaces" Bothby and Weiss (eds.), Dekker New York (1972).

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