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Irreducible Unitary Representations of the Group of Maps with Values in a Free Product Group

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Dedicated to Professor R. Takahashi for his 60th birthday

Introduction

In [30], Vershik, Gelfand and Graev studied the construction of irreducible unitary representations of the group $C^\infty(X, G)$ of smooth maps of a compact manifold X with values in a Lie group G. Following the physical terminology such a group is called a current group. In case $G=SL(2, \boldsymbol{R})$, they afforded factorizable irreducible unitary representations of the current group which depend upon measures on X. Their method reveals that the structure of a measure space is important rather than the structure of a manifold. In fact they started with the construction of those representations of the weak current group $G^{(X)}$. A weak current group is the group of maps of a measurable space X with only finitely many values in a topological group G. Furthermore their method relies deeply on the structure of the neighborhood of the trivial representation of $G=SL(2, \boldsymbol{R})$. In other words, it is essential that there exists a canonical state on $SL(2, \boldsymbol{R})$ (see [32] for its definition).

Apart from the representation theory of current groups, there has been a remarkable progress in harmonic analysis on free groups. In [10], Figà-Talamanca and Picardello found a close resemblance between harmonic analysis on free groups and that of $SL(2, \boldsymbol{R})$. Their results are known to be extended to certain free product groups (cf. [15]).

Based on the above stated resemblance, we consider in this paper the construction of factorizable irreducible unitary representations of the weak current group $G^{(X)}$. Here X is a measurable space and G is the free product of a countable family $(G_i)_{i\in I}$ of countable groups. Note that if all G_i are infinite cyclic then G is a free group. In Section 1 we show that a length function ℓ on G is negative definite, which yields a canonical state $\psi_t(x)=t^{\ell(x)}$ where $x\in G$ and $0<t<1$. The cyclic unitary representation L_t defined by ψ_t is called the canonical representation. We remark

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that when G is a free group our length function is different from the ordinary one. The ordinary length function on a free group is known to be negative definite (cf. [14]) and the corresponding canonical representation was recently considered in [25] for distinct purposes. In Section 2 we show that the canonical representation L_t is irreducible if the cardinality of I is infinite (see Theorem 1). In Section 3 we consider the case when G is the free product of an r family of finite groups of the same order s with $q=(r-1)(s-1)\geqq 2$. In this case we show in Theorem 2 that L_t is not irreducible; if $q^{-1/2}<t<1$, L_t contains a unique irreducible subrepresentation L_t^o which is not weakly contained in the regular representation of G (so called a complementary series representation), and its orthogonal complement is a subrepresentation of the regular representation. The existence of the unique irreducible summand L_t^o plays an important role for the construction of irreducible representations of $G^{(X)}$. While if $0<t\leqq q^{-1/2}$, L_t is weakly contained in the regular representation of G. The pure state ϕ_t corresponding to L_t^o is given explicitly in (3.7). The similar results are obtained if we start with a canonical state Ψ_t given in (3.16) (see Theorem 2′). Section 4 is devoted to reviewing the general facts about unitary representations of direct limit groups. The reason is that $G^{(X)}$ can be viewed as a certain direct limit group. Applying the results in Sections 2, 3 and 4, we construct in Section 5 the factorizable irreducible unitary representations of $G^{(X)}$ parametrized by finite positive measures on X (see Theorem 3, Theorem 4 and Theorem 4′). When the cardinality of I is infinite no restriction is needed for a measure space (X, μ). The pure state for our representation is given by Φ_μ (see (5.2)). While if G is the free product of a finite family of finite groups prescribed above, we need a certain condition on (X, μ) to get irreducible representations of $G^{(X)}$. Such a condition is fulfilled if (X, μ) is a nonatomic Lebesgue space, in which case the pure state of our representation is given by Ψ_μ (see (5.10)). The knowledge of the pure state enables us to see the possibility of the extension of the representations to those groups which contain $G^{(X)}$ as a dense subgroup (cf. [30]).

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§ 1. The canonical representations of a free product group

Throughout the paper, let $(G_i)_{i\in I}$ be a countable family of countable discrete nontrivial groups. We denote the free product group of $(G_i)_{i\in I}$

by G, that is, $G=*_{i\in I}G_i$. Put $G'=G-(e)$ (resp. $G_i'=G_i-(e)$ for $i\in I$) where e is the unit element of G. Every $x\in G'$ can be written uniquely as a reduced word

$$x=g_{i_1}\cdots g_{i_n} \tag{1.1}$$

where $g_{i_j}\in G'_{i_j}$ such that $i_j\in I\,(1\leqq j\leqq n)$ and $i_j\neq i_{j+1}$ for $1\leqq j\leqq n-1$. We put $\ell(x)=n$ and call it the length of x. We further define $\ell(e)=0$. Note that $\ell(x^{-1})=\ell(x)$. For $x\in G'$ in (1.1), we put

$$x(k)=g_{i_1}\cdots g_{i_k}\ (1\leqq k\leqq n) \quad \text{and} \quad x(0)=e. \tag{1.2}$$

In particular we write $\tau(x)=x(n-1)$ for $x\in G'$. Let $n\in N$, where N is the set of nonnegative integers. We put $G(n)=\{x\in G;\ \ell(x)=n\}$. Then $G(0)=\{e\}$, $G(1)=\bigcup_{i\in I}G_i'$ and in general

$$G(n)=\bigcup G'_{i_1}\cdots G'_{i_n} \tag{1.3}$$

where $i_1,\cdots,i_n$ run through I with the property $i_j\neq i_{j+1}$. For $m,n\in N$, we write $m\wedge n=\min\{m,n\}$. Let $x=g_{i_1}\cdots g_{i_m}$ and $y=g_{j_1}\cdots g_{j_n}$ be the reduced word expression of x and y respectively where $m,n\geqq 1$. Then there exists a unique h with $0\leqq h\leqq m\wedge n$ such that $g_{i_1}=g_{j_1},\cdots,g_{i_h}=g_{j_h}$ while $g_{i_{h+1}}\neq g_{j_{h+1}}$. If $i_{h+1}\neq j_{h+1}$, then $\ell(y^{-1}x)=m+n-2h$. On the other hand if $i_{h+1}=j_{h+1}$, then $\ell(y^{-1}x)=m+n-2h-1$. Hence we conclude that for $x,y\in G$ there exists a unique k with $0\leqq k\leqq 2\,(\ell(x)\wedge\ell(y))$ such that

$$\ell(y^{-1}x)=\ell(x)+\ell(y)-k. \tag{1.4}$$

Let $\mathscr{C}_o(G)$ be the space of all complex valued functions on G with finite support. We denote by $\delta[x]$ where $x\in G$ the element of $\mathscr{C}_o(G)$ given by $\delta[x](y)=1$ or 0 according as $y=x$ or not. Clearly $\delta[x]$ where $x\in G$ yield a basis of $\mathscr{C}_o(G)$. Let L be the representation of G on $\mathscr{C}_o(G)$ defined by $(L(x)f)(y)=f(x^{-1}y)$. Note that $L(x)\delta[y]=\delta[xy]$. Let $\mathscr{C}_{oo}(G)$ be the G-invariant subspace of $\mathscr{C}_o(G)$ consisting of all functions having total mass 0. If we put

$$\alpha[x]=\delta[x]-\delta[\tau(x)] \qquad \text{where } x\in G', \tag{1.5}$$

then $\alpha[x]\in\mathscr{C}_{oo}(G)$. Since $\delta[x]=\sum_{k=1}^{\ell(x)}\alpha[x(k)]+\delta[e]$, it follows that $\{\alpha[x];\ x\in G'\}$ provides a basis of $\mathscr{C}_{oo}(G)$. We introduce a G-invariant hermitian form on $\mathscr{C}_o(G)$ by

$$\langle f_1,f_2\rangle=-\sum_{x,y\in G}\ell(y^{-1}x)f_1(x)\overline{f_2(y)}. \tag{1.6}$$

From now on, we parametrize the elements of each G_i as follows;

$$(1.7)\qquad G_i=\{a_{i0}=e,\ a_{i1},\ a_{i2},\ \cdots\}.$$

Then we may rewrite $x\in G'$ in (1.1) as

$$(1.8)\qquad x=a_{i_1p_1}\cdots a_{i_np_n}$$

where $i_j\in I\ (1\leqq j\leqq n)$ with $i_j\neq i_{j+1}\ (1\leqq j\leqq n-1)$ and $p_j\geqq 1\ (1\leqq j\leqq n)$. Put $b(p)=(p(1+p))^{-1/2}$ for $p\geqq 1$. For $x\in G'$ given by (1.8), we put

$$(1.9)\qquad \varepsilon[x]=b(p_n)\{p_n\alpha[x]-\textstyle\sum_{k=1}^{p_n-1}\alpha[\tau(x)a_{i_nk}]\}.$$

Lemma 1.1. (i) *For $x\in G'$ in* (1.8), *we have*

$$(1.10)\qquad \alpha[x]=(1+p_n)b(p_n)\varepsilon[x]+\textstyle\sum_{k=1}^{p_n-1}b(k)\varepsilon[\tau(x)a_{i_nk}].$$

(ii) *For $x, y\in G'$, $\langle\varepsilon[x],\varepsilon[y]\rangle=1$ or 0 according as $x=y$ or not. Hence $\{\varepsilon[x];\ x\in G'\}$ provides an orthonormal basis of $\mathscr{C}_{oo}(G)$ with respect to* (1.6).

Proof. (i) By (1.9) we can get (1.10) immediately.

(ii) We note that

$$\langle\alpha[x],\alpha[y]\rangle=-\ell(y^{-1}x)+\ell(y^{-1}\tau(x))+\ell(\tau(y)^{-1}x)-\ell(\tau(y)^{-1}\tau(x)).$$

From this and (1.4), it follows that if $\tau(x)\neq\tau(y)$ then $\langle\alpha[x],\alpha[y]\rangle=0$. Hence by (1.9) we have $\langle\varepsilon[x],\varepsilon[y]\rangle=1$ or 0 according as $x=y$ or not. From (1.10) and the fact that $\{\alpha[x];\ x\in G'\}$ is a basis of $\mathscr{C}_{oo}(G)$, we conclude that $\{\varepsilon[x];\ x\in G'\}$ forms an orthonomal basis of $\mathscr{C}_{oo}(G)$.

