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Geometric Constructions of Representations

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§ 1. Introduction

Beginning with the work of Gelfand, it has become apparent that there is a close connection between representations of a Lie group G and its coadjoint orbits, i.e., G-orbits in the dual of the Lie algebra. In the case of a nilpotent group, unitary representations correspond to coadjoint orbits equipped with real polarizations, and the correspondence was used by Kirillov [11] to actually construct the representations. Harish-Chandra's parametrization of those unitary representations which enter the Plancherel decomposition of $L^2(G)$, with G semisimple, can also be phrased in terms of coadjoint orbits, though his construction ties the representations only indirectly to the orbits in question. A direct geometric construction via coadjoint orbits was conjectured by Langlands [14] and carried out in [17–20, 25] — at least for the discrete series, but implicitly for the various other non-degenerate series as well. In this connection I should mention also Duflo's synthesis of the nilpotent and semisimple cases [5], which attaches unitary representations to coadjoint orbits for algebraic groups over $\boldsymbol{R}$.

A short note of Kostant [13] suggests a method for associating representations — not necessarily unitary representation — to G-orbits in the dual of the complexified Lie algebra. Attempts to carry out his program in practice quickly lead to major analytic difficulties, especially if the orbits carry polarizations that are neither maximally real nor maximally complex (the terminology will be explained in Section 3 below). Perhaps for this reason, among others, coadjoint orbits with arbitrary polarizations have received little attention. Zuckerman's derived functor construction [23] mimics the "orbit method" (for semisimple coadjoint orbits of semisimple Lie groups) algebraically, and thus avoids all analytic difficulties. The derived functor construction, too, has been used almost exclusively in the setting of maximally real or maximally complex polarizations; indeed, these very special polarizations suffice to obtain all irreducible Harish-Chandra modules [15, 23]. Nonetheless a case can be made

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for the importance of arbitrary polarizations, particularly in view of the duality between the Zuckerman modules and the Beilinson-Bernstein modules [6].

In this paper I shall describe recent results of J. A. Wolf and myself, on the "orbit method" for semisimple coadjoint orbits of semisimple Lie groups, without restriction on the type of polarization: properly interpreted, the procedure outlined in [13] yields global representations on Fréchet spaces; moreover, the underlying Harish-Chandra modules coincide with the derived functor modules which correspond to the same data[1]. To put these facts into perspective, I shall discuss also the various canonical globalizations of Harish-Chandra modules, and recall the connection between the Zuckerman derived functor construction and the $\mathscr{D}$-module construction of Beilinson-Bernstein.

§ 2. Canonical globalizations of Harish-Chandra modules

From now on, G shall denote a connected, linear, semisimple Lie group, and K a particular maximal compact subgroup. The assumption of linearity is merely a convenience: everything that will be said remains correct, with minor modifications, for reductive groups in Harish-Chandra's class. Unless there is an indication to the contrary, the word "representation" shall mean a continuous representation on a complete, locally convex, Hausdorff topological vector space. If every chain of closed invariant subspaces breaks off after finitely many steps, the representation is said to be of finite length. The space of K-finite vectors[2] V for any representation π of finite length is dense in the representation space and consists entirely of C^∞ vectors; in particular, the complexified Lie algebra $\mathfrak{g}$ and its universal enveloping algebra $U(\mathfrak{g})$ act on V by differentiation. The group K also acts on V, via the restriction of π. These two actions turn V into a Harish-Chandra module, i.e.,

(2.1)
a) V is finitely generated over $U(\mathfrak{g})$;
b) as K-module, V is a direct sum of finite dimensional irreducibles, each occuring only finitely often; and
c) the actions of $U(\mathfrak{g})$ and K are compatible,

in the sense that the derivative of the K-action agrees with the restriction of the $\mathfrak{g}$-action to $\mathfrak{k}$ (= complexified Lie algebra of K). If a Harish-Chandra module V arises as the space of K-finite vectors for a representation π, I

[1] The note [22] is a brief announcement of our work; full details will appear elsewhere. See Hecht-Taylor [8] for related results.

[2] i.e., the linear span of all finite dimensional, K-invariant subspaces.

call π a globalization of V.

Every Harish-Chandra module can be globalized — in a variety of ways, if the module is infinite dimensional. Functorial globalizations do exist: the C^∞ and distribution globalizations of Casselman-Wallach [24], and the minimal and maximal globalizations [21]. I shall concentrate on the latter, since for our purposes it is the most important of the four.

To define the maximal globalization of a Harish-Chandra module V, I embed V as the space of K-finite vectors into $\tilde{V}$, the representation space of a particular globalization $(\pi, \tilde{V})$. Every vector τ in the dual Harish-Chandra module

(2.2) $V' =$ space of K-finite vectors in the algebraic dual V^*

extends to a bounded linear functional $\tilde{\tau}$ on $\tilde{V}$. Hence, to each $v \in V$ and $\tau \in V'$, one can associate a "matrix coefficient" $f_{v,\tau}$, with

$$f_{v,\tau}(g) = \langle \tilde{\tau}, \pi(g)v \rangle. \tag{2.3}$$

Functions of this type satisfy elliptic differential equations, and consequently are real analytic. The Taylor series of $f_{v,\tau}$ at the identity depends solely on the $U(\mathfrak{g})$-action. Thus, contrary to appearance, $f_{v,\tau}$ is an invariant of the Harish-Chandra module V.

Now let $\{\tau_1, \tau_2, \cdots, \tau_n\}$ be a finite set of generators for V' over $U(\mathfrak{g})$. The map

$$v \longmapsto (f_{v,\tau_1}, f_{v,\tau_2}, \cdots, f_{v,\tau_n}) \tag{2.4}$$

injects V into $C^\infty(G)^n$, equivariantly with respect to the actions of $\mathfrak{g}$ and K — the obvious actions on V, and by right translation on $C^\infty(G)$. The induced topology on V does not depend on the choice of generators $\{\tau_1, \tau_2, \cdots, \tau_n\}$: any other set of generators is related to the original one by a matrix with entries in $U(\mathfrak{g})$, and this matrix, acting as a matrix of right invariant differential operators, intertwines the two mappings (2.4). It can be shown that the representation of $\mathfrak{g}$ on V lifts to a representation of G on

(2.5) $V_{\max} =$ completion of V in the induced topology,

and that $V_{\max}$ is a globalization, the maximal globalization, of V. If $\tilde{V}$ is some other globalization, the identity map on V extends to a G-equivariant, continuous inclusion $\tilde{V} \hookrightarrow V_{\max}$ — essentially because the definition (2.3) makes sense for $v \in \tilde{V}$. In other words, $V_{\max}$ contains every other globalization; hence its name.

To give a concrete example, I suppose that G has a non-empty discrete series; equivalently, $\mathrm{rk}\,K = \mathrm{rk}\,G$. I fix a maximal torus $T \subset K$, which is then also a Cartan subgroup of G. The quotient manifold G/T can be realized as an open G-orbit in the flag variety X of $\mathfrak{g}$ (in several different ways), and thus inherits a G-invariant complex structure. Every character e^λ of T determines a G-homogeneous holomorphic line bundle $L_\lambda \to G/T$, i.e., a holomorphic line bundle to which the action of G lifts. In particular, G operates on the cohomology groups $H^*(G/T, \mathcal{O}(L_\lambda))$ of the sheaf of holomorphic sections. If L_λ is negative in the appropriate sense[3], the cohomology vanishes except in dimension $s = \frac{1}{2}\dim_{\mathbb{R}} K/T$, $H^s(G/T, \mathcal{O}(L_\lambda))$ is an irreducible Fréchet G-module, and the resulting representation of G has the same underlying Harish-Chandra module as the discrete series representation with character $\Theta_{\lambda+\rho}$ (in Harish-Chandra's notation; here ρ denotes the half sum of the roots which have a negative inner product with λ). The proof of these facts in [18] identifies $H^s(G/T, \mathcal{O}(L_\lambda))$ with a space of matrix coefficients, and consequently with the maximal globalization of its space of K-finite vectors.

