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The Space of Eisenstein Series in the Case of GL_2

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Introduction

It is known in the classical cases and also expected to be true in general that every automorphic form orthogonal to cusp forms is a linear combination of Eisenstein series. Among the classical and recent references are Hecke [6], Kloosterman [8], Gundlach [4], Maass [11], Roelcke [13], Shimizu [14], Shimura [15]. [6], [8], [4] and [14] treat holomorphic cases, while [11] and [13] treat real analytic cases. [15] proves the most general results known so far for Hilbert modular groups (it discusses also the case of half-integral weights).

In this note we consider the group GL_2 over an arbitrary number field, to show that the assertion in the biginning is valid for automorphic forms on that group which are eigenfunctions of bi-invariant differential operators; here we understand that 'a linear combination' of Eisenstein series includes a process of taking derivatives or residues with respect to a parameter.

We do not try to make our exposition self-contained. In fact, the automorphic representation theory and the fundamental property of Eisenstein series (analytic continuation etc.) are assumed. As to the first subject the basic reference is Jacquet-Langlands [7]. As to the second subject there are many references: Langlands [10], Harish-Chandra [5], Kubota [9], Gelbart-Jacquet [3], Arthur [1], Shimura [15].

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§ 1. Automorphic forms

1. Throughout this note F denotes an algebraic number field of finite degree. Let G be the group GL_2 viewed as an algebraic group over F so that $G_F = GL_2(F)$. Let P be the set of all places of F and P_f (resp. P_∞) the set of all finite (resp. infinite) places in P. For $v \in P$ we write

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simply G_v for $G_{F_v}=GL_2(F_v)$ where F_v is the completion of F with respect to v. If K_v is a standard maximal compact subgroup of G_v, the adelized group G_A of G is by definition the restricted direct product of G_v for v in P with respect to K_v.

The groups K_v can be defined as follows. Let $\mathfrak{o}$ be the ring of integers in F and $\mathfrak{o}_v$ the closure of $\mathfrak{o}$ in F_v for v in P_f. If v is in P_f, we set $K_v=GL_2(\mathfrak{o}_v)$. If v is in P_∞, K_v is the orthogonal or unitary group of degree 2 according as v is real or imaginary.

2. Definition of the Hecke algebra associated with G. For v in P_f, let $\mathscr{H}_v$ be the space of all $\boldsymbol{C}$-valued, locally constant and compactly supported functions on G_v (a function is said to be locally constant, if it is constant on a neighborhood of each point). For v in P_∞, let $\mathscr{H}_v$ be the space of all $\boldsymbol{C}$-valued, compactly supported C^∞ functions f such that the system of functions

$$\{g \longrightarrow f(kg) | k \in K_v\} \cup \{g \longrightarrow f(gk) | k \in K_v\}$$

on G_v spans a finite-dimensional space. In either case, $\mathscr{H}_v$ forms a $\boldsymbol{C}$-algebra, the multiplication being the convolution

$$f_1 * f_2(g) = \int_{G_v} f_1(gh) f_2(h^{-1}) dh.$$

Here dh is a Haar measure on G_v. $\mathscr{H}_v$ is called the Hecke algebra on G_v.

Let us fix a certain notation. Let f be a function on an abstract group G and h an element in G. The right (resp. left) translate $\rho(h)f$ (resp. $\lambda(h)f$) of f is a function

$$(\rho(h)f)(g) = f(gh)$$
$$(\text{resp. } (\lambda(h)f)(g) = f(h^{-1}g))$$

on G. H being a subgroup of G, we say that f is right H-finite, if $\{\rho(h)f | h \in H\}$ spans a finite-dimensional space. Left H-finiteness is defined similarly.

Let K be a compact group. For a finite-dimensional irreducible representation σ of K, we set

$$\xi_\sigma(k) = (\dim \sigma) \operatorname{tr} \sigma(k^{-1}) \qquad (k \in K).$$

A function on K of the form $\xi = \sum \xi_\sigma$ (where σ runs through a finite set of distinct irreducible representations of K) is called elementary idempotent. In fact, it is an idempotent with respect to the convolution product on K, i.e. $\xi * \xi = \xi$. This follows from the orthogonality relations of matrix entries of irreducible representations. If D_1, D_2 are finite sets of distinct irreducible representations of K such that $D_1 \subset D_2$ and if

$$\xi_1 = \sum_{\sigma \in D_1} \xi_\sigma, \qquad \xi_2 = \sum_{\sigma \in D_2} \xi_\sigma,$$

then we have $\xi_1 * \xi_2 = \xi_2 * \xi_1 = \xi_1$.

Assume that K is a compact subgroup in a topological group G. For continuous functions f and ξ on G and K, respectively, we put

$$\xi * f(g) = \int_K \xi(k^{-1}) f(kg) dk,$$

$$f * \xi(g) = \int_K f(gk) \xi(k^{-1}) dk,$$

where dk is a Haar measure on K with the total volume 1. It is easy to see that f is right (resp. left) K-finite if and only if there exists an elementary idempotent ξ on K such that $f * \xi = f$ (resp. $\xi * f = f$).

Now let v be in P_f and f an element in $\mathscr{H}_v$. Since f is locally constant and compactly supported, we can find an open subgroup H_v of K_v such that f is constant on the cosets of H_v. In particular f is both right and left K_v-finite. Note that the same property of f is implied in the definition if $v \in P_\infty$.

For v in P_f, denote by f_v^0 the characteristic function of K_v; it belongs to $\mathscr{H}_v$, since K_v is open and compact. Let

$$\mathscr{H} = \otimes_{v \in P} \mathscr{H}_v$$

be the restricted tensor product of $\mathscr{H}_v$ for v in P with respect to $\{f_v^0 | v \in P_f\}$. It is the set of all linear combinations of $\otimes_v f_v$ such that $f_v \in \mathscr{H}_v$ for all $v \in P$ and $f_v = f_v^0$ for almost all v. An element $f = \otimes_v f_v$ may be identified with a function

$$f(g) = \prod_v f_v(g_v) \qquad (g = (g_v) \in G_A)$$

on G_A so that $\mathscr{H}$ may be viewed as a function space on G_A. We call $\mathscr{H}$ the Hecke algebra on G_A.

Put $K = \prod_{v \in P} K_v$. An irreducible representation σ of K is a tensor product of irreducible representations σ_v of K_v for $v \in P$. Then we have

$$\xi_\sigma(k) = \prod_v \xi_{\sigma_v}(k_v) \qquad (k \in K).$$

It follows that, if ξ is an elementary idempotent of K, then $\xi * f$ and $f * \xi$ belong to $\mathscr{H}$ for all f in $\mathscr{H}$.

Let φ be a continuous function on G_A and f in $\mathscr{H}$. We set

$$\rho(f)\varphi(g)=\int_{G_A}\varphi(gh)f(h)dh,$$

dh being a Haar measure on G_A. The integral above converges, since f is compactly supported. If ξ is an elementary idempotent of K, we often write $\rho(\xi)\varphi=\varphi*\check{\xi}$, where $\check{\xi}(k)=\xi(k^{-1})$ for $k\in K$.

3. Definition of automorphic forms on G_A. Let η be a character of $A^\times/F^\times$, i.e. a Grössencharacter of F. An automorphic form (with a character η) is a continuous function φ on G_A satisfying the following conditions.

(i) $\varphi(\gamma zg)=\eta(z)\varphi(g)$ ($\gamma\in G_F$, $z\in A^\times$, $g\in G_A$).

(ii) φ is right K-finite.

(iii) For every elementary idempotent ξ of K, the space $\{\rho(\xi*f)\varphi|f\in\mathscr{H}\}$ is finite-dimensional.

(iv) For every compact subset C of G_A, there exist real constants M, N such that

$$\left|\varphi\left(\begin{pmatrix}a&0\\0&1\end{pmatrix}g\right)\right|\leqq M|a|_A^N$$

for all $a\in A^\times$ with $|a|_A\geqq 1$ and $g\in C$.
The space of all automorphic forms (with a character η) is denoted by $\mathscr{A}(\eta)$.

Let $\boldsymbol{R}_+$ be the set of all positive real numbers. Identify $t\in\boldsymbol{R}_+$ with an element $g=(g_v)$ in $A^\times$ such that $g_v=1$ ($v\in P_f$), $g_v=t$ ($v\in P_\infty$). Put $A^1=\{a\in A^\times||a|_A=1\}$; then we have $A^\times=A^1\times\boldsymbol{R}_+$.

Let ω be a compact subset of A, ω^1 a comapct subset of A^1 and c a positive real number. Let $\mathfrak{S}$ be the set of all elements in G_A of the form

$$z\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}a&0\\0&1\end{pmatrix}k$$

such that $z\in A^\times$, $x\in\omega$, $a\in A^\times$, $|a|_A\geqq c$, the projection of a to A^1 is in ω^1, and $k\in K$. $\mathfrak{S}$ is called Siegel domain. It is well known that there exists a Siegel domain $\mathfrak{S}$ such that $G_A=G_F\mathfrak{S}$. Hence the condition (iv) above gives an estimation of $|\varphi|$ on a Siegel domain. We say that a left G_F-invariant and $A^\times$-finite function φ on G_A is slowly increasing, if it satisfies (iv).

For an automorphic form φ, we set

$$\varphi^0(g)=\int_{A/F}\varphi\left(\begin{pmatrix}1&x\\0&1\end{pmatrix}g\right)dx\qquad(g\in G_A).$$

φ is called cusp form if $\varphi^0(g)=0$ for all g in G_A. The space of all cusp forms in $\mathscr{A}(\eta)$ is denoted by $\mathscr{A}_0(\eta)$.

4. In the following we collect some results on automorphic forms, supplying a proof whenever it is convenient for our purpose.

Let G_f and G_∞ be the finite and infinite part of G_A, respectively; namely

$$G_f=\{g \in G_A | g_v=1 \text{ for all } v \in P_\infty\},$$

$$G_\infty=\{g \in G_A | g_v=1 \text{ for all } v \in P_f\}.$$

We write an element g in G_A as $g=g_f g_\infty$ with $g_f \in G_f$, $g_\infty \in G_\infty$.

Lemma 1. *For every* φ *in* $A(\eta)$, *there exists an element* f *in* $\mathscr{H}$ *such that* $\varphi=\rho(f)\varphi$.

Proof. Since φ is right K-finite, there exists an elementary idempotent ξ of K such that $\rho(\xi)\varphi=\varphi*\check{\xi}=\varphi$. $V=\rho(\xi*\mathscr{H})\varphi$ is finite-dimensional by the definition of automorphic forms.

If $h \in \xi*\mathscr{H}*\xi$, then $\rho(h)V\subset V$. We denote by $\bar{\rho}(h)$ the endomorphism of V induced by $\rho(h)$. Now there exists a sequence $\{f_n\}$ of compactly supported continuous functions on G_A with the following properties.

1) $\operatorname{supp} f_n$ converges to the unit element 1 of G_A,

2) $f_n \geqq 0$,

3) $\int_{G_A} f_n dg=1$,

4) f_n can be written as $f_n(g)=f_n'(g_f)f_n''(g_\infty)$, where f_n' is a locally constant function on G_f and f_n'' is a C^∞ function on G_∞.

For any continuous function ϕ on G_A, $\rho(f_n)\phi$ converges to ϕ uniformly on a compact set. Especially, if $\rho(\xi)\phi=\phi$, then $\rho(h_n)\phi$ converges to ϕ for $h_n=\xi*f_n*\xi$. We see that there exists an element h in $\xi*\mathscr{H}*\xi$ such that $\bar{\rho}(h)$ is as close as we wish to the identity transformation of V so that $\det \bar{\rho}(h)\neq 0$. Let $\sum_{i=0}^m a_i X^i$ be the characteristic polynomial of $\bar{\rho}(h)$. Then

$$f=-a_0^{-1}\sum_{i=1}^m a_i h^i$$

($h^i=h*\cdots*h$ (i times)) belongs to $\xi*\mathscr{H}*\xi$ and $\bar{\rho}(f)=1$. Put $\varphi_n=\rho(h_n)\varphi \in V$; then $\rho(f)\varphi_n=\varphi_n$. Letting $n\to\infty$, we have $\rho(f)\varphi=\varphi$. q.e.d.

Let $\mathfrak{g}$ be the Lie algebra of G_∞, $\mathscr{U}(\mathfrak{g}_C)$ the universal envelopping algebra of $\mathfrak{g}\otimes C$ and $\mathscr{Z}$ the center of $\mathscr{U}(\mathfrak{g}_C)$. For a C^∞ function φ on G_∞ (or on G_A, regarded as a function of g_∞) and for $X \in \mathfrak{g}$, we put

$$\rho(X)\varphi(g)=\frac{d}{dt}\varphi(g\exp tX)|_{t=0},$$

$$\lambda(X)\varphi(g)=\frac{d}{dt}\varphi(\exp(-tX)g)|_{t=0}.$$

It is well known that ρ (resp. λ) can be extended to a homomorphism of $\mathscr{U}(\mathfrak{g}_C)$ onto the algebra of left (resp. right) invariant differential operators on the space of C^∞ functions on G_∞. If $Z\in\mathscr{Z}$, then $\rho(Z)$ is bi-invariant, i.e. commuting with right and left translations.