Corollary 1.2. (i) *The length function ℓ on G is negative definite.*

(ii) *Let K be the completion of $\mathscr{C}_{oo}(G)$ with respect to* (1.6) *Let U be the unitary representation of G on K which comes from the restriction of L to $\mathscr{C}_{oo}(G)$. Define a map β of G into K by*

$$(1.11)\qquad \beta(x)=\delta[x]-\delta[e]=\textstyle\sum_{k=1}^{\ell(x)}\alpha[x(k)].$$

Then β is a total cocycle on G for the representation U, namely,

$$(1.12)\qquad \beta(xy)=U(x)(\beta(y))+\beta(x)\quad\text{and}\quad 2^{-1}\|\beta(x)\|^2=\ell(x).$$

Let $0\leqq t\leqq 1$. We define a function ψ_t on G by

$$(1.13)\qquad \psi_t(x)=t^{\ell(x)}\qquad\text{for } 0<t\leqq 1 \text{ and } \psi_o=\delta[e].$$

Since ℓ is negative definite, it follows that ψ_t is a positive definite function on G. We consider the cyclic unitary representation L_t of G on a Hilbert space H_t defined by ψ_t (*GNS* construction). Clearly L_o is the left regular representation of G on $H_0=l^2(G)$, and L_1 is the trivial representation on $H_1=\boldsymbol{C}$. For $0<t<1$ we recall the construction of (L_t, H_t). Define a G-invariant hermitian form on $\mathscr{C}_0(G)$ by

$$(f_1 | f_2)_t=\sum_{x,y\in G}\psi_t(y^{-1}x)f_1(x)\overline{f_2(y)}. \tag{1.14}$$

Dividing $\mathscr{C}_0(G)$ by the G-invariant subspace of functions having norm 0, we get a prehilbert space and take its completion H_t. Let L_t be the unitary representation of G canonically obtained by L on $\mathscr{C}_0(G)$. Each $\delta[x]$ provides a nontrivial element of H_t, which is denoted by the same letter. For $x\in G$, we define $\gamma[x]\in H_t$ as follows. Put $\gamma[e]=\delta[e]$. For $x\in G'$ in (1.8), we put

$$\gamma[x]=A(p_n, t)\{(1+(p_n-1)t)\delta[x]-t\sum_{h=0}^{p_n-1}\delta[\tau(x)a_{i_nh}]\}. \tag{1.15}$$

Here and in the sequel we write

$$A(p, t)=((1-t)(1+(p-1)t)(1+pt))^{-1/2} \qquad \text{for } p\geqq 1. \tag{1.16}$$

Lemma 1.3. (i) *For* $x\in G'$ *in* (1.8), *we have*

$$\begin{aligned}\delta[x]=t^n\gamma[e]+(1-t)\sum_{k=1}^{n}t^{n-k}\{&(1+p_kt)A(p_k, t)\gamma[x(k)]\\ &+t\sum_{h=1}^{p_k-1}A(h, t)\gamma[x(k-1)a_{i_kh}]\}.\end{aligned} \tag{1.17}$$

(ii) $\{\gamma[x]; x\in G\}$ *yields an orthonormal basis of* H_t.

Proof. (i) We shall show (1.17) by induction argument. Suppose (1.17) holds for all $x\in G(k)$ with $1\leqq k\leqq n-1$. Let $x\in G(n)$ written as (1.8). We have only to see that

$$\delta[x]=t\delta[\tau(x)]+(1-t)\{(1+p_nt)A(p_n, t)\gamma[x]+t\sum_{h=1}^{p_n-1}A(h, t)\gamma[\tau(x)a_{i_nh}]\}.$$

This can be derived from (1.15) by induction on p_n.

(ii) Since $\gamma[x]=L_t(\tau(x))\gamma[a_{i_np_n}]$ (see (1.15)) and (1.14) is G-invariant, it is enough to consider $(\gamma[x] | \gamma[y])_t$ for $x, y\in G(1)$. Using (1.15), we can see $(\gamma[x] | \gamma[y])_t=1$ or 0 either $x=y$ or not by direct computations.

The above constructed unitary representation (L_t, H_t) of G is cyclic with cyclic vector $\gamma[e]$ such that $(L_t(x)\gamma[e] | \gamma[e])_t=\psi_t(x)$ for $x\in G$. We call (L_t, H_t) where $0<t<1$ as the canonical representation of G in analogy with the canonical representation of $SL(2, \boldsymbol{R})$ in [30].

§ 2. The canonical representations of an infinitely generated free product

In this section, we assume that the cardinality of I is infinite. We may set $I=\{1, 2, \cdots\}$. Let $0<t<1$, and (L_t, H_t) be the canonical representation of G introduced in Section 1. Our aim is to proving the irreducibility of it. For $N \geqq 1$, we define bounded operators $P_t^{(N)}$ on H_t by

$$P_t^{(N)} = (tN)^{-1} \sum_{j=1}^{N} L_t(a_{j1}). \tag{2.1}$$

We denote the orthogonal projection of H_t onto $C\gamma[e]$ by P_t.

Lemma 2.1. *For any $u \in H_t$, we have $\lim_{N\to\infty} \|P_t^{(N)}u - P_t u\|_t = 0$.*

Proof. First we show the lemma for $\gamma[x]$ ($x \in G$). From Lemma 1.3, it follows that

$$L_t(a_{j1})\gamma[e] = t\gamma[e] + (1-t^2)^{1/2}\gamma[a_{j1}]$$

and hence

$$P_t^{(N)}\gamma[e] = \gamma[e] + (tN)^{-1}(1-t^2)^{1/2} \sum_{j=1}^{N} \gamma[a_{j1}],$$

which implies

$$\|P_t^{(N)}\gamma[e] - \gamma[e]\|_t^2 = t^{-2}(1-t^2)^{1/2}N^{-1}.$$

Thus the lemma holds for $\gamma[e]$. If $\ell(x) \geqq 2$, then by (1.15) $L_t(a_{j1})\gamma[x] = \gamma[a_{j1}x]$. For such x, we have

$$P_t^{(N)}\gamma[x] = (tN)^{-1} \sum_{j=1}^{N} \gamma[a_{j1}x] \quad \text{and} \quad P_t\gamma[x] = 0.$$

Again the lemma holds for $\gamma[x]$ with $\ell(x) \geqq 2$. Finally assume that $x = a_{ip} \in G(1)$. If $i \neq j$, then $L_t(a_{j1})\gamma[a_{ip}] = \gamma[a_{j1}a_{ip}]$ by (1.15). Hence

$$P_t^{(N)}\gamma[a_{ip}] = (tN)^{-1} \sum_{j\neq i} \gamma[a_{j1}a_{ip}] + (tN)^{-1}L_t(a_{i1})\gamma[a_{ip}].$$

From (1.15) and (1.17), $L_t(a_{i1})\gamma[a_{ip}]$ is written as a finite linear combination of $\gamma[x]$ with $x \in G_i$. Consequently

$$\|P_t^{(N)}\gamma[a_{ip}]\|_t^2 = t^{-2}N^{-1}.$$

This means that the lemma holds for $\gamma[x]$ with $\ell(x) = 1$. Hence the lemma also holds for all u which are finite linear combinations of $\gamma[x]$ ($x \in G$). Since these u form a dense subset of H_t and the operator norms of $P_t^{(N)}$ where $N \geqq 1$ are uniformly bounded by t^{-1}, we conclude that the lemma holds for all $u \in H_t$.

Theorem 1. *Let G be the free product of a countable family $(G_i)_{i\in I}$ of countable groups. Assume that the cardinality of I is infinite. Let $0<$*

$t<1$. *Then the canonical representations* (L_t, H_t) *of* G *are irreducible and pairwise inequivalent.*

Proof. Let H be a closed nonzero invariant subspace of H_t. First we show that there exists $u \in H$ such that $P_t u \neq 0$. Let $u = \sum_{y \in G} c_y \gamma[y]$ be a nonzero element of H. Put $S(u) = \{y \in G; c_y \neq 0\}$. If $e \in S(u)$, then $P_t u = c_e \gamma[e] \neq 0$. Suppose $e \notin S(u)$. Put $n = \min\{\ell(y); y \in S(u)\}$. Then $n \geqq 1$ and we can choose $x = a_{i_1 p_1} \cdots a_{i_n p_n} \in S(u)$ such that $a_{i_1 p_1} \cdots a_{i_n p} \notin S(u)$ for $p < p_n$. We shall see $P_t(L_t(x^{-1})u) \neq 0$. Using (1.15) and (1.17), we have $P_t(L_t(x^{-1})\gamma[x]) = (1 + (p_n - 1)t) A(p_n, t) \gamma[e]$, which is nonzero. For $y \in S(u)$ such that $\tau(y) \neq \tau(x)$, we have $L_t(x^{-1})\gamma[y] = \gamma[x^{-1}y]$ and hence $P_t(L_t(x^{-1})\gamma[y]) = 0$. If $y \in S(u)$ such that $y = a_{i_1 p_1} \cdots a_{i_n p}$ with $p > p_n$, then $L_t(x^{-1})\gamma[y] = L_t(a_{i_n p_n}^{-1})\gamma[a_{i_n p}]$. It can be written as

$$A(p, t)\{(1 + (p-1)t)\delta[a_{i_n p_n}^{-1} a_{i_n p}] - t \textstyle\sum_{h=0}^{p-1} \delta[a_{i_n p_n}^{-1} a_{i_n h}]\}$$

by (1.15). Applying P_t, we obtain that $P_t(L_t(x^{-1})\gamma[y]) = 0$. Consequently H contains an element u such that $P_t u \neq 0$. Take such u and consider the sequence $\{P_t^{(N)} u; N \geqq 1\}$. Since H is invariant, each $P_t^{(N)} u \in H$. By Lemma 2.1, this sequence converges to $P_t u$. Since H is closed, it follows that $P_t u \in H$. Hence $\gamma[e] \in H$. On the other hand $\gamma[e]$ is a cyclic vector for (L_t, H_t). This implies $H = H_t$. Let $0 < t_1 \neq t_2 < 1$. Since $\psi_{t_1} \neq \psi_{t_2}$, L_{t_1} and L_{t_2} are inequivalent.