Now let $(\pi, \tilde{V})$ be a globalization of a Harish-Chandra module V, on a Banach space $\tilde{V}$. The space of analytic vectors

$$(2.6) \qquad \tilde{V}^\omega = \{v \in \tilde{V} \mid g \longmapsto \pi(g)v \text{ is a real analytic map from } G \text{ to } \tilde{V}\}$$

has a natural complete, locally convex topology; G acts continuously on $\tilde{V}^\omega$, and thus $\tilde{V}^\omega$ becomes a globalization of V. This construction can be dualized if the Banach space $\tilde{V}$ is reflexive: the dual representation π' on the dual Banach space $\tilde{V}'$ is then continuous, and the natural action of G on the space of "hyperfunction vectors"

$$(2.7) \qquad \tilde{V}^{-\omega} = \text{strong topological dual of } (\tilde{V}')^\omega$$

turns $\tilde{V}^{-\omega}$ into another globalization of V. It injects canonically into $V_{\max}$, because of the maximality property of the latter.

(2.8) **Theorem** ([21]). *The inclusion* $\tilde{V}^{-\omega} \hookrightarrow V_{\max}$ *is an isomorphism of topological vector spaces.*

Unlike the construction and basic properties of $V_{\max}$ — which are relatively straightforward — the proof of (2.8) requires some effort. The crux is a lower bound for the asymptotic behavior of K-finite matrix coefficients, in terms of their K-types.

Short exact sequences of Harish-Chandra modules can be lifted to short exact sequence of representations on reflexive Banach spaces. Also, the passage from a representation on a reflexive Banach space to the space

[3] see §§ 4–5 below.

of hyperfunction vectors is exact. Hence:

(2.9) **Corollary.** *$V \mapsto V_{\max}$ is an exact functor.*

The isomorphism asserted by the theorem arises typically as a boundary value map. To see this, I consider a homogeneous vector bundle $\boldsymbol{E} \to G/P$, over the quotient of G by a minimal parabolic subgroup P. Then $C^\infty(G/P, \boldsymbol{E})_{(K)}$, the Harish-Chandra module of K-finite sections, has obvious refiexive Banach globalizations, namely the spaces of L^p sections, $1<p<\infty$. The subspaces of analytic vectors may be identified with $C^\omega(G/P, \boldsymbol{E})$, the space of analytic sections of $\boldsymbol{E}$. Dualizing, one finds:

(2.10) **Corollary.** *The maximal globalization of $C^\infty(G/P, \boldsymbol{E})_{(K)}$ coincides with the space of hyperfunction sections $C^{-\omega}(G/P, \boldsymbol{E})$.*

Any Harish-Chandra module V can be realized as a quotient module of $C^\infty(G/P, \boldsymbol{E})_{(K)}$, for some appropriately chosen $\boldsymbol{E}$ [4]. When V arises as the space of K-finite solutions of a G-invariant system of differential equations, the quotient map is given by an integral kernel — for example, the Poisson kernel in the setting of Helgason's conjecture [10]. Corollary (2.10) asserts, in effect, that every such integral kernel induces a topological isomorphism between a space of hyperfunction "boundary values" on G/P on the one hand, and the full solution space of the system of differential equations on the other. In particuar, (2.10) implies Helgason's conjecture.

The construction of the minimal globalization $V_{\min}$ is formally dual to that of $V_{\max}$ [21]: $V_{\min}$ can be described as a quotient of $C_0^\infty(G)^n$, it injects canonically and continuously into every other globalization, and coincides with the space of analytic vectors in any Banach globalization of V. Thus $C^\infty(G/P, \boldsymbol{E})_{(K)}$ has $C^\omega(G/P, \boldsymbol{E})$ as minimal globalization.

The C^∞ and distribution globalizations were introduced by Casselman-Wallach [24] (and served as motivation for the definition of $V_{\min}$ and $V_{\max}$, which came later). Casselman-Wallach showed that the C^∞ and distribution topologies on any quotient V of submodules of $C^\infty(G/P, \boldsymbol{E})_{(K)}$ are intrinsic: they do not depend on $\boldsymbol{E}$ or the particular presentation of V as sub-quotient in $C^\infty(G/P, \boldsymbol{E})_{(K)}$. The completions with respect to these topologies, to be denoted by V^∞ and $V^{-\infty}$, are functorial in V. If $(\pi, \tilde{V})$ is a Banach globalization, the identity map on V extends to topological, G-equivariant isomorphisms $V^\infty \cong \tilde{V}^\infty$, $V^{-\infty} \cong \tilde{V}^{-\infty}$.

§ 3. Coadjoint orbits and polarizations

I shall be concerned primarily with regular, semisimple orbits in the dual space $\mathfrak{g}^*$ of the complexified Lie algebra. As homogeneous space, any such orbit can be identified with the quotient G/H, by a Cartan subgroup $H \subset G$. I assume, as I may, that H is stable under the Cartan involution. In addition to the homogeneous structure, the datum of the coadjoint orbit specifies a linear functional λ on the complexified Lie algebra $\mathfrak{h}$ of H. The orbit is said to be integral if λ lifts to a character $e^\lambda: H \to \boldsymbol{C}^*$. The lifting need not be unique. To avoid complicated terminology, I consider the lifting as part of the datum of integral coadjoint orbit.

The regularity of the orbit translates into the regularity of $\lambda \in \mathfrak{h}^*$. However, I shall also allow λ to be singular: representations which the various constructions assign to the pair $(G/H, e^\lambda)$, with λ singular, should be thought of as belonging to a singular semisimple coadjoint orbit, i.e., the orbit through λ.

An invariant polarization for the coadjoint orbit in question amounts to the choice of a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$, with $\mathfrak{h} \subset \mathfrak{b}$. I let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ denote the nilradical of $\mathfrak{b}$, so that $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. If the Cartan subgroup H splits over $\boldsymbol{R}$, every polarization is real, in the sense that $\mathfrak{n} = \bar{\mathfrak{n}}$ (=complex conjugate of $\mathfrak{n}$); if H is compact, all polarizations are totally complexes, i.e., $\mathfrak{n} \cap \bar{\mathfrak{n}} = 0$. Orbits corresponding to Cartan subgroups H which are neither split nor compact have various types of intermediate polarizations, among them maximally real and maximally complex polarizations: the former maximize the dimension of $\mathfrak{n} \cap \bar{\mathfrak{n}}$, the latter minimize it, subject to the condition $\mathfrak{b} \supset \mathfrak{h}$, of course. Henceforth I shall work interchangeably with the triple $(G/H, e^\lambda, \mathfrak{b})$ and the polarized, integral, coadjoint orbit which it specifies.

