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On Diophantine Inequalities of Real Indefinite Quadratic Forms of Additive Type in Four Variables

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Chapter 0. Introduction and Statements of the Result

0.1. We have a famous result of H. D. Kloosterman [17] on the solubility of the Diophatine equation

$$ax_1^2+bx_2^2+cx_3^2+dx_4^2=n.$$

There, the so-called "Kloosterman sum", with $e(\xi)=\exp(2\pi\sqrt{-1})$ for real ξ,

$$\sum_{x;\ 1\leqq x\leqq q,\quad (x,q)=1} e\left(\frac{1}{q}(ax+b\bar{x})\right) \qquad (x\bar{x}\equiv 1 \bmod q)$$

was estimated non-trivially. The error term in his asymptotic expansion of the number of Diophantine solutions would have been of the same order as the expected main term, thereby giving no positive result, if the Kloosterman sums had been estimated trivially. On the other hand, we have a result of H. Davenport and H. Heilbronn [5], acertaining the non-trivial solubility of the Diophantine inequality

$$|\lambda_1x_1^2+\lambda_2x_2^2+\lambda_3x_3^2+\lambda_4x_4^2+\lambda_5x_5^2|<\varepsilon$$

of a real indefinite quadratic form, for an arbitrarily given positive small ε. The proof in [5] was based on an extention of the so-called "circle method" of G. H. Hardy and J. E. Littlewood and on a lemma (Cf. 4.3.9) on simultaneous Diophantine approximations with small denominators. Then, is it possible to treat the Diophantine inequality

$$(*)\qquad |\lambda_1x_1^2+\lambda_2x_2^2+\lambda_3x_3^2+\lambda_4x_4^2|<\varepsilon$$

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of a real indefinite quadratic form, for an arbitrarily given small positive ε? G. L. Watson treated certain special cases in three or four variables, using Pell's equations [27]. H. Iwaniec showed that the Diophantine inequality

$$|(x_1^2+x_2^2)-\theta(x_3^2+x_4^2)|<\varepsilon$$

is soluble non-trivially for a non-zero irrational real positive θ, using his half-dimensional sieve [15]. We show here, in Corollary 0.2.1 with 0.2.1.2, that we can combine the methods in [17] and [5], if $\lambda_1, \cdots, \lambda_4$ satisfy certain complicated conditions. Our method seems, however, to be applicable to more general situations. So we have treated here the proof of Theorem 0.2 rather luxiously up to the end of 4.4.

We have another probable approach to our problem. First we approximate each λ_i by suitable irreducible rational fractions V_i/U_i, and then, we transform the problem to a Diophantine equation in $(x_1, \cdots, x, y)$ of the shape

$$(V_1U_2U_3U_4)x_1^2+\cdots+(U_1U_2U_3V_4)x_4^2=y$$

with $y\ll\varepsilon(U_1\cdots U_4)$. If $V_1/U_1, \cdots, V_4/U_4$ and y are regarded to be fixed, the problem was treated by T. Estermann in [34], Theorem 1. The fundamental lemma 1, which uses A. Weil's result, in [34] plays a similar role as our Propositions 2.3.9 and 3.1.4 combined. (Estermann's proof of Theorem 1 gives the right-hand side in (4) in [34] the estimate, with the notations in [34], $O_\varepsilon(|a_1\cdots a_4|\cdot(|\kappa|+1)n^{3/4+2\varepsilon}+|a_1\cdots a_4|^{3/2+s_0}n^{3/4+\varepsilon})$ for $n\gg(\max_m|a_m|)^{3/(2\varepsilon)}$).) The author was pointed out this paper of Estermann's by Professor T. Tatuzawa, after completing his work. When he had begun his calculations on our problem, it had seemed, and still seems, difficult to him how to obtain directly corresponding local solutions in our case (Cf. Theorem 2 in [34]), so he had not taken up this approach.