§ 3. The canonical representations of a finitely generated free product

Let $G = *_{i \in I} G_i$ be the free product of a countable family of countable groups. In this section, we assume that $I = \{1, 2, \cdots, r\}$ and all G_i are finite groups of the same order s. Put

$$q = (r-1)(s-1) \tag{3.1}$$

and assume $q \geqq 2$. It follows from (1.3) that

$$|G(n)| = r(s-1)q^{n-1} \qquad \text{for } n \geqq 1. \tag{3.2}$$

Let $x \in G(m)$ and $n \geqq 1$. We set $G(n, k; x) = \{y \in G(n); \ell(xy) = m + n - k\}$ for $0 \leqq k \leqq 2(m \wedge n)$ (see (1.4)). The following lemma is an immediate consequence of the argument below (1.3). So we leave the proof to the reader.

Lemma 3.1. *Let* $x \in G(m)$. *The set* $G(n)$ *can be decomposed into the disjoint union of* $G(n, k; x)$ *where* $0 \leqq k \leqq 2(m \wedge n)$. *Moreover if* $m \wedge n \geqq 1$, *the cardinalities of* $G(n, k; x)$ *are given as follows.*

$$|G(n, 0; x)|=q^{n}, \qquad |G(n, 2(m\wedge n); x)|=q^{n-m\wedge n},$$

$|G(n, 2k; x)|=(r-2)(s-1)q^{n-k-1}$ *for* $1\leqq k\leqq m\wedge n-1$ *and*

$$|G(n, 2k+1; x)|=(s-2)q^{n-k-1} \qquad \textit{for } 0\leqq k\leqq m\wedge n-1.$$

Lemma 3.2. *Let* $0\leqq t\leqq 1$. *For fixed* $y, z \in G$, *the function* $\psi_t(z^{-1}xy)$ *of* $x \in G$ *belongs to* $l^2(G)$ *if and only if* $0\leqq t<q^{-1/2}$.

Proof. Put $l=\ell(y)$ and $m=\ell(z)$. To see the lemma, it is enough to show that $\sum_{n\geqq l+m}\sum_{x\in G(n)}\psi_t(z^{-1}xy)^2$ is finite. Put $G(n, k; y, z)=\{x \in G(n); \ell(z^{-1}xy)=l+m+n-k\}$. Applying Lemma 3.1, we have for $n\geqq l+m$

$$\sum_{x\in G(n)}\psi_t(z^{-1}xy)^2=\sum_{k=0}^{2(l+m)} t^{2(l+m+n-k)}|G(n, k; y, z)|.$$

Since $|G(n, k; y, z)|\leqq|G(n)|$, the right hand side is dominated by

$$r(s-1)q^{-1}t^{2(l+m)}(qt^2)^n \sum_{k=0}^{2(l+m)} t^{2k}.$$

Hence we conclude that $\psi_t(z^{-1}xy) \in l^2(G)$ if and only if $qt^2<1$.

Let ϕ be a positive definite function on G. ϕ is said to be associated with the regular representation if it is of the form $\phi=f*\tilde{f}$ where $f \in l^2(G)$ and $\tilde{f}(x)=\overline{f(x^{-1})}$. This means that the cyclic unitary representation of G defined by ϕ is a subrepresentation of the regular representation (cf. [7]). A positive definite function ϕ on G is said to be weakly associated with the regular representation if ϕ is in the closure of $\{f*\tilde{f}; f \in l^2(G)\}$ with respect to simple convergence on G. This means that the cyclic unitary representation of G defined by ϕ is weakly contained in the regular representation (cf. [8]). One can show the following lemma without any essential change of the argument in [14], where the case of free groups is considered.

Lemma 3.3. *A positive definite function* ϕ *on* G *is weakly associated with the regular representation of* G *if and only if for any* $0<t<1$ *the function* $\phi\psi_t$ *belongs to* $l^2(G)$. *In particular* ψ_t *is weakly associated with the regular representation if and only if* $0\leqq t\leqq q^{-1/2}$.

Let $0<t\leqq 1$. For $t\neq q^{-1/2}$ we put

$$c(t)=1+r^{-1}(r-1)(1-t)(1+(s-1)t)(qt^2-1)^{-1}. \tag{3.3}$$

We find that $c((qt)^{-1})=1-c(t)$ and $c(t)$ is a monotone decreasing function for $q^{-1/2}<t\leqq 1$ such that $c(1)=1$ and $c'(1)=-s(r-1)/r(q-1)$. For

$0<t\leqq 1$ such that $t\neq q^{-1/2}$, we put

(3.4) $$C_t(n)=c(t)+(1-c(t))(qt^2)^{-n} \qquad \text{for } n\in \boldsymbol{N}.$$

If $t=q^{-1/2}$, we put

(3.5) $$C_t(n)=1+r^{-1}(s-1)^{-1}(2q+(s-2)q^{1/2}-r(s-1))n \qquad \text{for } n\in \boldsymbol{N}.$$

We note that if $q^{-1/2}<t\leqq 1$

(3.6) $$\lim_{n\to\infty} C_t(n)=c(t).$$

We define a function ϕ_t on G by

(3.7) $$\phi_t(x)=\psi_t(x)C_t(\ell(x)).$$

One can verify $\phi_{(qt)^{-1}}=\phi_t$ and for $t\neq q^{-1/2}$

(3.8) $$\phi_t=c(t)\psi_t+(1-c(t))\psi_{(qt)^{-1}}.$$

Lemma 3.4 (cf. [15]). *ϕ_t is a pure positive definite function on G for $q^{-1/2}<t\leqq 1$. The irreducible unitary representation of G defined by ϕ_t for $q^{-1/2}<t\leqq 1$ is not weakly contained in the regular representation.*

Proof. The first assertion is proved in [15] for the case when all G_i are finite cyclic groups of the same order. It is quite easy to extending their results to our case (cf. [5]). The second assertion is a direct consequence of Lemma 3.3.

Let χ_n be the characteristic function of $G(n)$.

Lemma 3.5. *Let $0<t\leqq 1$. For $x\in G$ and $n\geqq 1$, we have*

(3.9) $$(\delta[x]\,|\,\chi_n)_t=r(r-1)^{-1}(qt)^n t^{\ell(x)}C_t(\ell(x)\wedge n)$$

and for $m\wedge n\geqq 1$, we have

(3.10) $$(\chi_m\,|\,\chi_n)_t=r^2(r-1)^{-2}(qt)^{m+n}C_t(m\wedge n).$$

Proof. Note that $(\delta[x]\,|\,\chi_n)_t=\sum_{y\in G(n)} t^{\ell(x^{-1}y)}$. Using Lemma 3.1, we get, by putting $m=\ell(x)$, $(\delta[x]\,|\,\chi_n)_t=\sum_{k=0}^{2(m\wedge n)} t^{m+n-k}\,|G(n,k;x)|$. Again by Lemma 3.1, it can be written as

$$(qt)^n t^m\{1+q^{-1}(r-2)(s-1)\textstyle\sum_{k=1}^{m\wedge n-1}(qt^2)^{-k} \\ +(s-2)(qt)^{-1}\textstyle\sum_{k=0}^{m\wedge n-1}(qt^2)^{-k}+(qt^2)^{-m\wedge n}\}.$$

This agrees with $r(r-1)^{-1}(qt)^n t^m C_t(m\wedge n)$ by simple computations. Since

$(\chi_m|\chi_n)_t=\sum_{x\in G(m)}(\delta[x]|\chi_n)_t$, (3.10) follows from (3.9) and (3.2).

Lemma 3.6. *Suppose $q^{-1/2}<t<1$. Then*

(i) *the sequence $\{\|\chi_n\|_t^{-1}\chi_n;\, n\geqq 1\}$ in H_t converges strongly to an element γ_t^o.*

(ii) *We have $\|\gamma_t^o\|_t=1$ and for $x\in G$*

$$(\delta[x]|\gamma_t^o)_t=c(t)^{-1/2}\phi_t(x). \tag{3.11}$$

(iii) *If we put*

$$\gamma_t^1=(1-c(t)^{-1})^{-1/2}(\gamma[e]-c(t)^{-1/2}\gamma_t^o) \tag{3.12}$$

then we get

$$\|\gamma_t^1\|_t=1,\ (\gamma_t^o|\gamma_t^1)_t=0 \quad \text{and} \quad \gamma[e]=c(t)^{-1/2}\gamma_t^o+(1-c(t)^{-1})^{1/2}\gamma_t^1. \tag{3.13}$$

Proof. (i) It follows from (3.10) that $(\|\chi_m\|_t^{-1}\chi_m|\,\|\chi_n\|_t^{-1}\chi_n)_t=C_t(m\wedge n)(C_t(m)C_t(n))^{-1/2}$. Since $q^{-1/2}<t<1$, it follows from (3.6) that $\lim_{m,n\to\infty}(\|\chi_m\|_t^{-1}\chi_m|\,\|\chi_n\|_t^{-1}\chi_n)_t=1$. This yields that $\{\|\chi_n\|_t^{-1}\chi_n;\, n\geqq 1\}$ is a Cauchy sequence in H_t. Hence it has a limit, which we denote by γ_t^o.

(ii) Since the norms of $\|\chi_n\|_t^{-1}\chi_n$ are 1, we get $\|\gamma_t^o\|_t=1$. By (i) we have $(\delta[x]|\gamma_t^o)_t=\lim_{n\to\infty}\|\chi_n\|_t^{-1}(\delta[x]|\chi_n)_t$, which equals

$$\lim_{n\to\infty} t^{\ell(x)}C_t(\ell(x)\wedge n)C_t(n)^{-1/2}$$

by (3.9) and (3.10). Using (3.6) and (3.7), we obtain (3.11). The assertion (iii) is evident from (i) and (ii).