The character e^λ associates a G-invariant line bundle[4] $\boldsymbol{L}_\lambda \to G/H$ to the principal H-bundle $G \to G/H$; its fibre at the identity coset is a complex line L_λ, on which the isotropy group H acts according to e^λ. This line bundle $\boldsymbol{L}_\lambda$ can be continued to a G-invariant complex of vector bundles $\boldsymbol{L}_\lambda \otimes \wedge^{\cdot} \boldsymbol{N}^* \to G/H$, with fibre $L_\lambda \otimes \wedge^{\cdot} \mathfrak{n}^*$ at the identity coset. Let q denote the quotient map form $(\mathfrak{g}/\mathfrak{h})^*$ (=complexified cotangent space of G/H at eH) onto $\mathfrak{n}^*$; then

$$(3.1) \quad (\mathfrak{g}/\mathfrak{h})^* \otimes L_\lambda \otimes \wedge^{\cdot} \mathfrak{n}^* \longrightarrow L_\lambda \otimes \wedge^{\cdot+1} \mathfrak{n}^*, \qquad \phi \otimes l \otimes \omega \longmapsto l \otimes (q(\phi) \wedge \omega),$$

commutes with the natural actions of H. Since $G \to G/H$ has a unique G-invariant connection, the map (3.1) induces a G-invariant, first order differential operator

$$(3.2) \qquad d_{\mathfrak{n}}: C^\infty(G/H, \boldsymbol{L}_\lambda \otimes \wedge^{\cdot} \boldsymbol{N}^*) \longrightarrow C^\infty(G/H, \boldsymbol{L}_\lambda \otimes \wedge^{\cdot+1} \boldsymbol{N}^*),$$

[4] the notation is slightly deceptive, since the line bundle depends not only on λ but also on the lifting.

the differential for the complex of vector bundles $L_\lambda \otimes \wedge^{\boldsymbol{\cdot}} N^*$. Pulling back sections from G/H to G, one obtains a G-invariant isomorphism

$$C^\infty(G/H, L_\lambda \otimes \wedge^{\boldsymbol{\cdot}} N^*) \cong \{C^\infty(G) \otimes L_\lambda \otimes \wedge^{\boldsymbol{\cdot}} \mathfrak{n}^*\}^H; \tag{3.3}$$

here $\{\cdots\}^H$ refers to the space of H-invariants, with H acting on $C^\infty(G)$ on the right. I view $C^\infty(G) \otimes L_\lambda$ as $\mathfrak{n}$-module, by infinitesimal right translation on $C^\infty(G)$ and trivial action on L_λ. The coboundary map in the standard complex $(C^\infty(G) \otimes L_\lambda \otimes \wedge^{\boldsymbol{\cdot}} \mathfrak{n}^*, \delta)$ commutes with the action of H; the isomorphism (3.3) relates the differential $d_{\mathfrak{n}}$ to δ.

In the special case of a compact Cartan subgroup G, the quotient manifold G/H carries a G-invariant complex structure such that $\mathfrak{n} \subset \mathfrak{g}/\mathfrak{h}$ corresponds to the antiholomorphic tangent space at the identity coset. Recall the discussion in Section 2: the line bundle L_λ can be turned into a G-invariant holomorphic line bundle. The original description of the complex (3.2) shows that

$$0 \longrightarrow \mathcal{O}(L_\lambda) \longrightarrow \mathcal{C}^\infty(L_\lambda) \longrightarrow \mathcal{C}^\infty(L_\lambda \otimes N^*) \longrightarrow \cdots \tag{3.4}$$

($\mathcal{C}^\infty(\cdots)$ = sheaf of C^∞ sections of $\cdots$) is the Dolbeault resolution of the sheaf of holomorphic sections. In particular,

$$H^*(G/H, \mathcal{O}(L_\lambda)) \cong H^*(C^\infty(G/H, L_\lambda \otimes \wedge^{\boldsymbol{\cdot}} N^*), d_{\mathfrak{n}}). \tag{3.5}$$

The vanishing theorem mentioned in Section 2 — a precise statement will appear below, in a more general context — asserts the vanishing of the cohomology except in one dimension s, under an appropriate negativity hypothesis on the parameter λ;

$$H^s(C^\infty(G/H, L_\lambda \otimes \wedge^{\boldsymbol{\cdot}} N^*), d_{\mathfrak{n}}) \cong H^s(G/H, \mathcal{O}(L_\lambda))$$

can then be identified with the maximal globalization of a discrete series representation. It should be noted that the Dolbeault complex is elliptic and remains a resolution of $\mathcal{O}(L_\lambda)$ if smooth forms are replaced by forms with distribution or hyperfunction coefficients: the three complexes $(C^?(G/H, L_\lambda \otimes \wedge^{\boldsymbol{\cdot}} N), d_{\mathfrak{n}})$, where "?" stands for any one of the symbols ∞, $-\infty$, $-\omega$, all have the same cohomology.

At the opposite extreme, when H splits over $\boldsymbol{R}$, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ becomes the complexified Lie algebra of a minimal parabolic subgroup (equivalently, in this particular case, Borel subgroup) $B \subset C$. The line bundle $L_\lambda \to G/H$ drops to a G-invariant line bundle $L_\lambda \to G/B$, over the real flag variety G/B. Since $G/H \to G/B$ has Euclidean fibres $B/H \cong \mathfrak{n}$, an application of the Poincaré lemma proves

$$H^p(C^\infty(G/H, L_\lambda \otimes \wedge^{\boldsymbol{\cdot}} N^*), d_{\mathfrak{n}}) \cong \begin{cases} C^\infty(G/B, L_\lambda) & \text{if } p=0, \\ 0 & \text{if } p \neq 0, \end{cases} \tag{3.6}$$

without any special assumptions on λ. The Poincaré lemma applies equally in the context of distributions or hyperfunctions, so (3.6) carries over to those two settings. Now, however, the cohomology changes when we pass from C^∞ sections to distribution or hyperfunction sections: as was remarked in Section 2, $C^\infty(G/B, L_\lambda)$, $C^{-\infty}(G/B, L_\lambda)$, and $C^{-\omega}(G/B, L_\lambda)$ are, respectively, the C^∞ globalization, the distribution globalization, and the maximal globalization of the Harish-Chandra module $C^\infty(G/B, L_\lambda)_{(K)}$.

Next I consider the case of an arbitrary Cartan subgroup H and a maximally complex polarization $\mathfrak{b}$. If λ satisfies an appropriate negativity condition, one can combine the arguments in the real and totally complex situations: the cohomology of $(C^?(G/H, L_\lambda \otimes \wedge^\cdot N^*), d_\mathfrak{n})$, with either ∞ or $-\infty$ in place of the symbol "?", vanishes in all but one degree, the operator $d_\mathfrak{n}$ has closed range also in the remaining degree, and the cohomology group in that degree is a globalization of a Harish-Chandra module obtained by parabolic induction [25]. Topologically the representation is induced from the maximal globalization of a discrete series representation, with C^∞ and distribution coefficients, respectively. Thus both "$?=\infty$" and "$?=-\infty$" lead to "mixed" topologies, and not to any one of the four canonical globalizations. For general polarizations the situation appears to be much worse: although examples are difficult to work out explicitly, there is evidence suggesting that $d_\mathfrak{n}$ need not have closed range in the C^∞ or distribution topologies.

The space of hyperfunctions on a non-compact real analytic manifold carries no natural Hausdorff topology. Thus, at first glance, the complex $(C^{-\omega}(G/H, L_\lambda \otimes \wedge^\cdot N^*), d_\mathfrak{n})$ does not seem useful for constructing global representations. In the case of a real polarization, the problem is overcome by integrating out the dependence on certain variables and re-interpreting the cohomology as a space of hyperfunctions on a compact manifold; equivalently, a non-Hausdorff toplogy for the complex produces a Fréchet topology on the cohomology. A similar procedure puts a good topology on the cohomology groups $H^*(C^{-\omega}(G/H, L_\lambda \otimes \wedge^\cdot N^*), d_\mathfrak{n})$, for any polarization $\mathfrak{b}$; I shall make this precise in Section 5, following a discussion of the infinitesimal analogue of the complex $(C^{-\omega}(G/H, L_\lambda \otimes \wedge^\cdot N^*), d_\mathfrak{n})$.

In one important respect, the examples of real or maximally complex polarizations are misleading: for these special types of polarizations, the complex of sheaves of sections of $L_\lambda \otimes \wedge^\cdot N^*$ is acyclic, and hence provides a soft, respectively flabby resolution of the single sheaf

$$(3.7) \qquad \mathcal{S}^?_\mathfrak{n}(L_\lambda) = \ker \{d_\mathfrak{n} : \mathscr{C}^?(L_\lambda) \longrightarrow \mathscr{C}^?(L_\lambda \otimes N^*)\}$$

($\mathscr{C}^?(\cdots)$ = sheaf of $C^?$ sections of $\cdots$). Simple examples show that the complex $(\mathscr{C}^?(L_\lambda \otimes \wedge^\cdot N^*), d_\mathfrak{n})$ fails to be acyclic in general. Thus

$(C^?(G/H, L_\lambda \otimes \wedge^\cdot N^*), d_\mathfrak{n})$ computes not the cohomology of one sheaf, but rather the hypercohomology of a complex of sheaves.

