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A Classification Theory of Prehomogeneous Vector Spaces

Tatsuo Kimura

This is a survey of a classification theory of prehomogeneous vector spaces including some unpublished results of Professor Mikio Sato around 1962 with proofs under his permission (Sections 8, 10 and 15) and some results by the author (Section 9) not published elsewhere. This paper consists of the following 15 sections.

S. Kasai, Xiao-wei Zhu, M. Inuzuka, M. Taguchi and others are trying to classify some P.V.'s respectively, but since they are not completed yet, we do not contain their result here. About other aspects of the theory of P.V.'s originated by M. Sato and developed by many other mathematicians, one can see the papers in the references. The author would like to

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express his hearty thanks to Professor M. Sato who explained his results to the author and gave a permission to introduce his result with proofs.

§ 1. Basic definitions

Let Ω be an algebraically closed field of characteristic zero, and we shall consider everything over Ω in this section. Let G be a connected linear algebraic group, ρ a rational representation of G on a finite-dimensional vector space V. When V has a Zariski-dense G-orbit Y, we say that a triplet (G, ρ, V) (or simply (G, ρ)) is *a prehomogeneous vector space* (abbrev. *P.V.*). A point of Y is called *a generic point.* For $x \in V$, we denote by G_x the isotropy subgroup $\{g \in G; \rho(g)x=x\}$. Put $\mathfrak{g}=\mathrm{Lie}(G)$ and $\mathfrak{g}_x=\mathrm{Lie}(G_x)=\{A \in \mathfrak{g}; d\rho(A)x=0\}$ where $d\rho$ is the infinitesimal representation of ρ.

Theorem 1.1. *If there exists* $x \in V$ *satisfying* $\dim \mathfrak{g}_x = \dim G - \dim V$, *then a triplet* (G, ρ, V) *is a P.V.*

Proof. Since $\dim \rho(G)\cdot x=\dim G-\dim G_x=\dim G-\dim \mathfrak{g}_x=\dim V$, we have $\overline{\rho(G)\cdot x}=V$. Q.E.D.

A rational function $f(x)$ on V is called *a relative invariant* of a triplet (G, ρ, V) if there exists a rational character $\chi: G \to \Omega^\times$ satisfying $f(\rho(g)x) =\chi(g)f(x)$ for all $g \in G$. If $\chi=1$, $f(x)$ is called *an absolute invariant.*

Theorem 1.2. *If a triplet* (G, ρ, V) *has a non-constant absolute invariant, it cannot be a P.V.*

Proof. Assume that it is a P.V. Then $f(x)$ is constant on the Zariski-dense orbit Y, and hence it is constant on $\overline{Y}=V$. Q.E.D.

Theorems 1.1 and 1.2 are fundamental tools to check the prehomogeneity of a given triplet (G, ρ, V).

The complement $S=V-Y$ of Y is called *the singular set* of a P.V. (G, ρ, V), which is Zariski-closed. Let $S=S^1\cup\cdots\cup S^N\cup S''$ be the irreducible decomposition of S where each $S^i=\{x \in V; f_i(x)=0\}$ is an irreducible hypersurface $(1\leqq i\leqq N)$ and S'' is the union of irreducible components of codimension $\geqq 2$. All $f_i(x)$ $(1\leqq i\leqq r)$ are relatively invariant irreducible polynomials, which are called *basic relative invariants* of a P.V. (G, ρ, V). Any relative invariant $f(x)$ is uniquely expressed as $f(x)= cf_1(x)^{m_1}\cdots f_N(x)^{m_N}$ where $c \in \Omega^\times$ and $(m_1, \cdots, m_N) \in \boldsymbol{Z}^N$. A relative invariant $f(x)$ is called *non-degenerate if* its Hessian $\mathrm{Hess}_f(x)=\det(\partial^2 f/\partial x_i \partial x_j)$ is not identically zero. A P.V. (G, ρ, V) is called *regular* if it has a non-degenerate relative invariant. A P.V. (G, ρ, V) with a reductive algebraic group G is called *a reductive P.V.*

Theorem 1.3 (§ 4 in [S-K], [Servedio 3]). *Let (G, ρ, V) be a reductive P.V. Then the following conditions are equivalent.*

(1) *(G, ρ, V) is a regular P.V.*

(2) *The generic isotropy subgroup G_x $(x \in Y)$ is reductive.*

(3) *The singular set S is a (not necessarily irreducible) hypersurface.*

By Theorem 1.3, a reductive P.V. (G, ρ, V) is regular if and only if the generic isotropy subalgebra $\mathfrak{g}_x$ $(x \in Y)$ is reductive.

Now two triplets (G, ρ, V) and (G', ρ', V') are called *isomorphic* if there exists a rational isomorphism $\sigma: \rho(G) \to \rho'(G')$ and an isomorphism $\tau: V \to V'$ satisfying $(\sigma\rho)(g) \cdot \tau(v) = \tau \cdot \rho(g)v$ for all $g \in G$ and $v \in V$. In this case, we shall write $(G, \rho, V) \simeq (G', \rho', V')$. For example, we have $(SL_2, 2\Lambda_1) \simeq (SO_3, \Lambda_1)$ although SL_2 and SO_3 are not isomorphic. If G is reductive, then we have $(G, \rho, V) \simeq (G, \rho^*, V^*)$ where ρ^* is the contragredient representation on the dual vector space V^* of V. In our classification, we identify isomorphic triplets.

§ 2. Trivial P.V.'s and P.V.-equivalences

In this section, we shall show some general methods to construct infinitely many P.V.'s.

Theorem 2.1. *Let ρ be any representation of any group H on an m-dimensional vector space V. Then the triplet $(H \times GL_n, \rho \otimes \Lambda_1, V \otimes \Omega^n)$ is a P.V. for all $n \geqq m = \dim V$.*

Proof. It is enough to prove the prehomogeneity when $H = \{1\}$. In this case, this triplet is isomorphic to $(GL_n, \Lambda_1 \oplus \cdots \oplus \Lambda_1, M_{n,m})$. Put $x = \begin{pmatrix} I_m \\ 0 \end{pmatrix}$. Then the isotropy subgroup G_x is given by $G_x = \left\{ \left(\begin{array}{c|c} I_m & * \\ \hline 0 & A \end{array} \right); A \in GL_{n-m} \right\}$. Since $\dim G_x = m(n-m) + (n-m)^2 = \dim GL_n - \dim M_{n,m}$. it is a P.V. by Theorem 1.1. Q.E.D.

Definition 2.2. Any P.V. of the type in Theorem 2.1 is called *a trivial P.V.*

Now various P.V.-equivalences can be proved by using the following lemma.

Key Lemma 2.3. *Let G be a connected linear algebraic group and let W, W' be irreducible algebraic varieties on which G acts. Let $\psi: W \to W'$ be a dominant (i.e., $\overline{\psi(W)} = W'$) G-equivariant (i.e., compatible with the action of G) morphism. Then the following conditions are equivalent:*

(i) *W is G-prehomogeneous, i.e., it has a Zariski-dense G-orbit.*

(ii) *W' is G-prehomogeneous, and for a point y of a Zariski-dense orbit, the fiber $\psi^{-1}(y)$ is G_y-prehomogeneous.*

Proof. (i)$\Rightarrow$(ii): Let x be a point of the Zariski-dense G-orbit in W and put $y=\psi(x)$. Since $\overline{G\cdot y}=\overline{\psi(G\cdot x)}\supset\psi(\overline{G\cdot x})=\psi(W)$ and $W'=\overline{\psi(W)}$, we have $W'=\overline{G\cdot y}$, i.e., W' is G-prehomogeneous, and hence $\dim G_y=\dim G-\dim W'$. Since $\dim (G_y)_x=\dim G_x=\dim G-\dim W=\dim G_y+\dim W'-\dim W=\dim G_y-\dim\psi^{-1}(y)$, the fibre $\psi^{-1}(y)$ of y is G_y-prehomogeneous.

(ii)$\Rightarrow$(i): Since $\dim G_x=\dim (G_y)_x=\dim G_y-\dim\psi^{-1}(y)=(\dim G-\dim W')-\dim\psi^{-1}(y)=\dim G-\dim W$, W is G-prehomogeneous.

Q.E.D.

Theorem 2.4. *A triplet $(G, \rho_1\oplus\rho_2, V_1\oplus V_2)$ is a P.V. if and only if* (i) *(G, ρ_1, V_1) is a P.V. and* (ii) *$(H, \rho_2|H, V_2)$ is a P.V. where H is a generic isotropy subgroup of (G, ρ_1, V_1).*

Proof. By Key Lemma 2.3 for $W=V_1\oplus V_2$, $W'=V_1$ and $\psi=$the projection to V_1, we have our result. Q.E.D.

Theorem 2.5. *The following conditions are equivalent.*

(i) *All (G_i, ρ_i, V_i) are P.V.'s $(1\leqq i\leqq r)$.*

(ii) *A triplet $(G_1\times\cdots\times G_r\times GL_n, (\rho_1+\cdots+\rho_r)\otimes 1+\sigma\otimes\Lambda_1, (V_1\oplus\cdots\oplus V_r)\oplus(V\otimes\Omega^n))$ is a P.V. where σ is any representation of the group $G_1\times\cdots\times G_r$ on V and n is any natural number satisfying $n\geqq\dim V$.*

Proof. By Theorems 2.1 and 2.4, we have our result. Q.E.D.

Definition 2.6. A triplet in (ii) in Theorem 2.5 is called *a generalized direct sum of (G_i, ρ_i, V_i) $(1\leqq i\leqq r)$*. When $V=\{0\}$, it is called *the direct sum* and denoted by $\oplus(G_i, \rho_i, V_i)$.

Theorem 2.7. *Let ρ be a representation of an algebraic group H on an m-dimensional vector space V. For any n satisfying $m>n\geqq 1$, the following conditions are equivalent.*

(i) *$(H\times GL_n, \rho\otimes\Lambda_1, V\otimes\Omega^n)$ is a P.V.*

(ii) *$(H\times GL_{m-n}, \rho^*\otimes\Lambda_1, V^*\otimes\Omega^{m-n})$ is a P.V. where ρ^* is the contragredient representation of ρ on the dual space V^* of V. Note that if H is reductive, then this triplet is isomorphic to $(H\times GL_{m-n}, \rho\otimes\Lambda_1, V\otimes\Omega^{m-n})$.*

Proof. Put $W=\{x=(x_1, \cdots, x_n)\in M_{m,n}=V\otimes\Omega^n;\ \operatorname{rank} x=n\}$ and $W'=\mathrm{Grass}_n(V)$, i.e., the Grassmann variety consisting of n-dimensional subspaces of V. For $x \in W$, let $\psi(x)$ be the n-dimensional subspace of V spanned by $x_1, \cdots, x_n$. Then we obtain a dominant $H \times GL_n$-equivariant map $\psi: W \to W'$, and GL_n acts on each fibre transitively while it acts on W' trivially. Hence, by Key Lemma 2.3, (i) is equivalent to the condition: (i)$'$ $W' = \mathrm{Grass}_n(V)$ is H-prehomogeneous. Since $\mathrm{Grass}_n(V) \simeq \mathrm{Grass}_{m-n}(V^*)$, we have our result. Q.E.D.

We say the triplets (i) and (ii) in Theorem 2.7 are *castling transforms* of each other. Two triplets are called *castling-equivalent* if one is obtained from the other by a finite number of castling transformations. Let (G, ρ, V) be any given P.V. with $m = \dim V$. Clearly $(G \times GL_1, \rho \otimes \Lambda_1, V \otimes \Omega)$ is also a P.V., and hence so is its castling transform $(G \times GL_{m-1}, \rho^* \otimes \Lambda_1, V^* \otimes \Omega^{m-1})$. By repeating this procedure, one sees that there exist infinitely many non-isomorphic P.V.'s which are castling-equivalent to a given P.V. when $m = \dim V \geqq 3$. There are many other P.V.-equivalences (See [S-K], [K-K-T-I]). We shall finish this section by giving one more example of P.V.-equivalence.

Theorem 2.8. *Let ρ_1 (resp. ρ_2) be a representation of an algebraic group H on an m_1 (resp. m_2)-dimensional vector space V_1 (resp. V_2). Then the following conditions are equivalent.*

(i) *$(H, \rho_1 \otimes \rho_2, V_1 \otimes V_2)$ is a P.V.*

(ii) *$(H \times GL_n, \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1^*, V_1 \otimes \Omega^n + V_2 \otimes \Omega^n)$ is a P.V. for all n satisfying $n \geqq \max\{m_1, m_2\}$.*

(iii) *There exists n satisfying $n \geqq \max\{m_1, m_2\}$ such that $(H \times GL_n, \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1^*, V_1 \otimes \Omega^n + V_2 \otimes \Omega^n)$ is a P.V.*

By using Key Lemma 2.3, we can prove Theorem 2.8 (See § 1 in [K-K-T-I], § 4 in [K-K-H]). Thus, for any given P.V. (G, ρ, V), we can construct a new P.V. $(G \times GL_n, \rho \otimes \Lambda_1 + 1 \otimes \Lambda_1^*, V \otimes \Omega^n + \Omega^n)$ for any $n \geqq \dim V$.

§ 3. A classification of irreducible P.V.'s

The starting point of a classification is the following lemma.

Lemma 3.1. *If (G, ρ, V) is a P.V., then we have* $\dim G \geqq \dim V$.

Proof. If x is a generic point, we have $\dim G_x = \dim G - \dim V \geqq 0$. Q.E.D.

By a well-known theorem of E. Cartan, $\rho(G)$ is a reductive algebraic group with at most one-dimensional center when ρ is irreducible. Hence we may assume that $G = (GL_1 \times) G_1 \times \cdots \times G_k$ where each G_i $(1 \leqq i \leqq k)$ is a simple algebraic group of dim $\geqq 3$. It is also well-known that if σ is an irreducible representation of a group $H=H_1\times H_2$ over Ω, then we have $\sigma=\sigma_1\otimes\sigma_2$ where each σ_i is an irreducible representation of H_i $(i=1, 2)$. Hence we may assume that $\rho=(\Lambda_1\otimes)\rho_1\otimes\cdots\otimes\rho_k$, $V=V(d_1)\otimes\cdots\otimes V(d_k)$ $(d_1\geqq d_2\geqq\cdots\geqq d_k\geqq 2)$ where each ρ_i is a d_i-dimensional irreducible representation of G_i on $V(d_i)$ $(1\leqq i\leqq k)$ and Λ_1 denotes the scalar multiplication of GL_1 on V. Thus, by Lemma 3.1, we have $1+g_1+\cdots+g_k\geqq d_1d_2\cdots d_k$ where $g_i=\dim G_i$ $(1\leqq i\leqq k)$. The following lemma due to M. Sato is important for our purpose.

Lemma 3.2 (M. Sato) (p. 43 in [S-K]). *If* $k\geqq 3$, *then* $1+g_1\geqq 4d_1-6$.

Now what we have to do is first to classify all triplets (G, ρ, V) satisfying $\dim G\geqq\dim V$ by using castling transformations, case by case when $G_1=A_n, B_n, C_n, D_n, (G_2), F_4, E_6, E_7, E_8$. For example, Lemma 3.2 says that $k=1$ or 2 when G_1 is an exceptional algebraic group (G_2), F_4, E_6, E_7, E_8. Next, by using Theorems 1.1 and 1.2, we check the prehomogeneity of triplets (G, ρ, V) satisfying $\dim G\geqq\dim V$. Note that the property of $\dim G\geqq\dim V$ and the regularity are invariant property under castling transformations. The results are given as follows.

Theorem 3.3 ([S-K]). *Any irreducible P.V.* (G, ρ, V) *is castling-equivalent to one of the following P.V.'s.*

(I) Regular P.V.'s.

(1) A trivial P.V., i.e., $(H\times GL_n, \rho\otimes\Lambda_1, M_n)$ where ρ is an n-dimensional irreducible representation of a connected semisimple algebraic group H.

(2) (GL_n, ρ) where $\rho=2\Lambda_1$; $3\Lambda_1(n=2)$; $\Lambda_2(n=\text{even})$; $\Lambda_3(n=6, 7, 8)$.

(3) $(SL_3\times GL_2, 2\Lambda_1\otimes\Lambda_1, V(6)\otimes V(2))$.

(4) $(SL_6\times GL_2, \Lambda_2\otimes\Lambda_1, V(15)\otimes V(2))$.

(5) $(SL_5\times GL_n, \Lambda_2\otimes\Lambda_1, V(10)\otimes V(n))$ $(n=3, 4)$.