Lemma 3.7. *Suppose $q^{-1/2}<t<1$. Then for $x\in G$ we have*

$$\begin{aligned}&(L_t(x)\gamma_t^o|\gamma_t^o)_t=\phi_t(x),\ (L_t(x)\gamma_t^o|\gamma_t^1)_t=0 \quad \text{and}\\ &(L_t(x)\gamma_t^1|\gamma_t^1)_t=\psi_{(qt)^{-1}}(x).\end{aligned} \tag{3.14}$$

Proof. Let $m=\ell(x)$ and $n\geqq m$. From (3.11) we find that

$$(L_t(x)\chi_n|\gamma_t^o)_t=c(t)^{-1/2}\sum_{y\in G(n)}t^{\ell(xy)}C_t(\ell(xy)).$$

Using Lemma 3.1, we get

$$\sum_{y\in G(n)}t^{\ell(xy)}C_t(\ell(xy))=\sum_{k=0}^{2m}t^{n+m-k}C_t(n+m-k)|G(n,k;x)|,$$

which can be written as

$$\begin{aligned}t^m(qt)^n\{&C_t(n+m)+q^{-1}(r-2)(s-1)\sum_{k=1}^{m-1}(qt^2)^{-k}C_t(n+m-2k)\\ &+(qt)^{-1}(s-2)\sum_{k=0}^{m-1}(qt^2)^{-k}C_t(n+m-2k-1)+(qt^2)^{-m}C_t(n-m)\}.\end{aligned}$$

Since $\|\chi_n\|_t = r(r-1)^{-1}(qt)^n C_t(n)^{1/2}$ and

$$(L_t(x)\gamma_t^o \mid \gamma_t^o)_t = \lim_{n\to\infty} \|\chi_n\|_t^{-1}(L_t(x)\chi_n \mid \gamma_t^o)_t,$$

it follows from (3.6) that the right hand side is equal to

$$r^{-1}(r-1)t^m\{1+q^{-1}(r-2)(s-1)\sum_{k=1}^{m-1}(qt^2)^{-k} + (qt)^{-1}(s-2)\sum_{k=0}^{m-1}(qt^2)^{-k}+(qt^2)^{-m}\}.$$

The last expression coincides with $\phi_t(x)$. Note that by (3.11)

$$(L_t(x)\gamma_t^o \mid \gamma[e])_t = (\gamma_t^o \mid \delta[x^{-1}])_t = c(t)^{-1/2}\phi_t(x).$$

Since γ_t^1 is given by (3.12) and $(L_t(x)\gamma_t^o \mid \gamma_t^o)_t = \phi_t(x)$, we can deduce $(L_t(x)\gamma_t^o \mid \gamma_t^1)_t = 0$. Finally

$$(L_t(x)\gamma_t^1 \mid \gamma_t^1)_t = (1-c(t)^{-1})^{-1/2}(L_t(x)\gamma[e] \mid \gamma_t^1)_t - c(t)^{-1/2}(L_t(x)\gamma_t^o \mid \gamma_t^1)_t.$$

Using (3.12), we find that the right hand side agrees with $(1-c(t)^{-1})^{-1}(\psi_t(x) - c(t)^{-1}\phi_t(x))$, which equals $\psi_{(qt)^{-1}}(x)$ by (3.8).

By virtue of Lemma 3.7, we can define for $q^{-1/2}<t<1$ two closed invariant subspaces of H_t as follows. Let H_t^o be the closure of the linear span of $\{L_t(x)\gamma_t^o;\, x \in G\}$ in H_t, and let H_t^1 be the orthogonal complement of H_t^o in H_t. We often denote the restriction of L_t to H_t^o (resp. H_t^1) by L_t^o (resp. L_t^1).

Theorem 2. *Let $(G_i)_{1\leq i\leq r}$ be the family of finite groups of the same order s, and assume $q=(r-1)(s-1)\geqq 2$. Let $0<t<1$, and denote the canonical representation of the free product G of $(G_i)_{1\leq i\leq r}$ by (L_t, H_t).*

(i) *If $q^{-1/2}<t<1$, it can be decomposed into the direct sum of two subrepresentations (L_t^o, H_t^o) and (L_t^1, H_t^1). Furthermore L_t^o is the irreducible unitary representation defined by ϕ_t and hence it is not weakly contained in the regular representation. On the contrary, L_t^1 is the cyclic unitary representation with cyclic vector γ_t^1, which is defined by $\psi_{(qt)^{-1}}$. Hence it is a subrepresentation of the regular representation.*

(ii) *If $0<t\leqq q^{-1/2}$, the canonical representation L_t is weakly contained in the regular representation.*

Proof. (i) By definition, L_t^o is the cyclic unitary representation with cyclic vector γ_t^o and $(L_t^o(x)\gamma_t^o \mid \gamma_t^o)_t = \phi_t(x)$. Since ϕ_t is pure, L_t^o is irreducible and since ϕ_t is not weakly associated with the regular representation, L_t^o is not weakly contained in it. We shall show that γ_t^1 is a cyclic vector for L_t^1. Suppose that there exists $u \in H_t^1$ orthogonal to any finite linear combination of $\{L_t^1(x)\gamma_t^1; x \in G\}$. Since H_t^1 is the orthogonal complement of H_t^0, it follows from (3.13) that u is orthogonal to any finite linear combination of $\{L_t(x)\gamma[e]; x \in G\}$. Since $\gamma[e]$ is a cyclic vector for L_t, u must be zero. Therefore γ_t^1 is a cyclic vector for L_t^1. Using (3.14) and Lemma 3.3, we conclude that L_t^1 is a subrepresentation of the regular representation.

(ii) If $0<t\leqq q^{-1/2}$, ψ_t is weakly associated with the regular representation by Lemma 3.3. Hence L_t is weakly contained in the regular representation.

In the following, we consider the cyclic unitary representations of G, which possess the properties quite similar to those of the canonical representations. Let $\ell'(x)=[d\phi_t(x)/dt]_{t=1}$, that is,

$$\ell'(x)=\ell(x)+c'(1)(1-q^{-\ell(x)}) \qquad \text{where } x \in G. \tag{3.15}$$

Since ϕ_t is positive definite, we find that ℓ' is negative definite. For $0<t\leqq 1$, we define a positive definite function Ψ_t on G by

$$\Psi_t(x)=t^{\ell'(x)}. \tag{3.16}$$

Let $(\Pi_t, \mathscr{H}_t)$ be the cyclic unitary representation of G defined by Ψ_t. We denote the inner product of $\mathscr{H}_t$ by $(\ ,\ )_t$. Note that $\delta[e]$ induces a cyclic vector for $\mathscr{H}_t$. Put

$$a(n)=-c'(1)q^{-n}=s(r-1)r^{-1}(q-1)^{-1}q^{-n} \qquad \text{for } n \in N \tag{3.17}$$

and

$$\begin{aligned} B_t(m, n)=t^{a(m+n)}&+(r-2)(s-1)q^{-1}\textstyle\sum_{k=1}^{m\wedge n-1}(qt^2)^{-k}t^{a(m+n-2k)}\\ &+(s-2)(qt)^{-1}\textstyle\sum_{k=0}^{m\wedge n-1}(qt^2)^{-k}t^{a(m+n-2k-1)}\\ &+(qt^2)^{-m\wedge n}t^{a(m+n-2m\wedge n)}. \end{aligned} \tag{3.18}$$

As in Lemma 3.5, we can get

$$(\delta[x], \chi_n)_t=t^{c'(1)}t^m(qt)^nB_t(m, n) \qquad \text{where } x \in G(m) \tag{3.19}$$

and hence

$$(\chi_m, \chi_n)_t=r(r-1)^{-1}t^{c'(1)}(qt)^{m+n}B_t(m, n). \tag{3.20}$$

Since $t^{a(n)}$ is a monotone increasing function of $n \in N$ and $\lim_{n\to\infty} t^{a(n)}=1$, we conclude from (3.18) that if $q^{-1/2}<t\leqq 1$ $\lim_{m,n\to\infty} B_t(m, n)$ exists. While we find that for fixed $m \in N$ $\lim_{n\to\infty} B_t(m, n)=r(r-1)^{-1}C_t(m)$ (cf. Lemma 3.5). Hence by (3.6) we get $\lim_{m,n\to\infty} B_t(m, n)=r(r-1)^{-1}c(t)$. This implies (cf. Lemma 3.6) that $\{\|\chi_n\|_t^{-1}\chi_n; n\geqq 1\}$ is a Cauchy sequence in $\mathscr{H}_t$

and has a limit $u_t^o \in \mathscr{H}_t$ for $q^{-1/2} < t \leqq 1$. As in the proof of (3.11), we have

$$(\delta[x], u_t^o)_t = d(t)\phi_t(x) \tag{3.21}$$

where

$$d(t) = t^{c'(1)/2} c(t)^{-1/2}. \tag{3.22}$$

Put $u_t^1 = (1 - d(t)^2)^{-1/2}(\delta[e] - d(t)u_t^o)$. Then as in Lemma 3.7, we get

$$(\Pi_t(x)u_t^o, u_t^o)_t = \phi_t(x), \qquad (\Pi_t(x)u_t^o, u_t^1)_t = 0 \tag{3.23}$$

and

$$(\Pi_t(x)u_t^1, u_t^1)_t = O((qt)^{-\ell(x)}) \qquad \text{as } \ell(x) \to \infty. \tag{3.24}$$

Let $\mathscr{H}_t^o$ be the closure of the linear span of $\{\Pi_t(x)u_t^o;\, x \in G\}$ and let $\mathscr{H}_t^1$ be the orthogonal complement of $\mathscr{H}_t^o$ in $\mathscr{H}_t$. We denote the restriction of the representation Π_t to $\mathscr{H}_t^o$ (resp. $\mathscr{H}_t^1$) by Π_t^o (resp. Π_t^1). In conclusion, we obtain the following theorem, whose proof is quite similar to that of Theorem 2.

Theorem 2′. *Let G be the free product of a family $(G_i)_{1 \leqq i \leqq r}$ of finite groups of the same order s such that $q = (r-1)(s-1) \geqq 2$. Let $(\Pi_t, \mathscr{H}_t)$ be the cyclic unitary representation of G defined by Ψ_t (see* (3.16)).