The fundamental step of our proof in Chapter 4 is Lemma 4.4.14 (or its fore-runner 4.4.12). Lemma 4.4.14 tells us that we can obtain some informations on the convergents of $\lambda\alpha$ if we know convergents of both of real numbers λ and α satisfying certain conditions. This step was very easy in [17] as a, b, c and d were constant integers. Then, applications of our Propositions 3.2.4 and 2.3.9 settle down our proof of Theorem 0.2 in 4.5. Until we apply 3.2.4 and 2.3.9, we can proceed, in Chapter 4, under fairly general conditions ((i), (ii) and (iii) of Theorem 0.2). It is only in the final steps, in the applications of the above-stated two propositions, where we need to restrict ourselves so badly under (iv) and (v) of Theorem 0.2. So it can be hoped that these last two conditions of the Theorem may be considerably relaxed.

Returning to general $\lambda_1, \cdots, \lambda_4$ in (*), the author expects that a sum, which corresponds to the so-called "singular series" in case of Diophantine equations, must be figured out, when the products of Jacobi's symbols in the left hand-side sum in Proposition 2.3.9 are trivial with respect to $\ddot{A}^4$. Then what will be the "local nature" of our problem? Also what will be a criterion on $(\lambda_1, \cdots, \lambda_4)$ for the solubility of $(*)$? The author wishes someone will find them. Our theorem will be improved correspondingly as estimates on sums of Kloosterman sums are improved. (The main body of the present work was completed in the spring of 1982. Cf. The Linnik's conjecture. Kuznecov's work [29] may be suggestive. Also [30] and [31].) Our method of proof will be adaptable essentially unchanged when the partial fractions (or a_k in 1.2.3.1 (i) for $|\lambda_1\lambda_2^{-1}|$) are unbounded, but a new idea will be needed if they are bounded. The author does not know how to treat the problem in three variables, except [27]. Nor he knows how to treat general real indefinite quadratic forms which are not of additive type, whose number of variables are smaller than 21. (Oppenheim conjecture [28]. That 21 is sufficient is the result of [2′], [7] and [32] combined.)

Our method of proof of Theorem 0.2 goes as follows. In Chapter 1, we will see, in three propositions, that the measure of such α, that $\lambda_1\alpha$ and $\lambda_2\alpha$ have rational approximations whose irreducible denominators are of the same order and whose approximations are also of the same order, is small, if $\lambda_1\lambda_2^{-1}$ is irrational (with some conditions). We need Selberg's sieve at Lemma 1.4.3.3. In Chapter 2, we estimate our theta-Weyl sums (=finite theta series), Proposition 2.2.14. Using these estimates and circle method of Davenport and Heilbronn, we see, in the first three sections in Chapter 4, that we can cut off such α, the variable of integration, that $|\alpha|$ is not $\gg\ll 1$ or that the denominators near P of convergents of $|\alpha|$ and of $|\lambda_i\alpha|$ $(i=1, \cdots, 4)$ are not all $\gg\ll P$. For the α left, we will find relations between convergents of $|\alpha|$ and $|\lambda_i\alpha|$, Lemmas 4.4.12 and 4.4.14 and Proposition 4.4.19. (Strictly speaking, we must deprive the convergents of some small prime divisors.) We use, then, Propositions 2.3.11.5 and 3.2.4 concerning sums containing Kloosterman sums, and, then, estimates on sums of products of Jacobi's symbols (Propositions 2.3.11.5 and 2.3.9). All of α's which are $|\alpha|\gg P^{-1}$ will give minor contributions to the number of integer solutions of our problem.