(6) $(SL_3\times SL_3\times GL_2, \Lambda_1\otimes\Lambda_1\otimes\Lambda_1, V(3)\otimes V(3)\otimes V(2))$.

(7) $(Sp_n\times GL_{2m}, \Lambda_1\otimes\Lambda_1, V(2n)\otimes V(2m))$ $(n>m\geqq 1)$

(8) $(GL_1\times Sp_3, \Lambda_1\otimes\Lambda_3, V(14))$.

(9) $(SO_n\times GL_m, \Lambda_1\otimes\Lambda_1, V(n)\otimes V(m))$ $(n>m\geqq 1)$.

(10) $(\mathrm{Spin}_7\times GL_n$, the spin rep.$\otimes\Lambda_1)$ $(1\leqq n\leqq 3)$.

(11) $(GL_1\times\mathrm{Spin}_n$, the spin rep.$)$ $(n=9, 11)$.

(12) $(\mathrm{Spin}_{10}\times GL_n$, a half-spin rep.$\otimes\Lambda_1)$ $(n=2, 3)$.

(13) $(GL_1\times\mathrm{Spin}_n$, a half-spin rep.$)$ $(n=12, 14)$.

(14) $(G_2\times GL_n, \Lambda_2\otimes\Lambda_1, V(7)\otimes V(n))$ $(n=1, 2)$.

(15) $(E_6\times GL_n, \Lambda_1\otimes\Lambda_1, V(27)\otimes V(n))$ $(n=1, 2)$.

(16) $(GL_1\times E_7, \Lambda_1\otimes\Lambda_6, V(56))$.

(II) Non-regular P.V.'s.

(1) $(Sp_n \times GL_2, \Lambda_1 \otimes 2\Lambda_1, V(2n) \otimes V(3))$.

(2) $((GL_1 \times)H \times SL_n, (\Lambda_1 \otimes)\rho \otimes \Lambda_1, V(m) \otimes V(n))$ where ρ is an m-dimensional irreducible representations of a semisimple algebraic group H with $1 \leqq m < n$.

(3) $((GL_1 \times)SL_{2m+1}, (\Lambda_1 \otimes)\Lambda_2, V(m(2m+1)))$ $(m \geqq 2)$.

(4) $((GL_1 \times)SL_{2m+1} \times SL_2, (\Lambda_1 \otimes)\Lambda_2 \otimes \Lambda_1, V(m(2m+1)) \otimes V(2))$ $(m \geqq 2)$.

(5) $((GL_1 \times)Sp_n \times SL_{2m+1}, (\Lambda_1 \otimes)\Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(2m+1))$.

(6) $((GL_1 \times)\mathrm{Spin}_{10}, (\Lambda_1 \otimes)$ a half-spin rep., $V(16))$.

Remark. A classification of irreducible P.V.'s $(GL_1 \times G, \Lambda_1 \otimes \rho)$ with a simple algebraic group G was completed by M. Sato and T. Shintani (See P. 144 in [S-S]).

§ 4. A classification of simple P.V.'s

E.B. Vinberg has classified a P.V. (G, ρ, V) when G is a simple algebraic group ([V]). T. Shintani completed a classification of irreducible simple P.V.'s with the scalar multiplication ([S-S]). This result is included in Section 3. T. Kimura has classified simple P.V.'s with scalar multiplications ([Kimura 5]).

Theorem 4.1 (E.B. Vinberg [V]). *A P.V. (G, ρ, V) with a simple algebraic group is given as follows.*

(1) $G = SL_n$; $\rho = \Lambda_1 \oplus \cdots \oplus \Lambda_1$ $(1 \leqq k < n)$, Λ_2 $(n = odd)$, $\Lambda_2 \oplus \Lambda_1^*$ $(n = odd)$, $\Lambda_2 \oplus \Lambda_2$ $(n = odd)$.

(2) (Sp_n, Λ_1), $(\mathrm{Spin}_{10}$, *a half-spin rep.*).

Let G' be a simple algebraic group, $\rho' = \rho_1 \oplus \cdots \oplus \rho_k$ a rational representation of G' where each ρ_i is an irreducible representation. Put $G = GL_1^k \times G'$ and let ρ be the composition of ρ' and the scalar multiplications GL_1^k on each irreducible component ρ_i $(1 \leqq i \leqq k)$. A P.V. (G, ρ, V) of such type is called *a simple P.V.* For simplicity, we write $(GL_1^k \times G', \rho')$ or $(G', \rho')'$ instead of $(GL_1^k \times G', \rho)$.

Theorem 4.2 ([Kimura 5] with a correction [K-K-I-Y]). *All non-irreducible simple P.V.'s (with scalar multiplications) are given as follows.*

(1) $G' = SL_n$, $\rho = \Lambda_1 \oplus \cdots \oplus \Lambda_1 \oplus \Lambda_1^{(*)}$ $(2 \leqq k \leqq n+1, n \geqq 2)$, $\Lambda_2 \oplus \Lambda_1^{(*)} \oplus \cdots \oplus \Lambda_1^{(*)}$ $(2 \leqq k \leqq 4, n \geqq 4)$ *except* $\Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1^*$ $(n = odd)$, $2\Lambda_1 \oplus \Lambda_1^{(*)}$, $\Lambda_2 \oplus \Lambda_2$ $(n = odd)$, $\Lambda_2 \oplus \Lambda_2 \oplus \Lambda_1^*$ $(n = 5)$, $\Lambda_3 \oplus \Lambda_1^{(*)}$ $(n = 6, 7)$, $\Lambda_3 \oplus \Lambda_1 \oplus \Lambda_1$ $(n = 6)$.

(2) $G' = Sp_n$, $\rho = \Lambda_1 \oplus \Lambda_1$, $\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1$, $\Lambda_2 \oplus \Lambda_1$ $(n = 2)$, $\Lambda_3 \oplus \Lambda_1$ $(n = 3)$.

(3) $G' = \mathrm{Spin}_n$, $\rho =$ *the spin rep.* $\oplus$ *the vector rep.* $(n = 7)$, *a half-spin rep.* $\oplus$ *the vector rep.* $(n = 8, 10, 12)$, $\Lambda \oplus \Lambda$ *where* $\Lambda =$ *the even half-spin representation.*

Theorem 4.3. *All non-irreducible simple regular P.V.'s are given as follows.*

(1) $G'=SL_n$, $\rho=\Lambda_1\oplus\Lambda_1^*$, $\Lambda_1\oplus\cdots\oplus\Lambda_1$ $(k=n)$ $(\oplus\Lambda_1^{(*)})$, $2\Lambda_1\oplus\Lambda_1^{(*)}$, $\Lambda_2\oplus\Lambda_1^{(*)}\oplus\Lambda_1^{(*)}$ $(n=even)$, $\Lambda_2\oplus\Lambda_1(\oplus(\Lambda_1\oplus\Lambda_1)^{(*)})$ $(n=odd)$, $\Lambda_3\oplus\Lambda_1^{(*)}$ $(n=7)$.

(2) $G'=Sp_n$, $\rho=\Lambda_1\oplus\Lambda_1$, $\Lambda_3\oplus\Lambda_1$ $(n=3)$.

(3) *All P.V.'s given in* (3) *in Theorem* 4.2.

§ 5. A classification of 2-simple P.V.'s

In this section, we shall consider a triplet $(GL_1^{k+s+t}\times G_1\times G_2, (\sigma_1+\cdots+\sigma_s)\otimes 1+(\rho_1\otimes\rho_1'+\cdots+\rho_k\otimes\rho_k')+1\otimes(\tau_1+\cdots+\tau_t))$ where G_1, G_2 are simple algebraic groups; σ_i, ρ_i (resp. ρ_j', τ_j) are non-trivial irreducible representation of G_1 (resp. G_2), and GL_1^{k+s+t} acts on each irreducible component as scalar multiplications. A P.V. of such type is called *a 2-simple P.V.* If one of $(GL_1\times G_1\times G_2, \rho_i\otimes\rho_i')$ $(1\leq i\leq k)$ is a non-trivial P.V., it is called *a 2-simple P.V. of type* I. On the other hand, if all $(GL_1\times G_1\times G_2, \rho_i\otimes\rho_i')$ $(1\leq i\leq k)$ are trivial P.V.'s (See Definition 2.2), it is called *a 2-simple P.V. of type* II. Note that if $k=0$, it is just the direct sum of simple P.V.'s, and hence we shall assume $k\geq 1$. By using the results in Sections 3 and 4, one can complete the classification of 2-simple P.V. of type I.

Theorem 5.1 ([K-K-I-Y]). *Any 2-simple P.V. of type* I *is castling-equivalent to a simple P.V. or one of the following P.V.'s* $(GL_1^k\times G, \rho(=\rho_1\oplus\cdots\oplus\rho_k))$.

(I) *Regular P.V.'s*

(1) $G=SL_m\times SL_n$, (a) $m=4$; $\rho=\Lambda_2\otimes\Lambda_1+T$ $(n=2)$ *with* $T=\Lambda_1\otimes\Lambda_1$, $(\Lambda_1+\Lambda_1)\otimes 1$; $\Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1(+1\otimes\Lambda_1^{(*)})$ $(n=3)$, $\Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1+1\otimes\Lambda_1^*$ $(n=4)$, (b) $m=5$; $\rho=\Lambda_2\otimes\Lambda_1+(\Lambda_1^*+\Lambda_1^{(*)})\otimes 1$ $(n=2)$, $\Lambda_2\otimes\Lambda_1+1\otimes\Lambda_1^{(*)}$ $(n=3)$, $\Lambda_2\otimes\Lambda_1+1\otimes\Lambda_1^*$ $(n=8)$, $\Lambda_2\otimes\Lambda_1+1\otimes\Lambda_1^*$ $(n=9)$.

(2) $G=Sp_m\times SL_n$, (a) $\rho=\Lambda_1\otimes\Lambda_1+1\otimes(\Lambda_1^{(*)}+\Lambda_1^{(*)})$ $(n=even)$; $\Lambda_1\otimes\Lambda_1+1\otimes T$ $(n=2)$ *with* $T=2\Lambda_1$, $3\Lambda_1$, $(2\Lambda_1+\Lambda_1)$; $\Lambda_1\otimes\Lambda_1+\Lambda_1\otimes 1(+1\otimes(\Lambda_1+\Lambda_1)^{(*)})$ $(n=odd)$. (b) $m=2$; $\rho=\Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1(+1\otimes\Lambda_1)$ $(n=2)$, $\Lambda_1\otimes\Lambda_1+\Lambda_2\otimes 1+1\otimes\Lambda_1^*$ $(n=3)$, $\Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1+1\otimes\Lambda_1^*$ $(n=4)$.

(3) $G=SO_n\times SL_m$, $\rho=\Lambda_1\otimes\Lambda_1+1\otimes\Lambda_1^{(*)}$.

(4) $G=\mathrm{Spin}_7\times SL_n$, (a) $\rho=\Lambda\otimes\Lambda_1+1\otimes\Lambda_1$ $(n=2, 3)$, $\Lambda\otimes\Lambda_1+1\otimes\Lambda_1^*$ $(n=3, 6, 7)$ *where* $\Lambda=$ *the spin representation of* Spin_7. (b) $\rho=\chi\otimes\Lambda_1+\Lambda\otimes 1(+1\otimes\Lambda_1)$ $(n=2)$, $\chi\otimes\Lambda_1+\Lambda\otimes 1+1\otimes\Lambda_1^*$ $(n=6)$ *where* $\chi=$ *the vector representation of* Spin_7, *i.e.* $\chi(\mathrm{Spin}_7)=SO_7$.

(5) $G=\mathrm{Spin}_m\times SL_n$, (*a*) $m=8$, $\rho=\chi\otimes\Lambda_1+\Lambda'\otimes 1(+1\otimes\Lambda_1)$ $(n=2, 3)$, $\chi\otimes\Lambda_1+\Lambda'\otimes 1+1\otimes\Lambda_1^*$ $(n=3, 6, 7)$ *where* $\Lambda'=$ *a half-spin representation.* (b) $m=10$, $\rho=\Lambda'\otimes\Lambda_1+1\otimes T$ $(n=2)$ *with* $T=2\Lambda_1$, $3\Lambda_1$, $\Lambda_1+\Lambda_1$, $2\Lambda_1+\Lambda_1$,

$\Lambda_1+\Lambda_1+\Lambda_1$; $\Lambda'\otimes\Lambda_1+1\otimes\Lambda_1$ $(n=3)$, $\Lambda'\otimes\Lambda_1+1\otimes\Lambda_1^*$ $(n=3, 14, 15)$, $\chi\otimes\Lambda_1+\Lambda'\otimes 1$ $(n=2, 3, 4)$.

(6) $G=(G_2)\times SL_n$, $\rho=\Lambda_2\otimes\Lambda_1+1\otimes\Lambda_1$ $(n=2)$, $\Lambda_2\otimes\Lambda_1+1\otimes\Lambda_1^*$ $(n=6)$ *where* $\deg\Lambda_2=7$.

(II) *Non-regular P.V.'s*

(7) $G=SL_n\times SL_2$; $\rho=\Lambda_2\otimes\Lambda_1+1\otimes T$ $(n=odd)$ *with* $T=t\Lambda_1$ $(t=1,2,3)$, $\Lambda_1+t\Lambda_1$ $(t=1, 2)$, $\Lambda_1+\Lambda_1+\Lambda_1$; $\Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1(+1\otimes\Lambda_1)$ $(n=4)$, $\Lambda_2\otimes\Lambda_1+\Lambda_1^{(*)}\otimes 1(+1\otimes\Lambda_1)$ $(n=5)$, $\Lambda_2\otimes\Lambda_1+\Lambda_1^*\otimes 1+1\otimes T$ $(n=5)$ *with* $T=2\Lambda_1, \Lambda_1+\Lambda_1$; $\Lambda_2\otimes\Lambda_1+\Lambda_1^{(*)}\otimes 1$ $(n=6, 7)$, $\Lambda_2\otimes\Lambda_1+\Lambda_1^*\otimes 1+1\otimes\Lambda_1$ $(n=7)$, $\Lambda_2\otimes\Lambda_1+\Lambda_1^*\otimes 1$ $(n=9)$.

(8) $G=SL_4\times SL_5$, $\rho=\Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1+1\otimes\Lambda_1^*$.

(9) $G=SL_5\times SL_9$, $\rho=\Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1+1\otimes\Lambda_1^*$.

(10) $G=Sp_n\times SL_m$, (a) $m=even$, $\rho=\Lambda_1\otimes\Lambda_1(+T(+1\otimes(\Lambda_1^{(*)}+\Lambda_1^{(*)})))$ *with* $T=\Lambda_1\otimes 1, 1\otimes\Lambda_1^{(*)}$; $\Lambda_1\otimes\Lambda_1+\Lambda_1\otimes 1+1\otimes\Lambda_1^{(*)}$. (b) $m=odd$, $\rho=\Lambda_1\otimes\Lambda_1+1\otimes T$ *with* $T=\Lambda_1^{(*)}, \Lambda_2$, $\rho=\Lambda_1\otimes\Lambda_1+S+T$ *with* $S, T=\Lambda_1\otimes 1, 1\otimes\Lambda_1^{(*)}$. $\rho=\Lambda_1\otimes\Lambda_1+T$ *with* $T=1\otimes\Lambda_1+1\otimes(\Lambda_1+\Lambda_1)^{(*)}$, $1\otimes(\Lambda_1^*+\Lambda_1^*+\Lambda_1^*)$. (c) $m=2$, $\rho=\Lambda_1\otimes\Lambda_1+\Lambda_1\otimes 1+1\otimes 2\Lambda_1$, $\Lambda_1\otimes 2\Lambda_1+1\otimes\Lambda_1$, (d) $m=3$, $\rho=\Lambda_1\otimes\Lambda_1+1\otimes 2\Lambda_1$, (e) $m=5$, $\rho=\Lambda_1\otimes\Lambda_1+1\otimes(\Lambda_2+\Lambda_1^*)$.

(11) $G=\mathrm{Spin}_{10}\times SL_2$, $\rho=\Lambda'\otimes\Lambda_1+1\otimes\Lambda_1$.

Now let us consider 2-simple P.V.'s of type II.

Theorem 5.2. *For any simple P.V.* $(GL_1^r\times G, \rho_1\oplus\cdots\oplus\rho_r)$, *one can obtain the following 2-simple P.V.'s of type* II.

(1) $(GL_1^{r+s}\times G\times SL_n, (\sigma_1+\cdots+\sigma_s)\otimes\Lambda_1+(\rho_1+\cdots+\rho_r)\otimes 1)$ *for any representation* $\sigma_1+\cdots+\sigma_s$ *of* G *and any natural number* n *satisfying* $n\geqq\deg\sigma_1+\cdots+\deg\sigma_s$.