(i) *If $q^{-1/2} < t < 1$, it can be decomposed into the direct sum of subrepresentations Π_t^o and Π_t^1. Moreover Π_t^o is the irreducible unitary representation defined by ϕ_t. While Π_t^1 is the cyclic unitary representation with cyclic vector u_t^1, which is a subrepresentation of the regular representation.*

(ii) *If $0 < t \leqq q^{-1/2}$, Π_t is weakly contained in the regular representation.*

In what follows, we use the notational convention that $H_1^o = H_1$ and $H_t^o = \{0\}$ for $0 \leqq t \leqq q^{-1/2}$, whose orthogonal complement is denoted by H_t^1. Let $0 < t_1, \cdots, t_n \leqq 1$ and put $t = t_1 \cdots t_n$. Let $T = \otimes_{1 \leqq i \leqq n} L_{t_i}$ be the tensor representation of G on the tensor product Hilbert space $H = \otimes_{1 \leqq i \leqq n} H_{t_i}$. We define a G-equivariant isometry j of H_t into H as follows. We put $j(L_t(x)\gamma[e]) = T(x)(\otimes_{1 \leqq i \leqq n} \gamma[e])$ for $x \in G$, and extend it linearly on the dense subspace of H_t spanned by finite linear combinations of $L_t(x)\gamma[e]$ $(x \in G)$. Note that for $x, y \in G$ $(j(L_t(x)\gamma[e]) \,|\, j(L_t(y)\gamma[e]))$ is equal to $\prod_{1 \leqq i \leqq n} \psi_{t_i}(y^{-1}x)$, which agrees with $\psi_t(y^{-1}x) = (L_t(x)\gamma[e] \,|\, L_t(y)\gamma[e])_t$ since $t = t_1 \cdots t_n$. Hence j can be extended to a G-equivariant isometry of H_t into H. The next lemma will be used in Section 5.

Lemma 3.8. *Suppose that $q^{-1/2} < t_i \leqq 1$ for $1 \leqq i \leqq n$ and $q^{-1/2} < t = t_1 \cdots t_n \leqq 1$. Then j maps H_t^o into $\otimes_{1 \leqq i \leqq n} H_{t_i}^o$.*

Proof. If $t=1$, then all $t_i=1$ so that the lemma is clearly true. Assume $t<1$. It follows from Theorem 2 that T can be decomposed into the direct sum of G-invariant closed subspaces $H(\varepsilon_1, \cdots, \varepsilon_n)=\otimes_{1\leqq i\leqq n} H_{ti}^{\varepsilon_i}$ where $(\varepsilon_1, \cdots, \varepsilon_n) \in \{0, 1\}^n$. Since H_t^o is irreducible and occurs with multiplicity one, $j(H_t^o)$ must be contained in exaclty one component of the above decomposition. Since H_t^o is never contained weakly in the regular representation, we have only to see that each $H(\varepsilon_1, \cdots, \varepsilon_n)$ except $(\varepsilon_1, \cdots, \varepsilon_n)=(0, \cdots, 0)$ is weakly contained in the regular representation. By rearranging, we may assume that $\varepsilon_1=\cdots=\varepsilon_k=1$ and $\varepsilon_{k+1}=\cdots=\varepsilon_n=0$ where $k\geqq 1$. For $(y_1, \cdots, y_n) \in G^n$, we put

$$u(y_1, \cdots, y_n)=(\bigotimes_{1\leqq i\leqq k} L_{t_i}(y_i)\gamma_{ti}^1)\otimes(\bigotimes_{k<i} L_{t_i}(y_i)\gamma_{ti}^o).$$

Then the set of finite linear combinations of $u(y_1, \cdots, y_n)$ where $(y_1, \cdots, y_n) \in G^n$ is dense in $H(1, \cdots, 1, 0, \cdots, 0)$. Every matrix coefficient for the representation T restricted to $H(1, \cdots, 1, 0, \cdots, 0)$ is in the closure of the linear spans of the matrix coefficients of the form

$$f(x)=(T(x)u(y_1, \cdots, y_n)\,|\,u(z_1, \cdots, z_n))$$

with respect to simple convergence on G. Hence it is enough to show that the above f belongs to $l^2(G)$. By Lemma 3.7, we have

$$f(x)=\prod_{1\leqq i\leqq k} \psi_{(qt_i)^{-1}}(z_i^{-1}xy_i) \prod_{k<i} \phi_{t_i}(z_i^{-1}xy_i).$$

Since $|\phi_{t_i}(z_i^{-1}xy_i)|\leqq\phi_{t_i}(e)=1$, $|f(x)|\leqq \prod_{1\leqq i\leqq k} \psi_{(qt_i)^{-1}}(z_i^{-1}xu_i)$. Note that $q^{-1/2}<t<t_i<1$ and therefore $(qt_i)^{-1/2}<(qt)^{-1}<1$ for $1\leqq i\leqq n$. Hence we have $|f(x)|\leqq \prod_{1\leqq i\leqq k} \psi_{(qt)^{-1}}(z_i^{-1}xy_i)$. Applying Lemma 3.2, we obtain $f \in l^2(G)$.

§ 4. Unitary representations of direct limit groups

Let $(\Xi, <)$ be a directed ordered set. Let $(H_\xi, j_{\eta\xi})$ be a Ξ-direct system of Hilbert spaces. This means that each H_ξ is a Hilbert space with inner product $(\,|\,)_\xi$, and each $j_{\eta\xi}$ for $\xi<\eta$ is an isometry of H_ξ into H_η such that $j_{\xi\xi}=\text{id.}$ and $j_{\zeta\xi}=j_{\zeta\eta}\circ j_{\eta\xi}$ for $\xi<\eta<\zeta$. The direct limit Hilbert space (H_∞, j_ξ) is defined as follows. Let $(\varinjlim H_\xi, j_\xi)$ be the set-theoretical direct limit of $(H_\xi, j_{\eta\xi})$. Note that $\varinjlim H_\xi=\bigcup_{\xi\in\Xi} j_\xi(H_\xi)$ and j_ξ is the canonical map of H_ξ into $\varinjlim H_\xi$. We remark that $j_\xi=j_\eta\circ j_{\eta\xi}$ for $\xi<\eta$ and if $j_\xi(u)=j_\eta(v)$ then there exists $\zeta \in \Xi$ with $\xi<\zeta$ and $\eta<\zeta$ such that $j_{\zeta\xi}(u)=j_{\zeta\eta}(v)$. We define an inner product on $\varinjlim H_\xi$ as follows. Let $h_i \in \varinjlim H_\xi$ $(i=1, 2)$. Then we can select $\xi \in \Xi$ and $v_i \in H_\xi$ such that $h_i=j_\xi(v_i)$ $(i=1, 2)$. Put

$$(4.1) \qquad (h_1 \mid h_2) = (v_1 \mid v_2)_\xi,$$

which is easily seen to be well-defined. Let H_∞ be the completion of $\varinjlim H_\xi$ with respect to $(\,|\,)$. It follows from the definition of the inner product that j_ξ can be extended to an isometry of H_ξ into H_∞ and $j_\xi(H_\xi)$ yields a closed subspace of H_∞. Let $(G_\xi, i_{\eta\xi})$ be a Ξ-direct system of topological groups. We denote the corresponding direct limit group by $(G_\infty = \varinjlim G_\xi, i_\xi)$. Assume that for each $\xi \in \Xi$ there is given a unitary representation (π_ξ, H_ξ) of G_ξ, and for $\xi < \eta$ there is given a G_ξ-equivariant isometry $j_{\eta\xi}$ of H_ξ into H_η such that $(H_\xi, j_{\eta\xi})$ provides a Ξ-direct system of Hilbert spaces. In this setting, we say that $(\pi_\xi, H_\xi, j_{\eta\xi})$ is a Ξ-direct system of unitary representations of a Ξ-direct system of topological groups $(G_\xi, i_{\eta\xi})$. Let (G_∞, i_ξ) (resp. (H_∞, j_ξ)) be the direct limit group (resp. the direct limit Hilbert space). Then we are able to define a unitary representation π_∞ of G_∞ on H_∞ in the following manner. Let $g \in G_\infty$ and $h \in \varinjlim H_\xi$. Select $\xi \in \Xi$, $x \in G$ and $u \in H_\xi$ such that $g = i_\xi(x)$ and $h = j_\xi(u)$. Put

$$(4.2) \qquad \pi_\infty(g)h = j_\xi(\pi_\xi(x)u).$$

It is obvious that the right hand side does not depend on the choice of ξ, x and u. Since j_ξ is an isometry, we can obtain quickly $(\pi_\infty(g)h_1 \mid \pi_\infty(g)h_2) = (h_1 \mid h_2)$ for $h_1, h_2 \in \varinjlim H_\xi$ and $g \in G_\infty$. Since $\varinjlim H_\xi$ is dense in H_∞, $\pi_\infty(g)$ can be extended to a unitary operator on H_∞. Furthermore we can see that π_∞ yields a unitary representation of G_∞ on H_∞. The following lemma (which is probably known) was pointed out to the author by Dr. Obata and Dr. Yamashita.

Lemma 4.1. *Keep the notations and assumptions. Suppose further that each (π_ξ, H_ξ) is irreducible. Then (π_∞, H_∞) is irreducible.*

Proof. Let A be a G_∞-equivariant continuous linear operator on H_∞. We have only to see that A is a scalar operator. Since j_ξ is a G_ξ-equivariant isometry of H_ξ into H_∞ and (π_ξ, H_ξ) is irreducible, it follows that $(\pi_\infty \circ i_\xi, j_\xi(H_\xi))$ is an irreducible unitary representation of G_ξ equivalent to (π_ξ, H_ξ). For each $\xi \in \Xi$, let P_ξ be the orthogonal projection of H_∞ onto $j_\xi(H_\xi)$. Then $P_\xi A P_\xi$ (viewed as an operator on $j_\xi(H_\xi)$) is G_ξ-equivariant. Since $(\pi_\infty \circ i_\xi, j_\xi(H_\xi))$ is irreducible, there exists $\lambda_\xi \in \boldsymbol{C}$ such that $P_\xi A P_\xi = \lambda_\xi P_\xi$ for each $\xi \in \Xi$. Since $j_\xi(H_\xi)$ is contained in $j_\zeta(H_\zeta)$ for $\xi < \zeta$, it follows that $P_\xi P_\zeta = P_\zeta P_\xi = P_\xi$. Hence for $\xi < \zeta$ we have $\lambda_\xi P_\xi = P_\xi A P_\xi = P_\xi P_\zeta A P_\zeta P_\xi = \lambda_\zeta P_\xi$. This implies $\lambda_\xi = \lambda_\zeta$ for $\xi < \zeta$. Let $\xi, \eta \in \Xi$. By taking $\zeta \in \Xi$ such that $\xi < \zeta$ and $\eta < \zeta$, we conclude that $\lambda_\xi = \lambda_\zeta = \lambda_\eta$. Consequently λ_ξ is independent of $\xi \in \Xi$. From now on, we write $\lambda = \lambda_\xi$.