Lemma 2. *Every φ in $A(\gamma)$ is $\mathscr{Z}$-finite; namely $\{\rho(Z)\varphi\,|\,Z\in\mathscr{Z}\}$ is a finite-dimensional space.*

Proof. Let ξ be an elementary idempotent of K such that $\rho(\xi)\varphi=\varphi$. Since $\rho(Z)$ commutes with right translations, we have $\rho(\xi)\rho(Z)\varphi=\rho(Z)\rho(\xi)\varphi=\rho(Z)\varphi$. By Lemma 1 there exists a f in $\mathscr{H}$ such that $\rho(f)\varphi=\varphi$, then we have

$$\rho(Z)\varphi(g)=\rho(Z)\int\varphi(gh)f(h)dh$$

$$=\int\varphi(gh)\lambda(Z)f(h)dh=\rho(\lambda(Z)f)\varphi(g).$$

Evidently $\lambda(Z)f\in\mathscr{H}$. Hence $\{\rho(Z)\varphi\,|\,Z\in\mathscr{Z}\}$ is contained in $\rho(\xi*\mathscr{H})\varphi$, and the latter space is finite-dimensional. q.e.d.

5. $\rho(\mathscr{Z})$ can be described as follows. G_∞ is the direct product of G_v for $v\in P_\infty$ and $G_v=GL_2(\boldsymbol{R})$ or $GL_2(\boldsymbol{C})$ according as v is real or complex. If $\mathfrak{g}_v$ is the Lie algebra of G_v and $\mathscr{Z}_v$ the center of $\mathscr{U}(\mathfrak{g}_{vC})$, then

$$\mathscr{Z}=\otimes_{v\in P_\infty}\mathscr{Z}_v.$$

Hence it is enough to consider the action of $\mathscr{Z}$ component-wise.

1) The case of real v. Let $\mathfrak{gl}_2$ denote the Lie algebra of 2 by 2 matrices. The Lie algebra of $G_{\boldsymbol{R}}=GL_2(\boldsymbol{R})$ is identified with $\mathfrak{gl}_2(\boldsymbol{R})$ and $\mathfrak{gl}_2(\boldsymbol{R})\otimes C=\mathfrak{gl}_2(\boldsymbol{C})$. Put

$$J=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad X_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad X_2=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad X_3=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

and define an element D in $\mathscr{U}(\mathfrak{gl}_2(\boldsymbol{R})_C)$ by

$$D=\tfrac{1}{2}(X_1^2+X_2^2-X_3^2).$$

The center $\mathscr{Z}_{\boldsymbol{R}}$ of $\mathscr{U}(\mathfrak{gl}_2(\boldsymbol{R})_C)$ is a polynomial ring over $\boldsymbol{C}$ generated by J and D.

The action of J is obvious. To express $\rho(D)$, put

$$n(x)=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix},\quad a(y)=\begin{pmatrix}y & 0\\ 0 & y^{-1}\end{pmatrix},\quad k(\theta)=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$

and write $g \in G_{\boldsymbol{R}}$, $\det g>0$, as

$$g=zn(x)a(y^{1/2})k(\theta) \qquad (z>0,\ y>0).$$

With these coordinates, we have

$$\rho(D)=2y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-2y\frac{\partial^2}{\partial x\partial\theta}. \tag{1.1}$$

2) The case of imaginary v. The Lie algebra of $G_{\boldsymbol{C}}=GL_2(\boldsymbol{C})$ is identified with $\mathfrak{gl}_2(\boldsymbol{C})$ regarded as a real Lie algebra. We have

$$\mathfrak{gl}_2(\boldsymbol{C})\otimes\boldsymbol{C}=\mathfrak{gl}_2(\boldsymbol{C})\oplus\mathfrak{gl}_2(\boldsymbol{C}),$$

where $X=X\otimes 1$ is identified with $(X, \bar{X})$ for $X \in \mathfrak{gl}_2(\boldsymbol{C})$. From the embeddings i_1: $X\to(X, 0)$ and i_2: $X\to(0, X)$ we obtain the isomorphisms i_1 and i_2 of $\mathscr{U}(\mathfrak{gl}_2(\boldsymbol{C}))$ into $\mathscr{U}(\mathfrak{gl}_2(\boldsymbol{C})\oplus\mathfrak{gl}_2(\boldsymbol{C}))$. Then the isomorphism

$$\mathscr{U}(\mathfrak{gl}_2(\boldsymbol{C}))\otimes\mathscr{U}(\mathfrak{gl}_2(\boldsymbol{C}))\xrightarrow{\sim}\mathscr{U}(\mathfrak{gl}_2(\boldsymbol{C})\oplus\mathfrak{gl}_2(\boldsymbol{C}))$$

is induced by $X\otimes Y\to i_1(X)i_2(Y)(X, Y \in \mathscr{U}(\mathfrak{gl}_2(\boldsymbol{C}))$. Identifying the both sides by this isomorphism, we get

$$\rho(X\otimes 1)=\rho(i_1(X))=\frac{1}{2}\rho(X)-\frac{i}{2}\rho(iX),$$

$$\rho(1\otimes X)=\rho(i_2(X))=\frac{1}{2}\rho(\bar{X})+\frac{i}{2}\rho(i\bar{X}),$$

for $X \in \mathfrak{gl}_2(\boldsymbol{C})$, since

$$(X, 0)=\frac{1}{2}(X, \bar{X})-\frac{i}{2}(iX, -i\bar{X}),$$

$$(0, X)=\frac{1}{2}(\bar{X}, X)+\frac{i}{2}(i\bar{X}, -iX).$$

Hence

$$\rho(X\otimes 1)\varphi(g)=\frac{1}{2}\frac{d}{dt}\varphi(g\exp tX)_{t=0}-\frac{i}{2}\frac{d}{dt}\varphi(g\exp tiX)_{t=0},$$

$$\rho(1\otimes X)\varphi(g)=\frac{1}{2}\frac{d}{dt}\varphi(g\exp t\bar{X})_{t=0}+\frac{i}{2}\frac{d}{dt}\varphi(g\exp ti\bar{X})_{t=0},$$

or regarding t as a complex variable, we have

$$\rho(X\otimes 1)\varphi(g)=\frac{\partial}{\partial t}\varphi(g\exp tX)_{t=0},$$

$$\rho(1\otimes X)\varphi(g)=\frac{\partial}{\partial\bar{t}}\varphi(g\exp t\bar{X})_{t=0}.$$

The center $\mathscr{Z}_C$ of $\mathscr{U}(\mathfrak{gl}_2(C)_C)$ is a polynomial ring over C generated by $J\otimes 1$, $D\otimes 1$, $1\otimes J$, $1\otimes D$. Let B (resp. N) be the group of upper triangular (resp. unipotent) matrices in G. Let R be a complete system of representatives of $B_C\backslash G_C$ in G_C and write $g\in G_C$ as

$$g=znah,\quad n=n(x),\quad a=a(y^{1/2})$$

with $z, x, y\in C$, $h\in R$ (here we set $y^{1/2}=\exp(\frac{1}{2}\log y)$, taking a certain branch of $\log y$). Then it follows from the bi-invariance of $\rho(D\otimes 1)$ that

$$\begin{aligned}\rho(D\otimes 1)\varphi(g)&=\rho(h)\,(\rho(D\otimes 1)\varphi)(zna)\\&=\rho(D\otimes 1)(\rho(h)\varphi)(zna).\end{aligned}$$

Put $U=\begin{pmatrix}0&1\\0&0\end{pmatrix}$. We have

$$\rho(X_1\otimes 1)\rho(h)\varphi(zna)=\frac{\partial}{\partial t}\varphi(zna(y^{1/2})a(e^t)h)_{t=0}=2y\frac{\partial}{\partial y}\varphi(g),$$

$$\begin{aligned}\rho(U\otimes 1)\rho(h)\varphi(zna)&=\rho\,(\mathrm{Ad}\,(a)U\otimes 1)\rho(ah)\varphi(zn)\\&=\frac{\partial}{\partial t}\varphi(zn(x)n(yt)ah)_{t=0}=y\frac{\partial}{\partial x}\varphi(g),\end{aligned}$$

because $\mathrm{Ad}\,(a)U=aUa^{-1}=yU$.

Suppose that the representatives in R are taken from $SU(2)$. We have

$$\begin{aligned}X_2^2&=(2U-X_3)^2=4U^2+X_3^2-2(UX_3+X_3U)\\&=4U^2+X_3^2-4UX_3-2X_1\end{aligned}$$

(since $X_3U-UX_3=X_1$), and

$$D=\tfrac{1}{2}(X_1^2+X_2^2-X_3^2)=\tfrac{1}{2}X_1^2-X_1+2U(U-X_3).$$

Note further that

$$\begin{aligned}\rho((U-X_3)\otimes 1)&=\frac{1}{2}\rho(U-X_3)-\frac{i}{2}\rho(i(U-X_3))\\&=\frac{1}{2}\rho(U-X_3)-\frac{i}{2}\rho(i(X_2-U))\\&=\rho(1\otimes U)-\frac{1}{2}\rho(X_3)-\frac{i}{2}\rho(iX_2).\end{aligned}$$

We finally obtain

$$\begin{aligned}\rho(D\otimes 1)\varphi(g)=&\left[2\left(y\frac{\partial}{\partial y}\right)^2-2y\frac{\partial}{\partial y}+2y\frac{\partial}{\partial x}\left(\bar{y}\frac{\partial}{\partial \bar{x}}\right)\right]\varphi(g)\\&-y\frac{\partial}{\partial x}[\rho(X_3)+i\rho(iX_2)]\rho(h)\varphi(zna),\end{aligned}\tag{1.2}$$

where, by definition,

$$\rho(X_3)\rho(h)\varphi(zna)=\frac{d}{dt}\varphi(znak(t)h)_{t=0},$$

$$\rho(iX_2)\rho(h)\varphi(zna)=\frac{d}{dt}\varphi(znaw_0k(t)w_0^{-1}h)_{t=0}$$

with $w_0=\begin{pmatrix}0 & 1\\ i & 0\end{pmatrix}$. Especially, if φ is left N_C-invariant, then

$$\rho(D\otimes 1)\varphi(g)=2\left[\left(y\frac{\partial}{\partial y}\right)^2-y\frac{\partial}{\partial y}\right]\varphi(g).\tag{1.3}$$

A similar expression is valid for $\rho(1\otimes D)$.

Let $v\in P_\infty$. If v is real (resp. imaginary), denote by D_v (resp. D'_v, D''_v) an element $\otimes_{w\in P_\infty}Z_w$ in $\mathscr{Z}=\otimes_{w\in P_\infty}\mathscr{Z}_w$ such that $Z_w=1$ $(w\neq v)$, $Z_v=D$ (resp. $D\otimes 1$, $1\otimes D$).

6. We fix a non-trivial character ψ of A/F.

If $\varphi\in\mathscr{A}(\eta)$ and $g\in G_A$, then

$$x\longrightarrow\varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right)$$

in a function on A invariant under the translations $x\to x+\xi$ $(\xi\in F)$. Therefore it has a Fourier expansion of the form

$$\varphi\left(\begin{pmatrix}1 & x\\0 & 1\end{pmatrix}g\right)=\sum_{\alpha\in F}c(\alpha, g)\psi(\alpha x),$$

$$c(\alpha, g)=\int_{A/F}\varphi\left(\begin{pmatrix}1 & x\\0 & 1\end{pmatrix}g\right)\psi(-\alpha x)dx,$$

where dx is a Haar measure of A such that the total volume of A/F is 1. Obviously

$$c(0, g)=\varphi^0(g),\ c(\alpha, g)=c\left(1, \begin{pmatrix}\alpha & 0\\0 & 1\end{pmatrix}g\right) \qquad (\alpha\neq 0)$$

so that, putting $W_\varphi(g)=c(1, g)$, we have

$$\varphi\left(\begin{pmatrix}1 & x\\0 & 1\end{pmatrix}g\right)=\varphi^0(g)+\sum_{\alpha\in F^\times}W_\varphi\left(\begin{pmatrix}\alpha & 0\\0 & 1\end{pmatrix}g\right)\psi(\alpha x).$$

It is evident that the mappings $\varphi\to\varphi^0$ and $\varphi\to W_\varphi$ commute with the right translations.