(2) $(GL_1^{r+t}\times G\times SL(\sum_{i=1}^k\deg\rho_i+t-1), (\rho_1+\cdots+\rho_k)\otimes\Lambda_1+(\rho_{k+1}^*+\cdots+\rho_r^*)\otimes 1+1\otimes(\overbrace{\Lambda_1+\cdots+\Lambda_1}^{t}))$ $(1\leqq k\leqq r)$ *for any* $t\geqq 0$.

(3) $(GL_1^{r+t}\times G\times SL_n, (\rho_1+\cdots+\rho_k)\otimes\Lambda_1+(\rho_{k+1}+\cdots+\rho_r)\otimes 1+1\otimes(\overbrace{\Lambda_1+\cdots+\Lambda_1}^{t-1}+\Lambda_1^*))$ $(1\leqq k\leqq r)$ *for any pair of natural numbers* (t, n) *satisfying* $t\geqq 1$ *and* $n\geqq t-1+\deg\rho_1+\cdots+\deg\rho_k$.

Proof. By Theorem 2.5 (resp. Theorem 2.7, Theorem 2.8), we have (1) (resp. (2), (3)). Q.E.D.

A classification of 2-simple P.V.'s of type II is more difficult than that of type I because we have to classify 2-simple P.V.'s of type $(GL_1^{k+s+t}\times SL_m\times SL_n, (\overbrace{\Lambda_1^{(*)}+\cdots+\Lambda_1^{(*)}}^{s})\otimes 1+(\overbrace{\Lambda_1^{(*)}\otimes\Lambda_1^{(*)}+\cdots+\Lambda_1^{(*)}\otimes\Lambda_1^{(*)}}^{t})+1\otimes(\overbrace{\Lambda_1^{(*)}+\cdots+\Lambda_1^{(*)}}^{t}))$ where $\Lambda_1^{(*)}$ stands for the standard representation

Λ_1 or its dual $\Lambda_1^{(*)}$. One can see after using some P.V.-equivalences that the most essential part is to investigate the prehomogeneity of $(GL_1^{k+s+t}\times SL_m\times SL_n, (\overbrace{\Lambda_1+\cdots+\Lambda_1}^{s})\otimes 1+(\overbrace{\Lambda_1\otimes\Lambda_1+\cdots+\Lambda_1\otimes\Lambda_1}^{k})+1\otimes(\overbrace{\Lambda_1^*+\cdots+\Lambda_1^*}^{t}))$ with $km>n>m\geqq 2$ and $t\geqq 1$. Actually this was the most difficult part of a classification of 2-simple P.V.'s. Before stating the result, we need some definitions.

Definition 5.3. Let R be the set of triplets (k, m, n) of natural numbers satisfying $k\geqq 2$, $n>m\geqq 2$ and $k+m^2+n^2>kmn+2$. We define a map $\Psi: R\to \boldsymbol{Z}_+=\{0, 1, 2, \cdots\}$ by $\Psi(k, m, n)=\min\{i;\ c_i\leqq 0\}$ where $c_{-2}=n$, $c_{-1}=m$, $c_i=kc_{i-1}-c_{i-2}$ $(i\geqq 0)$. Define a sequence $\{a_i\}$ by $a_{-1}=-1$, $a_0=0$, $a_i=ka_{i-1}-a_{i-2}$ and put $b_i=a_i/a_{i+1}$ $(i\geqq 0)$.

Theorem 5.4 (See Theorem 4.13 in [K-K-T-I]). *A triplet* $(GL_1^{k+s+t}\times SL_m\times SL_n,\ (\overbrace{\Lambda_1+\cdots+\Lambda_1}^{s})\otimes 1+(\overbrace{\Lambda_1\otimes\Lambda_1+\cdots+\Lambda_1\otimes\Lambda_1}^{k})+1\otimes(\overbrace{\Lambda_1^*+\cdots+\Lambda_1^*}^{t}))$ $(km>n>m\geqq 2,\ t\geqq 1)$ *is a P.V. if and only if* $(k, m, n)\in R$ *and* $s+kt\leqq m-b_j(n-t)$ *where* $j=\Psi(k, m, n)$.

The basic idea of Theorem 5.4 is due to M. Inuzuka, and developed by T. Kimura. S. Kasai and M. Taguchi improved its proof. The scalar multiplications have a delicate role in our classification. For example, we have a following proposition due to T. Kimura and M. Taguchi.

Proposition 5.5 (Theorem 4.18 in [K-K-T-I]). *The following conditions are equivalent.*

(1) $(GL_1^{k+s+t}\times SL_m\times SL_n, (\Lambda_1^*+\overbrace{\Lambda_1+\cdots+\Lambda_1}^{s-1})\otimes 1+(\overbrace{\Lambda_1\otimes\Lambda_1+\cdots+\Lambda_1\otimes\Lambda_1}^{k})+1\otimes(\Lambda_1^*+\overbrace{\Lambda_1+\cdots+\Lambda_1}^{t-1}))$ $(s\geqq 1,\ t\geqq 1,\ km>n>m)$ *is a P.V.*

(2) $(GL_{m-1}\times GL_{n-1},\ (\overbrace{\Lambda_1+\cdots+\Lambda_1}^{k+s-2})\otimes 1+(\overbrace{\Lambda_1\otimes\Lambda_1+\cdots+\Lambda_1\otimes\Lambda_1}^{k})+1\otimes(\overbrace{\Lambda_1+\cdots+\Lambda_1}^{k+t-2}))$ *is a P.V.*

From the classification of 2-simple P.V.'s of type I, we can prove the following theorem.

Theorem 5.6 (M. Inuzuka and T. Kimura). *Let G_1 and G_2 be simple algebraic groups. Assume that $(G_1\times G_2, \rho, V)$ is a non-irreducible P.V. which contains a non-trivial P.V. as an irreducible component. Then it must be one of the following P.V.*'s.

(1) $(Sp_n\times SL_{2m+1}, \Lambda_1\otimes\Lambda_1+1\otimes\Lambda_1^*)$ $(n>m)$.

(2) $(Sp_n\times SL_{2m+1}, \Lambda_1\otimes\Lambda_1+1\otimes\Lambda_2)$ $(n>m\geqq 2)$.

Moreover, they have infinitely many orbits.

Proof. If $(G_1 \times G_2, \rho, V)$ is a P.V., then $(GL_1{}^k \times G_1 \times G_2, \rho(=\rho_1 \oplus \cdots \oplus \rho_k), V)$ is a 2-simple P.V. of type I without any relative invariant and hence we obtain (1) and (2) from Theorem 5.1 by checking their generic isotropy subalgebras. It is easy to prove that (1) and (2) are actually P.V.'s. By Proposition 1.4 in [K-K-Y], if (1) and (2) are F.P.'s, then $(SP_n \times SL_{2m+1}, \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_1)$ and $(Sp_n \times SL_{2m+1}, \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2^*)$ are also F.P.'s. But they are not P.V.'s, i.e., a contradiction. Q.E.D.

§ 6. A classification of reductive P.V.'s with finitely many orbits

A triplet (G, ρ, V) such that V decomposes into a finitely many G-orbits must be clearly a P.V. However the converse is not true in general. Actually since this condition is very strong, we can classify all such P.V.'s under the assumption that G is reductive without assuming the irreducibility of ρ ([K-K-Y]). A P.V. with a finitely many orbits is called *a finite P.V.* (abbrev *F.P.*).

Lemma 6.1 (cf. p. 148 in [S-K]). *If $(H \times GL_n, \rho \otimes \Lambda_1, V \otimes \Omega^n)$ $(m = \dim V > n \geq 1)$ is a F.P., then $(H \times GL_k, \rho \otimes \Lambda_1, V \otimes \Omega^k)$ is also a F.P. for any $k \leq n$.*

Proof. We identify $V \otimes \Omega^n$ with $M_{m,n}$. Define a map ψ of $M_{m,n}$ to the set $T = \cup_{r=0}^n \mathrm{Grass}_r(V)$ by $\psi(v)=$ the vector subspace of V spanned by column vectors of v. Since GL_n acts on each fibre transitively, there is a one-to-one correspondence between the orbits of $(H \times GL_n, \rho \otimes \Lambda_1, M_{m,n})$ and the H-orbits in T. Hence $\cup_{r=0}^k \mathrm{Grass}_r(V)$ has a finitely many H-orbits. This implies our assertion. Q.E.D.

As an example, we prove that a castling transform $(SL_2 \times GL_3, 3\Lambda_1 \otimes \Lambda_1, V(4) \otimes V(3))$ of $(GL_2, 3\Lambda_1, V(4))$ has infinitely many orbits. If it is a F.P., then $(SL_2 \times GL_2, 3\Lambda_1 \otimes \Lambda_1, V(4) \otimes V(2))$ must be a F.P. by Lemma 6.1. Since $\dim SL_2 \times GL_2 = 7 < \dim V(4) \otimes V(2) = 8$, it cannot be a P.V. by Lemma 3.1. This is a contradiction.

Proposition 6.2. *An irreducible P.V. $(GL_1 \times G_1 \times \cdots \times G_k, \rho_1 \otimes \cdots \otimes \rho_k, V(d_1) \otimes \cdots \otimes V(d_k))$ $(d_1 \geq \cdots d_k \geq 2)$ with $k \geq 4$ has infinitely many orbits.*

Proof. If it is a F.P., $(GL_1 \times SL(d_1) \times SL(d_2) \times SL(d_3) \times SL(d_4 \cdots d_k), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1)$ must be also a F.P. Applying Lemma 6.1 several times, we see that $(GL_1 \times SL_2 \times SL_2 \times SL_2 \times SL_2, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(2) \otimes V(2) \otimes V(2) \otimes V(2))$ is also a F.P. However since $\dim GL_1 \times SL_2 \times SL_2 \times SL_2 \times SL_2 = 13 < 16$, it cannot be even a P.V., i.e., a contradiction. Q.E.D.

By this way, T. Kimura determined all irreducible F.P.'s ([Kimura 1], also partly in [S-K]). V.G. Kac has also determined all irreducible F.P.'s independently ([Kac 1], [Kac 2]).

Theorem 6.3 (A classification of irreducible F.P.'s).

(1) *An irreducible trivial P.V.* $(H\times GL_n,\ \rho\otimes\Lambda_1,\ M_{m,n})$ $(m\leqq n)$ *is a F.P. if and only if* (H, ρ) *is one of* (SL_m, Λ_1), (SO_m, Λ_1), $(Sp_{m'}, \Lambda_1)$ $(m=2m')$.

(2) $((GL_1\times)SL_{2m+1}\times SL_2,\ (\Lambda_1\otimes)\Lambda_2\otimes\Lambda_1,\ V(m(2m+1))\otimes V(2))$ $(m\geqq 2)$ *is a F.P. if and only if* $m=2, 3$.

(3) *Under above restriction* (1) *and* (2), *all irreducible F.P.'s are given in the list of Theorem* 3.3. *Note that property to be a F.P. is not invariant under castling transformations as we saw above.*

Also, in [Kac 3], V.G. Kac has completed the classification of F.P.'s when each irreducible component is $(GL_n\times GL_m,\ \Lambda_1\otimes\Lambda_1,\ V(n)\otimes V(m))$ which is the development of the results of P. Gabriel ([G]). T. Kimura classified simple F.P.'s.

Theorem 6.4 ([K-K-Y]). *All simple F.P.'s* $(GL_1^k\times G,\ \rho\ (=\rho_1\oplus\cdots\oplus\rho_k))$ *are given as follows.*

(1) $G=SL_n$, $\rho=\Lambda_1\oplus\Lambda_1^{(*)}(\oplus\Lambda_1)$, $\Lambda_2\oplus\Lambda_1^{(*)}(\oplus\Lambda_1^{(*)})$, $2\Lambda_1\oplus\Lambda_1^{(*)}$, $\Lambda_3\oplus\Lambda_1^{(*)}$ $(n=6, 7)$.

(2) $G=Sp_n$, $\rho=\Lambda_1\oplus\Lambda_1(\oplus\Lambda_1)$, $\Lambda_2\oplus\Lambda_1$ $(n=2)$, $\Lambda_3\oplus\Lambda_1$ $(n=3)$.

(3) $G=Spin_n$, $\rho=$ *the spin rep.* $\oplus$ *the vector rep.* $(n=7)$, *a half-spin rep.* $\oplus$ *the vector rep.* $(n=8, 10, 12)$.

Now we denote a triplet $(G\times G',\ \Lambda\otimes\Lambda',\ V\otimes V')$ by a diagram $\overset{G}{\circ}\overset{\Lambda\quad\Lambda'}{\text{———}}\overset{G'}{\circ}$. If $G=GL_n$ or SL_n, we simply write $\overset{n}{\circ}\overset{\Lambda\quad\Lambda'}{\text{———}}\overset{G'}{\circ}$. Moreover, if $\Lambda=\Lambda_1$ (also for Sp_n), we write $\overset{n}{\circ}\overset{\quad\Lambda'}{\text{———}}\overset{G'}{\circ}$ (resp. $\overset{Sp_n}{\circ}\overset{\quad\Lambda'}{\text{———}}\overset{G'}{\circ}$). Any diagram should be assumed that on each irreducible component, the scalar multiplications act independently. Now the key point of the classification of the general case is the following "Basic Theorem" due to T. Kimura.

Theorem 6.5 (Basic Theorem). *Let* $\overset{H}{\circ}\overset{\rho}{\text{——}}\overset{m}{\circ}$ $(m\geqq 2)$ *be a F.P. such that* $\overset{1}{\circ}\overset{\rho}{\text{——}}\overset{H}{\circ}\overset{\rho}{\text{——}}\overset{1}{\circ}$ *is not a F.P. If* $\overset{H}{\circ}\overset{\rho}{\text{——}}\overset{m}{\circ}\text{- - - -}\overset{\sigma}{\text{——}}\overset{G}{\circ}$ *is a F.P., it must be one of the following type.*

(1) $\overset{H}{\circ}\overset{\rho}{\text{——}}\overset{m}{\circ}\text{——}\circ\text{- - - - -}\circ$

(2) $\overset{H}{\circ}\overset{\rho}{\text{——}}\overset{m}{\circ}\text{——}\circ\text{- - - - -}\circ\text{——}\overset{Sp_n}{\circ}$ $(n\geqq 2)$

(3) $\overset{H}{\circ}\overset{\rho}{\text{——}}\overset{m}{\circ}\text{——}\circ\text{- - - - -}\overset{n}{\circ}\overset{\Lambda_2}{\text{——}}\overset{1}{\circ}$ $(n\geqq 4)$

(4) $\overset{H}{\circ}\overset{\rho}{\text{——}}\overset{m}{\circ}\text{——}\circ\text{- - - - -}\circ\text{——}\overset{2}{\circ}\text{——}\overset{Sp_n}{\circ}\text{——}\overset{1}{\circ}$

Moreover, if $m=2$ *and* $\overset{H\ \rho}{\circ}\!\!-\!\!-\!\!\overset{2}{\circ}\!\!-\!\!-\!\!\overset{1}{\circ}$ *is a F.P., then* (1)$\sim$(4) *are actually F.P.'s.*

We can apply the basic theorem to all irreducible F.P.'s except $(SL_n \times GL_m, \Lambda_1 \otimes \Lambda_1)$, $(Sp_n \times GL_m, \Lambda_1 \otimes \Lambda_1)$ and (GL_n, Λ_2). O. Yasukura investigated F.P.'s with these irreducible components. S. Kasai did a lot of calculation for orbital decompositions to prove the finiteness of the number of orbits ([Kasai]).

§ 7. Some generalization of castling transformations and a classification of certain P.V.'s (Y. Teranishi's result)

Let $d_1, \cdots, d_r$ be positive integers and put $n=d_1+\cdots+d_r$. We denote by $GL(d_1, \cdots, d_r)$ the parabolic subgroup of GL_n consisting of all matrices of the form

$$g=\begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1r} \\ 0 & g_{22} & & \\ 0 & \cdots\cdots & 0 & g_{rr} \end{pmatrix} \qquad \text{where } g_{ii} \in GL_{d_i}.$$

Now let ρ be a rational representation of a connected linear algebraic group G on $V=\Omega^m$. Define representations $\rho_1, \rho_2, \rho_1^*, \rho_2^*$ of $G \times GL(d_1, \cdots, d_r)$ on $M_{m,n}$ by $\rho_1(g, a)x=\rho(g)xa^{-1}$, $\rho_2(g, a)x=\rho(g)x{}^ta$, $\rho_1^*(g, a)x={}^t\rho(g)^{-1}x{}^ta$, $\rho_2^*(g, a)x={}^t\rho(g)^{-1}xa^{-1}$ respectively for $g \in G$, $a \in GL(d_1, \cdots, d_r)$ and $x \in M_{m,n}$. Clearly $(G \times GL(d_1, \cdots, d_r), \rho_1, M_{m,n}) \simeq (G \times GL(d_r, \cdots, d_1), \rho_2, M_{m,n})$ and $(G \times GL(d_1, \cdots, d_r), \rho_1^*, M_{m,n}) \simeq (G \times GL(d_r, \cdots, d_1), \rho_2^*, M_{m,n})$.