For the remainder of this section, the polarization $\mathfrak{b}$ of G/H will be arbitrary. I let D denote the G-orbit through $\mathfrak{b}$ in the flag variety X (=variety of Borel subalgebras of $\mathfrak{g}$). Since H normalizes $\mathfrak{b}$, there is a natural G-equivariant fibration $G/H \to D$. Its fibres can be identified with the connected, simply connected, nilpotent Lie group which has $\mathfrak{n} \cap \bar{\mathfrak{n}}$ as complexified Lie algebra. In particular, the line bundle L_λ drops to a G-invariant, real analytic line bundle $L_\lambda \to D$. The pair (D, L_λ) completely determines the polarized, integral, semisimple coadjoint orbit that corresponds to the triple $(G/H, e^\lambda, \mathfrak{b})$.

As homogeneous submanifold of the complex manifold X, D has a global CR-structure: an induced $\bar\partial$ operator

$$\bar\partial_D : C^\infty(D, \wedge^\cdot N_D^*) \longrightarrow C^\infty(D, \wedge^{\cdot+1} N_D^*). \tag{3.8}$$

Here $N_D \to D$ denotes the intersection of the anti-holomorphic tangent bundle of X with the complexified tangent bundle of D; its fibre at $\mathfrak{b}$ is isomorphic to $\mathfrak{n}/\mathfrak{n} \cap \bar{\mathfrak{n}}$. Since $L_\lambda \to D$ is real analytic, it extends to a $\mathfrak{g}$-invariant holomorphic line bundle $L_\lambda \to \tilde{D}$, over some neighborhood $\tilde{D}$ of D in X. Twisting the operator (3.8) by this bundle, one obtains a complex $(C^\infty(D, L_\lambda \otimes \wedge^\cdot N_D^*), \bar\partial_D)$. It can be identified with a subcomplex of $(C^\infty(G/H, L_\lambda \otimes \wedge^\cdot N^*), d_\mathfrak{n})$, via pull-back from D to G/H. Under the isomorphism (3.3), this complex corresponds to the subcomplex of H-invariants in the standard complex for relative Lie algebra cohomology with respect to the pair $(\mathfrak{n}, \mathfrak{n} \cap \bar{\mathfrak{n}})$:

$$(C^\infty(D, L_\lambda \otimes \wedge^\cdot N_D^*), \bar\partial_D) \cong (\{C^\infty(G) \otimes L_\lambda \otimes \wedge^\cdot(\mathfrak{n}/\mathfrak{n} \cap \bar{\mathfrak{n}})^*\}^{\mathfrak{n}\cap\bar{\mathfrak{n}}, H}, \delta). \tag{3.9}$$

The inclusion of the $\bar\partial_D$-complex into the $d_\mathfrak{n}$-complex induces an isomorphism of cohomology, as follows from an application of the Poincaré lemma along the fibres of $G/H \to D$. Everything that has just been said remains correct when one works with hyperfunction coefficients. Consequently

$$\begin{aligned} H^*(C^{-\omega}(G/H, L_\lambda \otimes \wedge^\cdot N^*), d_\mathfrak{n}) &\cong H^*(C^{-\omega}(D, L_\lambda \otimes \wedge^\cdot N_D^*), \bar\partial_D) \\ &\cong H^*(\{C^{-\omega}(G) \otimes L_\lambda \otimes \wedge^\cdot(\mathfrak{n}/\mathfrak{n} \cap \bar{\mathfrak{n}})^*\}^{\mathfrak{n}\cap\bar{\mathfrak{n}}, H}, \delta). \end{aligned} \tag{3.10}$$

These cohomology groups are the subject of the main result, stated in Section 5 below.

§ 4. Infinitesimal constructions

The derived functor construction [23] associates a family of Harish-

Chandra modules $A^p(\mathfrak{b}, L_\lambda)$ to the data $(G/H, e^\lambda, \mathfrak{b})$: view $U(\mathfrak{g})$ as $\mathfrak{b}$-module by left multiplication, as $H\cap K$-module by conjugation, and as (left!) $\mathfrak{g}$-module via right multiplication, twisted by the standard anti-automorphism; then

$$M_{\mathfrak{b},\lambda}=\mathrm{Hom}_{\mathfrak{b}}(U(\mathfrak{g}), L_\lambda)_{(H\cap K)} \tag{4.1}$$

(=space of $H\cap K$-finite maps in $\mathrm{Hom}_{\mathfrak{b}}(U(\mathfrak{g}), L_\lambda)$) becomes a $(\mathfrak{g}, H\cap K)$-module — i.e., simultaneously a module for both $\mathfrak{g}$ and $H\cap K$, acting in a compatible fashion (recall that L_λ has been turned into a $\mathfrak{b}$-module with trivial $\mathfrak{n}$-action, and that H was assumed to be stable under the Cartan involution). Zuckerman's functor Γ, from the category of $H\cap K$-finite $(\mathfrak{g}, H\cap K)$-modules to the category of $(\mathfrak{g}, K)$-modules, associates to any module in its domain the largest $\mathfrak{k}$-finite, $\mathfrak{k}$-semisimple subspace on which the action of $\mathfrak{k}$ lifts to K. Since Γ is left exact, and since the domain contains enough injectives, one can define the right derived functors $R^p\Gamma$ [23]. The modules

$$A^p(\mathfrak{b}, L_\lambda)=R^p\Gamma(M_{\mathfrak{b},\lambda}) \tag{4.2}$$

turn out to be Harish-Chandra modules. They have been extensively studied in the situation of a maximally real or maximally complex polarization [23].

To interpret the modules (4.2) geometrically, I let $C^f(L_\lambda\otimes\wedge^\cdot N^*)$ denote the space of formal power series at the identity coset in G/H, with values in the bundle $L_\lambda\otimes\wedge^\cdot N^*$. The operator $d_\mathfrak{n}$ turns $C^f(L_\lambda\otimes\wedge^\cdot N^*)$ into a complex of $(\mathfrak{g}, H)$-modules. The $U(\mathfrak{g})$-action, followed by evaluation at the identity coset, induces a natural isomorphism

$$C^f(L_\lambda\otimes\wedge^\cdot N^*)\cong\mathrm{Hom}_{\mathfrak{b}}(U(\mathfrak{g}), L_\lambda\otimes\wedge^\cdot\mathfrak{n}^*), \tag{4.3}$$

which leads to the identification

$$M_{\mathfrak{b},\lambda}\cong\mathrm{Ker}\,\{d_\mathfrak{n}\colon C^f(L_\lambda)_{(H\cap K)}\longrightarrow C^f(L_\lambda\otimes N^*)_{(H\cap K)}\}. \tag{4.4}$$

The modules $C^f(L_\lambda\otimes\wedge^\cdot N^*)_{(H\cap K)}$ are injective in the category of $H\cap K$-finite $(\mathfrak{g}, H\cap K)$-modules, and they resolve $M_{\mathfrak{b},\lambda}$, hence

$$A^p(\mathfrak{b}, L_\lambda)\cong H^p(C^f(L_\lambda\otimes\wedge^\cdot N^*)_{(K)}, d_\mathfrak{n}) \tag{4.5}$$

[1, 6]. C^∞ functions have formal Taylor series, so the K-finite part of the complex $C^\infty(G/H, L_\lambda\otimes\wedge^\cdot N^*)$ maps into the complex of formal power series. The induced map on cohomology,

$$H^*(C^\infty(G/H, L_\lambda\otimes\wedge^\cdot N^*)_{(K)}, d_\mathfrak{n})\longrightarrow H^*(C^f(L_\lambda\otimes\wedge^\cdot N^*)_{(K)}, d_\mathfrak{n}), \tag{4.6}$$

will be a crucial ingredient of the proof of the main result.