0.1.1. Let $\lambda_1, \cdots, \lambda_4$ be non-zero real numbers, which are not in the same signature, and ε be an arbitrarily given positive small number. If we want to solve the inequality

$$|\lambda_1 x_1^2 + \cdots + \lambda_4 x_4^2| < \varepsilon$$

with arbitrarily large positive integers $x_1, \cdots, x_4$, it is sufficient to find arbitrarily large positive integer solutions $y_1, \cdots, y_4$ of

$$|\lambda_1' y_1^2 + \cdots + \lambda_4' y_4^2| < 2$$

with $\lambda_i' = 2\varepsilon^{-1}\lambda_i 4^{t_i}$, where t_i ($\in N \cup \{0\}$*⁾) are so chosen that we have

$$\tfrac{1}{2}|\lambda_1'| \leqq |\lambda_i'| \leqq 2|\lambda_1'| \qquad (i=2, 3, 4).$$

It is sufficient, therefore, to consider the Diophantine inequality

$$|\lambda_1 x_1^2 + \cdots + \lambda_4 x_4^2| < 2,$$

under additional restrictions that

$$\tfrac{1}{2}E_{100} < |\lambda_i| < 2E_{100}$$

with an arbitrarily given large constant E_{100}, which is not considered as absolute. Also we may regard as λ_i is an integer if it is rational. Now we can state our Theorem in 0.2.

0.2. Theorem. *Let $\eta_1, \cdots, \eta_4$ be ± 1, which are not in the same signature. There exist, then, positive numerical constants c_i', c_i'' ($i=1, 2, 3, 4$) (small, $c_i'' > c_i' > 0$), h_0 (large), c_{200}'' (small) and c (large), for which the following statements hold: Let E_{100} ($\geqq 100$) be an arbitrarily given large positive integer. There exist, then, positive constants G_0, G_0', L_0 and P_0, depending on E_{100} (and constants given at first), with the following properties: Let $\lambda_1, \cdots, \lambda_4$ be non-zero real number and P be $> P_0$, which satisfy the five assumptions* (i)$\sim$(v) *stated below. We have, then, the number of such solutions $(x_1, \cdots, x_4)$, that*

$$x_1, \cdots, x_4 \in N,$$
$$c_i' |\lambda_i|^{-1/2} P < x_i < c_i'' |\lambda_i|^{-1/2} P \qquad (i=1, \cdots, 4)$$

and

$$|\lambda_1 x_1^2 + \cdots + \lambda_4 x_4^2| < 2,$$

is

$$\geqq c_{200}'' |\lambda_1 \cdots \lambda_4|^{-1/2} P^2.$$

The five assumptions are;

(i) $\tfrac{1}{2}E_{100} < |\lambda_i| < 2E_{100}$, $\operatorname{sgn} \lambda_i = \eta_i$, ($i=1, \cdots, 4$),
λ_1 *is irrational*, $\lambda_2 = \pm E_{100}$,

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*⁾ N is the set of all natural numbers, Z, that of all rational integers, Q, that of all rational numbers, R, that of all real numbers, and C that of all complex numbers.

and, if λ_i *is rational, then* λ_i *is an integer.*

(ii) *Let us suppose that* $|\lambda_1\lambda_2^{-1}|$ *has two consecutive convergents* $R'Q'^{-1}$ *and* RQ^{-1} *of* $|\lambda_1\lambda_2^{-1}|$, $(R', Q', R, Q \in N, (R, Q)=1, (R', Q')=1)$, *obtained by the regular continued fraction expansion of* $|\lambda_1\lambda_2^{-1}|$, *satisfying*

$$L_0^3(\log Q)^{2h_0} < Q' < E_{100}^{-1}Q.$$

(*They are supposed to satisfy* (iv) *and* (v) *below.*) *We choose a real* P *satisfying*

$$(L_0^{-2}QQ')^{1/2} \geqq P \geqq Q^{1/2}(\log Q)^{h_0}.$$

We put the reduced fraction of $E_{100}R'Q'^{-1}$ *as* $V_1U_1^{-1}$, V_1 *being the numerator and* U_1 *the denoninator. We also put* $V_2=E_{100}$ *and* $U_2=1$.