Let $h_i \in \varinjlim H_\xi$ $(i=1, 2)$. Then there exists $\xi \in \Xi$ such that $h_i \in j_\xi(H_\xi)$, that is, $P_\xi h_i = h_i$ $(i=1, 2)$. We have $(Ah_1 | h_2) = (AP_\xi h_1 | P_\xi h_2) = (P_\xi AP_\xi h_1 | h_2) = \lambda(h_1 | h_2)$. Since $\varinjlim H_\xi$ is dense in H_∞ and A is continuous, we conclude that $A = \lambda$ id.

Remark. We explicate an example that even if (π_ξ, H_ξ) $(\xi \in \Xi)$ are not necessarily irreducible, the resulting representation π_∞ happens to be irreducible. Let $I = \{1, 2, 3, \cdots\}$. Let $(G_i)_{i \in I}$ be a family of finite groups G_i, each of which has the same order s. Let G be the free product of $(G_i)_{i \in I}$. By Theorem 1 the canonical representation (L_t, H_t) of G where $0 < t < 1$ is irreducible. For $a > 1$ we put $G^{(a)} = G_1 * G_2 * \cdots * G_a$, which can be viewed as a subgroup of G. Furthermore $G^{(a)}$ is regarded as a subgroup of $G^{(b)}$ for $a \leqq b$. Consequently $\{G^{(a)}; a > 1\}$ forms a direct system of discrete groups, whose direct limit agrees with G. For $a > 1$ and $0 < t < 1$, let $(L_t^{(a)}, H_t^{(a)})$ be the canonical representation of $G^{(a)}$. Then $H_t^{(a)}$ is isomorphic to the closed $G^{(a)}$-invariant subspace of H_t spanned by $\{\gamma[x]; x \in G^{(a)}\}$. Moreover $L_t^{(a)}$ is equivalent to the representation of $G^{(a)}$ obtained by the restriction of L_t to it. Evidently $\{(L_t^{(a)}, H_t^{(a)}); a > 1\}$ yields a direct system of unitary representations of the direct system $\{G^{(a)}; a > 1\}$. The corresponding representation of $G = \varinjlim G^{(a)}$ defined in (4.2) is identified with L_t. $(L_t^{(a)}, H_t^{(a)})$ for $a > 1$ are not necessarily irreducible (see Theorem 2), but the resulting representation L_t is irreducible (see Theorem 1).

Now we review a construction of a Ξ-direct system of unitary representations of a Ξ-direct system of topological groups $(G_\xi, i_{\eta\xi})$ (cf. [13] and [30]). Suppose that we are given a positive definite function Φ_ξ on G_ξ for each $\xi \in \Xi$ satisfying

$$\Phi_\eta \circ i_{\eta\xi} = \Phi_\xi \qquad \text{for } \xi < \eta. \tag{4.3}$$

This assures the existence of a positive definite function Φ on the direct limit group (G_∞, i_ξ) such that $\Phi \circ i_\xi = \Phi_\xi$ for $\xi \in \Xi$. Let (π_ξ, H_ξ) be the cyclic unitary representation of G_ξ with cyclic vector γ_ξ defined by Φ_ξ, so that

$$\Phi_\xi(x) = (\pi_\xi(x)\gamma_\xi | \gamma_\xi) \qquad \text{for } x \in G_\xi. \tag{4.4}$$

For $\xi < \eta$, we can define a G_ξ-equivariant isometry $j_{\eta\xi}$ of H_ξ into H_η as follows. For $x \in G_\xi$ we put

$$j_{\eta\xi}(\pi_\xi(x)\gamma_\xi) = \pi_\eta(i_{\eta\xi}(x))\gamma_\eta. \tag{4.5}$$

Then by (4.3) and (4.4) we have

(4.6) $$(j_{\eta\xi}(\pi_\xi(x)\gamma_\xi)\,|\,j_{\eta\xi}(\pi_\xi(y)\gamma_\xi))_\eta=(\pi_\xi(x)\gamma_\xi\,|\,\pi_\xi(y)\gamma_\xi)_\xi$$

for $x, y \in G_\xi$. Since $\{\pi_\xi(x)\gamma_\xi;\, x \in G_\xi\}$ is total in H_ξ, we conclude from (4.5) and (4.6) that $j_{\eta\xi}$ can be extended to a G_ξ-equivariant isometry of H_ξ into H_η. Furthermore we can check easily that $(\pi_\xi, H_\xi, j_{\eta\xi})$ provides a Ξ-direct system of unitary representations of a Ξ-direct system of topological groups $(G_\xi, i_{\eta\xi})$. Let $(\pi_\infty, H_\infty, j_\xi)$ be the resulting unitary representation of (G_∞, i_ξ). Put

(4.7) $$\gamma_\infty=j_\xi(\gamma_\xi) \qquad \text{for some } \xi \in \Xi.$$

It follows immediately that γ_∞ is an element of H_∞, which does not depend on the choice of ξ. Moreover γ_∞ is a cyclic vector for $(\pi_\infty, H_\infty, j_\xi)$ and

(4.8) $$(\pi_\infty(g)\gamma_\infty\,|\,\gamma_\infty)=\Phi(g) \qquad \text{for } g \in G_\infty.$$

§ 5. Construction of irreducible unitary representations of $G^{(X)}$

Let $(X, \mathscr{B})$ be a measurable space. Let $G^{(X)}$ be the group of maps of X having finitely many values in a topological group G. Such a group $G^{(X)}$ is sometimes called a weak current group (cf. [13]). Let Ξ be the set of all finite partitions of $(X, \mathscr{B})$. For $\xi, \eta \in \Xi$, we write $\xi<\eta$ if η is a refinement of ξ. Then $(\Xi, <)$ provides a directed ordered set. For $\xi=\{X_1, \cdots, X_n\} \in \Xi$, let G_ξ be the subgroup of $G^{(X)}$ consisting of maps which take constant values on each X_i. Every element of G_ξ is of the form

(5.1) $$f_{x_1,\cdots,x_n} \qquad \text{for } (x_1, \cdots, x_n) \in G^n$$

where $f_{x_1,\cdots,x_n}(X_i)=\{x_i\}$ for $1\leqq i\leqq n$. This implies that G_ξ is canonically isomorphic to the n copies G^n of G. Let $\eta \in \Xi$ such that $\xi<\eta$. Then there exists a natural monomorphism $i_{\eta\xi}$ of G_ξ into G_η, and $(G_\xi, i_{\eta\xi})$ yields a Ξ-direct system of topological groups. The direct limit group agrees with $(G^{(X)}, i_\xi)$ where i_ξ is the natural inclusion of G_ξ into $G^{(X)}$.

Now we take G as the free product of a countable family $(G_i)_{i\in I}$ of countable groups. Let μ be a finite measure on $(X, \mathscr{B})$. Define a function Φ_μ on $G^{(X)}$ by

(5.2) $$\Phi_\mu(f)=\exp\left\{-\int_X \ell(f(\omega))\mu(d\omega)\right\}.$$

The restriction of Φ_μ to G_ξ is denoted by Φ_ξ. Then (4.3) holds for $\{\Phi_\xi;\, \xi \in \Xi\}$. For $\xi=\{X_1, \cdots, X_n\} \in \Xi$, we put

(5.3) $$t_i=\exp\{-\mu(X_i)\} \qquad \text{for } 1\leqq i\leqq n.$$

Note that $\exp\{-\mu(X)\} \leqq t_i \leqq 1$ for $1 \leqq i \leqq n$. Let (L_{t_i}, H_{t_i}) be the canonical representations of G introduced in Section 1. Put

$$H_\xi = H_{t_1} \otimes \cdots \otimes H_{t_n}. \tag{5.4}$$

The canonical inner product of H_ξ is denoted by $(\,|\,)_\xi$. Define the representation L_ξ of G_ξ on H_ξ by

$$L_\xi(f_{x_1,\cdots,x_n}) = L_{t_1}(x_1) \otimes \cdots \otimes L_{t_n}(x_n). \tag{5.5}$$

Define $\gamma_\xi \in H_\xi$ by

$$\gamma_\xi = \gamma[e] \otimes \cdots \otimes \gamma[e]. \tag{5.6}$$

Lemma 5.1. *For $\xi \in \Xi$, (L_ξ, H_ξ) is a cyclic unitary representation of G_ξ with cyclic vector γ_ξ such that*

$$(L_\xi(f)\gamma_\xi \,|\, \gamma_\xi)_\xi = \Phi_\xi(f) \qquad \textit{for } f \in G_\xi. \tag{5.7}$$

Proof. Since each L_{t_i} is a cyclic unitary representation of G with cyclic vector $\gamma[e]$, the first assertion is obvious. Let $f = f_{x_1,\cdots,x_n} \in G_\xi$. Then $(L_\xi(f)\gamma_\xi \,|\, \gamma_\xi)_\xi = \prod_{1 \leqq i \leqq n} \psi_{t_i}(x_i)$ which is, by (5.3), equal to

$$\exp\{-\textstyle\sum_{i=1}^n \ell(x_j)\mu(X_i)\}.$$

We can rewrite it as

$$\exp\left\{-\int_X \ell(f(\omega))\mu(d\omega)\right\}.$$

Combining the lemma with the result in Section 4, we get a Ξ-direct system $(L_\xi, H_\xi, j_{\eta\xi})$ of cyclic unitary representations of $(G_\xi, i_{\eta\xi})$. Here $j_{\eta\xi}$ is given by (4.5). We denote the resulting representation of $(G^{(X)}, i_\xi)$ by (L_μ, H_μ). Note that $\gamma_\infty = j_\xi(\gamma_\xi)$ is a cyclic vector for L_μ and $(L_\mu(f)\gamma_\infty \,|\, \gamma_\infty) = \Phi_\mu(f)$ for $f \in G^{(X)}$.