The complexification K_C of K is an algebraic group, which operates on the flag variety X with finitely many orbits; the orbits are locally closed in the Zariski topology. A G-orbit D and a K_C-orbit Q are said to be dual to each other if K acts transitively on $D \cap Q$. This notion of duality establishes an order reversing bijection between the two types of orbits [16]. The restriction to $D \cap Q$ of any G-invariant line bundle $L_\lambda \to D$ extend uniquely to a K_C-invariant algebraic line bundle over Q. Typically the isotropy subgroup of K at a point in $D \cap Q$ is smaller than the isotropy subgroup of G. For this reason the passage from the G-invariant line bundle L_λ on D to the induced algebraic K_C-invariant bundle on Q is not one-to-one. Nonetheless, for simplicity, I denote the induced bundle by the same symbol L_λ. In addition to $L_\lambda \to Q$, the datum of λ (or its lifting e^λ) determines a $\mathfrak{g}$-invariant twisted sheaf of differential operators[5] $\mathscr{D}_\lambda$ on X (in the algebraic category) [3], whose restriction of to Q operates on sections of $L_\lambda \to Q$. The $\mathscr{D}$-module direct image $j_+\mathcal{O}_Q(L_\lambda)$ under the inclusion $j\colon Q \hookrightarrow X$ is a K_C-invariant sheaf of $\mathscr{D}_\lambda$-modules. Its cohomology groups $H^p(X, j_+\mathcal{O}_Q(L_\lambda))$ become $\mathfrak{g}$-modules — in fact, Harish-Chandra modules — via the natural map from $\mathfrak{g}$ to the space of global sections of $\mathscr{D}_\lambda$ [2].

Let Φ^+ denote the positive root system in $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ (=root system of $(\mathfrak{g}, \mathfrak{h})$), such that $\mathfrak{n}$ is spanned by root spaces corresponding to negative roots. I write ρ for the half-sum of the positive roots, as usual. According to a very general vanishing theorem of Beilinson-Bernstein, any quasi-coherent sheaf of $\mathscr{D}_\lambda$-modules on X has cohomology only in degree zero, unless $2(\lambda+\rho, \alpha)/(\alpha, \alpha)$ is a strictly negative integer for some $\alpha \in \Phi^+$ [2]. In particular,

$$\begin{aligned} &H^p(X, j_+\mathcal{O}_Q(L_\lambda)) = 0 \qquad \text{if } p > 0, \\ &\text{provided } 2\frac{(\lambda+\rho, \alpha)}{(\alpha, \alpha)} \neq -1, -2, \cdots \quad \text{for every } \alpha \in \Phi^+. \end{aligned} \tag{4.7}$$

If λ satisfies the stronger hypothesis $\mathrm{Re}\,((\lambda+\rho, \alpha)/(\alpha, \alpha)) > 0$ for $\alpha \in \Phi^+$, the 0-th cohomology group is non-zero and contains exactly one irreducible submodule. Every irreducible Harish-Chandra module with regular infinitesimal character occurs as such an irreducible submodule, corresponding to uniquely determined data $(Q, L_\lambda, \mathscr{D}_\lambda)$. A similar, but more involved statement describes the irreducible Harish-Chandra modules with singular infinitesimal character. This, in effect, is Beilinson-Bernstein's classification of irreducible Harish-Chandra modules [2].

[5] This parameterization of the sheaves $\mathscr{D}_\lambda$ differs from Beilinson-Bernstein's: $\lambda = 0$ corresponds to the non-twisted sheaf $\mathscr{D}_X$.

As a sum of roots, $-2\rho \in \mathfrak{h}^*$ lifts to a character $e^{-2\rho}$ of H which is trivial on the center of G. The corresponding line bundle $L_{-2\rho}$ — which makes sense as Aut(g)-invariant line bundle over all of X — coincides with the canonical bundle of X. Now let $L_\lambda \to D$ be a G-invariant line bundle over a G-orbit $D \subset X$, Q the dual K_C-orbit, $L_{-\lambda-2\rho} \to Q$ the bundle induced by the reciprocal of L_λ, tensored with $L_{-2\rho}$. The duality theorem of [6] asserts:

(4.8) $A^p(\mathfrak{b}, L_\lambda)$ and $H^{s-p}(X, j_+\mathcal{O}_Q(L_{-\lambda-2\rho}))$ are canonically dual in the category of Harish-Chandra modules,

with $s=\dim_R(Q \cap D)-\dim_C Q$. Via the duality theorem, (4.7) translates into a vanishing theorem for the derived functor modules $A^p(\mathfrak{b}, L_\lambda)$.

There exist two earlier, quite general vanishing theorems for the modules $A^p(\mathfrak{b}, L_\lambda)$, which can also be deduced from (4.7) and (4.8). First, when $\mathfrak{b}$ is maximally real, the modules $A^p(\mathfrak{b}, L_\lambda)$ are obtained by parabolic induction from derived functor modules associated to totally complex polarizations of coadjoint orbits of proper subgroups of G. This implies:

(4.9) $A^p(\mathfrak{b}, L_\lambda)=0$ for $p \neq s$ if $\mathfrak{b}$ is maximally real and $2((\lambda+\rho, \alpha)/(\alpha, \alpha)) \neq 1, 2, \cdots$, for every positive imaginary root α;

here $s=s(\mathfrak{b})$ has the same meaning as in (4.8). Next, I suppose that $\mathfrak{b}$ is maximally complex. Let $\Sigma^+ \subset (\mathfrak{h} \cap \mathfrak{k})^*$ be the set of positive restricted roots, i.e., the set of non-zero images of positive roots under the projection $\mathfrak{h}^* \to (\mathfrak{h} \cap \mathfrak{k})^*$ (orthogonal projection, relative to the Killing form). Then

(4.10) $A^p(\mathfrak{b}, L_\lambda)=0$ for $p \neq s$ if $\mathfrak{b}$ is maximally complex and $\mathrm{Re}(\lambda+\rho, \alpha) \leq 0$, for every $\alpha \in \Sigma^+$

[23]. Both (4.9) and (4.10) can be strengthened — more on this later — at the expense of making the statements considerably more complicated.

In the setting of either of the two vanishing theorems, certain additional inequalities imply that the one remaining module $A^s(\mathfrak{b}, L_\lambda)$ is non-zero and has a unique irreducible quotient; moreover, every irreducible Harish-Chandra module occurs among these irreducible quotients, with parameters that are unique up to conjugacy. For maximally real polarizations, this amounts to a paraphrase of Langlands' classification [15], and for maximally complex polarizations, of the Vogan-Zuckerman classification [23]. Thus three types of polarizations lead to classifications of irreducible Harish-Chandra modules in terms of semisimple coadjoint orbits: negative polarizations, characterized by the inequlities $\mathrm{Re}(\lambda+\rho, \alpha) \geq 0$ for $\alpha \in \Phi^+$, which enter the statement of the Beilinson-Bernstein classification (or rather its implication for the $A^p(\mathfrak{b}, L_\lambda)$ via the duality theorem (4.8)), as well as maximally real and maximally complex polarizations.