The constant term φ^0 of the Fourier expansion plays a principal role in our investigation. φ^0 is $\mathscr{Z}$-finite, since φ is so (Lemma 2), and it is left N_A-invariant, where N is the group of upper unipotent matrices. Let A be the group of diagonal matrices in G. Then we have $G_A=N_A A_A K$. Fix a place v in P_∞ and identify $a(e^t)$ for $t\in F_v$ with an element in A_A such that the v-component is $a(e^t)$ and all the other components are 1. For $k\in K$ and $a\in A_A$ with $a_v=1$, we consider a function

$$u(t)=\varphi^0(a(e^t)ak).$$

If v is real, then

$$\rho(D_v)\varphi^0(a(e^t)ak)=\left[\frac{1}{2}\left(\frac{\partial}{\partial t}\right)^2-\frac{\partial}{\partial t}\right]u(t). \tag{1.4}$$

If v is imaginary, write $t=\tau+i\theta$ with $\tau, \theta\in \boldsymbol{R}$; then we have

(1.5)
$$\rho(D'_v)\varphi^0(a(e^t)ak)=\frac{1}{8}\left[\left(\frac{\partial}{\partial\tau}\right)^2-4\frac{\partial}{\partial\tau}-\left(\frac{\partial}{\partial\theta}\right)^2-2i\left(\frac{\partial}{\partial\tau}-2\right)\frac{\partial}{\partial\theta}\right]u(t),$$

$$\rho(D''_v)\varphi^0(a(e^t)ak)=\frac{1}{8}\left[\left(\frac{\partial}{\partial\tau}\right)^2-4\frac{\partial}{\partial\tau}-\left(\frac{\partial}{\partial\theta}\right)^2+2i\left(\frac{\partial}{\partial\tau}-2\right)\frac{\partial}{\partial\theta}\right]u(t).$$

Recall that φ^0 is $\mathscr{Z}$-finite and right K-finite. The above equalities imply that, if we put

$$L=\frac{1}{2}\left(\frac{\partial}{\partial t}\right)^2-\frac{\partial}{\partial t}\qquad \left(\text{resp. } \left(\frac{\partial}{\partial \tau}\right)^2-4\frac{\partial}{\partial \tau}\right)$$

for real (resp. imaginary) v, then $L^n u$ $(n=0, 1, 2, \cdots)$ span a finite dimensional space V.

Let $f(x)$ be the characteristic polynomial of L on V. It is easy to see that every solution of the differential equation $f(L)u=0$ is a finite linear combination of $|e^{pt}|(\mathrm{Re}\, t)^m$ $(p \in \boldsymbol{C}, m \in \boldsymbol{Z}, m \geqq 0)$ as a function of Re t.

Lemma 3. *φ^0 is left A_A-finite.*

Proof. φ^0 is left $(A_A \cap K)$-finite, since it is right K-finite (if $n \in N_A$, a $\in A_A$, $k \in K$, $a_0 \in A_A \cap K$, then $\varphi^0(a_0 nak)=\varphi^0(aa_0k)$). For $v \in P_\infty$, put

$$A_v^+=\left\{\begin{pmatrix} x & 0\\ 0 & y\end{pmatrix} \in A_v \,|\, x>0,\ y>0\right\}.$$

It follows from the preceding remark that

$$\varphi^0\left(\begin{pmatrix} x & 0\\ 0 & y\end{pmatrix}ak\right)\qquad (a \in A_A,\ k \in K)$$

is, as a function of x and y, a finite linear combination of $x^p y^q (\log x)^m \times (\log y)^n$ $(p, q \in \boldsymbol{C}, m, n \in \boldsymbol{Z}, m, n \geqq 0)$. Therefore, φ^0 is left A_v^+-finite. Since $A_v=A_v^+(A_v \cap K_v)$ for $v \in P_\infty$ and $A_A/A_F(A_A \cap K)A_\infty$ is a finite group, our assertion follows. q.e.d.

Denote by $|\ \ |_v$ the normalized valuation of $F_v (v \in P)$ and put

$$|x|_A=\prod_{v \in P}|x_v|_v \qquad (x \in A).$$

We write occasionally $\alpha(x)=|x|_A$.

Theorem 1. *For every φ in $\mathscr{A}(\eta)$ and g in G_A, $\varphi^0\left(\begin{pmatrix} x & 0\\ 0 & y\end{pmatrix}g\right)$ is, as a function of x and y in $A^\times$, a finite linear combination of $\alpha(xy^{-1})^{1/2}\mu(x)\nu(y)\times(\log \alpha(xy^{-1}))^m$, where $m \in \boldsymbol{Z} \geqq 0$ and μ, ν are quasi-characters of $A^\times/F^\times$ such that $\mu\nu=\eta$; in other words we have an expression of the form*

$$\varphi^0\left(\begin{pmatrix} x & 0\\ 0 & y\end{pmatrix}g\right)=\sum_{\mu\nu=\eta,\, m\geqq 0}\alpha(xy^{-1})^{1/2}\mu(x)\nu(y)\,(\log \alpha(xy^{-1}))^m f_{\mu\nu m}(g)$$

with certain functions $f_{\mu\nu m}$.

Proof. Put

$$A_A^1=\left\{\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \in A_A \,\middle|\, |a_1|_A=|a_2|_A=1\right\}.$$

We identity a positive real number t with an element in $A^\times$ such that the v-component is t for any $v \in P_\infty$ and all the other components are 1. Then, putting

$$A_\infty^+=\left\{\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \,\middle|\, t_1,\ t_2 \in \boldsymbol{R}_+\right\},$$

we have $A_A=A_A^1\times A_\infty^+$. $\varphi^0(ak)$ $(a \in A_A,\ k \in K)$ is, as a function on A_∞^+, a linear combination of

$$t_1^p t_2^q\,(\log t_1)^m\,(\log t_2)^n \qquad (p,\ q \in \boldsymbol{C},\ m,\ n \in \boldsymbol{Z},\ \geqq 0)$$

(cf. the proof of Lemma 3). By Lemma 3, if it is regarded as a function on A_A^1, it is $(A_A^1/F^\times)$-finite and hence is a linear combination of

$$\chi_1(a_1)\chi_2(a_2),$$

where χ_1 and χ_2 are characters of $A_A^1/F^\times$. Noting that $\varphi^0(zg)=\eta(z)\varphi^0(g)$ for $z \in A^\times$, we get our assertion for $g=k$. Evidently, k may be replaced by any element in G_A. q.e.d.

7. For every φ in $\mathscr{A}(\eta)$, the space $\{\rho(Z)\varphi \mid Z \in \mathscr{Z}\}$ is finite-dimensional by Lemma 2. Hence $Z\to\rho(Z)\varphi$ defines a homomorphism of $\mathscr{Z}$ into the endomorphism algebra of this space, whose kernel is an ideal of finite codimension. $\mathfrak{a}$ being any such ideal of $\mathscr{Z}$, we set

$$\mathscr{A}(\eta,\ \mathfrak{a})=\{\varphi \in \mathscr{A}(\eta) \mid \rho(Z)\varphi=0 \ \text{ for } Z \in \mathfrak{a}\}.$$

Then $\mathscr{A}(\eta)$ is a union of $\mathscr{A}(\eta,\ \mathfrak{a})$ if $\mathfrak{a}$ runs through all ideals of $\mathscr{Z}$ of finite codimension. Let $\mathscr{A}_0(\eta,\ \mathfrak{a})$ be the space of all cusp forms in $\mathscr{A}(\eta,\ \mathfrak{a})$.

Theorem 2. *For every elementary idempotent ξ of K, the space*

$$\rho(\xi)A_0(\eta,\ \mathfrak{a})=\{\varphi \in \mathscr{A}_0(\eta,\ \mathfrak{a}) \mid \rho(\xi)\varphi=\varphi\}$$

is finite-dimensional.

The theorem asserts that the cusp forms of a given 'type' make up a finite-dimensional space. cf. [7, Proposition 10.8], [5, Theorem 1].

We say that a $A^\times$-finite and left G_F-invariant function φ on G_A is rapidly decreasing, if for every compact subset C of G_A and for every $N>0$, there exists a $M>0$ such that

$$\left|\varphi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right)\right| \leqq M|a|_A^{-N} \qquad (a \in A^\times, |a|_A \geqq 1, g \in C).$$

It is known (cf. [7, § 10], [5, § 4]) that every cusp form is rapidly decreasing so that if $\varphi_1 \in \mathscr{A}_0(\eta)$ and $\varphi_2 \in \mathscr{A}(\eta)$, then $|\varphi_1\varphi_2|$ is bounded on G_A. Hence the inner product

$$(\varphi_1, \varphi_2) = \int_{A^\times G_F \backslash G_A} \varphi_1(g)\overline{\varphi_2(g)}dg$$

can be defined for φ_1, $\varphi_2 \in \mathscr{A}(\eta)$ whenever either one of φ_1, φ_2 is a cusp form.

Lemma 4. *Put*

$$\mathscr{A}_1(\eta, \mathfrak{a}) = \{\varphi \in \mathscr{A}(\eta, \mathfrak{a}) | (\varphi, \varphi_0) = 0 \textit{ for all } \varphi_0 \in \mathscr{A}_0(\eta, \mathfrak{a})\};$$

then we have

$$\mathscr{A}(\eta, \mathfrak{a}) = \mathscr{A}_0(\eta, \mathfrak{a}) \oplus \mathscr{A}_1(\eta, \mathfrak{a}).$$

Proof. Let ξ be an elementary idempotent of K. For $\varphi \in \mathscr{A}(\eta)$ and $\varphi_0 \in \mathscr{A}_0(\eta)$, we have $(\rho(\xi)\varphi, \varphi_0) = (\varphi, \rho(\xi)\varphi_0)$ and hence $(\rho(\xi)\varphi, (1-\rho(\xi))\varphi_0) = (\varphi, \rho(\xi)(1-\rho(\xi))\varphi_0) = 0$. Let $\{\varphi_1, \cdots, \varphi_n\}$ be an orthonormal basis of $\rho(\xi)\mathscr{A}_0(\eta, \mathfrak{a})$. If φ is in $\rho(\xi)\mathscr{A}(\eta, \mathfrak{a})$, then

$$\psi = \varphi - \sum_{i=1}^{n} (\varphi, \varphi_i)\varphi_i$$

is orthogonal to $\rho(\xi)\mathscr{A}_0(\eta, \mathfrak{a})$. Consequently, it is also orthogonal to

$$\mathscr{A}_0(\eta, \mathfrak{a}) = \rho(\xi)\mathscr{A}_0(\eta, \mathfrak{a}) + (1-\rho(\xi))\mathscr{A}_0(\eta, \mathfrak{a}).$$

This proves that

$$\rho(\xi)\mathscr{A}(\eta, \mathfrak{a}) \subset \rho(\xi)\mathscr{A}_0(\eta, \mathfrak{a}) + \mathscr{A}_1(\eta, \mathfrak{a})$$

and, since $\mathscr{A}(\eta, \mathfrak{a})$ is a union of $\rho(\xi)\mathscr{A}(\eta, \mathfrak{a})$ for all ξ,

$$\mathscr{A}(\eta, \mathfrak{a}) = \mathscr{A}_0(\eta, \mathfrak{a}) + \mathscr{A}_1(\eta, \mathfrak{a}).$$

That the sum above is direct is obvious. q.e.d.

The Hecke algebra $\mathscr{H}$ is made to act on $\mathscr{A}(\eta)$ by $\varphi \to \rho(f)\varphi$ ($f \in \mathscr{H}$, $\varphi \in \mathscr{A}(\eta)$). $\mathscr{A}_0(\eta)$ is then a $\mathscr{H}$-invariant subspace.

Theorem 3. *Regard $\mathscr{A}_0(\eta)$ as a representation space of $\mathscr{H}$. Then $\mathscr{A}_0(\eta)$ is a direct sum of irreducible subspaces, on each of which the representation of $\mathscr{H}$ is admissible. Moreover, the multiplicity of every irreducible representation of $\mathscr{H}$ in $\mathscr{A}_0(\eta)$ is at most* 1.

cf. [7, Proposition 10.9], [2]. As for the multiplicity one theorem, cf. [7, Proposition 11.1.1], [12].

§ 2. Induced representations

8. In this section we quote from [3, 7] several results needed later. Let (μ, ν) be a pair of quasi-characters of $A^\times/F^\times$. Let $\mathscr{B}(\mu, \nu)$ be the space of continuous functions φ on G_A satisfying the following conditions.

(i) $\varphi\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g\right) = \left|\frac{a}{b}\right|_A^{1/2} \mu(a)\nu(b)\varphi(g)$ for $a, b \in A^\times$, $x \in A$, $g \in G_A$.

(ii) φ is right K-finite.

Let $\pi(\mu, \nu)$ denote the representation of $\mathscr{H}$ on $\mathscr{B}(\mu, \nu)$ defined by the right translation ρ.

A space analogous to the above can be defined locally; namely, (μ_v, ν_v) being a pair of quasi-characters of $F_v^\times$ for $v \in P$, let $\mathscr{B}(\mu_v, \nu_v)$ be the space of continuous functions φ on G_v such that

(i) $\varphi\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g\right) = \left|\frac{a}{b}\right|_v^{1/2} \mu_v(a)\nu_v(a)\varphi(g)$ for $a, b \in F_v^\times$, $x \in F_v$, $g \in G_v$,

(ii) φ is right K_v-finite.

We then obtain a representation $\pi(\mu_v, \nu_v)$ of $\mathscr{H}_v$ on $\mathscr{B}(\mu_v, \nu_v)$ in the same way.

If μ_v and ν_v denote the v-components of μ and ν, respectively, then μ_v and ν_v are unramified for almost all v. For such a v, there exists a function φ_v^0 in $\mathscr{B}(\mu_v, \nu_v)$ such that $\varphi_v^0 = 1$ on K_v. We see that

$$\mathscr{B}(\mu, \nu) = \otimes_{v \in P} \mathscr{B}(\mu_v, \nu_v).$$

where the right hand side is the restricted tensor product with respect to $\{\varphi_v^0\}$. Also it is evident that

$$\rho(f)\varphi = \otimes_{v \in P} \rho(f_v)\varphi_v$$

if $f = \otimes f_v \in \mathscr{H}$ and $\varphi = \otimes \varphi_v \in \mathscr{B}(\mu, \nu)$ (note that $\rho(f_v^0)\varphi_v^0 = \varphi_v^0$, f_v^0 being the same as in no. 2). In this sense the representation $\pi(\mu, \nu)$ of $\mathscr{H}$ is the tensor product of the representations $\pi(\mu_v, \nu_v)$ of $\mathscr{H}_v$.