Theorem 3. *Let $(G_i)_{i \in I}$ be a countable family of countable groups and G be its free product. Assume that the cardinality of I is infinite. Then the unitary representation (L_μ, H_μ) of $G^{(X)}$ is irreducible. Moreover if μ_1 and μ_2 are different finite measures on $(X, \mathscr{B})$, then L_{μ_1} and L_{μ_2} are inequivalent.*

Proof. It follows from Theorem 1 that each L_{t_i} is an irreducible representation of G and hence L_ξ is an irreducible representation of G_ξ for every $\xi \in \Xi$. Therefore L_μ is irreducible by Lemma 4.1. If $\mu_1 \neq \mu_2$, then there exists $E \in \mathscr{B}$ such that $\mu_1(E) \neq \mu_2(E)$. Let f in $G^{(X)}$ such that $f(E) = \{x\}$ where $x \neq e$ and $f(X-E) = \{e\}$. Then $\Phi_{\mu_1}(f) \neq \Phi_{\mu_2}(f)$. This implies that L_{μ_1} and L_{μ_2} are inequivalent.

From now on, we consider the case of $G^{(X)}$ where G is the free product of $(G_i)_{1\leqq i\leqq r}$ such that all G_i are finite groups of the same order s with $q=(r-1)(s-1)\geqq 2$. Let (L_t, H_t) be the canonical representation of G studied in Section 3. Let $\xi=\{X_1, \cdots, X_n\}$ in Ξ. By Theorem 2 H_ξ (see (5.4)) can be written as a direct sum of two closed invariant subspaces H_ξ^o and H_ξ^1. Here

$$(5.8) \qquad H_\xi^o=\bigotimes_{1\leqq i\leqq n} H_{t_i}^o \quad \text{and} \quad H_\xi^1=\bigoplus \bigotimes_{1\leqq i\leqq n} H_{t_i}^{\varepsilon_i}$$

where $(\varepsilon_1, \cdots, \varepsilon_n)$ runs through $\{0, 1\}^n-(0, \cdots, 0)$. We denote by L_ξ^o the restriction of L_ξ to H_ξ^o. Put

$$(5.9) \qquad \mu(\xi)=\max\{\mu(X_i);\ 1\leqq i\leqq n\}.$$

If $\mu(\xi)<2^{-1}\ln(q)$, then $q^{-1/2}<t_i\leqq 1$ for $1\leqq i\leqq n$ and hence H_ξ^o is a nontrivial irreducible subspace of H_ξ (see Theorem 2).

Lemma 5.2. *Let $\xi, \eta \in \Xi$ such that $\xi<\eta$. Then $j_{\eta\xi}$ maps H_ξ^o into H_η^o and consequently $(L_\xi^o, H_\xi^o, j_{\eta\xi})$ yields a subdirect system of $(L_\xi, H_\xi, j_{\eta\xi})$.*

Proof. It is enough to consider the case when $\mu(\xi)<2^{-1}\ln(q)$. Write $\xi=\{X_1, \cdots, X_n\}$. Then η is of the form $\eta=\{X_{ij};\ 1\leqq i\leqq n,\ 1\leqq j\leqq n_i\}$ where $X_i=\bigcup_{1\leqq j\leqq n_i} X_{ij}$ (a disjoint union) for $1\leqq i\leqq n$. Put $t_{ij}=\exp\{-\mu(X_{ij})\}$, $H_{\eta,i}=\otimes_{1\leqq j\leqq n_i} H_{t_{ij}}$, $\Upsilon_{\eta,i}=\otimes_{1\leqq j\leqq n_i}\Upsilon[e]\in H_{\eta,i}$ and $L_{\eta,i}(x)=\otimes_{1\leqq j\leqq n_i} L_{t_{ij}}(x)$ for $x\in G$. Then $t_i=t_{i1}\cdots t_{in_i}$, $H_\eta=\otimes_{1\leqq i\leqq n} H_{\eta,i}$, $\Upsilon_\eta=\otimes_{1\leqq i\leqq n}\Upsilon_{\eta,i}$ and $L_\eta(i_{\eta\xi}(f_{x_1,\dots,x_n}))=\otimes_{1\leqq i\leqq n} L_{\eta,i}(x_i)$. Define the G-equivariant isometry $j_{\eta,i}$ of H_{t_i} into $H_{\eta,i}$ by putting $j_{\eta,i}(L_{t_i}(x)\Upsilon[e])=L_{\eta,i}(x)\Upsilon_{\eta,i}$ (cf. Lemma 3.8). Then we find that $j_{\eta\xi}=\otimes_{1\leqq i\leqq n} j_{\eta,i}$. Since $\mu(\eta)\leqq\mu(\xi)<2^{-1}\ln(q)$, it follows that $q^{-1/2}<t_{ij}\leqq 1$ and $q^{-1/2}<t_i\leqq 1$. Applying Lemma 3.8, we conclude that $j_{\eta,i}(H_{t_i}^o)$ lies in $H_{\eta,i}^o$ for each i. Here $H_{\eta,i}^o=\otimes_{1\leqq j\leqq n_i} H_{t_{ij}}^o$. This yields that $j_{\eta\xi}(H_\xi^o)$ is contained in H_η^o.

Theorem 4. *Let $(X, \mathscr{B}, \mu)$ be a measure space with a finite measure μ. Assume that there exists $\xi\in\Xi$ such that $\mu(\xi)<2^{-1}\ln(q)$. Then the unitary representation (L_μ^o, H_μ^o) of $G^{(X)}$ defined by the direct system $(L_\xi^o, H_\xi^o, j_{\eta\xi})$ of unitary representations of the direct system $(G_\xi, i_{\eta\xi})$ is irreducible. In particular if $(X, \mathscr{B}, \mu)$ is a nonatomic Lebesgue space, then L_μ^o is irreducible.*

Proof. By assumption, $(L_\xi^o, H_\xi^o, j_{\eta\xi})$ is a nontrivial subdirect system of $(L_\xi, H_\xi, j_{\eta\xi})$. For each $\xi\in\Xi$ satisfying $\mu(\xi)<2^{-1}\ln(q)$, L_ξ^o is an irreducible unitary representation by Theorem 2. Hence by Lemma 4.1 L_μ^o is irreducible.

In what follows, we shall give an another construction of the irreducible representation equivalent ot L_μ^o. Let $\xi=\{X_1, \cdots, X_n\}$ and $t_i=$

$\exp\{-\mu(X_i)\}$ for $1\leqq i\leqq n$. Let $(\Pi_{t_i}, \mathscr{H}_{t_i})$ be the cyclic unitary representations of G defined by Ψ_{t_i} (see (3.16)). Put $\mathscr{H}_\xi=\otimes_{1\leqq i\leqq n}\mathscr{H}_{t_i}$ and $\Pi_\xi(f_{x_1,\dots,x_n})=\otimes_{1\leqq i\leqq n}\Pi_{t_i}(x_i)$. Then $(\Pi_\xi, \mathscr{H}_\xi)$ is a cyclic unitary representation of G_ξ with cyclic vector $u_\xi=\otimes_{1\leqq i\leqq n}\delta[e]$ such that $(\Pi_\xi(f)u_\xi, u_\xi)_\xi=\Psi_\mu(f)$ for $f\in G_\xi$ where Ψ_μ is a function on $G^{(X)}$ defined by

$$\Psi_\mu(f)=\exp\left\{-\int_X \ell'(f(\omega))\mu(d\omega)\right\}. \tag{5.10}$$

Using the results in Section 4, we get a Ξ-direct system $(\Pi_\xi, \mathscr{H}_\xi, k_{\eta\xi})$ of cyclic unitary representations of $(G_\xi, i_{\eta\xi})$. Here $k_{\eta\xi}$ is an G_ξ-equivariant isometry of $\mathscr{H}_\xi$ into $\mathscr{H}_\eta$ defined by $k_{\eta\xi}(\Pi_\xi(f)u_\xi)=\Pi_\eta(i_{\eta\xi}(f))u_\eta$. The resulting representation of $G^{(X)}$ is denoted by $(\Pi_\mu, \mathscr{H}_\mu)$. Suppose that $\xi\in\Xi$ such that $\mu(\xi)<2^{-1}\ln(q)$. Put $\mathscr{H}^\circ_\xi=\otimes_{1\leqq i\leqq n}\mathscr{H}^\circ_{t_i}$ and denote the restriction of Π_ξ to $\mathscr{H}^\circ_\xi$ by Π°_ξ. Then by Theorem 2′ Π°_ξ is irreducible. We notice that Π°_t is equivalent to L°_t where $q^{-1/2}<t\leqq 1$ and hence Π°_ξ is equivalent to L°_ξ for each $\xi\in\Xi$. As in Lemma 5.2, we obtain that $k_{\eta\xi}(\mathscr{H}^\circ_\xi)$ is contained in $\mathscr{H}^\circ_\eta$ for $\xi<\eta$. Therefore we get a subdirect system $(\Pi^\circ_\xi, \mathscr{H}^\circ_\xi, k_{\eta\xi})$ of $(\Pi_\xi, \mathscr{H}_\xi, k_{\eta\xi})$. We denote by $(\Pi^\circ_\mu, \mathscr{H}^\circ_\mu)$ the representation of $G^{(X)}$ defined by $(\Pi^\circ_\xi, \mathscr{H}^\circ_\xi, k_{\eta\xi})$. From the argument above, we have

Theroem 4′. *Under the same assumption as in Theorem* 4, Π°_μ *is an irreducible unitary representation of* $G^{(X)}$ *equivalent to* L°_μ.

The construction of the representations Π_μ and Π°_μ leads to the following remarkable fact.