Proposition 7.1 ([T1]). *For $m>n$, the following conditions are equivalent.*

(1) $(G \times GL(d_1, \cdots, d_r), \rho_1, M_{m,n})$ *is a P.V.*

(2) $(G \times GL(m-n, d_r, \cdots, d_2), \rho_2^*, M_{m,m-d_1})$ *is a P.V.*

When $r=1$, then $GL(d_1, \cdots, d_r)=GL_n$, $GL(m-n, d_r, \cdots, d_2)=GL_{m-n}$, $M_{m,m-d_1}=M_{m,m-n}$ and we obtain Theorem 2.7. Hence Proposition 7.1 is a some generalization of a castling transformation.

Proposition 7.2 ([T1]). *Let G be a connected linear semisimple algebraic group, B_n the group of all $n \times n$ upper triangular matrices, ρ an irreducible n-dimensional rational representation of G. Then a triplet $(G \times B_n, \rho_1, M_n)$ is a P.V. if and only if $(G, \rho)=(SL_n, \Lambda_1)$, (Sp_m, Λ_1) $(n=2m)$, (SO_n, Λ_1).*

Remark 7.3. Y. Yeranishi ([T1] [T2]) investigated P.V.'s in Proposition 7.2 in detail.

§ 8. A classification of certain reductuve P.V.'s (M. Sato's unpublished result I)

Let V be a d-dimensional vector space over Ω and G a connected reductive subgroup of $GL(V)$. Then we have $V=V_1\oplus\cdots\oplus V_m$ with $d_i=\dim V_i$ and $d=d_1+\cdots+d_m$ where G acts on each V_i irreducibly $(1\leqq i\leqq m)$. By a remark above Lemma 3.2, we have $V_\mu=V_{\mu 1}\otimes\cdots\otimes V_{\mu k_\mu}$ with $d_{\mu\nu}=\dim V_{\mu\nu}$ and $d_\mu=d_{\mu 1}\cdots d_{\mu k_\mu}(k_\mu\geqq 0,\ \mu=1,\cdots,m;\ 2\leqq d_{\mu\nu}\leqq d_{11}(=\delta))$ where some simple component of G acts on $V_{\mu\nu}$ irreducibly. Now let G_0 be a connected semisimple subgroup of SL_n. In this situation, let us consider $(G_0\times G,\ \Omega^n\otimes V)$.

Theorem 8.1 (M. Sato). *Assume that* $(G_0\times G,\ \Omega^n\otimes V)$ *is a P.V. Then we have the following assertions.*

(i) *If* $\delta\leqq n\leqq d-\delta$, *then we have* $k_1\leqq 2$.

(ii) *If* $\delta\leqq n\leqq 2d-\delta-1$ *and* $\dim G_0\leqq\frac{1}{2}n(n+1)$, *then we have* $k_1\leqq 2$.

(iii) *If* $\delta\leqq n\leqq 2d-\delta+1$ *and* $\dim G_0\leqq\frac{1}{2}n(n-1)$, *then we have* $k_1\leqq 2$.

Proof. By Lemma 3.1, we have $\dim G_0+\dim G\geqq\dim(\Omega^n\otimes V)=nd$. Since $n^2-1\geqq\dim G_0$ (resp. $\frac{1}{2}n(n+1)\geqq\dim G_0$, $\frac{1}{2}n(n-1)\geqq\dim G_0$) in (i) (resp. (ii), (iii)), we have $\dim G\geqq n(d-n)+1$ (resp. $\frac{1}{2}n(2d-1-n)$, $\frac{1}{2}n(2d+1-n)$). In general, we have $x(a-x)\geqq x_0(a-x_0)$ for $x_0\leqq x\leqq a-x_0$ since $x(a-x)-x_0(a-x_0)=(x-x_0)(a-x_0-x)\geqq 0$. Hence we have $\dim G\geqq\delta(d-\delta)+1$ (resp. $\dim G\geqq\frac{1}{2}\delta(2d\mp 1-\delta)$) in (i) (resp. (ii) (iii)). Let G_μ be the image of $G\to GL(V_\mu)$. Since $G\to G_1\times\cdots\times G_m$ is injective, we have $\dim G\leqq\dim G_1+\cdots+\dim G_m$. Hence we have $N_1+N_2+\cdots+N_m\leqq 0$ for (i) and $N_1^\pm+N_2+\cdots+N_m\leqq 0$ for (ii), (iii) with $N_1=-\dim G_1+\delta(d_1-\delta)+1$, $N_\mu=-\dim G_\mu+\delta d_\mu$ $(2\leqq\mu\leqq m)$ and $N_1^\pm=-\dim G_1+\delta(d_1-\delta)+\frac{1}{2}\delta(\delta\mp 1)$. Since $N_1^-\geqq N_1^+\geqq N_1$, we have $N_1+N_2+\cdots+N_m\leqq 0$ also for (ii), (iii). We shall show that $N_\mu\geqq 0$ $(2\leqq\mu\leqq m)$. When $k_\mu=0$, then $\dim G_\mu=1$, $d_\mu=1$ and $N_\mu=-1+\delta\geqq 0$. When $k_\mu\geqq 1$, we may assume $2\leqq d_{\mu\kappa}\leqq d_{\mu 1}$ without loss of generality. Since $\dim G_\mu\leqq 1+(d_{\mu 1}{}^2-1)+\cdots+(d_{\mu k_\mu}{}^2-1)\leqq 1+k_\mu\cdot(d_{\mu 1}{}^2-1)$ and $d_\mu\geqq d_{\mu 1}\cdot 2^{k_\mu-1}$, we have $N_\mu=-\dim G_\mu+\delta d_\mu\geqq -1-k_\mu\cdot(d_{\mu 1}{}^2-1)+d_{\mu 1}{}^2\cdot 2^{k_\mu-1}=(2^{k_\mu-1}-k_\mu)\cdot d_{\mu 1}{}^2+k_\mu-1$. Since $2^{k-1}-k\geqq 0$ for any $k\geqq 1$, we have $N_\mu\geqq 0$ $(2\leqq\mu\leqq m)$. Now $N_1+N_2+\cdots+N_m\leqq 0$ implies $0\geqq N_1=-\dim G_1+\delta(d_1-\delta)+1\geqq -k_1(\delta^2-1)+\delta^2(2^{k_1-1}-1)=(2^{k_1-1}-1-k_1)\delta^2+k_1$. Thus we have $2^{k_1-1}-1-k_1<0$ and hence $k_1\leqq 2$.
Q.E.D.

Now we shall classify all P.V.'s $(G_0\times G,\ \Omega^n\otimes V)$ when $k_1=2$. First we assume that, by the action of G, the scalar multiplications act on each V_i independently $(1\leqq i\leqq m)$. But as a result (Theorem 8.4), we will see that there is no P.V.'s without this assumption.

Lemma 8.2. *Assume that* $k_1=2$. *Put* $N_1=-\dim G_1+\delta(d_1-\delta)+1$ *and* $N_\mu=-\dim G_\mu+\delta d_\mu$ $(2\leqq\mu\leqq m)$. *Then we have*:

(1) $N_1=-2$ *and* $(G_1, V_1)=(SL_\delta\times GL_2, V(\delta)\otimes V(2))$.

(2) $0\leqq N_\mu\leqq 2$, $0\leqq N_2+\cdots+N_m\leqq 2$, $0\leqq k_\mu\leqq 2$ *and* $(G_\mu, V_\mu)=(GL_1\times SL(d_{\mu1})\times\cdots\times SL(d_{\mu k_\mu}), V(d_{\mu1})\otimes\cdots\otimes V(d_{\mu k_\mu}))$ *for* $2\leqq\mu\leqq m$.

(3) $(G, V)=(G_1, V_1)\oplus\cdots\oplus(G_m, V_m)$, *i.e*, $G=G_1\times\cdots\times G_m$.

Proof. (1) Since $G_1\subset SL_\delta\times GL(d_{12})$, we have $N_1=-\dim G_1+\delta(d_1-\delta)+1\geqq-(\delta^2+d_{12}^2-1)+\delta(\delta d_{12}-\delta)+1=\delta^2(d_{12}-2)-d_{12}^2+2$. If $d_{12}\geqq 3$, then $N_1\geqq\delta^2-d_{12}^2+2\geqq 2>0$. Since $0\geqq N_1$ (See the proof of Theorem 8.1), it is a contradiction, and hence we have $d_{12}=2$, $\dim G_1\leqq\delta^2+3$, and $0\geqq N_1=-\dim G_1+\delta^2+1$, i.e. $\delta^2+3=\dim SL_\delta\times GL_2\geqq\dim G_1\geqq\delta^2+1$. There is no proper non-abelian simple subgroup of SL_n with codimension at most 2, and hence we have $G_1=SL_\delta\times GL_2$ and $N_1=-2$.

(2) In the proof of Theorem 8.1, we have $0\leqq N_\mu$ $(2\leqq\mu\leqq m)$ and $N_1+\cdots+N_m\leqq 0$, i.e., $N_2+\cdots+N_m\leqq 2$. Now $0\leqq\dim GL_1\times SL(d_{\mu1})\times\cdots\times SL(d_{\mu k_\mu})-\dim G_\mu\leqq 1+k_\mu\cdot(d_{\mu1}^2-1)+(N_\mu-\delta d_{\mu1}2^{k_\mu-1})\leqq(3-k_\mu)+d_{\mu1}^2(k_\mu-2^{k_\mu-1})\,(=A)$. If $k_\mu=1$, then $A=2$. If $k_\mu=2$, then $A=1$. If $k_\mu\geqq 3$, then $0\leqq A<0$, i.e., a contradiction. Hence $0\leqq k_\mu\leqq 2$ and $A\leqq 2$. Thus, as (1), we have $G_\mu=GL_1\times SL(d_{\mu1})\times\cdots\times SL(d_{\mu k_\mu})$.

(3) Since $0\leqq\dim G_1\times\cdots\times G_m-\dim G\leqq\dim G_1+\cdots+\dim G_m-\delta(d_1+\cdots+d_m-\delta)-1=-(N_1+N_2+\cdots+N_m)=2-(N_2+\cdots+N_m)\leqq 2$, we have $G=G_1\times\cdots\times G_m$ by the same reason as (1). Q.E.D.

Lemma 8.3. *We have only the following possibilities for* $2\leqq\mu\leqq m$ *where* $N_2+\cdots+N_m\leqq 2$.

(I) $N_\mu=2$, $(G_1, V_1)=(SL_3\times GL_2, V(3)\otimes V(2))$,

(I–1) $(G_\mu, V_\mu)=(GL_1, V(1))$, (I–2) $(G_\mu, V_\mu)=(GL_2, V(2))$.

(II) $N_\mu=1$, $(G_1, V_1)=(SL_2\times GL_2, V(2)\otimes V(2))$,

(II–1) $(G_\mu, V_\mu)=(GL_1, V(1))$, (II–2) $(G_\mu, V_\mu)=(SL_2\times GL_2, V(2)\otimes V(2))$.

(III) $N_\mu=0$, $(G_\mu, V_\mu)=(GL_\delta, V(\delta))$.

Proof. We have $0\leqq k_\mu\leqq 2$ by Lemma 8.2. If $k_\mu=0$, then $N_\mu=-\dim G_\mu+\delta d_\mu=\delta-1\leqq 2$ and hence $\delta=2$, i.e. (II–1) or $\delta=3$, i.e., (I–1). If $k_\mu=1$, then $N_\mu=-d_\mu^2+\delta d_\mu=(\delta-d_\mu)d_\mu\leqq 2$ and hence $\delta=3$, $d_\mu=d_{\mu1}=2$, i.e., (I–2) or $\delta=d_\mu=d_{\mu1}$ i.e., (III). If $k_\mu=2$, then $N_\mu=(\delta-d_{\mu1})d_\mu+(d_{\mu2}-2)d_{\mu1}^2+(d_{\mu1}^2-d_{\mu2}^2)+1\leqq 2$, and hence $\delta=d_{\mu1}=d_{\mu2}=2$, i.e., (II–2). Q.E.D.

Since $N_2+\cdots+N_m\leqq 2$, we have only four possibilities $(N_2,\cdots,N_m)=$(i) $(0,\cdots,0)$, (ii) $(1,0,\cdots,0)$, (iii) $(1,1,0,\cdots,0)$, (iv) $(2,0,\cdots,0)$. We shall check the prehomogeneity of each case.

(i) The case for $(N_2, \cdots, N_m)=(0, \cdots, 0)$.

By Lemma 8.3, we have $(G, V)=(GL_\delta\times SL_2,\ V(\delta)\otimes V(2))\oplus(GL_\delta,\ V(\delta))^{m-1}$. Since $\dim G=m\delta^2+3\geqq n(d-n)+1$ with $d=(m+1)\delta$, we have $2\geqq(n-\delta)(m\delta-n)$ with $\delta\leqq n\leqq m\delta$. Thus we have (i-a) $n=\delta$ or $n=m\delta$, or (i-b) $m=2$; $\delta=2$ with $n=3$ or $\delta=3$ with $n=4, 5$. In the case (i-a), we have $\dim G_0\geqq\dim SL_n-2$ by Lemma 3.1, and hence $G_0=SL_n$ (cf. the proof of (1) in Lemma 8.2). Hence, to check the prehomogeneity, we may assume that $n=\delta$ since they are castling-equivalent. Then, by Theorem 2.5, the prehomogeneity of $(G_0\times G,\ \Omega^n\otimes V)$ reduces to that of an irreducible triplet $(SL_\delta\times SL_\delta\times GL_2,\ \Lambda_1\otimes\Lambda_1\otimes\Lambda_1)$. By Section 3, it is a regular P.V. for $\delta=2, 3$ and a non P.V. for $\delta\geqq 4$. In the case (i-b), we have also $G_0=SL_n$ similarly as (i-a). Hence, by a castling transformation, we may assume that $n=\delta+1$ with $\delta=2, 3$. If we denote by $\sim$ a castling-equivalence, we have $(G_0\times G,\ \Omega^n\otimes V)=(SL(\delta+1)\times(GL_\delta\times SL_2\times GL_\delta),\ V(\delta+1)\otimes(V(\delta)\otimes V(2)+V(\delta)))\sim(SL(\delta+1)\times(GL_\delta\times SL_2\times GL_1),\ V(\delta+1)\otimes(V(\delta)\otimes V(2)+V(1))\sim(SL_\delta\times(GL_\delta\times SL_2\times GL_1),\ V(\delta)\otimes(V(\delta)\otimes V(2)+V(1))=(GL_\delta\times SL_\delta\times GL_2,\ \Lambda_1\otimes\Lambda_1\otimes\Lambda_1+\Lambda_1\otimes 1\otimes 1)$ $(\delta=2, 3)$. They are regular P.V.'s (cf. p. 99 in [S-K] for $\delta=3$ and (3) with $n=4$, $m=2$ in Theorem 5.1 for $\delta=2$). Note that $(SL_2\times SL_2,\ \Lambda_1\otimes\Lambda_1)\simeq(SO_4,\ \Lambda_1)$.

(ii) The case for $(N_2, \cdots, N_m)=(1, 0, \cdots, 0)$.