For $\varphi_1 \in \mathscr{B}(\mu, \nu)$ and $\varphi_2 \in \mathscr{B}(\bar{\mu}^{-1}, \bar{\nu}^{-1})$ we set

$$(\varphi_1, \varphi_2) = \int_K \varphi_1\bar{\varphi}_2(k)dk = \int_{B_A \backslash G_A} \varphi_1\bar{\varphi}_2(g)d\dot{g},$$

$d\dot{g}$ being a right invariant mesaure on $B_A \backslash G_A$. It defines a non-degenerate pairing on $\mathscr{B}(\mu, \nu) \times \mathscr{B}(\bar{\mu}^{-1}, \bar{\nu}^{-1})$ and we have

$$(\pi_1(f)\varphi_1, \pi_2(f)\varphi_2) = (\varphi_1, \varphi_2)$$

for $f \in \mathscr{H}$, where $\pi_1 = \pi(\mu, \nu)$, $\pi_2 = \pi(\bar{\mu}^{-1}, \bar{\nu}^{-1})$.

Put $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Write $\mu\nu^{-1} = |\ |_A^s \chi$ with $s \in \boldsymbol{C}$ and a character χ of $A^\times / F^\times$. Assuming that $\mathrm{Re}\, s > 1$, define an operator $M(\lambda, \mu)$ on $\mathscr{B}(\mu, \nu)$ by

$$M(\lambda, \mu)\varphi(g) = \int_A \varphi\left(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx.$$

It is easy to see that $M(\lambda, \mu)$ maps $\mathscr{B}(\lambda, \mu)$ into $\mathscr{B}(\nu, \mu)$. Furthermore we have

$$M(\mu, \nu)\pi_1(f) = \pi_2(f) M(\mu, \nu)$$

for $f \in \mathscr{H}$, $\pi_1 = \pi(\mu, \nu)$, $\pi_2 = \pi(\nu, \mu)$ and

$$(M(\mu, \nu)\varphi_1, \varphi_2) = (\varphi_1, M(\bar{\nu}^{-1}, \bar{\mu}^{-1})\varphi_2)$$

for $\varphi_1 \in \mathscr{B}(\mu, \nu)$, $\varphi_2 \in \mathscr{B}(\bar{\nu}^{-1}, \bar{\mu}^{-1})$.

9. We recall a few facts on the zeta functions of local fields. Let V be a vector space of finite dimension over F_v. Let $\mathscr{S}(V)$ denote the space of Schwartz-Bruhat functions on V (if $v \in P_f$, it consists of all locally constant and compactly supported functions on V; if $v \in P_\infty$, it consists of all rapidly decreasing functions on V).

Let f be in $\mathscr{S}(F_v)$, χ a quasi-character of $F_v^\times$ and $s \in \boldsymbol{C}$. We set

$$Z(f, \chi, s) = \int_{F_v^\times} f(t)\chi(t)|t|_v^s d^\times t.$$

If χ is a character, the integral converges for $\mathrm{Re}\, s > 0$. There exists an Euler factor $L(s, \chi)$ such that $Z(f, \chi, s)/L(s, \chi)$ is continued to an entire function for all f in $\mathscr{S}(F_v)$. Fixing a character ψ of F_v, we obtain a functional equation

$$\frac{Z(\hat{f}, \chi^{-1}, 1-s)}{L(1-s, \chi^{-1})} = \varepsilon(s, \chi, \psi) \frac{Z(f, \chi, s)}{L(s, \chi)},$$

where $\varepsilon(s, \chi, \psi)$ is an exponential function of s and

$$\hat{f}(x) = \int_F f(y)\psi(xy) dy.$$

$L(s, \chi)$ is explicitly known.

(1) $v \in P_f$

$$L(s, \chi)=\begin{cases}(1-\chi(\varpi_v)|\varpi_v|_v^s)^{-1} & \text{if } \chi \text{ is unramified,}\\ 1 & \text{otherwise.}\end{cases}$$

Here ϖ_v is a prime element of F_v.

(2) $v \in P_\infty$

If v is real and $\chi(x)=|x|^r (\operatorname{sgn} x)^m$ with $r \in \boldsymbol{C}$, $m=0, 1$, then

$$L(s, \chi)=\pi^{-(s+r+m)/2}\Gamma\left(\frac{s+r+m}{2}\right).$$

If v is imaginary and $\chi(x)=|x|_v^r x^m \bar{x}^n$ with $r \in \boldsymbol{C}$, $m, n \in \boldsymbol{Z}$, $mn=0$, then

$$L(s, \chi)=2(2\pi)^{-(s+r+m+n)}\Gamma(s+r+m+n).$$

Let $\chi(x)=\prod_v \chi_v(x_v)$ be a quasi-character of $A^\times/F^\times$ and $\psi(x)=\prod_v \psi_v(x_v)$ a character of A/F. Put

$$L(s, \chi)=\prod_{v \in P} L(s, \chi_v),$$
$$\varepsilon(s, \chi)=\prod_{v \in P} \varepsilon(s, \chi_v, \psi_v).$$

Then $L(s, \chi)$ can be analytically continued to the whole s-plane and satisfies the following functional equation.

$$L(s, \chi)=\varepsilon(s, \chi)L(1-s, \chi^{-1}).$$

10. For $\Phi \in \mathscr{S}(F_v \times F_v)$ and $g \in G_v$, put

$$\varphi(g; \mu_v, \nu_v, \Phi)=\frac{\mu_v(\det g)|\det g|_v^{1/2}}{L(1, \mu_v\nu_v^{-1})}\int_{F_v^\times}\Phi((0, t)g)\mu_v\nu_v^{-1}(t)|t|_v d^\times t.$$

The right hand side may be written as

$$\mu_v(\det g)|\det g|_v^{1/2} Z(f_{\rho(g)\Phi}, \mu_v\nu_v^{-1}, 1)/L(1, \mu_v\nu_v^{-1})$$

with $f_\Phi(t)=\Phi((0, t))$ and $\rho(g)\Phi(x, y)=\Phi((x, y)g)$. In this form it makes sense for all μ_v, ν_v.

Lemma 5. *Let Φ be an element in $\mathscr{S}(F_v \times F_v)$ such that the functions $\rho(k)\Phi$ $(k \in K_v)$ span a finite-dimensional space. Then $\varphi(\ ; \mu_v, \nu_v, \Phi)$ belongs to $\mathscr{B}(\mu_v, \nu_v)$. Conversely, assume that $\mu_v\nu_v^{-1}=|\ |_v^s\chi$ with a character χ of $F_v^\times$ and $s \in \boldsymbol{C}$,* Re $s>-1$; *then, for every φ in $\mathscr{B}(\mu_v, \nu_v)$, there exists a Φ in $\mathscr{S}(F_v \times F_v)$ such that $\varphi=\varphi(\ ; \mu_v, \nu_v, \Phi)$.*

Proof. The first assertion is obvious if the integral defining

$\varphi(\ ;\mu_v,\nu_v,\Phi)$ converges. It holds in general by analytic continuation.

To prove the second assertion, we first assume that $v\in P_f$. For a given φ, define Φ as follows:

$$\Phi(x,y)=\mu_v^{-1}(\det g)\varphi(g)$$

if $(x,y)=(0,1)g$ for $g\in GL_2(\mathfrak{o}_v)$ and equals 0 otherwise. It is easy to see that the function Φ has a required property.

Next assume that $v\in P_\infty$ is real. Write $\mu_v\nu_v^{-1}(t)=|t|_v^s(\operatorname{sgn} t)^m$ with $s\in\boldsymbol{C}$, $m=0, 1$. Let $\varphi_n(n\in\boldsymbol{Z})$ be an element in $\mathscr{B}(\mu_v,\nu_v)$ such that $\varphi_n(gk(\theta))=e^{in\theta}\varphi_n(g)$ for $g\in G_v$, $k(\theta)\in SO(2)$. Since $\{\varphi_n|n\equiv m \pmod 2\}$ forms a basis of $\mathscr{B}(\mu_v,\nu_v)$, it is enough to prove the assertion for each φ_n. Put

$$\Phi(x,y)=e^{-\pi(x^2+y^2)}(x+i(\operatorname{sgn} n)y)^{|n|};$$

then

$$\Phi((x,y)k(\theta))=e^{in\theta}\Phi(x,y).$$

By a simple calculation we see that $\varphi(\ ;\mu_v,\nu_v,\Phi)$ is a constant multiple of φ_n.

Finally assume that $v\in P_\infty$ is imaginary. Write

$$\mu_v\nu_v^{-1}(t)=(t\bar{t})^{s-(a+b)/2}t^a\bar{t}^b$$

with $s\in\boldsymbol{C}$, $a, b\in\boldsymbol{Z}$, $\geqq 0$, $ab=0$. We note that $SU(2)$ acts on $\mathscr{B}(\mu_v,\nu_v)$ by the right translation. Denoting by ρ_n the n-th symmetric tensor representation of $SU(2)$, let $\mathscr{B}(\mu_v,\nu_v,\rho_n)$ be the space of all elements φ in $\mathscr{B}(\mu_v,\nu_v)$ such that the representation of $SU(2)$ in a linear span of $\rho(k)\varphi(k\in SU(2))$ decomposes into a direct sum of ρ_n. It is known that ρ_n occurs in $\mathscr{B}(\mu_v,\nu_v)$ with a multiplicity $\leqq 1$ so that the above subspace is irreducible. Further we have

$$\mathscr{B}(\mu_v,\nu_v)=\bigoplus_{n\geqq a+b,\,n\equiv a+b\,(2)}\mathscr{B}(\mu_v,\nu_v,\rho_n).$$

Put

$$\Phi(x,y)=e^{-2\pi(x\bar{x}+y\bar{y})}y^{b+m}\bar{y}^{a+m}$$

for $n=a+b+2m$ ($m\in\boldsymbol{Z}$, $\geqq 0$). We can show that $\varphi(\ ;\mu_v,\nu_v,\Phi)$ is a non-zero element in $\mathscr{B}(\mu,\nu_v,\rho_n)$. Since the mapping $\Phi\to\varphi(\ ;\mu_v,\nu_v,\Phi)$ from $\mathscr{S}(F_v\times F_v)$ into $\mathscr{B}(\mu_v,\nu_v)$ commutes with the action of $SU(2)$, our assertion follows. q.e.d.

11. Let $M(\mu_v, \nu_v)$ be the mapping from $\mathscr{B}(\mu_v, \nu_v)$ to $\mathscr{B}(\nu_v, \mu_v)$ defined by

$$M(\mu_v, \nu_v)\varphi(g)=\int_{F_v}\varphi\left(w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right)dx.$$

The integral converges for $\operatorname{Re} s>0$ in the notation of Lemma 5. Let $\hat{\Phi}$ be the Fourier transform of Φ in $\mathscr{S}(F_v\times F_v)$ with respect to the pairing $\langle(x, y), (x', y')\rangle=\psi_v(yx'-xy')$:

$$\hat{\Phi}(x, y)=\iint\Phi(x', y')\psi_v(yx'-xy')dx'dy'.$$

Assuming that $-1<\operatorname{Re} s<1$ in the notation of Lemma 5, consider $\varphi(\ ;\nu_v, \mu_v, \hat{\Phi})$ as well as $\varphi(\ ;\mu_v, \nu_v, \Phi)$. We are going to see that if $\varphi(\ ;\mu_v, \nu_v, \Phi)=0$ for $\Phi\in\mathscr{S}(F_v\times F_v)$, then $\varphi(\ ;\nu_v, \mu_v, \hat{\Phi})=0$ also so that

$$R(\mu_v, \nu_v)\colon \varphi(\ ;\mu_v, \nu_v, \Phi)\longrightarrow\mu_v\nu_v(-1)\varphi(\ ;\nu_v, \mu_v, \hat{\Phi})$$

is a well defined mapping from $\mathscr{B}(\mu_v, \nu_v)$ into $\mathscr{B}(\nu_v, \mu_v)$.

Observe that B_vwN_v is dense in G_v and hence an element in $\mathscr{B}(\mu_v, \nu_v)$ is determined by its values at $w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$ $(x\in F_v)$. It is easy to see that, for $M>0$,

$$\int_{|t|_v\leq M}\hat{\Phi}\left((0, t)w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\mu_v\nu_v^{-1}(t)|t|_vd^\times t$$
$$=\iint\left\{\int_{|t|_v\leq M}\Phi(ty, tz)\mu_v\nu_v^{-1}(t)|t|_vd^\times t\right\}\psi_v(z-xy)dydz.$$

If $\varphi(g;\mu_v, \nu_v, \Phi)=0$ for all $g\in G_v$, the right hand side can be written as

$$\iint\left\{\int_{|t|_v>M}\Phi(ty, tz)\mu_v\nu_v^{-1}(t)|t|_vd^\times t\right\}\psi_v(z-xy)dydz.$$

If $v\in P_f$, $\Phi(x, y)$ has a compact support and if $v\in P_\infty$, then

$$|\Phi(x, y)|\leq\text{const.}\,(|x|_v^2+1)^{-1}(|y|_v^2+1)^{-1}.$$

It follows that the above integral tends to 0 if $M\to\infty$.