Theorem 5. *Let* $(X, \mathscr{B}, \mu)$ *be a nonatomic Lebesgue space. Then* $\mathscr{H}_\mu=\mathscr{H}^\circ_\mu$ *and* $\Pi_\mu=\Pi^\circ_\mu$ *is equivalent to* L°_μ.

Proof. Since Π_μ is a cyclic unitary representation of $G^{(X)}$ with cyclic vector $u_\infty=k_\xi(u_\xi)$ for any $\xi\in\Xi$, we have only to show $u_\infty\in\mathscr{H}^\circ_\mu$. Let $\xi=\{X_1,\cdots,X_n\}$ and put $t_i=\exp\{-\mu(X_i)\}$ for $1\leqq i\leqq n$. Define $u^\circ_\xi\in\mathscr{H}^\circ_\xi$ by $u^\circ_\xi=\otimes_{1\leqq i\leqq n}u^\circ_{t_i}$ and $d(\xi)=\prod_{1\leqq i\leqq n}d(t_i)$ where $u^\circ_t=\lim_{n\to\infty}\|\chi_n\|^{-1}\chi_n$ in $\mathscr{H}_t$ with $q^{-1/2}<t\leqq 1$ and $d(t)$ is given by (3.22). By the assumption of the theorem, we can select a sequence $\{\xi_m; m\geqq 1\}$ in Ξ satisfying $\xi_m<\xi_{m+1}$ for $m\geqq 1$ and $\mu(\xi_m)\to 0$ as $m\to\infty$. Put $u_m=k_{\xi_m}(d(\xi_m)u^\circ_{\xi_m})$. Then $\{u_m; m\geqq 1\}$ is a sequence in $\mathscr{H}^\circ_\mu$ such that $\|u_\infty-u_m\|^2=1-d(\xi_m)^2$. Since $d(t)$ is monotone increasing, we have $d(\xi)\geqq d(\exp\{-\mu(\xi)\})^n$ and consequently $1-d(\xi)^2\leqq 1-d(\exp\{-\mu(\xi)\})^{2n}$. Since $d(t)=1+O((t-1)^2)$ as $t\to 1$, we conclude that $d(\xi_m)\to 1$ and hence $\|u_\infty-u_m\|\to 0$ as $m\to\infty$. This means that $u_\infty\in\mathscr{H}^\circ_u$.

Corollary 6. *Let* $(X, \mathscr{B}, \mu_i)$ $(i=1, 2)$ *be nonatomic Lebesgue spaces such that* $\mu_1\neq\mu_2$. *Then* Π_{μ_1} *and* Π_{μ_2} *are inequivalent.*

Proof. If $\mu_1 \neq \mu_2$, then we find that $\Psi_{\mu_1} \neq \Psi_{\mu_2}$, as in Theorem 3. Since Π_{μ_i} $(i=1, 2)$ are cyclic unitary representations defined by Ψ_{μ_i}, it follows that Π_{μ_1} and Π_{μ_2} are inequivalent.

Let σ be an invertible bi-measurable transformation of a finite measure space $(X, \mathscr{B}, \mu)$. Then σ induces an automorphism of $G^{(X)}$ by ${}^{\sigma}f(\omega)=f(\sigma^{-1}(\omega))$ where $f \in G^{(X)}$ and $\omega \in X$. We get new representations of $G^{(X)}$ by setting ${}^{\sigma}L_{\mu}(f)=L_{\mu}({}^{\sigma}f)$ and ${}^{\sigma}\Pi_{\mu}(f)=\Pi_{\mu}({}^{\sigma}f)$. Let $\mu \circ \sigma$ be the measure on $(X, \mathscr{B})$ such that $\mu \circ \sigma(E)=\mu(\sigma(E))$ for $E \in \mathscr{B}$. Since $\Phi_{\mu}({}^{\sigma}f)=\Phi_{\mu \circ \sigma}(f)$ (resp. $\Psi_{\mu}({}^{\sigma}f)=\Psi_{\mu \circ \sigma}(f)$), it follows that ${}^{\sigma}L_{\mu}$ and $L_{\mu \circ \sigma}$ (resp. ${}^{\sigma}\Pi_{\mu}$ and $\Pi_{\mu \circ \sigma}$) are equivalent. This yields the following theorem.

Theorem 7. *Let σ be a measure-preserving, invertible bi-measurable transformation on a finite measure space $(X, \mathscr{B}, \mu)$.*

(i) *If G is the free product of $(G_i)_{i \in I}$ such that the cardinality of I is infinite, then ${}^{\sigma}L_{\mu}$ and L_{μ} are equivalent.*

(ii) *If G is an r family of finite groups of the same order s with $q=(r-1)(s-1) \geqq 2$, and if $(X, \mathscr{B}, \mu)$ is a nonatomic Lebesgue space, then ${}^{\sigma}\Pi_{\mu}$ and Π_{μ} are equivalent.*

References

[1] S. Albeverio, R. Høegh-Kröhn and D. Testard, Irreducibility and reducibility for the energy representation of the group of mappings of a Riemannian manifold into a compact semisimple group, J. Funct. Anal., **41** (1981), 378–396.

[2] K. Aomoto, Spectral theory on a free group and algebraic curves, J. Fac. Sci. Univ. Tokyo. **31** (1984), 297–318.

[3] H. Araki, Factorizable representations of current algebra, Publ. Res. Inst. Math. Sci., **5** (1969/70), 361–422.

[4] P. Cartier, Fonctions harmoniques sur un arbre, Symp. Math., **9** (1972), 203–270.

[5] J. M. Cohen and A. R. Trenholme, Orthogonal polynomials with a constant recursion formula and an application to harmonic analysis, J. Funct. Anal., **59** (1984), 175–184.

[6] P. Delorme, Irréducibilité de certaines représentations de G^X, J. Funct. Anal., **30** (1978), 36–47.

[7] J. Dixmier, Les C^*-algèbres et leurs représentations, Gauthier-Villars, Paris, 1969.

[8] P. Eymard, L'algèbre de Fourier d'un groupe localment compact, Bull. Soc. Math. France, **92** (1964), 181–236.

[9] J. Faraut et K. Harzallah, Distances hilbertiennes invariantes sur un espace homogène, Ann. Inst. Fourier Grenoble, **24** (1974), 171–217.

[10] A. Figà-Talamanca and M. A. Picardello, Spherical functions and harmonic analysis on free groups, J. Funct. Anal., **47** (1982), 281–304.

[11] ——, Harmonic Analysis on free groups, Lecture Notes in Pure and Applied Math., Vol. 87, Marcel Dekker, New York, 1983.

[12] R. Godement, Les fonctions de type positif et la theorie des groupes, Trans. Amer. Math. Soc., **63** (1948), 1–84.

[13] A. Guichardet, Symmetric Hilbert spaces and related topics, Lecture Notes in Math., Vol. 261, Springer-Verlag, Berlin, 1972.
[14] U. Haagerup, An example of a non nuclear C^*-algebra, which has the metric approximation property, Invent. Math., **50** (1979), 279–293.
[15] A. Iozzi and M. A. Picardello, Spherical functions on symmetric graphs, in Conference in Harmonic Analysis, Lecture Notes in Math., Vol. 992, Springer-Verlag, Berlin, 1983.
[16] P. Julg and A. Valette, K-theoretic amenability for $SL_2(Q_p)$ and the action on the associated tree, J. Funct. Anal., **58** (1984), 194–215.
[17] T. Kajiwara, On irreducible decompositions of the regular representation of free groups, Boll. Un. Mat. It., **6** (1985), 425–431.
[18] A. M. Mantero and A. Zappa, Special series of unitary representations of groups acting on homogeneous trees, J. Austral. Math. Soc., **40** (1986), 83–88.
[19] H. Matsumoto, Analyse harmonique dans les systemes de Tits bornologiques de type affine, Lecture Notes in Math., Vol. 590, Springer-Verlag, Berlin, 1970.
[20] H. Matsushima, K. Okamoto and T. Sakurai, On a certain class of irreducible unitary representations of the infinite dimensional rotation group I, Hiroshima Math. J., **11** (1981), 181–193.
[21] N. Obata, Certain unitary representations of infinite symmetric groups, preprint.
[22] G. I. Ol'shanskii, Classification of irreducible representations of groups of automorphisms of Bruhat-Tits trees, Funct. Anal. Appl., **11** (1977), 26–34.
[23] ——, Unitary representations of the infinite dimensional classical groups $U(p, \infty)$, $SO_0(p, \infty)$, $Sp(p, \infty)$ and the corresponding motion groups, Funct. Anal. Appl., **12** (1978), 185–195.
[24] K. R. Parthasarathy and K. Schmidt, Positive definite kernels, continuous tensor products and central limit theorems of probability theory, Lecture Notes in Math., Vol. 272, Springer-Verlag, Berlin, 1972.
[25] T. Pytlik and R. Szwarc, An analytic family of uniformly bounded representations of free groups, Acta Math., **157** (1986), 287–309.
[26] M. Saito, Représentations unitaires monomiales d'un groupe discret, en particulier du groupe modulaire, J. Math. Soc. Japan, **26** (1974), 464–482.
[27] I. J. Schoenberg, Metric spaces and completely monotone functions, Ann. Math., **39** (1938), 811–841.
[28] J. P. Serre, Arbres, amalgames, SL_2, Astérisque, Vol. 46, 1977.
[29] R. F. Streater, Infinitely divisible representations of Lie algebras, Z. Wahrsheinlichkeitstheorie Verw. Geb., **19** (1971), 67–80.
[30] A. M. Vershik, I. M. Gelfand and M. I. Graev, Representations of the group $SL(2, R)$, where R is a ring of functions, Russ. Math. Surveys, **28** (1973), 87–132.
[31] ——, Irreducible representations of the group G^X and cohomology, Funct. Anal. Appl., **8** (1974), 67–68.
[32] A. M. Vershik and S. I. Karpushev, Cohomology of groups in unitary representations, the neighborhood of the identity and conditionally positive definite functions, Math. USSR Sbornik, **47** (1984), 513–526.
[33] H. Yoshizawa, Some remarks on unitary representations of the free group, Osaka Math. J.. **3** (1951), 55–63.

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