(iii) *We suppose that* $|\lambda_i|$ $(i=3, 4)$, *if it is irrational, has such a convergent* $V_iU_i^{-1}$, *that*

$$U_i,\ V_i \in N, \qquad (U_i, V_i)=1,$$
$$||\lambda_i| - V_iU_i^{-1}| < (L_0P^2)^{-1}$$

and

$$G_0' < U_i \ (\ll P^{1.9}).$$

If λ_i *is rational, therefore an integer, we put as*

$$|\lambda_i| = V_i, \quad U_i=1, \quad (\tfrac{1}{2}E_{100} \leqq V_i \leqq 2E_{100}).$$

(*They are supposed to satisfy* (iv) *and* (v) *below.*)

(iv) *Let us put, for a positive integer* X, *as*

$$\Delta^1(X; Z) = \prod p \qquad p;\ \text{prime}\ p \leqq Z,\ p \mid X$$

and)*

$$\Delta^1_{G_0}(X) = \Delta^1(X; G_0 \cdot \nu(X)).$$

We suppose, then, U_i *and* V_i $(i=1, \cdots, 4)$ *in* (ii) *and* (iii) *satisfy that*

$$\begin{aligned}&[U_1V_1, \cdots, U_4V_4] \times \Delta^1_{G_0}(U_1V_1 \cdots U_4V_4) \times \tau(\Delta^1_{G_0}(U_1V_1 \cdots U_4V_4))\\ &\quad \times \text{L.C.M. } of\ \{(U_{i_1}, U_{i_2});\ i_1, i_2 = 1, \cdots, 4\ and\ i_1 \neq i_2\}\\ &\quad < p^{1/4}(\log P)^{-5}\end{aligned}$$

*) $\nu(X)$ is the number of different prime divisors of X. Let us put the greatest common divisor and the least common multiple (L. C. M.) of $X_1, \cdots X_k$ as $(X_1, \cdots, X_k)$ and $[X_1, \cdots, X_k]$ respectively. Let $\tau(X)$ denote the number of divisors of X. $(X_1, \cdots, X_k)$ may be used as a vectorial notations, and $[\xi]$ may be used to denote the greatest integer such that it is $\leqq \xi$.

and that

$$[U_1V_1, \cdots, U_4V_4] > [V_1, \cdots, V_4]\cdot(\log P).$$

(v) *Let us put* $U_i^\sharp$ $(i=1, \cdots, 4)$ *as the square-free kernel of the odd divisor part of* U_i, *that is,*

$$U_i^\sharp = \prod p \qquad p;\ prime,\ p>2,\ p\,|\,U_i.$$

Let us put Z_0 *as*

$$Z_0 = \max\left(\begin{array}{l} p+1,\ G_0\cdot\nu(U_1V_1\cdots U_4V_4), \qquad G_0(\log\log P)^2; \\ \quad p \text{ are prime divisors of } (U_1^\sharp\cdots U_4^\sharp,\ V_1\cdots V_4) \\ \quad \text{or of } (U_{i_1}, U_{i_2}) \qquad (i_1, i_2=1, \cdots, 4 \text{ and } i_1\neq i_2) \end{array}\right).$$

We suppose, then, that there exist an i_0 $(i_0=1, \cdots, 4)$ *and a prime* p_0 *such that*

$$p_0\,|\,U_{i_0}^\sharp \quad \text{and} \quad p_0 \geqq Z_0.$$

We suppose also that

$$\begin{aligned}&[U_1V_1\cdots U_4V_4]^{-1}\cdot[V_1, \cdots, V_4]\cdot(U_1^\sharp\cdots U_4^\sharp)^{-1/2}\cdot[U_1^\sharp, \cdots, U_4^\sharp]\\ &\times \Delta_{G_0}^1(U_1)\cdots\Delta_{G_0}^1(U_4)\times\Delta^1(U_1; Z_0)\cdots\Delta^1(U_4; Z_0)\\ &<(\log P)^{-2}.\end{aligned}$$

Though very complicated the above assumptions are, we can see, from the theory of continued fractions, ([16], for instance), or as the following Corollary 0.2.1 with 0.2.1.2 gives an example, that the assumptions together are non-void. Apparently the conditions (iv) and (v) can be welded. They are left divided, however, as the condition (iv) concerns with 3.2.4, and (v) with 2.3.9, whose applications in 4.5.6 and 4.5.7.3, respectively, can be expected to be improved. In the proof of the Theorem 0.2 in Chapter 4, we need only the assumptions (i) ~ (iii) until the end of 4.4. We need (iv) and (v) in 4.5. Whole of this note is devoted to prove this Theorem.