By virtue of the functional equation of a local zeta function, it can be shown that

$$M(\mu_v, \nu_v)=\frac{L(0, \mu_v\nu_v^{-1})}{L(1, \mu_v\nu_v^{-1})\varepsilon(0, \mu_v\nu_v^{-1}, \psi_v)}R(\mu_v, \nu_v)$$

if $0<\mathrm{Re}\, s<1$. Note that, for $-1<\mathrm{Re}\, s<1$,

$$R(\nu_v, \mu_v)R(\mu_v, \nu_v)=\mathrm{id}.$$

In view of the above equalities, we infer that the both $R(\mu_v, \nu_v)$ and $M(\mu_v, \nu_v)$ can be analytically continued to all μ_v, ν_v and $R(\mu_v, \nu_v)$ is holomorphic for $\mathrm{Re}\, s>-1$.

Suppose that, for $v \in P_f$, μ_v and ν_v are unramified and the conductor of ψ_v is $\mathfrak{o}_v$. Let φ_v^0 (resp. $\tilde{\varphi}_v^0$) be the unique element in $\mathscr{B}(\mu_v, \nu_v)$ (resp. $\mathscr{B}(\nu_v, \mu_v)$) whose restriction to K_v is identically 1. If Φ is the characteristic function of $\mathfrak{o}_v \times \mathfrak{o}_v$, we have $\hat{\Phi}=\Phi$ and

$$\begin{aligned}\int_{F_v^\times} &\Phi((0, t))\mu_v\nu_v^{-1}(t)|t|_v d^\times t \\ &=\int_{\mathfrak{o}_v} \mu_v\nu_v^{-1}(t)|t|_v d^\times t \\ &=\sum_{n=0}^{\infty} \mu_v\nu_v^{-1}(\varpi_v)^n|\varpi_v|_v^n \\ &=L(1, \mu_v\nu_v^{-1}).\end{aligned}$$

Hence $\varphi(\ ; \mu_v, \nu_v, \Phi)=\varphi_v^0$; by definition we see that $R(\mu_v, \nu_v)\varphi_v^0=\tilde{\varphi}_v^0$.

Now, let μ, ν be quasi-characters of $A^\times/F^\times$. Let $R(\mu, \nu)$ be the mapping from $\mathscr{B}(\mu, \nu)$ to $\mathscr{B}(\nu, \mu)$ defined as a tensor product of $R(\mu_v, \nu_v)$ for $v \in P$:

$$R(\mu, \nu)\varphi=\otimes_v R(\mu_v, \nu_v)\varphi_v$$

for $\varphi=\otimes_v\varphi_v \in B(\mu, \nu)$. This definition makes sense because of the preceding remark. We have then

$$\begin{aligned}M(\mu, \nu)&=\frac{L(0, \mu\nu^{-1})}{L(1, \mu\nu^{-1})\varepsilon(0, \mu\nu^{-1})}R(\mu, \nu) \\ &=\frac{L(1, \nu\mu^{-1})}{L(1, \mu\nu^{-1})}R(\mu, \nu)\end{aligned}$$

and

$$R(\nu, \mu)R(\mu, \nu)=\mathrm{id}, \qquad M(\nu, \mu)M(\mu, \nu)=\mathrm{id}.$$

Theorem 4. *Write* $\mu=|\ |_A^{s/2}\chi_1$, $\nu=|\ |_A^{-s/2}\chi_2$ *with* $s \in \boldsymbol{C}$ *and characters* χ_1, χ_2 *of* $A^\times/F^\times$. *Then* $M(\mu, \nu)$ *can be analytically continued to a meromorphic function on the whole s-plane and satisfies the functional equation*

$$M(\nu, \mu)M(\mu, \nu)=\mathrm{id}.$$

In the region $\mathrm{Re}\, s > -1$, *it has a pole only at* $(\mu, \nu) = (|\ |_A^{1/2}\chi, |\ |_A^{-1/2}\chi)$, *where* χ *is a character of* $A^\times/F^\times$.

The last assertion follows from the known property of $L(s, \chi)$.

12. The notation being the same as in no. 8, consider, as before, $\mathscr{B}(\mu_v, \nu_v)$ as a representation space of $\mathscr{H}_v$.

(1) For $v \in P_f$, $\mathscr{B}(\mu_v, \nu_v)$ is reducible if and only if $\mu_v\nu_v^{-1} = |\ |_v$ or $|\ |_v^{-1}$ ([7, Theorem 3.3]).

(2) For a real v in P_∞, $\mathscr{B}(\mu_v, \nu_v)$ is reducible if and only if there exists a $p \in \boldsymbol{Z}, p \neq 0$ such that $\mu_v\nu_v^{-1}(x) = x^p \operatorname{sgn} x$ $(x \in F_v^\times)$ ([7, Theorem 5.11]).

(3) For an imaginary v in P_∞, $\mathscr{B}(\mu_v, \nu_v)$ is reducible if and only if there exist $p, q \in \boldsymbol{Z}, pq > 0$ such that $\mu_v\nu_v^{-1}(x) = x^p \bar{x}^q$ $(x \in F_v^\times)$ ([7, Lemma 6.1]).

In either case, if $\mathscr{B}(\mu_v, \nu_v)$ is reducible, $\mathscr{B}(\mu_v, \nu_v)$ has the only one irreducible subspace, which is denoted by $\mathscr{B}_f(\mu_v, \nu_v)$ or $\mathscr{B}_s(\mu_v, \nu_v)$ according as its dimension is finite or infinite.

Lemma 6. *Write* $\mu_v\nu_v^{-1} = |\ |_v^s \chi$ *with* $s \in \boldsymbol{C}$ *and a character* χ *of* $F_v^\times$. *If* $\mathrm{Re}\, s > 0$ *and* $\mathscr{B}(\mu_v, \nu_v)$ *is reducible, then* $R(\mu_v, \nu_v)$ *maps* $\mathscr{B}(\mu_v, \nu_v)$ *onto* $\mathscr{B}_f(\nu_v, \mu_v)$, *and its kernel is* $\mathscr{B}_s(\mu_v, \nu_v)$.

Proof. It is enough to prove that $R(\mu_v, \nu_v)$ or $M(\mu_v, \nu_v)$ has non-trivial image of finite dimension. Let $\Phi \in \mathscr{S}(F_v \times F_v)$ and write $\varphi = \varphi(\ ; \mu_v, \nu_v, \Phi)$ for simplicity. We have

$$M(\mu_v, \nu_v)\varphi(g) = \frac{\mu_v(\det g)|\det g|_v^{1/2}}{L(1, \mu_v\nu_v^{-1})} \int_{F_v}\int_{F_v^\times} \Phi\left((0, t)w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right)\mu_v\nu_v^{-1}(t)|t|_v d^\times t dx.$$

The integral on the right hand side equals

$$(*) \qquad \int_{F_v}\int_{F_v^\times} \Phi((t, u)g)\mu_v\nu_v^{-1}(-t)d^\times t du.$$

If $v \in P_f$, we have $\mu_v\nu_v^{-1} = |\ |_v$ by (1). Then $(*)$ is written as

$$|\det g|_v^{-1}\int_{F_v}\int_{F_v} \Phi(t, u)dtdu$$

so that the image of $M(\mu_v, \nu_v)$ is generated by a single function

$$g \longrightarrow \mu_v(\det g)|\det g|_v^{-1/2}.$$

If v is real, we have $\mu_v\nu_v^{-1}(t)=t^p \operatorname{sgn} t$ by (2), where $p \in \boldsymbol{Z}, >0$. Writing

$$g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ (t, u)=(x, y)g^{-1}=(\det g)^{-1}(xd-yc, \ -xb+ya),$$

we see that (*) equals

$$(\det g)^{1-p}|\det g|_v^{-1}\int_{F_v}\int_{F_v}\Phi(x, y)(yc-xd)^{p-1}dxdy.$$

Hence the image of $M(\mu_v, \nu_v)$ is generated by

$$\nu_v(\det g)|\det g|_v^{1/2}P(c, d),$$

where $P(c, d)$ is a homogeneous polynomial of degree $p-1$.

If v is imagenary, we have $\mu_v\nu_v^{-1}(t)=t^p\bar{t}^q$ by (3), where $p, q \in \boldsymbol{Z}, >0$. The proof proceeds in the same way as in the real case. The image of $M(\mu_v, \nu_v)$ is generated by

$$\nu_v(\det g)|\det g|_v^{1/2}P(c, d)\overline{Q(c, d)},$$

where $P(c, d)$ and $Q(c, d)$ are homogeneous polynomials of degree $p-1$ and $q-1$, respectively.

§ 3. Eisenstein series

13. Let μ and ν be quasi-characters of $A^\times/F^\times$ and φ an element in $\mathscr{B}(\mu, \nu)$. A function on G_A of the form

$$E(\varphi, g)=\sum_{\gamma \in B_F\backslash G_F}\varphi(\gamma g)$$

is called Eisenstein series. We often denote by $E(\varphi)$ the function $g\to E(\varphi, g)$.

We set $\delta(g)=|a_1/a_2|_A$ for

$$g=nak, \quad n \in N_A, \quad a=\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \in A_A, \quad k \in K.$$

Lemma 7. *μ, ν and φ being as above, write $|\mu\nu^{-1}(x)|=|x|_A^\sigma$ ($x \in A^\times$) with $\sigma \in \boldsymbol{R}$. If $\sigma>1$, then the Eisenstein series $E(\varphi, g)$ is uniformly convergent on every compact subset of G_A.*

Proof. We first assert that there exists an element f in $\mathscr{H}$ such that $\rho(f)\varphi=\varphi$. Since a function in $\mathscr{B}(\mu, \nu)$ is determined by its restriction to

K, $\rho(\xi)\mathscr{B}(\mu, \nu)$ is finite-dimensional for every elementary idempotent ξ of K. Therefore, the above assertion follows as in Lemma 1.

Let C_0 be any compact subset of G_A. Let C_1 be the support of f and M the maximum of $|f|$. If $g \in C_0$ and $\gamma \in G_F$, then

$$\begin{aligned}|\varphi(\gamma g)| &\leqq \int_{G_A} |\varphi(\gamma gh) f(h)| dh \\ &= \int_{G_A} |\varphi(\gamma h) f(g^{-1}h)| dh \\ &\leqq M \int_C |\varphi(\gamma h)| dh\end{aligned}$$

with $C = C_0 C_1$. Since C is compact, the number m of elements γ in G_F such that $\gamma C \cap C \neq \varnothing$ is finite. We can show that there exist positive constants c_1, c_2, c_3 such that

$$\delta(g) \leqq c_1, \quad c_2 \leqq |\det g|_A \leqq c_3$$

for all $g \in G_F C$. Then we have

$$\begin{aligned}&\sum_{\gamma \in B_F \backslash G_F} \int_C |\varphi(\gamma h)| dh \\ &\quad \leqq m \int_{B_F \backslash G_F C} |\varphi(h)| dh \\ &\quad \leqq m \int_K dk \int_{N_F \backslash N_A} dn \int_D d^\times z \int_E \left| \varphi\left(\begin{pmatrix} zx & 0 \\ 0 & x^{-1} \end{pmatrix} k \right) \right| |zx^2|_A^{-1} d^\times x,\end{aligned}$$

where $D = \{z \in A^\times | c_2 \leqq |z|_A \leqq c_3\}/F^\times$, $E = \{x \in A^\times |\ |x|_A^2 \leqq c_1 c_2^{-1}\}/F^\times$. Since

$$\left| \varphi\left(\begin{pmatrix} zx & 0 \\ 0 & x^{-1} \end{pmatrix} k \right) \right| = |\mu(z)|\ |z|_A^{1/2} |x|_A^{1+\sigma} |\varphi(k)|,$$

the above integral converges if $\sigma > 1$. q.e.d.

It is obvious that $E(\varphi)$ is left G_F-invariant if it converges, and that $\varphi \to E(\varphi)$ commutes with the action of $\mathscr{H}$ and $A^\times$; namely

$$\begin{aligned}E(\varphi, \gamma g) &= E(\varphi, g) && (\gamma \in G_F), \\ \rho(f) E(\varphi) &= E(\rho(f)\varphi) && (f \in \mathscr{H}), \\ \rho(z) E(\varphi) &= E(\rho(z)\varphi) && (z \in A^\times).\end{aligned}$$

Furthermore, we have

$$E^0(\varphi) = \varphi + M(\mu, \nu)\varphi.$$

In fact, since $G_F = B_F \cup B_F w N_F$, we have

$$E(\varphi, g) = \varphi(g) + \sum_{\xi \in F} \varphi\left(w \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} g\right)$$

and hence

$$E^0(\varphi, g) = \int_{A/F} E\left(\varphi, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx$$
$$= \varphi(g) + \int_A \left(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx.$$

If $\eta = \mu\nu$ is a character, then $E(\varphi)$ is orthogonal to $\mathscr{A}_0(\eta)$. In fact, we have

$$(E(\varphi), \varphi_0) = \int_{A^\times G_F \backslash G_A} E(\varphi, g) \bar{\varphi}_0(g) dg$$
$$= \int_{A^\times G_F \backslash G_A} \Big(\sum_{\gamma \in B_F \backslash G_F} \varphi(\gamma g)\Big) \bar{\varphi}_0(g) dg$$
$$= \int_{A^\times B_F \backslash G_A} \varphi(g) \bar{\varphi}_0(g) dg$$
$$= \int_{A^\times B_F N_A \backslash G_A} \varphi(g) \int_{N_F \backslash N_A} \bar{\varphi}_0(ng) dn dg = 0$$

for $\varphi_0 \in \mathscr{A}_0(\eta)$.