By Lemma 8.3, we have $\delta=2$. In our case, $\dim G\geqq n(d-n)+1$ (cf. the proof of Theorem 8.1) implies that $(n-2)\ (d-2-n)\leqq 1$. Hence we have $n=2$ or $n=d-2$. Note that $n=3$ and $d=6$ is not a solution because d is odd or $d\geqq 8$. If $n=2$, we have $G_0=SL_2$. If $n=d-2$, $\dim G_0+\dim G=\dim G_0+(2d-2)\geqq(d-2)d$ implies $\dim G_0\geqq(d-2)^2-2$ and hence $G_0=SL(d-2)$. By a castling transformation, we may assume that $G_0=SL_2$, and by Theorem 2.5, we may assume that $m=2$. When $(G_2, V_2)=(SL_2\times GL_2,\ V(2)\otimes V(2))$, it is not a P.V. Because GL_2-part of a generic isotropy subgroup of $(SL_2\times SL_2\times GL_2,\ \Lambda_1\otimes\Lambda_1\otimes\Lambda_1)=(SO_4\times GL_2,\ \Lambda_1\otimes\Lambda_1)$ is O_2 (p. 100 in [S-K]) and hence the prehomogeneity implies that of $(GL_1^2\times SO_4,\ \Lambda_1\oplus\Lambda_1)$, which is a non P.V. (Theorem 4.2). When $(G_2, V_2)=(GL_1,\ V(1))$ i.e. (II–1), it is a regular P.V. since $(GL_1\times SO_2,\ \Lambda_1)$ is a P.V. with a reductive generic isotropy subgroup.

(iii) The case for $(N_2, \cdots, N_m)=(1, 1, 0, \cdots, 0)$ with $m\geqq 3$.

Similarly as before, we have $(n-2)\ (d-2-n)\leqq 1$ and hence $n=2$ or $d-2$. We have $G_0=SL_n$ and hence we may assume that $n=2$ and $m=3$. If one of (G_μ, V_μ) $(\mu=2, 3)$ is (II–2) $(SL_2\times GL_2,\ V(2)\otimes V(2))$, it is not a P.V. by (ii). Hence $(G_\mu, V_\mu)=(GL_1,\ V(1))$ $(\mu=2, 3)$. But it is not a P.V. either. Because the projection to Lie (G_0) of a generic isotropy subalgebra of $(G_0\times G_1\times G_2,\ V_0\otimes(V_1+V_2))$ is $\{0\}$, its prehomogeneity implies that of $(GL_1,\ V(2))$, which is a contradiction.

(iv) The case for $(N_2, \cdots, N_m)=(2, 0, \cdots, 0)$.

In this case, we have $\delta=3$ by Lemma 8.3. Similarly as above, we have $(n-3)(d-3-n)\leqq 0$ $(3\leqq n\leqq d-3)$ and hence $n=3$ or $d-3$. We have $G_0=SL_n$ in this case by dimension reason, and hence we may assume $n=3$, and $m=2$. Also we may assume that $(G_2, V_2)=(GL_1, V(1))$ by a castling transformation. In this case, as we saw in (i), it is a regular P.V.

Thus we obtain the following theorem.

Theorem 8.4 (M. Sato). *Assume that $(G_0\times G, \Omega^n\otimes V)$ is a P.V. with $\delta\leqq n\leqq d-\delta$ and $k_1=2$ (cf. Theorem 8.1). Then it is one of the following regular P.V.'s.*

(1) $(SL_n\times((GL_\delta\times SL_2)\times GL_\delta^{m-1}), \Omega^n\otimes(V(\delta)\otimes V(2)+V(\delta)^{m-1}))$ $(m\geqq 1;\ n=\delta$ *or* $d-\delta;\ \delta=2, 3)$.

(2) $(SL_n\times((GL_\delta\times SL_2)\times GL_\delta), \Omega^n\otimes(V(\delta)\otimes V(2)+V(\delta)))$ $(\delta=2, n=3; \delta=3, n=4, 5)$.

(3) $(SL_n\times((GL_3\times SL_2)\times GL_k\times GL_3^{m-2}), \Omega^n\otimes(V(3)\otimes V(2)+V(k)+V(3)^{m-2}))$ $(m\geqq 2;\ n=3$ *or* $d-3$: $k=1$ *or* $2)$.

(4) $(SL_n\times((GL_2\times SL_2)\times GL_1\times GL_2^{m-2}), \Omega^n\otimes(V(2)\otimes V(2)+V(1)+V(2)^{m-2}))$ $(m\geqq 2;\ n=2$ *or* $d-2)$.

Remark. Although $\Omega^n=V(n)$, we use Ω^n as a representation space of G_0 to distinguish from that of G.

§9. Prehomogeneity of some reductive triplets

The starting point to prove Theorem 5.4 was to show that a triplet $(GL_1^k\times SL_m\times SL_n,\ \overbrace{\Lambda_1\otimes\Lambda_1+\cdots+\Lambda_1\otimes\Lambda_1}^{k})$ $(m\neq n)$ is a P.V. if and only if $\dim G\geqq\dim V$, i.e., $k+m^2+n^2\geqq kmn+2$, and when $\dim G>\dim V$, it is transformed to a trivial P.V. by j-times castling transformations where $j=\Psi(k, m, n)$ in Definition 5.3 (Theorem 4.5 in [K-K-T-I]). With this in mind, we shall consider the triplet $(GL_1^k\times SL_{m_1}\times\cdots\times SL_{m_t},\ \overbrace{(\overbrace{\Lambda_1\otimes\cdots\otimes\Lambda_1}^{t})+\cdots+(\overbrace{\Lambda_1\otimes\cdots\otimes\Lambda_1}^{t})}^{k})$ with $m_1\geqq\cdots\geqq m_t\geqq 2$, $k\geqq 2$ and $t\geqq 3$. If $m_1\geqq km_2\cdots m_t$, then it is clearly a trivial P.V. Hence we shall investigate its prehomogeneity when $km_2\cdots m_t>m_1$ $(\geqq m_2)$. If it is a P.V., we have $\dim G\geqq\dim V$ by Lemma 3.1 and hence we have $f(m_1)\geqq 0$ where $f(x)=x^2-(km_2\cdots m_t)x+(m_2^2+\cdots+m_t^2+k-t)$. Its discriminant $D=(km_2\cdots m_t)^2-4(m_2^2+\cdots+m_t^2+k-t)$ is positive because $D=(k^2-4)m_2^2\cdots m_t^2+4t$

$-4k+4(m_2^2\cdots m_t^2-m_2^2-\cdots-m_t^2)\geqq 16(k^2-4)+12-4k+4(4^{t-2}m_2^2-(t-1)m_2^2)\geqq 4\{k(4k-1)-13\}+16(4^{t-2}-t)+16\geqq 4+16+16>0$. We shall show that $m_1\geqq\frac{1}{2}\{(km_2\cdots m_t)+\sqrt{D}\}$. For this purpose, it is enough to show $\frac{1}{2}\{(km_2\cdots m_t)-\sqrt{D}\}<m_2$ $(\leqq m_1)$. L.H.S. $=\{(km_2\cdots m_t)^2-D\}/\{2(km_2\cdots m_t+\sqrt{D})\}=2(m_2^2+\cdots+m_t^2+k-t)/(km_2\cdots m_t+\sqrt{D})\leqq 2\{(t-1)m_2^2+k-t\}/2^{t-2}km_2<m_2$ if and only if $(A=)(2^{t-3}k-t+1)m_2^2+t-k>0$. However we have $A\geqq 4(2^{t-3}k-t)+4+t-k=(2^{t-1}-1)k-3t+4\geqq 2^t-3t+2>0$.

Lemma 9.1. *If* $\dim G\geqq\dim V$ *and* $km_2\cdots m_t>m_1$ *with* $t\geqq 3$, *then we have* $(m_1'=)km_2\cdots m_t-m_1<m_2$.

Proof. Assume that $m_1'\geqq m_2$. Then we have $km_2\cdots m_t-m_2\geqq m_1\geqq\frac{1}{2}\{(km_2\cdots m_t)+\sqrt{D}\}$, i.e., $km_2\cdots m_t-2m_2\geqq\sqrt{D}$. Hence we have $m_2^2+(m_2^2+\cdots+m_t^2+k-t)\geqq m_2^2(km_3\cdots m_t)$. Since $m_2\geqq\cdots\geqq m_t\geqq 2$, we have $tm_2^2+k-t\geqq 2^{t-2}\cdot km_2^2$ i.e., $0\geqq(2^{t-2}k-t)m_2^2-k+t\geqq 4(2^{t-2}k-t)-k+t=(2^t-1)k-3t\geqq 2^{t+1}-3t-2>0$ for $t\geqq 3$, which is a contradiction, and hence $m_1'<m_2$. Q.E.D.

Theorem 9.2 (T. Kimura). *Let* (G,ρ,V) *be a triplet such that* $G=GL_1^k\times SL_{m_1}\times\cdots\times SL_{m_t}$, $\rho=\overbrace{(\Lambda_1\otimes\cdots\otimes\Lambda_1)}^{t}\overbrace{+\cdots+}^{k}\overbrace{(\Lambda_1\otimes\cdots\otimes\Lambda_1)}^{t}$, $V=\overbrace{V(m_1\cdots m_t)\oplus\cdots\oplus V(m_1\cdots m_t)}^{k}$ *with* $m_1\geqq\cdots\geqq m_t\geqq 2$, $t\geqq 3$ *and* $k\geqq 2$.

(1) *If* $\dim G>\dim V$, *then it is castling-equivalent to a trivial P.V.*

(2) *If* $\dim G=\dim V$ *and* $k\geqq 3$, *it is castling-equivalent to a regular simple P.V.* $(GL_1^k\times SL_{k-1}, \overbrace{\Lambda_1\oplus\cdots\oplus\Lambda_1}^{k}, M_{k-1,k})$.

(3) *If* $\dim G=\dim V$ *and* $k=2$ (*such as* $m_1=7$, $m_2=m_3=2$, $t=3$), *it is not a P.V.*

Proof. First note that the number k and $\dim G-\dim V$ are invariant under castling transformations. We denote this (G,ρ,V) by $T(m_1,\cdots,m_t)$ with $m_1\geqq\cdots\geqq m_t\geqq 2$. (1) If it is not a trivial P.V., it is castling-equivalent to some $T(n_1,\cdots,n_s)$ with $n_1=m_2\geqq n_2\geqq\cdots\geqq n_s$ and $t\geqq s$ by Lemma 9.1. If $s\leqq 2$, it is castling-equivalent to a trivial P.V. (cf. Theorem 4.5 in [K-K-T-I]). If it is not a trivial P.V. with $s\geqq 3$, we can use Lemma 9.1 again. Repeating this procedure, finally we obtain our result. Note that $T(m_1,m_2)$ with $\dim G>\dim V$ implies $m_1\neq m_2$. (2) (3) First we show that $T(m_1,\cdots,m_t)(m_1\geqq\cdots\geqq m_t\geqq 2,\ t\geqq 2)$ with $\dim G=\dim V$ cannot be a trivial P.V. In fact, if $m_1\geqq km_2\cdots m_t$ and $k-t+m_1^2+\cdots+m_t^2=km_1\cdots m_t$ $(\leqq m_1^2)$, we have $4(t-1)\leqq(t-1)m_t^2\leqq m_2^2+\cdots+m_t^2\leqq t-k\leqq t-2$, i.e., $3t\leqq 2$, which is a contradiction. Hence, by Lemma 9.1, it is castling-equivalent to $T(m_1, \cdots, m_t)$ with $t \leqq 2$. $T(m_1, m_2)$ with $\dim G = \dim V$ and $k \geqq 3$ implies $m_1 \neq m_2$ and hence, by Theorem 4.5 in [K-K-T-I], we have (2). $T(m_1, m_2)$ with $\dim G = \dim V$ and $k=2$ implies $m_1 = m_2$, which is not a P.V. If $\dim G = \dim V$ and $k=2$, only $(GL_1^2, V(1)^2)$ is a P.V. which is not castling-equivalent to any $T(m_1, \cdots, m_t)$ with $t \geqq 3$. Q.E.D.

Proposition 9.3. *A triplet* $(GL_1^2 \times SL_{m_1} \times \cdots \times SL_{m_t}, \overbrace{\Lambda_1 \otimes \cdots \otimes \Lambda_1}^{t} + \Lambda_1^* \otimes \overbrace{\Lambda_1^{(*)} \otimes \cdots \otimes \Lambda_1^{(*)}}^{t-1}, V(m_1 \cdots m_t) \oplus V'(m_1 \cdots m_2))$ *with* $m_1 \geqq \cdots \geqq m_t \geqq 2$ *and* $t \geqq 2$, *is not a P.V.*

Proof. Assume that it is a P.V. If $2m_2 \cdots m_t > m_1$, then we have $2m_2 \cdots m_t - m_1 < m_2$ by Lemma 9.1 $(t \geqq 3)$, which is concerning only the dimensions of the representation spaces. Hence $m_1 > 2m_2 \cdots m_t - m_2 > m_2 \cdots m_t$. In any case, we have $m_1 \geqq m_2 \cdots m_t$. By Theorem 2.8, $(GL_1 \times SL_{m_2} \times \cdots \times SL_{m_t}, (\overbrace{\Lambda_1 \otimes \cdots \otimes \Lambda_1}^{t-1}) \otimes (\overbrace{\Lambda_1^{(*)} \otimes \cdots \otimes \Lambda_1^{(*)}}^{t-1}))$ must be a P.V. and hence $1 + (m_2^2 - 1) + \cdots + (m_2^2 - 1) \geqq (m_2 \cdots m_t)^2$. Thus we have $(t-1)m_2^2 - (t-2) \geqq m_2^2 \cdot 4^{t-2}$, i.e. $0 \geqq t - 2 + m_2^2(4^{t-2} - t + 1) > 0$ $(t \geqq 3)$, which is a contradiction. When $t=2$, it is known (cf. § 5). Q.E.D.

Theorem 9.4 (T. Kimura). *Assume that* $(GL_1^k \times SL_{m_1} \times \cdots \times SL_{m_t}, \overbrace{\Lambda_1^{(*)} \otimes \cdots \otimes \Lambda_1^{(*)}}^{t} \overbrace{+ \cdots +}^{k} \overbrace{\Lambda_1^{(*)} \otimes \cdots \otimes \Lambda_1^{(*)}}^{t})$ $(k \geqq 2, t \geqq 3)$ *is a P.V. Then it is castling-equivalent to one of* (1) *a trivial P.V.* (2) $(GL_1^k \times SL(k-1), \overbrace{\Lambda_1 \oplus \cdots \oplus \Lambda_1}^{k})$ (3) $(GL_1^k \times SL_m, \overbrace{\Lambda_1 \oplus \cdots \oplus \Lambda_1}^{k-1} \oplus \Lambda_1^*)$ $(k-1 \leqq m)$.

Proof. We may assume that $\rho = \overbrace{\Lambda_1 \otimes \cdots \otimes \Lambda_1}^{t} + \overbrace{\Lambda_1^{(*)} \otimes \cdots \otimes \Lambda_1^{(*)}}^{t} \overbrace{+ \cdots +}^{k-1} \overbrace{\Lambda_1^{(*)} \otimes \cdots \otimes \Lambda_1^{(*)}}^{t}$ with $m_1 \geqq \cdots \geqq m_t \geqq 2$. If one of $\Lambda_1^{(*)}$ for SL_{m_1} is Λ_1^*, then it is not a P.V. by Proposition 9.3, i.e., $\rho = \Lambda_1 \otimes (\overbrace{\Lambda_1 \otimes \cdots \otimes \Lambda_1}^{t-1} + \overbrace{\Lambda_1^{(*)} \otimes \cdots \otimes \Lambda_1^{(*)}}^{t-1} \overbrace{+ \cdots +}^{k-1} \overbrace{\Lambda_1^{(*)} \otimes \cdots \otimes \Lambda_1^{(*)}}^{t-1})$. If it is not trivial P.V., then by Lemma 9.1, it is castling-equivalent to $(GL_1^k \times SL_{m'_1} \times \cdots \times SL_{m'_t}, \overbrace{\Lambda_1^{(*)} \otimes \cdots \otimes \Lambda_1^{(*)}}^{t} \overbrace{+ \cdots +}^{k} \overbrace{\Lambda_1^{(*)} \otimes \cdots \otimes \Lambda_1^{(*)}}^{t})$ with $m_2 = m'_1 \geqq \cdots \geqq m'_t \geqq 1$. Hence if $m'_2 \geqq 2$, we may assume that any $\Lambda_1^{(*)}$ for $SL_{m'_1} = SL_{m_2}$ is Λ_1 by Proposition 9.3. Repeating this procedure, we have our result by Theorem 9.2.

Note that $(GL_1^k \times SL_m, \overbrace{\Lambda_1 \oplus \cdots \oplus \Lambda_1}^{k})$ $(m \geqq k)$ is a trivial P.V. Q.E.D.

Remark 9.5. In Theorems 9.2 and 9.4, the case for $k=1$ (resp. $t=1$, $t=2$) has been treated as an irreducible P.V. (resp. a simple P.V., a 2-simple P.V.).