0.2.1. Corollary. *Let* $\eta_1, \cdots, \eta_4$ *be* ± 1 *which are not in the same signature. Let* $\Lambda(\eta_1, \cdots, \eta_4)$ *be the subset of* $\boldsymbol{R}^4$ *defined below, where* P*'s are also defined. There exist, then, positive numerical constants* c_i' *and* c_i'' $(i=1, \cdots, 4)$ *and* c' *for which the following statements hold; Let* $(\lambda_1, \cdots, \lambda_4)$ *belong to* $\Lambda(\eta_1, \cdots, \eta_4)$, *and* ε *be arbitrarily given positive small real number. There exists, then, a (large) real* P_0' *such that, if* P, *defined in* (i) *below, is* $>P_0'$, *the number of such solutions* $(x_1, \cdots, x_4)$, *that*

$$x_1, \cdots, x \in N,$$
$$c_i'|\lambda_i|^{-1/2}P < x_i < c_i''|\lambda_i|^{-1/2}P \qquad (i=1, \cdots, 4)$$

and

$$|\lambda_1 x_1^2 + \cdots + \lambda_4 x_4^2| < \varepsilon,$$

is

$$\geqq c' (\max_i |\lambda_i|)^{-2} \varepsilon P^2.$$

The set $\Lambda(\eta_1, \cdots, \eta_4)$ *of* $(\lambda_1, \cdots, \lambda_4) \in \boldsymbol{R}^4$ *is defined as follows;*

(0) $\lambda_1, \cdots, \lambda_4$ *are* $\neq 0$, $\operatorname{sgn} \lambda_i = \eta_i \quad (i=1, \cdots, 4)$,
λ_1 *is irrational, and* $\lambda_2 = \pm 1$.

(The last one is only for covenience of the statements.) They are supposed to have infinitely many quintuples $(RQ^{-1}, V_1U_1^{-1}, \cdots, V_4U_4^{-1})$ *of convergents of* $|\lambda_i|$*'s satisfying the following assumptions* (i)$\sim$(iii). *Here* $R, Q, V_i, U_i \in \boldsymbol{N}$ *and* $(R, Q)=1$, $(V_i, U_i)=1$ $(i=1, \cdots, 4)$.

(i) $V_1U_1^{-1}$ *and* RQ^{-1} *are two consecutive convergents of* $|\lambda_1|$ *such that*

$$e^{(\log Q)^{1/3}} < U_1 < Q^{0.03},$$

that

$$\nu(U_1V_1) < (\log Q)^{1/4}.$$

and that U_1 *has a prime divisor* p_1, *dividing* U_1 *exactly to an odd power of* p_1, *such that*

$$p_1 > (\log Q)^{0.26}.$$

We define P by

$$P = (QU_1(\log Q)^{-1/2})^{1/2}.$$

(ii) $V_iU_i^{-1}$ *are convergents of* $|\lambda_i|$ $(i=3, 4)$ *such that*

$$||\lambda_i| - V_iU_i^{-1}| < (P^2(\log P)^{1/2})^{-1},$$

that

$$\nu(U_iV_i) < (\log Q)^{1/4},$$

that, if λ_i *is irrational, then*

$$(\log P)^{1/2} < U_i < P^{0.03}$$

and that, if λ_i *is rational, then*

$$|\lambda_i| = V_i U_i^{-1}, \qquad (U_i \textit{ and } V_i \ll 1).$$

We put $V_2 = U_2 = 1$, *for simplicity of statements.*

(iii) $V_1 U_1^{-1}, \cdots, V_4 U_4^{-1}$ *are supposed to satisfy that*

$$(U_{i_1}, V_{i_2}) = 1 \qquad \textit{for all } i_1, i_2 = 1, \cdots, 4,$$

and that

$$(U_{i_1}, U_{i_2}) = 1 \qquad \textit{for all } i_1, i_2 = 1, \cdots, 4 \textit{ and } i_1 \neq i_2.$$