14. Let $\varphi \in \mathscr{B}(\mu, \nu)$ and $s \in \boldsymbol{C}$. Put

$$\varphi(s, g) = [\varphi(s)](g) = \varphi(g) \delta(g)^{s/2} \qquad (g \in G_A).$$

For simplicity, write $\alpha = |\ |_A$. Then $\varphi(s)$ belongs to $\mathscr{B}(\mu\alpha^{s/2}, \nu\alpha^{-s/2})$. The basic property of the Eisenstein series can be resumed as follows.

Theorem 5. *Let μ, ν be quasi-characters of $A^\times/F^\times$ and $\varphi \in \mathscr{B}(\mu, \nu)$.*

(1) *$E(\varphi(s))$ can be analytically continued to a meromorphic function on the whole s-plane, whose pole occurs at most at the poles of $M(\mu\alpha^{s/2}, \nu\alpha^{-s/2})$.*

(2) *The following functional equation holds.*

$$E(\varphi) = E(M(\mu, \nu)\varphi).$$

(3) *If $M(\mu, \nu)$ is regular at (μ, ν), then $E(\varphi)$ is slowly increasing so that it is an automorphic form on G_A. To be more precise, let D be a compact subset of the s-plane such that $E(\varphi(s))$ is regular on a neighborhood of D. Let C be a compact subset of G_A. Then there exist M, $N > 0$ depending only on D and C such that*

$$\left|E\left(\varphi(s), \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right)\right| \leqq M\alpha(a)^N$$

for all $a \in A^\times$, $\alpha(a) \geqq 1$, $g \in C$ *and* $s \in D$.

Concerning this theorem, we refer to the references in the introduction. Especially, as to (3), cf. [5, Chap. IV], [15, Appendix].

§ 4. Maass-Selberg relations

15. We state the Maass-Selberg relations in Harish-Chandra [5] in an adelic form. The proof goes entirely in the same way.

Theorem 6. *Fix an infinite place* v. *For* C^∞ *functions* φ, ψ *on* G_A, *put*

$$[\varphi, \psi] = (\rho(D_v)\varphi)\psi - \varphi(\overline{\rho(D_v)\psi})$$

if v *is real and*

$$[\varphi, \psi] = (\rho(D'_v)\varphi)\psi - \varphi(\overline{\rho(D''_v)\psi})$$

if v *is imaginary. Regard*

$$a(e^t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \qquad (t \in F_v)$$

as an element in G_A *such that the* v*-component equals the above and all the other components* $=1$. *Put*

$$\Phi(t, g) = |e^{-t}|_v \varphi^0(a(e^t)g),$$
$$\Psi(t, g) = |e^{-t}|_v \psi^0(a(e^t)g),$$

for $t \in F_v$, $g \in G_A$. *Further, put*

$$J(\varphi, \psi, t) = \int_K \int_{A^1/F^\times} \left[\frac{d\Phi}{dt}\overline{\Psi} - \Phi\frac{d\overline{\Psi}}{dt}\right]\left(t, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right) da dk$$

if v *is real and*

$$J(\varphi, \psi, t) = \int_K \int_{A^1/F^\times} \left[\frac{1}{2}\left(\frac{\partial\Phi}{\partial\tau}\overline{\Psi} - \Phi\frac{\partial\overline{\Psi}}{\partial\tau}\right) - i\frac{\partial\Phi}{\partial\theta}\overline{\Psi}\right]\left(\tau, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right) da dk$$

if v *is imaginary.*

Let $\mathfrak{S}$ *be a Siegel domain as in no.* 3 *and* $\mathfrak{S}(r)$ *the set of all* g *in* $\mathfrak{S}$ *with*

$\delta(g) \geqq |e^{2r}|_v$. Let $S(r)$ be the projection of $\mathfrak{S}(r)$ on $D = A^\times G_F \backslash G_A$ and $U(r)$ the projection of $\mathfrak{S}(r)$ on $A^\times B_F \backslash G_A$.

Let $\varphi, \psi \in \mathscr{A}(\eta)$. Then, for a sufficiently large r, we have

$$\int_{D-S(r)} [\varphi, \psi] dg + \int_{U(r)} [\varphi^*, \psi^*] dg - J(\varphi, \psi, r) = 0. \tag{4.1}$$

Here $\varphi^* = \varphi - \varphi^0$ and dg is a Haar measure on $A^\times \backslash G_A$.

Proof. Note first that, if r is sufficiently large, $\gamma\mathfrak{S}(r) \cap \mathfrak{S}(r) \neq \emptyset$ ($\gamma \in G_F$) implies $\gamma \in B_F$. Hence the natural projection of $U(r)$ onto $S(r)$ is injective.

Assume for a moment that φ is a C^∞ function on G_A satisfying the conditions (i), (ii) in no. 3 and having a compact support modulo $A^\times G_F$. We have then

$$\int_D [\varphi, \psi] dg = 0$$

for all $\psi \in \mathscr{A}(\eta)$. Divide the integral above into two integrals each being taken over $S(r)$ and $D - S(r)$, respectively. However, by the preceding remark, the first one can be integrated over $U(r)$ instead of $S(r)$. Write

$$[\varphi, \psi] = [\varphi^0, \psi] + [\varphi^*, \psi^0] + [\varphi^*, \psi^*].$$

Putting $A(r) = \{a \in A^\times / F^\times |\ |a|_A > |e^{2r}|_v\}$, we have

$$\begin{aligned} &\int_{U(r)} [\varphi^0, \psi] dg \\ &= \int_K \int_{A/F} \int_{A(r)} [\varphi^0, \psi]\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right) |a|_A^{-1} d^\times a\, dx\, dk \\ &= \int_K \int_{A(r)} [\varphi^0, \psi^0]\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right) |a|_A^{-1} d^\times a\, dk \\ &= \int_{U(r)} [\varphi^0, \psi^0] dg. \end{aligned}$$

Similarly, we see that

$$\int_{U(r)} [\varphi^*, \psi^0] dg = \int_{U(r)} [(\varphi^*)^0, \psi^0] dg = 0,$$

since $(\varphi^*)^0 = 0$. Hence

$$\int_{U(r)} [\varphi, \psi] dg = \int_{U(r)} [\varphi^0, \psi^0] dg + \int_{U(r)} [\varphi^*, \psi^*] dg.$$

Suppose that v is real; then

$$\frac{\partial^2}{\partial t^2}\Phi(t, g)=e^{-t}\left(1-2\frac{\partial}{\partial t}+\frac{\partial^2}{\partial t^2}\right)\varphi^0(a(e^t)g)$$
$$=e^{-t}(1+2\rho(D_v))\varphi^0(a(e^t)g)$$

or

$$\rho(D_v)\varphi^0(a(e^t)g)=\frac{1}{2}e^t\left(\frac{\partial^2}{\partial t^2}-1\right)\Phi(t, g).$$

Consequently, we have

$$[\varphi^0, \psi^0](a(e^t)g)=\frac{1}{2}|e^{2t}|_v\left[\frac{\partial^2}{\partial t^2}\Phi\cdot\overline{\Psi}-\Phi\frac{\partial^2}{\partial t^2}\overline{\Psi}\right](t, g).$$

However, the same equality holds also for imaginary v.

To integrate $[\varphi^0, \psi^0]$ over $U(r)$, observe that the measure dg on $U(r)$ is written as $dg=2q|e^{-2t}|_v dtdadkdn$ for

$$g=na(e^t)\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}k \qquad (n\in N_A,\ a\in A^1,\ t\in \boldsymbol{R},\ k\in K),$$

where $q=[F_v: \boldsymbol{R}]$.

Assume v is real; then

$$\frac{d^2\Phi}{dt^2}\overline{\Psi}-\Phi\frac{d^2\overline{\Psi}}{dt^2}=\frac{d}{dt}\left[\frac{d\Phi}{dt}\overline{\Psi}-\Phi\frac{d\overline{\Psi}}{dt}\right].$$

The fact that the support of φ is compact modulo $A^\times G_F$ implies that $\varphi^0(g)=0$ for g as above if t is large enough. We see immediately

$$\int_{U(r)}[\varphi^0, \psi^0]dg=-J(\varphi, \psi, r).$$

Assume now v is imaginary. Let K_0 denote the subgroup $\{a(e^{i\theta})|\theta\in\boldsymbol{R}\}$ of K. We have $dk=d\theta d\dot{k}$, $d\dot{k}$ being a right invariant measure on $K_0\backslash K$. Let $t=\tau+i\theta(\tau, \theta\in\boldsymbol{R})$. A simple calculation shows that

$$\int_0^{2\pi}\left[\frac{\partial^2\Phi}{\partial t^2}\overline{\Psi}-\Phi\frac{\partial^2\overline{\Psi}}{\partial t^2}\right]d\theta$$
$$=\frac{1}{4}\frac{\partial}{\partial\tau}\int_0^{2\pi}\left[\frac{\partial\Phi}{\partial\tau}\overline{\Psi}-\Phi\frac{\partial\overline{\Psi}}{\partial\tau}\right]d\theta-\frac{i}{2}\frac{\partial}{\partial\tau}\int_0^{2\pi}\frac{\partial\Phi}{\partial\theta}\overline{\Psi}d\theta.$$

Hence

$$\int_{U(r)}[\varphi^0, \psi^0]dg$$
$$=2\int_{A^1/F^\times}\int_K\int_r^\infty\left[\frac{1}{4}\frac{\partial}{\partial\tau}\left(\frac{\partial\Phi}{\partial\tau}\overline{\Psi}-\Phi\frac{\partial\overline{\Psi}}{\partial\tau}\right)-\frac{i}{2}\frac{\partial}{\partial\tau}\left(\frac{\partial\Phi}{\partial\theta}\overline{\Psi}\right)\right]$$
$$\left(\tau, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}k\right)d\tau dkda$$
$$=-J(\varphi, \psi, r).$$

This concludes the first step of the proof.

The proof of the theorem can be completed by an approximation process. Let φ be any element in $\mathscr{A}(\eta)$. By Lemma 1, there exists a f in $\mathscr{H}$ such that $\rho(f)\varphi=\varphi$. Denote by C_0 the support of f, and let be ω a compact subset of N_A such that $N_A=N_F\omega$. Then we can find a sequence C_n ($n=1, 2, \cdots$) of compact subsets of G_A such that

$$\omega C_n C_0 \subset C_{n+1}, \qquad \bigcup_n \text{(the interior of } C_n)=G_A.$$

Let β_n be the characteristic function of the image of C_n on $A^\times G_F\backslash G_A$ and put $\varphi_n=\rho(f)(\beta_n\varphi)$. Then φ_n is a C^∞ function on G_A satisfying the conditions (i), (ii) in no. 3 and its support is contained in $A^\times G_F C_n C_0^{-1}$. We have $\varphi_n^0=\rho(f)(\beta_n\varphi^0)$ and hence $\varphi_n^*=\rho(f)(\beta_n\varphi^*)$.

Every compact subset C of G_A is contained in C_{n-1} for sufficiently large n. Then we have $\varphi_n=\varphi$, $\varphi_n^0=\varphi^0$ and $\varphi_n^*=\varphi^*$ on C. Therefore, if the integrals in the equality (4.1) are absolutely convergent, we obtain (4.1) by substituting φ_n for φ and letting $n\to\infty$. Since $D-S(r)$ is compact, we even have $\varphi_n=\varphi$ on $D-S(r)$ if n is large. By the same reason we have $J(\varphi_n, \psi, r)=J(\varphi, \psi, r)$. It is known that φ^* is rapidly decreasing so that the second integral in (4.1) converges absolutely. q.e.d.

16. Corollary. *In the notation of Theorem* 6, *assume that there exists a complex number* λ *such that*

$$\rho(D_v)\varphi=\lambda\varphi, \qquad \rho(D_v)\psi=\bar{\lambda}\psi$$

if v *is real and*

$$\rho(D'_v)\varphi=\lambda\varphi, \qquad \rho(D''_v)\psi=\bar{\lambda}\psi$$

if v *is imaginary. Then we have, for large* r,

$$J(\varphi, \psi, r)=0.$$

Proof. Since $[\varphi, \psi]=[\varphi^*, \psi^*]=0$, the assertion follows from (4.1). q.e.d.