Initially the three classification schemes appeared essentially distinct. It is now clear that they are really one and the same, dressed in three different disguises. The intertwining functors of Beilinson-Bernstein [3, 7] make it possible to pass back and forth between them. Let D be the G-orbit corresponding to the triple $(G/H, e^\lambda, \mathfrak{b})$, Q the dual K_C-orbit, α a simple root for the positive root system $\Phi^+=\Phi^+(D)$. Then X fibres Aut$(\mathfrak{g})$-equivariantly over X_α, the variety of parabolic subalgebra of type α, with fibre $\boldsymbol{C}P^1$. I denote the map of the fibration by p_α. Depending on the nature of the simple root α, $p_\alpha^{-1}(p_\alpha Q)$ is a union of one, two, or three K_C-orbits, and $p_\alpha^{-1}(p_\alpha D)$ a union of the same number of G-orbits. In the following I shall assume that α is a complex root, i.e., neither real nor imaginary, and that $\bar{\alpha}$ is a negative root. In this situation,

$$(4.11)\qquad \begin{aligned} p_\alpha^{-1}(p_\alpha D) &= D\cup D_0, \quad \text{with} \quad \dim_{\boldsymbol{R}} D_0=\dim_{\boldsymbol{R}} D-2,\\ p_\alpha^{-1}(p_\alpha Q) &= Q\cup Q_0, \quad \text{with} \quad \dim_{\boldsymbol{C}} Q_0=\dim_{\boldsymbol{C}} Q+1; \end{aligned}$$

D_0 corresponds to the polarization $\mathfrak{b}_\alpha$ obtained from $\mathfrak{b}$ by replacing the $(-\alpha)$-root space with the α-root space, and Q_0 is dual to D_0. The map p_α induces fibrations $D\to D_0$, $Q_0\to Q$, both with fibres $\boldsymbol{C}^1$. Only the polarization is changed in going from D to D_0, so $\boldsymbol{L}_\lambda$ exists also as G-invariant line bundle over D_0. I let $\boldsymbol{L}_{\lambda+\alpha}$ denote the tensor product of $\boldsymbol{L}_\lambda$ with $\boldsymbol{L}_\alpha$, the G-invariant line bundle associated to the character e^α of H; geometrically, $\boldsymbol{L}_\alpha$ is the bundle of vectors tangential to the fibres of p_α. An analysis of the Leray spectral sequences for $D\to D_0$, $Q_0\to Q$ shows[6]

$$(4.12)\qquad \begin{aligned} &H^p(X, j_+\mathcal{O}_Q(\boldsymbol{L}_\lambda))\cong H^p(X, j_+\mathcal{O}_{Q_0}(\boldsymbol{L}_{\lambda+\alpha})) \\ &\qquad\qquad\qquad\qquad \text{if } 2\frac{(\lambda+\rho,\alpha)}{(\alpha,\alpha)}\neq -1, -2, \cdots \\ &A^p(\mathfrak{b}, L_\lambda)\cong A^{p-1}(\mathfrak{b}_\alpha, L_{\lambda+\alpha}) \qquad \text{if } 2\frac{(\lambda+\rho,\alpha)}{(\alpha,\alpha)}\neq 1, 2, \cdots \end{aligned}$$

[7]. Both isomorphisms can be described in geometric terms — for example, via the description (4.5) of the modules $A^p(\mathfrak{b}, L_\lambda)$. Chains of such isomorphisms, coupled with the duality theorem (4.8), directly relate the three classifications [7].

The sheaf $j_+\mathcal{O}_Q(\boldsymbol{L}_\lambda)$ is supported on Q. Hence, if the K_C-orbit Q happens to be affine, the cohomology groups $H^p(X, j_+\mathcal{O}_Q(\boldsymbol{L}_\lambda))$, $p>0$, vanish

[6] Caution: the line bundles $\boldsymbol{L}_\lambda\to Q$, $\boldsymbol{L}_\lambda\to Q_0$ correspond to the same character e^λ of H, but to different Borel subalgebras; if they extend to $\mathfrak{g}$-invariant line bundles over all of X, the extensions will agree only if $(\lambda,\alpha)=0$.

even without any special assumptions on λ. More generally, Q may admit a fibration over an affine K_C-orbit, in which case the cohomology in strictly positive degrees vanishes as soon as L_λ restricts to a positive line bundle along the fibres. Alternatively, the cohomology may vanish because there exists a fibration with affine fibres and the bundle L_λ satisfies an appropriate positivity condition relative to the base. Vanishing statements can also be transferred from one orbit to another by isomorphisms like (4.12), or by analogous isomorphisms corresponding to simple roots α which are either imaginary or real. Considerations of this type lead to the refinements of the vanishing theorems (4.9–10) that were alluded to before [7].

§ 5. The main result

I recall that each G-invariant line bundle L_λ over a G-orbit $D\subset X$ extends $\mathfrak{g}$-equivariantly to a holomorphic line bundle $L_\lambda\to\tilde{D}$, over a neighborhood $\tilde{D}$ of D in X. The local cohomology groups of the extended bundle along D, $H^p_D(\tilde{D}, \mathcal{O}(L_\lambda))$, are computed by the Dolbeault complex over $\tilde{D}$, with hyperfunction coefficients which are supported on D [12]. These groups depend only on the original bundle $L_\lambda\to D$, not on the particular choice of $\tilde{D}$. If D is open orbit, one can take $\tilde{D}=D$; in this case $H^p_D(\tilde{D}, \mathcal{O}(L_\lambda))=H^p(D, \mathcal{O}(L_\lambda))$. At the opposite extreme, when X is a complexification of D — as happens in the situation of a real polarization —, the local cohomology vanishes except in degree $d=\dim_{\mathbf{R}} D$, and $H^d_D(\tilde{D}, \mathcal{O}(L_\lambda))$ coincides with the space of hyperfunction sections of $L_\lambda\to D$ [12].

For the statement of the main theorem, I fix an integral, polarized, semisimple coadjoint orbit, corresponding to the triple $(G/H, e^\lambda, \mathfrak{b})$, and let $D\subset X$ denote the G-orbit through $\mathfrak{b}$.

(5.1) **Theorem.** *There exist canonical isomorphisms*

$$H^*(C^{-\omega}(D, L_\lambda\otimes\wedge^{\cdot}N_D^*), \bar{\partial}_D)\cong H^*(C^{-\omega}(G/H, L_\lambda\otimes\wedge^{\cdot}N^*), d_{\mathfrak{n}})$$
$$\cong H_D^{*+c}(\tilde{D}, \mathcal{O}(L_\lambda)),$$

with $c=$ real codimension of D in X. These cohomology groups carry natural Fréchet topologies which make the action of G continuous. The resulting representations are canonically and topologically isomorphic to the maximal globalizations of the derived functor modules $A^*(\mathfrak{b}, L_\lambda)$.

In the case of a totally complex polarization, this is a result of Aguilar-Rodriguez [1].

The suggestion that various standard representations might be realized as local cohomology groups along G-orbits in X was made by

Zuckerman [26]. He also observed that a statement of this sort would lead to resolutions of finite dimensional representations by representations induced from various parabolic subgroups: the G-orbits induce a stratification, and thus a spectral sequence for local cohomology on X with support on the strata; an appropriate vanishing theorem should make the spectral sequence collapse, into a resolution for the global cohomology. The character formula which would follow from such a resolution was later proved by Vogan [23]. After Beilinson-Bernstein [2] had attached Harish-Chandra modules to K_C-orbits, Zuckerman constructed a resolution in the setting of Harish-Chandra modules, via local cohomology (in the algebraic category) supported along K_C-orbits [27]. The same resolution was obtained independently by J. Johnson [9], by purely algebraic methods. Theorem (5.1), the vanishing theorem (4.7) and the duality (4.8) immediately give a resolution on the level of global representations.