Now recall that $((GL_{m_1}\times\cdots\times GL_{m_k})\times GL_n,\ (V(m_1)\oplus\cdots\oplus V(m_k))\otimes V(n))$ is a F.P. (and hence P.V. See Section 6) for any $m_1, \cdots, m_k, n$ with $1\leqq k\leqq 3$ ([G], [Kac 3], [K-K-Y]). We shall study when $k=4$. First we shall prove the following Theorem.

Theorem 9.6. *$((GL_{m_1}\times GL_{m_2})\times(GL_{n_1}\times GL_{n_2}),\ (V(m_1)\oplus V(m_2))\otimes(V(n_1)\oplus V(n_2)))$ is a P.V. if and only if $m_1+m_2\neq n_1+n_2$.*

Proof. If $m_1+m_2\neq n_1+n_2$, we may assume that $m_1+m_2<n_1+n_2$. If $m_1+m_2\leqq n_1$ or $m_1+m_2\leqq n_2$, it is a P.V. by Theorem 2.5 and the above result. If $m_1+m_2>n_i$ $(i=1, 2)$, it is castling-equivalent to $((GL_{m_1}\times GL_{m_2})\times(GL_{n'_1}\times GL_{n'_2}),\ (V(m_1)\oplus V(m_2))\otimes(V(n'_1)\oplus V(n'_2)))$ with $n'_1+n'_2<m_1+m_2$ where $n'_i=m_1+m_2-n_i$ $(i=1, 2)$. Repeating this procedure, we have our result for $m_1+m_2\neq n_1+n_2$. To prove the case $m_1+m_2=n_1+n_2$, first we prove two lemmas.

Lemma 9.7. *$((GL_m\times GL_n)\times(GL_m\times GL_n),\ (V(m)\oplus V(n))\otimes(V(m)\oplus V(n)))$ is not a P.V.*

Proof. For $m\leqq n$; $A_1, A_2\in GL_m$; $B_1, B_2\in GL_n$; $X\in M_m$, $Y\in M_n$, $Z\in M_{m,n}$ and $W\in M_{n,m}$, the prehomogeneity of $(X, Y, Z, W)\to(A_1X{}^tA_2, B_1Y{}^tB_2, A_1Z{}^tB_2, B_1W{}^tA_2)$ reduces to that of $(Z, W)\to(A_1Z{}^tB_1{}^{-1}, B_1{}^{-1}WA_1{}^{-1})$ which is not a P.V. since it has a non-constant absolute invariant $\det(ZW)$ (cf. Theorem 1.2).

Lemma 9.8. *$((GL_m\times GL_m)\times(GL_{n_1}\times GL_{n_2}), (\rho=)\Lambda_1\otimes 1\otimes(\Lambda_1\otimes 1+1\otimes\Lambda_1)+1\otimes\Lambda_1\otimes(\Lambda_1^*\otimes 1+1\otimes\Lambda_1^*))$ $(m\leqq n_1, n_2)$ is not a P.V.*

Proof. For $g=(A, A', B_1, B_2)\in GL_m\times GL_m\times GL_{n_1}\times GL_{n_2}$ and $x=(X, Y, Z, W)\in M_{m,n_1}\oplus M_{m,n_2}\oplus M_{m,n_1}\oplus M_{m,n_2}$, we have $\rho(g)x=(AX{}^tB_1, AY{}^tB_2, A'ZB_1{}^{-1}, A'WB_2{}^{-1})$ and hence $f(x)=\det(X{}^tZ)/\det(Y{}^tW)$ is a non-constant absolute invariant. Q.E.D. for Lemma 9.8.

Now we shall prove Theorem 9.6 for $m_1+m_2=n_1+n_2$. We may assume that $m_1\leqq m_2$ and $n_1\leqq n_2$. If $m_1=n_1$, then $m_2=n_2$, and hence it reduces to Lemma 9.7. If $m_1<n_1\leqq n_2<m_2$, since $(n_1+n_2)-m_2=m_1$, it reduces to Lemma 9.8 by a castling transformation. Q.E.D. for Theorem 9.6.

Corollary 9.9. *For any algebraic group G, a triplet $(G, (\sigma_1+\cdots+\sigma_s)\otimes(\tau_1+\cdots+\tau_t),\ (V(m_1)\oplus\cdots\oplus V(m_s))\otimes(V(n_1)\oplus\cdots\oplus V(n_t)))$ with $m_1+\cdots+m_s=n_1+\cdots+n_t$ $(s\geqq 2, t\geqq 2)$ is not a P.V.*

Proof. If it is P.V., then $((GL(m_1)\times GL(m_2+\cdots+m_s))\times(GL(n_1)\times GL(n_2+\cdots+n_t)),\ (V(m_1)\oplus V(m_2+\cdots+m_s))\otimes(V(n_1)\oplus V(n_2+\cdots+n_t)))$ $(m_1+(m_2+\cdots+m_s)=n_1+(n_2+\cdots+n_t))$ must be a P.V., which is a contradiction. Q.E.D.

Theorem 9.10. *$((GL_{m_1}\times GL_{m_2}\times GL_{m_3}\times GL_{m_4})\times GL_n,\ (V(m_1)\oplus V(m_2)\oplus V(m_3)\oplus V(m_4))\otimes V(n))$ is a P.V. if and only if $m_1+m_2+m_3+m_4\neq 2n$ or $n\leqq \max\{m_i\}$.*

Proof. Assume that $m_1+m_2+m_3+m_4=2n$ with $n>m_1\geqq m_2\geqq m_3\geqq m_4$. Then we have $m_1'+m_2'=m_3+m_4\leqq n$ with $m_i'=n-m_i$ and it is castling-equivalent to $((GL_{m_1'}\times GL_{m_2'})\times(GL_{m_3}\times GL_{m_4})\times GL(n),\ (V(m_1')+V(m_2'))\otimes V(n)^*+(V(m_3)+V(m_4))\otimes V(n))$, which is P.V.-equivalent to $((GL_{m_1'}\times GL_{m_2'})\times(GL_{m_3}\times GL_{m_4}),\ (V(m_1')+V(m_2'))\otimes(V(n_3)\oplus V(n_4)))$ by Theorem 2.8, which is not a P.V. by Theorem 9.6. If some $m_i\geqq n$, it reduces to the case $k=3$ by Theorem 2.5 and hence it is a P.V. Now assume that $m_1+m_2+m_3+m_4\neq 2n$ and $n>\max\{m_i\}$. If $n\geqq m_1+m_2+m_3+m_4$, it is a trivial P.V., and if $m_1+m_2+m_3+m_4>n$, we may assume that $m_1+m_2+m_3+m_4>2n$ by a castling transformation. Put $m_i'=n-m_i$ $(1\leqq i\leqq 4)$. Then we have $m_1'+\cdots+m_4'=4n-(m_1+\cdots+m_4)<4n-2n=2n<m_1+\cdots+m_4$, and it is castling-equivalent to $((GL_{m_1'}\times\cdots\times GL_{m_4'})\times GL_n,\ (V(m_1')+\cdots+V(m_4'))\otimes V(n))$. If $m_1'+\cdots+m_4'\leqq n$, it is a trivial P.V. If $m_1'+\cdots+m_4'>n$, we have $n>n'=m_1'+\cdots+m_4'-n$ and $m_1'+\cdots+m_4'>2n'$. Repeating this procedure, we have our result. Q.E.D.

§ 10. P.V.'s of associative algebras (M. Sato's unpublished result II)

First we recall the definition of quasi-regularity of P.V.'s. Let (G, ρ, V) be a P.V. with the Zariski-dense orbit $Y=V-S$. Let G_1 be a subgroup of G generated by the commutator subgroup $[G, G]$ and a generic isotropy subgroup G_x $(x\in Y)$. This does not depend on a choice of a generic point x, and a rational character χ of G corresponds to some relative invariant if and only if it annihilates G_1 and the rank N of the character group of G/G_1 coincides with the number of basic invariants. Let $\mathfrak{g}$ (resp. $\mathfrak{g}_1$) be the algebra of G (resp. G_1) and $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$. Then we have the following lemma.

Lemma 10.1 (Lemma 1.1 in [S-K-K-O]). *For $\omega\in\mathfrak{g}^*$, there exists a rational map $\Psi_\omega\colon Y\to V^*$ satisfying (1) $\Psi_\omega(\rho(g)X)=\rho^*(g)\Psi_\omega(x)$ for $g\in G$, $x\in Y$, (2) $\langle\Psi_\omega(x), d\rho(A)x\rangle=\omega(A)$ for all $x\in Y$ and $A\in\mathfrak{g}$ if and only if $\omega(\mathfrak{g}_1)=0$.*

Definition 10.2. A P.V. (G, ρ, V) is called *quasi-regular* if there exists

$\omega \in (\mathfrak{g}/\mathfrak{g}_1)^*$ such that $\Psi_\omega: Y \to V^*$ is dominant. If there exists a rational character χ corresponding to a relative invariant f such that $\Psi_{\delta\chi} = \text{grad} \log f$ is dominant (i.e., f=non-degenerate), we say that (G, ρ, V) is regular. Hence regularity implies quasi-regularity. As we shall see, the converse is not true. However if G is reductive, it is equivalent.

Definition 10.3. Let $\mathscr{A}$ be a finite-dimensional associative algebra over $\boldsymbol{C}$ with the identity 1, and $\mathscr{A}^\times$ the multiplicative group of all invertible elements in $\mathscr{A}$. Then $G = \mathscr{A}^\times$ acts on $V = \mathscr{A}$ by $\rho(a)b = ab$ for $a \in G$, $b \in V$. Clearly the triplet (G, ρ, V) is a P.V. which is called *the P.V. of the associative algebra* $\mathscr{A}$. Let $\mathscr{A}^*$ be the dual vector space of $\mathscr{A}$. Then $\mathscr{A}^*$ becomes bi-$\mathscr{A}$-module by $\langle axb, y\rangle = \langle x, bya\rangle$ for $a, x, b \in \mathscr{A}$ and $y \in \mathscr{A}^*$.

Definition 10.4. We call $\mathscr{A}$ a *Frobenius algebra* when there exists an isomorphism $\Psi: \mathscr{A} \to \mathscr{A}^*$ satisfying $\Psi(ab) = a\Psi(b)$ for all $a, b \in \mathscr{A}$. This map Ψ induces the adjoint map $\Psi^*: (\mathscr{A}^*)^* = \mathscr{A} \to \mathscr{A}^*$. We call $\mathscr{A}$ *a symmetric algebra* if there exists Ψ satisfying $\Psi = \Psi^*$ and $\Psi(ab) = a\Psi(b)$ for all $a, b \in \mathscr{A}$. If we put $B(a, b) = \langle a, \Psi(b)\rangle$ and $y_0 = \Psi(1)$, then we have $B(a, b) = \langle ab, y_0\rangle$. Hence $\mathscr{A}$ is a Frobenius (resp. symmetric) algebra if and only if there exists $y_0 \in \mathscr{A}^*$ such that the bilinear form $\langle ab, y_0\rangle$ on $\mathscr{A}$ is non-degenerate (resp. non-degenerate and symmetric).

The remaining part of this section will be devoted to prove the following unpublished work of M. Sato around 1962.

Theorem 10.5 (M. Sato). *Let (G, ρ, V) be a P.V. of an associative algebra $\mathscr{A}$.*

(1) *The dual triplet (G, ρ^*, V^*) is a P.V. if and only if $\mathscr{A}$ is a Frobenius algebra.*

(2) *The triplet (G, ρ, V) is a quasi-reqular P.V. if and only if $\mathscr{A}$ is a symmetric algebra.*

(3) *The triplet (G, ρ, V) is a regular P.V. if and only if $\mathscr{A}$ is a semi-simple algebra.*

To prove this theorem, we shall prove several lemmas.

Lemma 10.6. *$\mathscr{A}$ is a Frobenius algebra if and only if there exists an element y_0 of $\mathscr{A}^*$ satisfying $\mathscr{A}^* = \mathscr{A}y_0$.*

Proof. Let $\mathscr{A}$ be a Frobenius algebra, and $\Psi: \mathscr{A} \to \mathscr{A}^*$ be a left $\mathscr{A}$-module isomorphism. Put $y_0 = \Psi(1)$. Then we have $\mathscr{A}^* = \Psi(\mathscr{A}) = \mathscr{A}y_0$ since $ay_0 = \Psi(a)$ for all $a \in \mathscr{A}$· Conversely, if $\mathscr{A}^* = \mathscr{A}y_0$, then $\Psi(a) = ay_0$ $(a \in \mathscr{A})$ is a left $\mathscr{A}$-module surjective homomorphism. Since $\dim \mathscr{A} =$

dim $\mathscr{A}^*$, it is injective.

Lemma 10.7. *For* $y_0 \in \mathscr{A}^*$, *we have* $\mathscr{A}^* = \mathscr{A} y_0$ *if and only if* $\mathscr{A}^* = y_0 \mathscr{A}$.

Proof. We have $B(a, a') = \langle a, a' y_0 \rangle = \langle a', y_0 a \rangle$ and this is non-degenerate if and only if $\det (B(a_i, a_j)) \neq 0$ where $\{a_1, \cdots, a_n\}$ is a basis of $\mathscr{A}$ over $\boldsymbol{C}$. Hence we have $\mathscr{A}^* = \mathscr{A} y_0 \Leftrightarrow B(a, a') = 0$ for all $a' \in \mathscr{A}$ implies $a = 0 \Leftrightarrow \det B(a_i, a_j)) \neq 0 \Leftrightarrow B(a, a') = 0$ for all $a \in \mathscr{A}$ implies $a' = 0 \Leftrightarrow \mathscr{A}^* = y_0 \mathscr{A}$. Q.E.D.

Now we ready to prove (1) of Theorem 10.5. Since ρ^* is defined by $\langle gx, \rho^*(g) y \rangle = \langle x, y \rangle$ for all $x \in \mathscr{A}$ and $y \in \mathscr{A}^*$, we have $\langle x, \rho^*(g) y \rangle = \langle g^{-1} x, y \rangle = \langle x, y g^{-1} \rangle$ for all $x \in \mathscr{A}$, i.e., $\rho^*(g) y = y g^{-1}$ for $y \in \mathscr{A}^*$ and $g \in G = \mathscr{A}^\times$. Hence the dual triplet (G, ρ^*, V^*) is a P.V. if and only if there exists an element y_0 in $\mathscr{A}^*$ such that $\rho^*(G) y_0 = y_0 \mathscr{A}^\times$ is dense in $V^* = \mathscr{A}^*$. Since $y_0 \mathscr{A}^\times \subset y_0 \mathscr{A}$ and $\mathscr{A}^\times$ is dense in $\mathscr{A}$, $y_0 \mathscr{A}^\times$ is dense in $\mathscr{A}^*$, if and only if $y_0 \mathscr{A} = \mathscr{A}^*$, i.e., $\mathscr{A}$ is a Frobenius algebra by Lemmas 10.6 and 10.7. This proves (1).

As we have seen above, $\mathscr{A}$ is a Frobenius algebra if and only if $B(a, a') = \langle a a', y_0 \rangle$ $(a, a' \in \mathscr{A})$ is non-degenerate for some $y_0 \in \mathscr{A}^*$. Moreover, $\mathscr{A}$ is a symmetric algebra if and only if B is a non-degenerate symmetric bilinear form for some y_0.

Let us prove (2) of Theorem 10.5. First assume that (G, ρ, V) is quasi-regular. Then by the definition and Lemma 10.1, there exists y_0 in $\mathfrak{g}^* = \mathscr{A}^*$ and a dominant G-admissible rational map $\Psi\colon V - S = \mathscr{A}^\times \to V^* = \mathscr{A}^*$ satisfying $\langle ax, \Psi(x) \rangle = \langle a, y_0 \rangle$ for any $a \in \mathfrak{g} = \mathscr{A}$ and $x \in V - S = \mathscr{A}^\times$. Moreover, by Lemma 10.1, we have $\langle \mathfrak{g}_1, y_0 \rangle = 0$ and hence $\langle ab, y_0 \rangle = \langle ba, y_0 \rangle$ for any $a, b \in \mathfrak{g} = \mathscr{A}$. On the other hand, we have $\Psi(1) = y_0$ because $\langle a \cdot 1, \Psi(1) \rangle = \langle a, y_0 \rangle$ for all $a \in \mathscr{A}$. This implies that $\Psi(\mathscr{A}^\times) = y_0 \mathscr{A}^\times$ is dense in $\mathscr{A}^*$, and hence $y_0 \mathscr{A} = \mathscr{A}^*$. Thus $\mathscr{A}$ is a symmetric algebra. Conversely, assume that $\mathscr{A}$ is a symmetric algebra. Then there exists y_0 in $\mathscr{A}^*$ such that $y_0 \mathscr{A} = \mathscr{A}^*$ and $\langle ab, y_0 \rangle = \langle ba, y_0 \rangle$ for any $a, b \in \mathscr{A}$. Define a map $\Psi\colon \mathscr{A}^\times \to \mathscr{A}^*$ by $\Psi(g) = y_0 g^{-1}$ for $g \in \mathscr{A}^\times$. Then Ψ is clearly G-admissible and dominant. By Lemma 10.1, it is enough to show that $\langle ax, \Psi(x) \rangle = \langle a, y_0 \rangle$ for any $x \in V - S = \mathscr{A}^\times$ and $a \in \mathfrak{g} = \mathscr{A}$. However, it is obvious since $\langle ax, \Psi(x) \rangle = \langle ax, y_0 x^{-1} \rangle = \langle x^{-1}(ax), y_0 \rangle = \langle (ax) x^{-1}, y_0 \rangle = \langle a, y_0 \rangle$. This proves (2).