§ 5. Main theorems

17. Let η be a character of $A^\times/F^\times$ as in no. 3. Let ω be a homomorphism of $\mathscr{Z}$ into $\boldsymbol{C}$. Consider the following subspaces of $\mathscr{A}(\eta)$.

$$\mathscr{A}(\eta, \omega)=\{\varphi \in \mathscr{A}(\eta) | \rho(Z)\varphi=\omega(Z)\varphi \quad \text{for } Z \in \mathscr{Z}\},$$

$$\mathscr{A}_0(\eta, \omega)=\mathscr{A}(\eta, \omega) \cap \mathscr{A}_0(\eta),$$

$$\mathscr{A}_1(\eta, \omega)=\{\varphi \in \mathscr{A}(\eta, \omega) | (\varphi, \varphi_0)=0 \quad \text{for } \varphi_0 \in \mathscr{A}_0(\eta, \omega)\}.$$

By Lemma 4 $\mathscr{A}(\eta, \omega)$ is the direct sum of $\mathscr{A}_0(\eta, \omega)$ and $\mathscr{A}_1(\eta, \omega)$. Our aim is to prove that $\mathscr{A}_1(\eta, \omega)$ is generated by Eisenstein series or certain functions derived from them.

Put $\omega(D_v)=c_v$ if v is real and $\omega(D'_v)=c'_v$, $\omega(D''_v)=c''_v$ if v is imaginary. Let φ be any element in $\mathscr{A}(\eta, \omega)$. Retaining the notation in no.6, we note that the function

$$u(t)=\varphi_0(a(e^t)ak)$$

satisfies the following differential equations.

Assume v is real; by (1.4) we have

$$\left[\frac{1}{2}\left(\frac{d}{dt}\right)^2-\frac{d}{dt}\right]u=c_v u. \tag{5.1}$$

A general solution of this equation is of the form $ae^{pt}+be^{qt}$ or $(a+bt)e^{pt}$ ($p, q \in \boldsymbol{C}$, a, b are constants) and the latter case occurs if and only if $c_v=-1/2$.

Assume v is imaginary. Since φ^0 is right K-finite, u is a linear combination of functions u_n such that $u_n(t+i\theta)=e^{in\theta}u_n(t)$ with $n \in \boldsymbol{Z}$. Suppose that u itself has this property. Then we have $(\partial/\partial\theta)u=inu$ and hence, by (1.5)

$$\frac{1}{8}\left[\left(\frac{\partial}{\partial\tau}-2\right)^2-4+n^2+2n\left(\frac{\partial}{\partial\tau}-2\right)\right]u=c'_v u,$$

$$\frac{1}{8}\left[\left(\frac{\partial}{\partial\tau}-2\right)^2-4+n^2-2n\left(\frac{\partial}{\partial\tau}-2\right)\right]u=c''_v u$$

or

$$(5.2)\qquad \frac{1}{4}\left[\left(\frac{\partial}{\partial\tau}-2\right)^2-4+n^2\right]u=(c_v'+c_v'')u,$$
$$\frac{1}{2}n\left(\frac{\partial}{\partial\tau}-2\right)u=(c_v'-c_v'')u.$$

We see that if the above equations have a non-zero solution, then the integer n has to satisfy

$$(5.3)\qquad n^4-4(c_v'+c_v''+1)n^2+4(c_v'-c_v'')^2=0.$$

A general solution of the equations (5.2) is of the form $ae^{p\tau}+be^{q\tau}$ or $(a+b\tau)e^{p\tau}$ and the latter case occurs if and only if $c_v'=c_v''=-1/2$.

The above results may be resumed as

Lemma 8. *Let φ be in $\mathscr{A}(\eta, \omega)$. Then, in the notation of Theorem* 1, *we have*

$$\varphi^0\left(\begin{pmatrix} x & 0 \\ 0 & y\end{pmatrix}g\right)=\sum_{\mu,\nu,m}\alpha(xy^{-1})^{1/2}\mu(x)\mu(y)\,(\log\alpha(xy^{-1}))^m f_{\mu\nu m}(g)$$

for $x, y \in A^\times$, $g \in G_A$, where $m=0$, 1 and μ, ν run through all quasi-characters of $A^\times/F^\times$ such that $\mu\nu=\eta$.

The term containing $\log\alpha(xy^{-1})$ *occurs only if*

$$(5.4)\qquad c_v=-1/2 \quad or \quad c_v'=c_v''=-1/2 \quad for\ all\ v \in P_\infty.$$

18. We fix any place v in P_∞ and apply Corollary of Theorem 6 to $\varphi \in \mathscr{A}(\eta, \omega)$ and $\psi \in \mathscr{A}(\eta, \omega')$, assuming that

$$\omega'(D_v)=\overline{\omega(D_v)} \quad \text{or} \quad \omega'(D_v'')=\overline{\omega(D_v')},\quad \omega'(D_v')=\overline{\omega(D_v'')}$$

according as v is real or imaginary.

Let us introduce the following notation. Let χ be a quasi-character of $A^\times/F^\times$. For $x \in F_v^\times$, set

$$\chi_v(x)=\begin{cases} x^s(x>0) & \text{if } v \text{ is real.} \\ |x|^s(x/|x|)^l & \text{if } v \text{ is imaginary,}\end{cases}$$

where $s \in \boldsymbol{C}$, $l \in \boldsymbol{Z}$. s and l will be denoted by $s(\chi)$ and $l(\chi)$, respectively. Further we set

$$(f, g)=\int_K f(k)\overline{g(k)}dk$$

for continuous functions f, g on K.

Let

$$\varphi^0\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} g\right) = \sum \alpha(xy^{-1})^{1/2}\mu(x)\nu(y)\,(\log \alpha(xy^{-1}))^m f_{\mu\nu m}(g),$$

$$\psi^0\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} g\right) = \sum \alpha(xy^{-1})^{1/2}\mu(x)\nu(y)\,(\log \alpha(xy^{-1}))^m g_{\mu\nu m}(g)$$

be the expression of φ^0, ψ^0 as in Lemma 8.

To calculate J in Theorem 6, we assume that v is imaginary, for the real case is similar. Put $t=\tau+i\theta(\tau, \theta \in \boldsymbol{R})$ and $q=[F_v: \boldsymbol{R}]$ as before. We have

$$2J(\varphi, \psi, \tau) = \sum (f_{\mu\nu m}, g_{\kappa\lambda n})(2q)^{m+n} \times [(s-s'+2l)\tau^{m+n} + (m-n)\tau^{m+n-1}]e^{(s+s')\tau}.$$

Here we have put $s=s(\mu\nu^{-1})$, $l=l(\mu\nu^{-1})$, $s'=\overline{s(\kappa\lambda^{-1})}$ and the sum is taken over all m, n, μ, ν, κ, λ such that $m, n=0, 1$, $\mu\nu=\kappa\lambda=\eta$, the restriction of $\mu\kappa^{-1}$ to $A^1=1$ and $l(\mu\nu^{-1})=l(\kappa\lambda^{-1})$.

Note that the left hand side is identically 0 for sufficiently large τ. In particular, the term with $s+s'=0$ must vanish identically, whence follows the equality

$$\sum (f_{\mu\nu 0}, g_{\kappa\lambda 0})(s+l) + q\sum [(f_{\mu\nu 1}, g_{\kappa\lambda 0}) - (f_{\mu\nu 0}, g_{\kappa\lambda 1})] = 0. \tag{5.5}$$

Here $\kappa=\bar{\mu}^{-1}$, $\lambda=\bar{\nu}^{-1}$, $s=s(\mu\nu^{-1})$, $l=l(\mu\nu^{-1})$ and the sum is taken over all pairs of quasi-characters μ, ν such that $\mu\nu=\eta$.

If we interchange the role of φ, ψ, the equality (5.5) turns to

$$\sum (g_{\kappa\lambda 0}, f_{\mu\nu 0})(-\bar{s}+l) + q\sum [(g_{\kappa\lambda 1}, f_{\mu\lambda 0}) - (g_{\kappa\lambda 0}, f_{\mu\nu 1})] = 0.$$

Combined with (5.5), it gives

$$\sum (f_{\mu\nu 0}, g_{\kappa\lambda 0})(s\pm l) + q\sum [(f_{\mu\nu 1}, g_{\kappa\lambda 0}) - (f_{\mu\nu 0}, g_{\kappa\lambda 1})] = 0, \tag{5.6}$$

where, as before, $\kappa=\bar{\mu}^{-1}$, $\lambda=\bar{\nu}^{-1}$, $s=s(\mu\nu^{-1})$, $l=l(\mu\nu^{-1})$ and (μ, ν) runs over all pairs of quasi-characters such that $\mu\nu=\eta$.

If v is real, we obtain the corresponding equality just putting $l=0$.

19. Let μ and ν be quasi-characters of $A^\times/F^\times$. A remark is necessary about the eigenvalue of $\rho(D_v)$, $\rho(D'_v)$ or $\rho(D''_v)$ on $\mathscr{B}(\mu, \nu)$. Put $s=s(\mu\nu^{-1})$ and $l=l(\mu\nu^{-1})$. If v is real, then

$$\rho(D_v) = \tfrac{1}{2}(s^2-1)\,\mathrm{id}. \tag{5.7}$$

on $\mathscr{B}(\mu, \nu)$. If v is imaginary, then

$$(5.8) \qquad \rho(D'_v)=\frac{1}{2}\left(\left(\frac{s+l}{2}\right)^2-1\right)\mathrm{id}, \qquad \rho(D''_v)=\left(\left(\frac{s-l}{2}\right)^2-1\right)\mathrm{id}.$$

on $\mathscr{B}(\mu, \nu)$. These formulas can be seen by the arguments in no. 17. Therefore, if (μ, ν) is replaced by $(\bar{\nu}^{-1}, \bar{\mu}^{-1})$, then the eigenvalue c (resp. c', c'') of $\rho(D_v)$ (resp. $\rho(D'_v)$, $\rho(D''_v)$) is replaced by $\bar{c}$ (resp, $\bar{c}''$, $\bar{c}'$).

20. We are going to prove that the space $\mathscr{A}_1(\eta, \omega)$ is generated by Eisenstein series. First assume that the condition (5.4) is not satisfied for some $v \in P_\infty$. Let φ be in $\mathscr{A}_1(\eta, \omega)$ and write φ as in Lemma 8. Then $f_{\mu\nu 1}=0$ for all μ, ν. Write $f_{\mu\nu}=f_{\mu\nu 0}$ for simplicity. It is immediate to see that $f_{\mu\nu}$ belongs to $\mathscr{B}(\mu, \nu)$. Note that $f_{\mu\nu}$ is an eigenfunction of $\rho(D_v)$ (or $\rho(D'_v)$, $\rho(D''_v)$) with the same eigenvalue as φ, if $f_{\mu\nu}\neq 0$.

It is convenient to assume always that, out of two pairs (μ, ν) and (ν, μ), (μ, ν) is the one satisfying $|\mu\nu^{-1}(x)|=\alpha(x)^\sigma (x \in A^\times)$ with $\sigma \geqq 0$. Assume further that $(\mu, \nu)\neq(\alpha^{1/2}\chi, \alpha^{-1/2}\chi)$ for all characters χ of $A^\times/F^\times$. For any element ϕ of $\mathscr{B}(\bar{\nu}^{-1}, \bar{\mu}^{-1})$, the Eisenstein series $E(\phi)$ belongs to $\mathscr{A}(\eta)$. Apply Corollary of Theorem 6 to φ and $E(\phi)$. Since

$$E^0(\phi)=\phi+M(\bar{\nu}^{-1}, \bar{\mu}^{-1})\phi,$$

(5.6) implies

$$(f_{\mu\nu}, M(\bar{\nu}^{-1}, \bar{\mu}^{-1})\phi)-(f_{\nu\mu}, \phi)=0,$$

for, if $s=l=0$ in the notation of (5.6) then $c_v=-1/2$ or $c'_v=c''_v=-1/2$ by (5.7) and (5.8), which contradicts our assumption. Since ϕ is arbitrary, we have

$$M(\mu, \nu)f_{\mu\nu}=f_{\nu\mu}.$$

Observe, for the same reason as above, that $f_{\mu\nu}=0$ if $\mu=\nu$.

21. An additional consideration is necessary if $f_{\mu\nu}\neq 0$ for $(\mu, \nu)=(\alpha^{1/2}\chi, \alpha^{-1/2}\chi)$, where χ is a character of $A^\times/F^\times$. In this case we must have $\eta=\chi^2$ and $c_v=0$, $c'_v=c''_v=0$ for all $v \in P_\infty$. The function

$$g \longrightarrow \chi(\det g) \qquad (g \in G_A)$$

belongs to $\mathscr{B}(\nu, \mu)$ and it is also an element of $\mathscr{A}(\eta)$. We can apply Corollary of Theorem 6 to φ and $\chi \circ \det$ and obtain, by (5.6),

$$(f_{\mu\nu}, \chi \circ \det)=0.$$

Lemma 9. *μ and ν being as above, put*

$$\mathscr{B}^*(\mu, \nu)=\{f \in \mathscr{B}(\mu, \nu) | (f, \chi \circ \det)=0\}.$$

Then we have

$$\mathscr{B}^*(\mu, \nu)=\sum_{v \in P} \mathscr{B}_s(\mu_v, \nu_v) \otimes (\bigotimes_{w \neq v} \mathscr{B}(\mu_w, \nu_w)).$$

Proof. Denote by U the right hand side of the above equality. It is known that $\mathscr{B}_s(\mu_v, \nu_v)$ is the subspace of all f in $\mathscr{B}(\mu_v, \nu_v)$ such that

$$\int_{K_v} f(k)\bar{\chi} \circ \det(k) dk=0$$

and it has the codimension 1 in $\mathscr{B}(\mu_v, \nu_v)$ (cf. [7]). Let f_v be an element in $\mathscr{B}(\mu_v, \nu_v)$ such that

$$\mathscr{B}(\mu_v, \nu_v)=Cf_v+\mathscr{B}_s(\mu_v, \nu_v).$$

We may assume that f_v is the characteristc function of K_v if χ_v is unramified. It is evident that, if $f^0=\otimes f_v$, then

$$\mathscr{B}(\mu, \nu)=\otimes_v \mathscr{B}(\mu_v, \nu_v)=Cf^0+U.$$

Since $U \subset \mathscr{B}^*(\mu, \nu)$ and $(f^0, \chi \circ \det) \neq 0$, we have $U=\mathscr{B}^*(\mu, \nu)$. q.e.d.