The topology of $H^*(C^{-\omega}(D, L_\lambda \otimes \wedge^\cdot N_D^*), \bar{\partial}_D)$ can be described explicitly, as follows. Let $P \subset G$ be a cuspidal parabolic subgroup associated to the Cartan subgroup H, such that P contains the isotropy subgroup of G at $\mathfrak{b}$. The fibres of the natural G-equivariant fibration $D \to G/P$ are complex analytic submanifolds of X. Applying the Dolbeault lemma locally along these submanifolds, one finds that the complex of sheaves $(\mathscr{C}^{-\omega}(L_\lambda \otimes \wedge^\cdot N_D^*), \bar{\partial}_D)$ is quasi-isomorphic to a subcomplex of forms whose hyperfunction coefficients are holomorphic in the fibre directions. The same argument shows that this latter complex is also quasi-isomorphic to the subcomplex of $(\mathscr{C}^{-\omega}(L_\lambda \otimes \wedge^\cdot N_D^*), \bar{\partial}_D)$ consisting of forms which are smooth along the fibres; notation: $(\mathscr{C}_P^{-\omega}(L_\lambda \otimes \wedge^\cdot N_D^*), \bar{\partial}_D)$. The sheaf $\mathscr{C}_P^{-\omega}$ of partially smooth hyperfunctions is fine in the fibre directions, and flabby transversely to the fibration, hence globally acyclic. This implies:

$$(5.2) \qquad H^*(C^{-\omega}(D, L_\lambda \otimes \wedge^\cdot N_D^*), \bar{\partial}_D) \cong H^*(C_P^{-\omega}(D, L_\lambda \otimes \wedge^\cdot N_D^*), \bar{\partial}_D),$$

with $C_P^{-\omega}(D, \cdots) =$ space of global sections of $\mathscr{C}_P^{-\omega}(\cdots)$. Since G/P is compact, the spaces $C_P^{-\omega}(D, L_\lambda \otimes \wedge^\cdot N_D^*)$ have natural Hausdorff topologies. The proof of (5.1) will show that the operator $\bar{\partial}_D$ has closed range, so the induced topology is also Hausdorff.

Because of the closed range property, every K-finite cohomology class has a K-finite representative, and a K-finite form can be exact only if it is the boundary of a K-finite form. Hence

$$(5.3) \quad H^*(C^{-\omega}(D, L_\lambda \otimes \wedge^\cdot N_D^*), \bar{\partial}_D)_{(K)} = H^*((C_P^{-\omega}(D, L_\lambda \otimes \wedge^\cdot N_D^*)_{(K)}, \bar{\partial}_D).$$

Since K acts transitively on G/P, K-finite forms is $C_P^{-\omega}(D, L_\lambda \otimes \wedge^\cdot N_D^*)$ are smooth:

(5.4) $$(C_P^{-\omega}(D, L_\lambda \otimes \wedge^{\cdot} N_D^*)_{(K)} = C^\infty(D, L_\lambda \otimes \wedge^{\cdot} N_D^*)_{(K)}.$$

These two identifications, combined with (4.4–5), provide morphisms

(5.5) $$H^*(C^{-\omega}(D, L_\lambda \otimes \wedge^{\cdot} N_D^*). \bar\partial_D)_{(K)} \longrightarrow A^*(\mathfrak{b}, L_\lambda);$$

isomorphisms, it will turn out, which identify the derived functor modules as the K-finite part of $H^*(C^{-\omega}(D, L_\lambda \otimes^{\cdot} N_D^*), \bar\partial_D)$.

The first of the isomorphisms asserted by theorem (5.1) was mentioned already in Section 3. The second depends on the nature of the embedding $D \subset X$. Near any point of D, there exist commuting holomorphic vector fields $Z_1, Z_2, \cdots, Z_c$ on X, whose real parts are tangential along D, and whose imaginary parts frame the normal bundle of D. The Z_j define a holomorphic foliation of X with c-dimensional leaves, which are complexifications of their intersections with D. A spectral sequence argument, using the description of hyperfunctions in terms of local cohomology, reduces the Dolbeault complex on $\tilde D$ (with hyperfunction coefficients supported on D), to the $\bar\partial_D$-complex on D. Thus

(5.6) $$H_D^{*+c}(\tilde D, \mathcal{O}(L_\lambda)) \cong H^*(C^{-\omega}(D, L_\lambda \otimes \wedge^{\cdot} N_D^*), \bar\partial_D)).$$

The local cohomology groups $H_D^*(\tilde D, \mathcal{O}(L_\lambda))$ can be computed from a complex of $\mathcal{O}(L_\lambda)$-valued cochains on a relative covering of $(\tilde D, D)$ [12]. One knows that $Z(\mathfrak{g})$ (=center of $U(\mathfrak{g})$) acts on $\mathcal{O}(L_\lambda)$ via the character $\chi_{\lambda+\rho}$ (in Harish-Chandra's notation), so

(5.7) $Z(\mathfrak{g})$ operates on $H^*(C^{-\omega}(D, L_\lambda \otimes \wedge^{\cdot} N_D^*), \bar\partial_D))$ via $\chi_{\lambda+\rho}$.

The analogous statement about $A^*(\mathfrak{b}, L_\lambda)$ follows directly from the definition (4.1–2).

It suffices to prove (5.1) for parameters λ in any one particular Weyl chamber: the isomorphisms (3.10), (5.6), the map (5.5), the maximal globalization all behave well with respect to the process of "tensoring across the walls"; the assertion (5.7) makes the argument go through. If the polarization $\mathfrak{b}$ happens to be maximally real, and if λ is negative with respect to the imaginary roots, the modules $H^*(C^{-\omega}(D, L_\lambda \otimes \wedge^{\cdot} N_D^*), \bar\partial_D))$ are obtained by parabolic induction from discrete series representations, as has been described in Section 3. In this situation, the statements of the theorem can be verified directly. In particular, (5.5) is an isomorphism which indentifies the $\bar\partial_D$-cohomology groups with the maximal globalization of the derived functor modules. To complete the proof — and this is really the crux of the matter — one needs an analogue of (4.12) on the level of the groups $H^*(C^{-\omega}(D, L_\lambda \otimes \wedge^{\cdot} N_D^*), \bar\partial_D))$.

As in (4.11–12), I fix a simple complex root α, whose complex conjugate is negative. Near any point of the base X_α, the fibration p_α is a product; I choose an open, relatively compact neighborhood U_α in $p_\alpha(D)$, so that

$$V_\alpha =_{\text{def}} p_\alpha^{-1}(U_\alpha) \cong U_\alpha \times \mathbf{C}P^1, \tag{5.8}$$

and set $U = V_\alpha \cap D$, $U_0 = V_\alpha \cap D_0$, $\bar{U}_\alpha$ = closure of U_α in $p_\alpha(D)$; similarly, $\bar{U}$ and $\bar{U}_0$ will denote the closures of U in D and of U_0 in D_0, respectively. Shrinking U_α if necessary, I may assume that $\bar{V}_\alpha = p_\alpha^{-1}(\bar{U}_\alpha)$ is still a product of $\bar{U}_\alpha$ with $\mathbf{C}P^1$, and that $\bar{U}_0$ does not meet the ∞-section, i.e., $\bar{U}_0 \subset \bar{U}_\alpha \times \mathbf{C}$. The argument will involve three induced $\bar{\partial}$ operators: $\bar{\partial}_\alpha$ on D_α, $\bar{\partial}_0$ on D_0, $\bar{\partial}_D$ on D. Let t be a coordinate on $\mathbf{C}P^1$; then

$$U_0 = \{(w, t) \in U_\alpha \times \mathbf{C}P^1 \mid t = \phi(w)\}, \tag{5.9}$$

for some function $\phi \in C^\omega(U_\alpha)$ with the property that $\bar{\partial}_\alpha \phi \neq 0$ at every point of U_α. Any (scalar-valued) differential form in the $\bar{\partial}_\alpha$-complex on U_α can be expressed uniquely as $\omega + \bar{\partial}_\alpha \phi \wedge \psi$, where ω, ψ are differential forms in the induced Dolbeault complex on U_0, viewed as forms on U_α via $p_\alpha \colon U_0 \cong U_\alpha$. In terms of this representation, the operators $\bar{\partial}_0$, $\bar{\partial}_\alpha$ are related by the identity

$$\bar{\partial}_\alpha(\omega + \bar{\partial}_\alpha \phi \wedge \psi) = \bar{\partial}_0 \omega + \bar{\partial}_\alpha \phi \wedge (\delta_Y \omega - \bar{\partial}_0 \psi); \tag{5.10}$$

here δ_Y denotes the Lie derivative in the direction of the (0, 1) vector field Y on U_α which is characterized by the two conditions i) $Y\phi = 1$, and ii) Y commutes with the pullback to U_α of any (0, 1) vector field on U_0.