Finally we shall prove (3) of Theorem 10.5. Assume that $\mathscr{A}$ is semi-simple. Then we have $\mathscr{A} = M(m_1, \boldsymbol{C}) \oplus \cdots \oplus M(m_k, \boldsymbol{C})$ where $M(m_i, \boldsymbol{C})$ denotes the totality of $m_i \times m_i$ matrices over $\boldsymbol{C}$. Let N (resp. Tr) be the norm (resp. trace) of $\mathscr{A}$, i.e., $\mathrm{N}(a) = (\det a_1)^{m_1} \cdots (\det a_k)^{m_k}$ (resp. $\mathrm{Tr}\,(a) =$

$m_1 \operatorname{tr} a_1 + \cdots + m_k \operatorname{tr} a_k$) for $a = (a_1, \cdots, a_k) \in \mathscr{A}$. We identify $\mathscr{A}$ and $\mathscr{A}^*$ by the bilinear form $\operatorname{Tr}(ab)$. We have $G = GL(m_1, \boldsymbol{C}) \times \cdots \times GL(m_k, \boldsymbol{C})$ and $S = \{a \in \mathscr{A};\ \mathrm{N}(a) = 0\}$. For $x \in GL(n, \boldsymbol{C})$, we have grad log det $x = (\text{grad det } x)/\det x = {}^t x^{-1}$ and hence, we see that grad log $\mathrm{N}(a)$ is dominant, i.e., the triplet (G, ρ, V) is a regular P.V. Conversely assume that $\mathscr{A}$ is not semi-simple and let $\mathscr{R}(\neq 0)$ be the radical of $\mathscr{A}$. Then there exists a semi-simple subalgebra $\mathscr{A}_0$ satisfying $\mathscr{A} = \mathscr{A}_0 \oplus \mathscr{R}$. Let $g = g_0 + n_0$ $(g_0 \in \mathscr{A}_0, n_0 \in \mathscr{R})$ be invertible in $\mathscr{A}$. Put $g^{-1} = g_0' + n_0'$. Then $1 = g_0 g_0' + n_0''$. Since $1 \in \mathscr{A}_0$, we have $n_0'' = 0$, $g_0' = g_0^{-1}$, and hence $g_0 + n_0 = g_0(1 + g_0^{-1} n_0)$. Put $U = \{1 + n;\ n \in \mathscr{R}\}$. Since $\mathscr{R}$ is a nilpotent ideal, U is a unipotent group. Let $f(x)$ be a relative invariant of (G, ρ, V) and χ its character. Let x_0 be a generic point satisfying $f(x_0) = 1$. Then for $x = gx_0 \in V - S$, we have $f(x) = \chi(g)$. If $g = g_0 + n_0$ $(g_0 \in \mathscr{A}_0, n_0 \in \mathscr{R})$ then $g = g_0 u$ $(u = 1 + g_0^{-1} n_0 \in U)$, and hence $\chi(g) = \chi(g_0)$. Note that $\chi | U = 1$ since U is unipotent. This implies that $f(x)$ is function only on $\mathscr{A}_0$ and hence Hess $\log f(x) = 0$, i.e., (G, ρ, V) is not regular. This proves (3) and this completes the proof of Theorem 10.5.

Let (G, ρ, V) be a P.V., and $\boldsymbol{C}[\rho(G)]$ the vector subspace of $\operatorname{End}(V)$ generated by $\rho(G)(\subset GL(V))$. We shall close this section by proving the following Remark due to T. Kimura.

Remark 10.8. Let (G, ρ, V) be a P.V. such that ρ is faithful. Then it is a P.V. of some associative algebra $\mathscr{A}$ if and only if (1) a generic isotropy subgroup H is the identity group, and (2) $\dim \boldsymbol{C}[\rho(G)] = \dim V$.

Proof. If $V = \mathscr{A}$, $G = \mathscr{A}^\times$ and $\rho(a)b = ab$ for $a, b \in \mathscr{A}$ where $\mathscr{A}$ is an associative algebra, then a generic isotropy subgroup H is the identity $\{1\}$ and $\boldsymbol{C}[\mathscr{A}^\times] \simeq \mathscr{A}$, and hence we have (1) and (2). Conversely, assume (1) and (2). Let x_0 be a generic point and define a map $\Psi: \boldsymbol{C}[\rho(G)] \to V$ by $\Psi(\sum c_i \rho(g_i)) = \sum c_i \rho(g_i) x_0 \in V$. Clearly it is surjective and hence an isomorphism by the condition (2). Since $\boldsymbol{C}[\rho(G)]$ is an associative algebra, V has a structure of an associative algebra which we denote by $\mathscr{A}$. Since $\boldsymbol{C}[\rho(G)]^\times$ is connected and $\dim \boldsymbol{C}[\rho(G)]^\times = \dim \rho(G)$, we have $\boldsymbol{C}[\rho(G)]^\times = \rho(G) \simeq G$, i.e., $\mathscr{A}^\times \simeq G$. Q.E.D.

§ 11. A classification of regular irreducible P.V.'s with universally transitive open orbits (J. Igusa's result)

Let k be a field of characteristic zero. Let $\tilde{G}$ be a connected linear k-split algebraic group, $\rho: \tilde{G} \to GL(X)$ with $X = \mathit{Aff}^n$ a k-homomorphism. Assume that $(\tilde{G}, \rho, X)$ is a regular irreducible P.V. In this case, the singular set S is an irreducible hypersurface and its complement Y is necessarily

k-open. Put $G=\rho(\tilde{G})$. The number $\ell=\ell_k(G, X)=|G(k)\backslash Y(k)|$ of $G(k)$-orbits in $Y(k)$ is finite [(Serre)]. We say that Y is a universally transitive open orbit if $\ell=|G(k)\backslash Y(k)|=1$ for all k satisfying $H^1(k, \mathrm{Aut}(SL_2))\neq 0$, i.e., there exists a nonsplit quaternion k-algebra. This condition is satisfied by every local field k other than $\boldsymbol{C}$.

Theorem 11.1 ([Igusa 2]). *The number $\ell=|G(k)\backslash Y(k)|$ is invariant under a castling transformation.*

By this theorem, it is enough to check the universal transitivity for all regular irreducible P.V.'s in Theorem 3.3 (Section 3). Using Galois cohomology, J. Igusa obtained the following result.

Theorem 11.2 ([Igusa 1], [Igusa 2]). *A regular irreducible P.V. has a universally transitive open orbit (, i.e., $\ell=1$) if and only if it is castling-equivalent to one of the following P.V.'s.*

(1) $(H\times GL_m, \rho\otimes\Lambda_1)$ *where ρ is an m-dimensional irreducible representation of H.*

(2) (GL_{2m}, Λ_2)

(3) $(Sp_n\times GL_{2m}, \Lambda_1\otimes\Lambda_1)$

(4) $(GL_1\times SO_n, \Lambda_1\otimes\Lambda_1)$ *where n is even and $n\geqq 4$.*

(5) $(GL_1\times \mathrm{Spin}_7, \Lambda_1\otimes$*the spin rep.*$)$

(6) $(GL_1\times \mathrm{Spin}_9, \Lambda_1\otimes$*the spin rep.*$)$

(7) $(\mathrm{Spin}_{10}\times GL_2$, *a half-spin rep.*$\otimes\Lambda_1)$

(8) $(GL_1\times E_6, \Lambda_1\otimes\Lambda_1)$ *with* $\deg(\Lambda_1)=27$ *for* E_6.

Theorem 11.3 ([Igusa 2]). (1) *If all octonion k-algebras split over k, e.g., if k is a p-adic field then not only P.V.'s in Theorem* 11.2 *but also $\ell=|G(k)\backslash Y(k)|=1$ for $(GL_7, \Lambda_3, V(35))$.* (2) *If $k=\boldsymbol{R}$, we have $\ell_k(G, X)=1$ if and only if all roots of the b-function of the reduced P.V.'s in the castling-equivalence classes of (G, X) are integers.*

Remark 11.4. For the b-function, see [S-K-K-O], [Kimura 4]. A P.V. (G, ρ, V) is called "reduced" if any castling transform (G', ρ', V') has a property that $\dim V'\geqq\dim V$. Each castling-equivalence class contains the unique reduced P.V. in the irreducible case (See p. 39 in [S-K]).

§ 12. Universal transitivity of simple P.V.'s and 2-simple P.V.'s

T. Kimura, S. Kasai and H. Hosokawa proved the invariance of ℓ (See Section 11) under various P.V.-equivalences (e.g. P.V.-equivalence in Theorem 2.8) and classified simple P.V.'s, 2-simple P.V.'s of type I, 2-simple P.V.'s of type II with universally transitive open orbits respectively.

Theorem 12.1 ([K-K-H]). *All non-irreducible simple P.V.'s with universally transitive open orbits are given as follows.*

(1) $(GL_1^{k+1}\times SL_n,\ \overbrace{\Lambda_1\oplus\cdots\oplus\Lambda_1}^{k}\oplus\Lambda_1^{(*)})$ $(1\leqq k\leqq n,\ n\geqq 2)$

(2) $(GL_1^{k+1}\times SL_n,\ \Lambda_2\oplus\overbrace{\Lambda_1^{(*)}\oplus\cdots\oplus\Lambda_1^{(*)}}^{k})$ $(1\leqq k\leqq 3,\ n\geqq 4)$ *except* $(GL_1^4\times SL_n,\ \Lambda_2\oplus\Lambda_1\oplus\Lambda_1\oplus\Lambda_1^*)$ *with* n=*odd.*

(3) $(GL_1^2\times SL_{2m+1},\ \Lambda_2\oplus\Lambda_2)$ *for* $m\geqq 2$.

(4) $(GL_1^3\times SL_5,\ \Lambda_2\oplus\Lambda_2\oplus\Lambda_1^*)$.

(5) $(GL_1^k\times Sp_n,\ \overbrace{\Lambda_1\oplus\cdots\oplus\Lambda_1}^{k})$ $(k=2,3)$.

(6) $(GL_1^2\times \mathrm{Spin}_n$, *a half-spin rep.* $\oplus$ *the vector rep.*) *with* $n=8, 10$.

(7) $(GL_1^2\times \mathrm{Spin}_{10},\ \Lambda\oplus\Lambda)$ *where* Λ=*the even half-spin representation.*

Corollary 12.2. *All non-irreducible regular simple P.V.'s with universally transitive open orbits are given as follows.*

(1) $(GL_1^2\times SL_n,\ \Lambda_1\oplus\Lambda_1^*)$.

(2) $(GL_1^n\times SL_n,\ \overbrace{\Lambda_1\oplus\cdots\oplus\Lambda_1}^{n})$.

(3) $(GL_1^{n+1}\times SL_n,\ \overbrace{\Lambda_1\oplus\cdots\oplus\Lambda_1}^{n}\oplus\Lambda_1^{(*)})$.

(4) $(GL_1^3\times SL_{2m},\ \Lambda_2\oplus\Lambda_1^{(*)}\oplus\Lambda_1^{(*)})$.

(5) $(GL_1^2\times SL_{2m+1},\ \Lambda_2\oplus\Lambda_1)$.

(6) $(GL_1^4\times SL_{2m+1},\ \Lambda_2\oplus\Lambda_1\oplus(\Lambda_1\oplus\Lambda_1)^{(*)})$.

(7) $(GL_1^2\times Sp_n,\ \Lambda_1\oplus\Lambda_1)$.

(8) $(GL_1^2\times \mathrm{Spin}_n$, *a half-spin rep.* $\oplus$ *the vector rep.*) *with* $n=8, 10$.

(9) $(GL_1^2\times \mathrm{Spin}_{10},\ \Lambda\oplus\Lambda)$ *where* Λ=*the even half-spin representation.*

Theorem 12.3. *Any non-irreducible 2-simple P.V.* $(GL_1^k\times G,\ \rho(=\rho_1\oplus\cdots\oplus\rho_k))$ *of type* I *with the universally transitive open orbit is castling-equivalent to one of the following P.V.'s.*

(1) $G=SL_{2m+1}\times SL_2$, $\rho=\Lambda_2\otimes\Lambda_1+1\otimes\Lambda_1(+T)$ *with* $T=1\otimes\Lambda_1(+1\otimes\Lambda_1)$.

(2) $G=SL_5\times SL_2$, $\rho=\Lambda_2\otimes\Lambda_1+\Lambda_1^*\otimes 1(+1\otimes\Lambda_1(+1\otimes\Lambda_1))$.

(3) $G=SL_5\times SL_2$, $\rho=\Lambda_2\otimes\Lambda_1+(\Lambda_1^*+\Lambda_1^*)\otimes 1$.

(4) $G=Sp_n\times SL_m$, $\rho=\Lambda_1\otimes\Lambda_1+T$, *with* $T=1\otimes(\Lambda_1^{(*)}+\cdots+\Lambda_1^{(*)})$ $(1\leqq k\leqq 3)$ *except* $1\otimes(\Lambda_1+\Lambda_1+\Lambda_1^*)$ *with* m=*odd*, $\Lambda_1\otimes 1+1\otimes(\Lambda_1^{(*)}+\cdots+\Lambda_1^{(*)})$ $(0\leqq k\leqq 2)$ *except* $\Lambda_1\otimes 1+1\otimes(\Lambda_1+\Lambda_1^*)$ *with* m=*odd*, $1\otimes\Lambda_2$ (m=*odd*), $1\otimes(\Lambda_2+\Lambda_1^*)$ $(m=5)$.

(5) $G=Sp_n\times SL_{2m+1}$, $\rho=\Lambda_1\otimes\Lambda_1+(\Lambda_1+\Lambda_1)\otimes 1$.

(6) $G=\mathrm{Spin}_{10}\times SL_2$, ρ=*a half-spin rep.* $\otimes\Lambda_1+1\otimes\Lambda_1(+T)$ *with* $T=1\otimes\Lambda_1(+1\otimes\Lambda_1)$.

Corollary 12.4. *Any non-irreducible regular 2-simple P.V. of type* I *with the universally transitive orbit is castling-equivalent to one of the following P.V.'s.*

(1) $(GL_1^3 \times SL_5 \times SL_2, \Lambda_2 \otimes \Lambda_1 + (\Lambda_1^* + \Lambda_1^*) \otimes 1)$.
(2) $(GL_1^3 \times Sp_n \times SL_{2m}, \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1^{(*)} + \Lambda_1^{(*)}))$.
(3) $(GL_1^2 \times Sp_n \times SL_{2m+1}, \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1)$.
(4) $(GL_1^4 \times Sp_n \times SL_{2m+1}, \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes (\Lambda_1 + \Lambda_1)^{(*)})$,
(5) $(GL_1^3 \times \mathrm{Spin}_{10} \times SL_2$, *a half-spin rep.* $\otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1))$.
(6) $(GL_1^4 \times \mathrm{Spin}_{10} \times SL_2$, a *half-spin rep.* $\otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1))$.

One can see also that any non-regular irreducible P.V., which is not castling-equivalent to $(Sp_n \times GL_2, \Lambda_1 \otimes 2\Lambda_1)$, has the universally transitive open orbit. Any 2-simple P.V. of type II in Theorem 5.2 has the universally transitive open orbit if and only if so is the corresponding simple P.V. There are many other 2-simple P.V.'s of type II with universally transitive open orbits (See [K-K-H]).