0.2.1.1.*) *Proof of* 0.2.1. We put as

$$x_i = 2^{t_i} y_i, \qquad \tilde{\lambda}_i = \varepsilon^{-1} 2^{2t_i+1} \lambda_i,$$

and

$$\tilde{P} = (2\varepsilon^{-1})^{1/2} P$$

with $t_i \in N \cup \{0\}$, where we choose t_i's so that

$$4^{t_i} |\lambda_i| \gg \ll \max |\lambda_i|,$$

and we may suppose that ε^{-1} is modified to be an integer, so that rational $\tilde{\lambda}_i$'s are integers. The Diophantine inequality is transformed into

$$c_i' |\tilde{\lambda}_i|^{-1/2} \tilde{P} < y_i < c_i'' |\tilde{\lambda}_i|^{-1/2} \tilde{P} \qquad (i = 1, \cdots, 4)$$

and

$$|\tilde{\lambda}_1 y_1^2 + \cdots + \tilde{\lambda}_4 y_4^2| < 2.$$

We see that $\tilde{\lambda}_i$, $(\varepsilon^{-1} 2^{2t_i+1} V_i) U_i^{-1}$ (or its reduced fraction) and RQ^{-1} satisfy the assumptions (i) $\sim$ (iii) in 0.2. Also

$$\textstyle\sum_i \nu(U_i V_i) \ll (\log Q)^{1/4}$$

which means that

$$\Delta_{G0}^1 (U_1 V_1 \cdots U_4 V_4) \ll \exp(c(\log Q)^{0.26}).$$

Therefore the assumption (iv) in 0.2 is satisfied also. We have also

$$\Delta_{G0}^1 (U_i) \mid \Delta_{G0}^1 (U_1 V_1 \cdots U_4 V_4),$$

therefore that

$$\Delta_{G0}^1 (U_i) \ll e^{(\log Q)^{0.27}}.$$

*) The use of the set $\Delta(\cdots)$ to state 0.2.1 follows the suggestion of Professor Tamotsu Murata.

If Z_0, in (v) of 0.2, is $=G_0\cdot\nu(U_1V_1\cdots U_4V_4)$, then

$$\Delta^1(U_i;\ Z_0)\,|\,\Delta^1_{G_0}(U_1V_1\cdots U_4V_4),$$

therefore

$$\Delta^1(U_i;\ Z_0)\ll e^{(\log Q)^{0.27}}.$$

If Z_0 is $=G_0(\log\log P)^2$, then

$$\nu(U_1V_1\cdots U_4V_4)\leqq(\log\log P)^2.$$

Therefore

$$\Delta^1(U_i;\ Z_0)\leqq\{G_0(\log\log P)^2\}^{(\log\log P)^2}\ll e^{(\log Q)^{0.01}}.$$

As for $U_i^{\#}$'s, we see that

$$\begin{aligned}&[U_1V_1,\ \cdots,\ U_4V_4]^{-1}\cdot[V_1,\ \cdots,\ V_4]\cdot(U_1^{\#}\cdots U_4^{\#})^{-1/2}\cdot[U_1^{\#},\ \cdots,\ U_4^{\#}]\\&\quad\leqq(U_1\cdots U_4)^{-1}\times(U_1^{\#}\cdots U_4^{\#})^{1/2}\\&\quad\leqq(U_1\cdots U_4)^{-1/2}\leqq e^{-(\log Q)^{0.33}}.\end{aligned}$$

We see, then, we can apply the Theorem, and obtain that the number of solutions $(y_1,\ \cdots,\ y_4)$ is

$$\gg(\varepsilon^{-1}\max_i|\lambda_i|)^{-2}((2\varepsilon^{-1})^{1/2}P)^2,$$

which gives the result.