It follows from Lemma 9 and Lemma 6 that

$$R(\mu, \nu)\phi=0$$

for $\phi \in \mathscr{B}^*(\mu, \nu)$. Putting $\varphi(s)=\phi\delta^{s/2}$, we see that $R(\mu\alpha^{s/2}, \nu\alpha^{-s/2})\phi(s)$ has a zero at $s=0$. Therefore,

$$M(\mu\alpha^{s/2}, \nu\alpha^{-s/2})\phi(s)=\frac{L(0, \alpha^{1+s})}{L(1, \alpha^{1+s})\varepsilon(0, \alpha^{1+s})} R(\mu\alpha^{s/2}, \nu\alpha^{-s/2})\phi(s)$$

is regular at $s=0$, because $L(0, \alpha^{1+s})$ has a pole of order 1 at the same point. In conclusion, $E(\phi)=E(\phi(s))_{s=0}$ is defined for $\phi \in \mathscr{B}^*(\mu, \nu)$ even if $M(\mu\alpha^{s/2}, \nu\alpha^{-s/2})$ has a pole at $s=0$.

Now $f_{\mu\nu}$ is in $\mathscr{B}^*(\mu, \nu)$ as we have seen. Taking $\varphi - E(f_{\mu\nu})$ in place of φ, we may assume $f_{\mu\nu}=0$. Let ϕ be any element in $\mathscr{B}^*(\mu, \nu)$ and apply Corollary of Theorem 6 to φ and $E(\phi)$. By (5.6) we have

$$(f_{\nu\mu}, \phi)=0,$$

which implies that $f_{\nu\mu}$ is a constant multiple of $\chi \circ \det$. In view of the arguments in no. 20 and no. 22, we infer that there exists a certain linear combination ψ of $E(f_{\mu\nu})$ and $\chi\circ\det$ with $\chi^2=\eta$ such that $\varphi-\psi$ is a cusp form.

22. Next assume that the condition (5.4) is satisfied for all $v\in P_\infty$. This time $f_{\mu\nu 1}$ belongs to $\mathscr{B}(\mu, \nu)$.

Let ϕ be any element in $\mathscr{B}(\bar{\nu}^{-1}, \bar{\mu}^{-1})$ and apply Corollary of Theorem 6 to φ and $E(\phi)$. By (5.6) we have

$$(f_{\mu\nu 1}, M(\bar{\nu}^{-1}, \bar{\mu}^{-1})\phi)+(f_{\nu\mu 1}, \phi)=0$$

and hence

$$M(\mu, \nu)f_{\mu\nu 1}=-f_{\nu\mu 1}.$$

Now let ϕ be in $\mathscr{B}(\mu, \nu)$ and put

$$E'(\phi)=\frac{d}{ds}E(\phi(s))_{s=0}$$

for $\phi(s)=\phi\delta^{s/2}$. Writing $M(\mu\alpha^{s/2}, \nu\alpha^{-s/2})\phi(s)=\phi_1(s)\delta^{-s/2}$, we have

$$\begin{aligned}(E'(\phi))^0&=\frac{d}{ds}[\phi(s)+\phi_1(s)\delta^{-s/2}]_{s=0}\\&=\frac{1}{2}\phi\log\delta-\frac{1}{2}\phi_1(0)\log\delta+\phi_1'(0)\\&=\frac{1}{2}[\phi-M(\mu, \nu)\phi]\log\delta+\phi_1'(0).\end{aligned}$$

Observe that $\phi_1(s)$ belongs to $\mathscr{B}(\nu, \mu)$ and so does $\phi_1'(0)$. Especially, if, $\mu=\nu$, we have $M(\mu, \nu)=-1$ so that

$$(E'(\phi))^0=\phi\log\delta+\phi_1'(0).$$

Replacing φ by $\varphi-2\sum_{\mu\neq\nu}E'(f_{\mu\nu 1})-\sum E'(f_{\mu\mu 1})$, we are led to the case where $f_{\mu\nu 1}=f_{\nu\mu 1}=0$ for all μ, ν.

Assuming the above, let ϕ be any element in $\mathscr{B}(\bar{\nu}^{-1}, \bar{\mu}^{-1})$. Apply Corollary of Theorem 6 to φ and $E(\phi)$. Then (5.6) gives

$$(f_{\mu\nu 0}, -M(\bar{\nu}^{-1}, \bar{\mu}^{-1})\phi)+(f_{\nu\mu 0}, \phi)=0$$

and hence

$$M(\mu, \nu)f_{\mu\nu 0}=f_{\nu\mu 0}.$$

It follows that

$$\varphi-\sum_{\mu,\nu}E(f_{\mu\nu 0})$$

is a cusp form, which has to vanish if $\varphi\in\mathscr{A}_1(\eta,\omega)$.

23. The preceding results can be resumed as follows.

Theorem 7. *$\mathscr{A}_1(\eta,\omega)$ is generated by all functions of the form*

$$E(\phi),\quad \frac{d}{ds}E(\phi(s))_{s=0}\quad \textit{and}\quad \chi\circ\det.$$

The functions of the second (resp. thrid) form appear if and only if

$$\omega(D_v)=\omega(D'_v)=\omega(D''_v)=-1/2\ \ (\textit{resp.}\ 0)$$

for all $v\in P_\infty$. ϕ and χ can be arbitrary so long as the following conditions are satisfied:

(i) *ϕ is an element of $\mathscr{B}(\mu,\nu)$, where (μ,ν) is a pair of quasi-characters of $A^\times/F^\times$ such that $\mu\nu=\eta$, $\rho(Z)\phi=\omega(Z)\phi$ ($\phi\in\mathscr{B}(\mu,\nu)$, $Z\in\mathscr{Z}$) and $|\mu\nu^{-1}(x)|$ $\alpha(x)^\sigma(x\in A^\times)$ with $\sigma\geqq 0$.*

(ii) *χ is a character of $A^\times/F^\times$ with $\chi^2=\eta$.*

(iii) *If $(\mu,\nu)=(\alpha^{1/2}\chi,\alpha^{-1/2}\chi)$, ϕ should be in $\mathscr{B}^*(\mu,\nu)$.*

Remark. Let (μ,ν) be as in (iii). $\chi\circ\det$ is the residue of $E(\phi(s))$ at $s=0$ for an element ϕ in $\mathscr{B}(\mu,\nu)$ not in $\mathscr{B}^*(\mu,\nu)$. We note also that $\chi\circ\det$ is an element in $\mathscr{B}(\nu,\mu)$ and $E(\chi\circ\det)=\chi\circ\det$.

24. The holomorphic case. Assume that F is a totally real number field. Let ω be a homomorphism of $\mathscr{Z}$ into $\boldsymbol{C}$ such that

$$\omega(D_v)=\tfrac{1}{2}m(m-2)$$

for all $v\in P_\infty$, where m is a given positive integer. Let η be a character of $A^\times/F^\times$. The homomorphism ω such that $\mathscr{A}(\eta,\omega)\neq\{0\}$ is uniquely determined by m and η.

It is well known that every holomorphic Hilbert modular form of weight m is contained in $\sum_\eta\mathscr{A}(\eta,\omega)$. In the notation of no.5, put

$$\sigma_m(k(\theta))=e^{im\theta}.$$

Let U be an open compact subgroup of G_f. Let $S_m(\eta,U)$ be the space of all φ in $\mathscr{A}_0(\eta,\omega)$ such that

(5.9) $$\rho(k_v)\varphi=\sigma_m(k_v)\varphi \qquad (k_v \in K_v,\ \det k_v=1)$$

for all $v \in P_\infty$ and

(5.10) $$\rho(u)\varphi=\varphi \qquad (u \in U).$$

Then the sum of $S_m(\eta, U)$ for all η and U is essentially the space of holomorphic cusp forms of weight m. However, to define holomorphic forms not necessarily cuspidal, we need some additional conditions. For instance, we let $H_m(\eta, U)$ be the space of all φ in $\mathscr{A}(\eta, \omega)$ satisfying (5.9) and (5.10) such that $f_{\mu\nu 1}=0$ for all μ, ν in the notation of Lemma 8 and $f_{\mu\nu 0}\neq 0$ only if

(5.11) $$\mu_v\nu_v^{-1}(x)=x^{m-1}\,(\operatorname{sgn} x) \qquad (x \in F_v^\times)$$

for all $v \in P_\infty$. Then the sum of $H_m(\eta, U)$ is the space of holomorphic forms of weight m.

By Theorem 7, every element in $H_m(\eta, U)\cap\mathscr{A}_1(\eta, \omega)$ is a linear combination of Eisenstein series. In this linear combination, the functions $E'(\phi)$ do not appear by definition, also the functions $\chi\circ\det$ are excluded by (5.9). Hence we obtain

Theorem 8. *Let I be the set of all pairs (μ, ν) of quasi-characters of $A^\times/F^\times$ satisfying $\mu\nu=\eta$ and* (5.11). *Let $\mathscr{B}(\mu, \nu)^U$ be the space of all right U-invariant elements in $\mathscr{B}(\mu, \nu)$. Then, every element in $H_m(\eta, U)$ orthogonal to cusp forms is a linear combination of Eisenstein series $E(\phi)$ such that $\phi \in \mathscr{B}(\mu, \nu)^U$, $(\mu, \nu) \in I$.*

If $m=1$, we find in [14] another proof based on the 'multiplicity one theorem'.

25. Theorem 9. *Every element in $\mathscr{A}(\eta)$ is a linear combination of a cusp form and*

$$\frac{d^n}{ds^n}E(\phi(s))_{s=0} \qquad (n=0, 1, 2, \cdots)$$

for certain functions ϕ in $\mathscr{B}(\mu, \nu)$ with $\mu\nu=\eta$.

Proof. Consider the subspace of all φ in $\mathscr{A}(\eta)$ satisfying

$$(\rho(D_v)-c_v)^N\varphi=0$$

or

$$(\rho(D'_v)-c'_v)^N\varphi=(\rho(D''_v)-c''_v)^N\varphi=0$$

for all $v \in P_\infty$, where $N \in \boldsymbol{Z}, >0$ and $c_v, c'_v, c''_v \in \boldsymbol{C}$. Denote this space for a moment by V_N. If (μ, ν) is such that

$$\begin{aligned} (5.12) \qquad & \rho(D_v)=c_v \text{ id.} \qquad \text{or } \rho(D'_v)=c'_v \text{ id.}, \\ & \rho(D''_v)=c''_v \text{ id.} \qquad \text{on } \mathscr{B}(\mu, \nu) \text{ for all } v \in P_\infty, \end{aligned}$$

then

$$E_n(\phi)=\frac{d^n}{ds^n}E(\phi(s))_{s=0} \qquad (\phi \in \mathscr{B}(\mu, \nu))$$

belongs to V_N (here $0 \leqq n < 2N$ if (5.4) is satisfied and $0 \leqq n < N$ otherwise).

Let φ be any element in V_N. Write φ^0 as in Theorem 1. If $f_{\mu\nu m} \neq 0$ and m is the largest integer with this property, then $f_{\mu\nu m} \in \mathscr{B}(\mu, \nu)$ and (μ, ν) has to satisfy (5.12).

First exclude the case where (5.4) is satisfied. Then it is easy to see that $m<N$ and that if $f_{\mu\nu N-1}=0$ for all μ, ν, then $\varphi \in V_{N-1}$. Fixing a $v \in P_\infty$, apply Corollary of Theorem 6 to $(\rho(D_v)-c_v)^{N-1}\varphi$ (or $(\rho(D'_v)-c'_v)^{N-1}\varphi$) and $E(\psi)$ with an arbitrary ψ in $\mathscr{B}(\bar{\nu}^{-1}, \bar{\mu}^{-1})$. It yields

$$M(\mu, \nu)f_{\mu\nu N-1}=(-1)^{N-1}f_{\nu\mu N-1}.$$

However, if $c_v=c'_v=c''_v=0$ for all v, we proceed as in no. 21; note that if $\phi=\chi \circ \det$, then

$$E^0_{N-1}(\phi)=2^{1-N}\phi\,(\log \delta)^{N-1}+\textstyle\sum_{n=0}^{N-2} \phi f_n\,(\log \delta)^n$$

with $f_n \in \mathscr{B}(\alpha^{1/2}, \alpha^{-1/2})$. In any case it can be shown that $\varphi-\sum E_{N-1}(\phi)$ belongs to V_{N-1} for a suitable choice of functions ϕ.

The case where (5.4) is satisfied can be treated similarly. By the induction on N we see that our assertion is true for the elements in V_N. Since $\mathscr{A}(\eta)$ is the sum of all subspaces like V_N, this completes the proof of the theorem.

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