§ 13. Irreducible P.V.'s of characteristic $p \geqq 3$ (Z. Chen's result)

In [C3], [C4], Z. Chen obtained the following result. Since all irreducible P.V.'s are defined over $\boldsymbol{Q}$ (Section 3), by the reduction modulo p, one obtains corresponding P.V.'s in characteristic p. If $p > 5$, every regular irreducible P.V. of characteristic 0 induces a regular irreducible one of characteristic p by the reduction modulo p([C4]). There is one exception if $p = 5$ and 6 exceptions if $p = 3$. But not every irreducible P.V.'s of characteristic p can be obtained by the reduction mudulo p. In fact, when $p > 2$, there are following 4 new types.

(1) $(GL_n, (1+p^s)\Lambda_1, V(n^2))$ $(s > 0, n \geqq 2)$.
(2) $(GL_n, \Lambda_1 + p^s \Lambda_{n-1}, V(n^2))$ $(s > 0, n \geqq 3)$.
(3) $(GL_1 \times SL_3, \Lambda_1 \otimes (\Lambda_1 + \Lambda_2), V(1) \otimes V(7))$ $(p = 3)$.
(4) $(GL_4, \Lambda_1 + \Lambda_2, V(16))$ $(p = 3)$.

Among them, Z. Chen investigated a P.V. (4) in detail. It is a regular irreducible P.V. with the irreducible relative invariant of degree 8.

§ 14. A classification of irreducible P.V.'s of parabolic type and their real forms (H. Rubenthaler's result)

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra. Let R be the root system w.r.t. $(\mathfrak{g}, \mathfrak{h})$ and fix a base Ψ of R. Let θ be a subset of Ψ. Put $h_\theta = \{x \in h, \alpha(x) = 0 \text{ for all } \alpha \in \theta\}$ and define an element H^θ of h_θ by $\alpha(H^\theta) = 0$ for $\alpha \in \theta$ and $\alpha(H^\theta) = 2$ for $\alpha \in \Psi - \theta$. For $n \in \boldsymbol{Z}$, put $d_n(\theta) = \{X \in \mathfrak{g};\ [H^\theta, X] = 2nX\}$. Then we have $[d_i(\theta), d_j(\theta)] \subset d_{i+j}(\theta)$ and hence we obtain Z-grading.

The space $d_0(\theta)$, which is denoted by ℓ_θ, is a reductive subalgebra of $\mathfrak{g}$. It operates on each $d_i(\theta)$ by the adjoint action. Let L_θ be the connected subgroup of G corresponding to ℓ_θ where G is adjoint group of $\mathfrak{g}$. Then we have the action of L_θ on $d_i(\theta)$ which corresponds to that of ℓ_θ on $d_i(\theta)$. It is known (Vinberg) that $(L_\theta, d_1(\theta))$ is a P.V. of paraboilc type. It is an irreducible P.V. if and only if Card $(\Psi - \theta) = 1$. H. Rubenthaler classified irreducible P.V.'s of parabolic type [R1]. Irreducible regular P.V.'s of parabolic type are given by (1) with $(H, \rho) = (SL_n, \Lambda_1)$, (2) $\sim$ (9), $(Sp_{n'}, \Lambda_1)$ $(n=2n')$, (SO_n, Λ_1). (10) with $n=1$, (12), (13), (15), (16), in (I) in Theorem 3.3. H. Rubenthaler also investigated their real forms ([R1], [R2]).

§ 15. Indecomposable commutative Frobenius algebras and δ-functions; Examples of quasi-regular, non-regular P.V.'s (M. Sato's unpublished result III)

Let $A = \boldsymbol{C}[x_1, \cdots, x_n]$ be the polynomial ring of n variables over $\boldsymbol{C}$. Any finite-dimensional commutative algebra $\bar{A}$ over $\boldsymbol{C}$ can be expressed as $\bar{A} = A/J$ where J is an ideal of A satisfying $\mathfrak{m} \supset J \supset \mathfrak{m}^k$ for some k with $\mathfrak{m} = Ax_1 + \cdots + Ax_n$. Let $\mathscr{B}_{pt} = \{P(D)\delta(x_1, \cdots, x_n);\ P(D)$ is a partial differential operator with constant coefficients$\}$ be the hyperfunctions with the support at the origin of $\boldsymbol{C}^n$, where $\delta(x_1, \cdots, x_n)$ is the Dirac's delta function. By the inner product $(f, \triangle(x)) = \int f(u)\triangle(u)du$, we can regard $\mathscr{B}_{pt}$ as the dual A^* of A canonically. Note that $\mathscr{B}_{pt}$ is an A-module. The dual $\bar{A}^*$ of $\bar{A}$ is given by $\bar{A}^* = \{\triangle(x) \in \mathscr{B}_{pt};\ (f, \triangle(x)) = 0$ for all $f \in J\}$. By Lemma 10.6, $\bar{A} = A/J$ is a Frobenius algebra if and only if $\bar{A}^* = A\triangle(x)$ for some $\triangle(x) \in \bar{A}^* \subset \mathscr{B}_{pt}$. Note that a commutative Frobenius algebra is a symmetric algebra. When $A^* = A\triangle(x)$, the ideal J is given by $J = \left\{f \in A;\ \langle f, \Psi\triangle(x)\rangle = \int f\Psi\triangle(u)du = \int \Psi f\triangle(u)du = \langle \Psi, f\triangle(x)\rangle = 0 \text{ for all } \Psi \in A\right\} = \{f \in A; f\triangle(x) = 0\}$. Hence, for any given $\triangle(x) \in \mathscr{B}_{pt}$, we can get a commutative Frobenius algebra $\bar{A} = A/J$ by the ideal $J = \{f \in A;\ f\triangle(x) = 0\}$ of $A = \boldsymbol{C}[x_1, \cdots, x_n]$. Conversely, any finite-dimensional indecomposable commutative Frobenius algebra can be obtained by this method.

Remark 15.1. $\{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}\}$ is a base of $A = \boldsymbol{C}[x_1, \cdots, x_n]$. The dual base of $\mathscr{B}_{pt}$ is given by $\{((-1)^{|\beta|}/\beta!)\delta^{(\beta)}(x)\}$, *i.e.*, $(x^\alpha, ((-1)^{|\beta|}/\beta!)\delta^{(\beta)}(x)) = 1$ $(\alpha = \beta)$ and $= 0$ $(\alpha \neq \beta)$.

Remark 15.2. Assume that $A = A_1 \oplus \cdots \oplus A_k$ where each A_i is a twosided ideal of A. Then A is a Frobenius algebra (resp. symmetric algebra) if and only if each A_i is a Frobenius algebra (resp. symmetric algebra).

Remark 15.3. Both $A=\boldsymbol{C}[x_1, \cdots, x_n]$ and $\mathscr{B}_{pt}$ are of infinite dimension. If one wants to consider a finite-dimensional vector space, one can modify our argument as follows. Let $\mathfrak{M}^k$ be the ideal of $A=\boldsymbol{C}[x_1,\cdots,x_n]$ generated by $x_1{}^k, \cdots, x_n{}^k$, i.e., $\mathfrak{M}^k=Ax_1{}^k+\cdots+Ax_n{}^k$ and put $A_k=A/\mathfrak{M}^k$. For $\bar{A}=A/J,$, we have $A_k\to\bar{A}\to 0$ (exact) for a sufficiently large k. Its kernel $\bar{J}$ is given by $\bar{J}=J/M^k$. Now A_k is a finite-dimensional vector space over $\boldsymbol{C}$. Let $\mathscr{B}_{pt}{}^{(k)}$ be the totality of less than k-th derivations of the δ-function. Then $\mathscr{B}_{pt}{}^{(k)}$ is the dual of A_k.

Now we shall investigate the correspondence between $\mathscr{B}_{pt}$ and $\bar{A}=A/J$ ($=$a Frobenius algebra) in detail. Assume that $\bar{A}$ has two expression: (1) $\bar{A}=\boldsymbol{C}[x_1, \cdots, x_n]/J_1$ which corresponds to $\triangle_1(x)$ with n-variables, (2) $\bar{A}=\boldsymbol{C}[x_1, \cdots, x_{n+r}]/J_2$ which corresponds to $\triangle_2(x)$ with $(n+r)$-variables. In this case, put $\triangle_1'(x)=\triangle_1(x)\ \delta(x_{n+1})\cdots\delta(x_{n+r})$ and $J_1'=\{f\in\boldsymbol{C}[x_1,\cdots, x_{n+r}]; f\triangle_1'(x)=0\}$. Since J_1' contains $x_{n+1}, \cdots, x_{n+r}$, we have (3) $\bar{A}=\boldsymbol{C}[x_1, \cdots, x_n, \cdots, x_{n+r}]/J_1'$. Thus we may assume that two expressions have the same number of variables, i.e., $A=\boldsymbol{C}[x_1, \cdots, x_n]/J_1\ (\Leftrightarrow\triangle_1(x))=\boldsymbol{C}[y_1, \cdots, y_n]/J_2(\Leftrightarrow\triangle_2(y))$. We have $\bar{A}=\boldsymbol{C}\oplus N$ where N is the radical of $\bar{A}$. Nakayama's lemma says that elements $u_1, \cdots, u_n$ of N generate $\bar{A}$ as an algebra if and only if $u_1 \bmod N^2, \cdots, u_n \bmod N^2$ generate N/N^2 as a vector space. Put $u_i=x_i \bmod J_1$ and $v_i=y_i \bmod J_2$ $(1\leqq i\leqq n)$. We may assume that (1) $u_i, v_i\in N$ $(1\leqq i, j\leqq n)$, (2) $\{u_1, \cdots, u_m\} \bmod N^2$ and $\{v_1, \cdots, v_m\} \bmod N^2$ are bases of N/N^2 so that $A=\boldsymbol{C}[u_1, \cdots, u_m, \cdots, u_n]=\boldsymbol{C}[u_1, \cdots, u_m]$ etc. Put $u_i=\varPsi_i(v)$ and $v_j=\varPhi(u)$. We have $v_i\equiv\sum a_{ij}u_j \bmod N^2$ $(1\leqq i\leqq n)$. If $n=m$, we have $\det(a_{ij})\neq 0$. Even if $n>m$, we may take $\varPsi$ and $\varPhi$ such that $\det(a_{ij})\neq 0$, and hence $u_i=\varPsi_i(v)$ and $v_j=\varPhi_j(u)$ are non-singular transformations at the origin. We say that $\triangle_1(x)$ and $\triangle_2(x)$ are equivalent (denoted by $\triangle_1(x)\simeq\triangle_2(x)$) if one is transformed from the other by (1) the coordinate transformation $\triangle(u)du\mapsto\triangle(\varPsi(v))(\partial\varPsi(u)/\partial v)\cdot dv$ and (2) $\triangle(u)du\mapsto f(u)\triangle(u)du$ where $f(f(0)\neq 0)$ is an invertible operator. If $\triangle_1\simeq\triangle_2$, then the corresponding Frobenius algebras are isomorphic, i.e., {equivalence class of $\mathscr{B}_{pt}$}$\Leftrightarrow${isomorphic class of indecomposable commutative Frobenius algebras}.

Example 15.4 (one variable). We have $a_0\delta^{(n)}(x)+a_1\delta^{(n-1)}(x)+\cdots+a_n\delta(x)\ (a_0\neq 0)\simeq\delta^{(n)}(x)$. It is transformed by (1) (also possible by (2)). Put $A=\boldsymbol{C}[x]$. Then $A\delta^{(n)}(x)=\boldsymbol{C}\delta^{(n)}(x)\oplus\cdots\oplus\boldsymbol{C}\delta^{(1)}(x)\oplus\boldsymbol{C}\delta(x)$, and $J=\{f\in\boldsymbol{C}[x]; f\delta^{(n)}(x)=0\}=\boldsymbol{C}[x]\cdot x^{n+1}$. Hence $A=A/J=\boldsymbol{C}\cdot 1\oplus\boldsymbol{C}\cdot x\oplus\cdots\oplus\boldsymbol{C}x^n \bmod x^{n+1}$. We shall consider a commutative Frobenius algebra obtained in such a way. Namely $A=\boldsymbol{C}\cdot 1\oplus\boldsymbol{C}\cdot u\oplus\cdots\oplus\boldsymbol{C}u^{n-1}$ with $u^n=0$.

This is obtained from $\delta^{(n-1)}(x)$. For an element $g=a_0+a_1u+\cdots+a_{n-1}u^{n-1}$ of $\bar{A}$, we have

$$(g, gu, \cdots, gu^{n-1})=(1, u, \cdots, u^{n-1})\begin{vmatrix} a_0 & & & \\ a_1 & a_0 & & 0 \\ \vdots & & \ddots & \\ a_{n-1} & \cdots & \cdots & a_0 \end{vmatrix}.$$

Hence $\det g=a_0^n$ and $G=\bar{A}^\times=\{a_0+a_1u+\cdots+a_{n-1}u^{n-1};\ a_0\neq 0\}$. By Section 10, $(\bar{A}^\times, \bar{A})$ is a quasi-regular, non-regular P.V.

Proposition 15.5. *There exist n G-invariant closed* 1*-forms on an n-dimensional commutative Frobenius algebra* $\mathscr{A}$.

Proof. For $x=x_1u_1+\cdots+x_nu_n\in\mathscr{A}^\times$ and $\xi\in\mathscr{A}$, put $x^{-1}\xi=\Psi_1(x,\xi)u_1+\cdots+\Psi_n(x,\xi)u_n$ where $\{u_1,\cdots,u_n\}$ is a base of $\mathscr{A}$. Since $d\log x=x^{-1}dx$ and $(gx)^{-1}(g\xi)=x^{-1}\xi$, $\Psi_1(x, dx),\cdots,\Psi_n(x, dx)$ are G-invariant closed 1-forms. Q.E.D.

Example 15.6. We shall consider $\mathscr{A}=\boldsymbol{C}\cdot 1\oplus\boldsymbol{C}\cdot u\oplus\boldsymbol{C}u^2$ $(u^3=0)$. For $x=x_0+x_1u+x_2u^2\in\mathscr{A}^\times$, we have $\log x=\log x_0+\log(1+(x_1/x_0)u+(x_2/x_0)u^2)=\log x_0+(x_1/x_0)u+(x_2/x_0-\frac{1}{2}(x_1/x_0)^2)u^2$ since $\log(1+t)=t-\frac{1}{2}t^2 \bmod t^3$. Hence $d\log x=1/x_0dx_0+(dx_1/x_0)u+(d(2x_0x_2-x_1^2)/2x_0^2)u^2$ and we obtain G-invariant closed 1-forms $\Psi_0=dx_0/x_0$, $\Psi_1=d(x_1/x_0)$ and $\Psi_2=d((2x_0x_2-x_1^2)/2x_0^2)$. Now Ψ_0 corresponds to a rational relative invariant $f_0(x)=x_0$, and Ψ_1 (resp. Ψ_2) corresponds to a transcendental relative invariant $f_1(x)=\exp(x_1/x_0)$ (resp. $f_2(x)=\exp((2x_0x_2-x_1^2)/2x_0^2)$). We have $f_0(gx)=a_0f_0(x)$, $f_1(gx)=(\exp a_1/a_0)f_1(x)$, $f_2(gx)=(\exp((2x_0x_0-x_1^2)/2x_0^2)f_2(x)$ for $g=a_0+a_1u+a_2u^2\in\mathscr{A}^\times$. Since $\operatorname{grad}\log f_0^{s_0}f_1^{s_1}f_2^{s_2}={}^t(s_0/x_0-s_1x_1/x_0^2-s_2(x_0x_2+x_1^2)/x_0^3,\ s_1/x_0-s_2x_1/x_0^2,\ s_2/x_0)$, we can see the quasi-regularity again. This quasi-regular P.V. can be considered as a deformation of a regular P.V. $(GL_1^3, V(3))$. We shall explain this fact for a 2-dimensional commutative associative algebra $A=\boldsymbol{C}\oplus\boldsymbol{C}u$. The structure of an algebra is determined by giving λ and ν for $u^2=\lambda+\nu u$. Using $u'=u-\nu/2$, we may assume that $\nu=0$ and $u^2=\lambda$. If $\lambda\neq 0$, then $(\mathscr{A}^\times, \mathscr{A})$ is a regular P.V. isomorphic to $(GL_1^2, V(2))$. In fact, putting $e_1=\frac{1}{2}+u/(2\sqrt{\lambda})$ and $e_2=\frac{1}{2}-u/(2\sqrt{\lambda})$, we have $g(e_1, e_2)=(e_1, e_2)\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}$ for $g=ae_1+be_2\in\mathscr{A}=\boldsymbol{C}e_1\oplus\boldsymbol{C}e_2$. However, as the limit of $\lambda\to 0$, we obtain a quasi-regular, non-regular P.V.

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The Institute of Mathematics
University of Tsukuba
Ibaraki, 305 Japan