To establish the analogue of (4.12), it suffices to define a morphism of complexes

$$C^{-\omega}(D_0, \mathbf{L}_{\lambda+\alpha} \otimes \wedge^{\cdot} \mathbf{N}^*_{D_0}) \longrightarrow C^{-\omega}(D, \mathbf{L}_\lambda \wedge^{\cdot+1} \mathbf{N}^*_D), \tag{5.11}$$

whose restriction to every sufficiently small neighborhood $U_0 \subset D_0$ induces isomorphisms

$$H^*(C^{-\omega}(U_0, \mathbf{L}_{\lambda+\alpha} \otimes \wedge^{\cdot} \mathbf{N}^*_{D_0}), \bar{\partial}_0) \cong H^{*+1}(C^{-\omega}(U, \mathbf{L}_\lambda \otimes \wedge^{\cdot} \mathbf{N}^*_D), \bar{\partial}_D); \tag{5.12}$$

in addition, the morphism must be compatible with (4.12), (5.5). It is easy to describe the morphism via the identification (3.9): let $\omega^\alpha \in (\mathfrak{n}/\mathfrak{n} \cap \bar{\mathfrak{n}})^*$ be dual to the $(-\alpha)$-root space; then (5.11) is induced by the assignment

$$\psi \longmapsto \psi \wedge \omega^\alpha, \text{ for } \psi \in \{C^{-\omega}(G) \otimes \mathbf{L}_{\lambda+\alpha} \otimes \wedge^{\cdot}(\mathfrak{n}_\alpha/\mathfrak{n}_\alpha \cap \bar{\mathfrak{n}}_\alpha)^*\}^{\mathfrak{n}_\alpha \cap \bar{\mathfrak{n}}_\alpha, H}, \tag{5.13}$$

with $\mathfrak{n}_\alpha = [\mathfrak{b}_\alpha, \mathfrak{b}_\alpha]$. The geometric interpretation of this map will show that it has the appropriate compatibility properties.

The formal duality between hyperfunctions and real analytic functions can be used to reduce (5.12) to a statement about forms with real analytic coefficients. Rather than justifying the passage to the dual statement in detail, I shall describe the relevant properties of hyperfunctions and of the induced Dolbeault complex. To begin with, an application of the Dolbeault lemma along the fibres of $U \to U_\alpha$ shows that the complex $(C^{-\omega}(U_0, \boldsymbol{L}_{\lambda+\alpha} \otimes \wedge^{\cdot} \boldsymbol{N}^*_{D_0}), \bar{\partial}_0)$ has the same cohomology as the subcomplex of forms which are holomorphic in the fibre directions and do not involve the differential $d\bar{t}$. The map (5.11,13) takes values in this subcomplex, so I shall work with the latter from now on. In analogy to

$$(5.14) \qquad C^{-\omega}(U_0) \cong C^{\omega}(\bar{U}_0)'/C^{\omega}(\partial U_0)'$$

[12] (the superscript prime stands for "strong topological dual"), the space of hyperfunctions on U which are holomorphic along the fibres is a quotient of duals of spaces of real analytic functions: functions, defined on the germ of a neighborhood in $\bar{V}_\alpha$ of, respectively, $\bar{U}_0$ and ∂U_0, and holomorphic along the fibres. In effect, this is the duality between holomorphic functions on $\boldsymbol{C}$ and germs of holomorphic functions at ∞, with parameters. The duality persists when scalar functions are replaced by sections of $\boldsymbol{L}_\lambda$, provided the parameter λ satisfies the hypothesis of the second half of (4.12). This hypothesis ensures that the space of holomorphic sections of L_λ over each fibre $F \subset D$ is irreducible under the action of the stabilizer of F in $\mathfrak{g}$.

The usual Dolbeault complex is formally self-dual, except for a shift by the canonical bundle. Restricted Dolbeault complexes have the same formal self-duality property; the shift is given by a CR line bundle: the top exterior power of the annihilator in the complexified cotangent bundle of the (0, 1) tangent subbundle. Recall that the complex on the right hand side of (5.11–13) has been replaced by a certain subcomplex — in effect, the restrited Dolbeault complex of $p_\alpha(D) \subset X_\alpha$, with parameters which are holomorphic sections of $\boldsymbol{L}_\lambda$ on the fibres of p_α. In the case of the restricted Dolbeault complexes over D_α and D_0, the compensating shifts in the selfduality statement are related by the line bundle of tangent vectors to $X \to X_\alpha$; this explains the appearance of $\boldsymbol{L}_\lambda$ on one side of (5.11), and of $\boldsymbol{L}_{\lambda+\alpha}$ on the other. Also, the lengths of the two complexes differ by one, which accounts for the shifted degree on the right hand side.

The arguments which have just been outlined reduce (5.12) to the following statement. Let $S \subset D_0$ be a sufficiently small compact subset, which may play the role of either $\bar{U}_0$ or ∂U_0; let $C^{\omega}(S, \bar{\partial}_0)$ denote the restricted Dolbeault complex over S with real analytic coefficients, and $C^{\omega}_h(\tilde{S}, \bar{\partial})$ the complex of real analytic forms, defined on the germ of a neighborhood of S in $p_\alpha^{-1}(S)$, holomorphic in the fibre directions, and not involving the differential $d\bar{t}$. Then, if the restriction map

$$C_h^\omega(\tilde{S}, \bar{\partial}) \longrightarrow C^\omega(S, \bar{\partial}_0) \tag{5.15}$$

induces an isomorphism in cohomology, (5.12) will follow. The line bundle has disappeared at this point, since it can be trivialized over a neighborhood of S.

Forms $\Omega \in C_h^\omega(\tilde{S}, \bar{\partial}_0)$ can be expanded as series in powers of $t-\phi$, with coefficients that are real analytic Dolbeault forms on $p_\alpha(S)$. Expressing the coefficients as $\omega_n + \bar{\partial}_\alpha\phi \wedge \psi_n$, as described above, one obtains a series expansion

$$\Omega = \sum_{n\geq 0} (\omega_n + \bar{\partial}_\alpha \phi \wedge \psi_n)(t-\phi)^n, \tag{5.16}$$

with positive but arbitrarily small radius of convergence. I now use (5.10) to calculate the coboundary of Ω:

$$\bar{\partial}\Omega = \sum_{n\geq 0} \{\bar{\partial}_0\omega_n + \bar{\partial}_\alpha\phi \wedge (\delta_Y\omega_n - (n+1)\omega_{n+1} - \bar{\partial}_0\psi_n)\}(t-\phi)^n. \tag{5.17}$$

The restriction of ϕ to D_0 conicides with that of the holomorphic function t, so $\bar{\partial}_\alpha\phi$ restricts to zero on D_0. Conclusion: the map (5.15) is given by the assignment $\Omega \mapsto \omega_0$. If $\omega_0 \in C^\omega(S, \bar{\partial}_0)$ is closed, one can inductively solve the relations $\omega_{n+1} = (n+1)^{-1}\delta_Y\omega_n$ to construct a closed form $\Omega = \sum_n \omega_n(t-\phi)^n \in C_h^\omega(\tilde{S}, \bar{\partial})$ which restricts to ω_0 on S; symbolically

$$\Omega = \exp((t-\phi)\delta_Y)\omega_0 \tag{5.18}$$

(note: $\bar{\partial}_0$ commutes with δ_Y). Convergence presents no problem, since ω_0 and Y are real analytic. Similarly, if Ω is closed and ω_0 exact, one can recursively construct the coefficients of a series Φ, such that $\Omega = \bar{\partial}_0\Phi$. Thus (5.15) induces an isomorphism in cohomology, as had to be shown.

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