0.2.1.2. The conditions (i), (ii) and (iii) in 0.2.1 together are non-void. We can construct $\lambda_1,\ \cdots,\ \lambda_4$'s satisfying the conditions as follows: To construct $|\lambda_1|$ let us suppose we have already obtained $V_1U_1^{-1}$ and RQ^{-1}. Let U_1° and V_1° be mutually different prime numbers such that

$$U_1^\circ>\exp(Q^{700}),\ V_1U_1^{-1}\lesssim V_1^\circ U_1^{\circ-1}\lesssim RQ^{-1}\ \text{ and }\ |RQ^{-1}-V_1^\circ U_1^{\circ-1}|<(2Q^2)^{-1}.$$

Then RQ^{-1} is one of the convergents to $V_1^\circ U_1^{\circ-1}$, owing to 1.2.3.2(i). We choose V_{1*}° and U_{1*}° so that

$$U_1^\circ V_{1*}^\circ-V_1^\circ U_{1*}^\circ=\pm1,\ U_1^\circ>U_{1*}^\circ\geqq Q\quad\text{and}\quad V_1^\circ U_1^{\circ-1}\lesssim V_{1*}^\circ U_{1*}^{\circ-1}\lesseqqgtr RQ^{-1}.$$

$V_1^\circ U_{1*}^{\circ-1}$ may be equal to RQ^{-1}. We choose a positive integer a° so that, putting $R^\circ Q^{\circ-1}$ as $R^\circ=a^\circ V_1^\circ+V_{1*}^\circ$ and $Q^\circ=a^\circ U_1^\circ+U_{1*}^\circ$, we have

$$\exp((\log Q^\circ)^{1/3})<U_1^\circ<Q^{\circ\,0.03}.$$

These $R^\circ Q^{\circ-1}$, $V_1^\circ U_1^{\circ-1}$, $V_{1*}^\circ U_{1*}^{\circ-1}$, RQ^{-1} and $V_1U_1^{-1}$ make a sequence of consecutive (except possibly between $V_{1*}^{\circ}U_{1*}^{\circ-1}$ and RQ^{-1}) convergents to $R^{\circ}Q^{\circ-1}$. We repeat the process beginning with $V_1^{\circ}U_1^{\circ-1}$ and $R^{\circ}Q^{\circ-1}$. Continuing in this way we obtain a Cauchy sequence $V_1U_1^{-1}$, RQ^{-1}, $V_1^{\circ}U_1^{\circ-1}$, $R^{\circ}Q^{\circ-1}$, $\cdots$, which defines a real number $|\lambda_1|$ (>0) satisfying (i) for each choice of $V_1U_1^{-1}$ and RQ^{-1}. Next we construct $|\lambda_3|$ and $|\lambda_4|$ in relation to this $|\lambda_1|$. As for rational λ_i, the construction is easy. So we may suppose that we construct irrational ones. Suppose we have constructed $V_3U_3^{-1}$ and $V_4U_4^{-1}$. By 1.2.3.1(iv), we have $\tilde{V}_3\tilde{U}_3^{-1}$ and $\tilde{V}_4\tilde{U}_4^{-1}$ satisfying, with $P=(QU_1(\log Q)^{-1/2})^{1/2}$,

$$\tilde{U}_iV_i-U_i\tilde{V}_i=\pm 1, \quad P^{100}>\tilde{U}_i>U_i, \quad \text{and} \quad \tilde{U}_iU_i>2P^2(\log P)^{1/2} \quad (i=3, 4).$$

Then, we can find mutually different prime numbers V_3°, U_3°, V_4° and U_4°, also different from V_1° and U_1°, so that we have, with

$$\begin{aligned}
&P^{\circ}=(Q^{\circ}U_1^{\circ}(\log Q^{\circ})^{-1/2})^{1/2},\\
&(\log P^{\circ})^{1/2}<U_i^{\circ}<P^{\circ 0.03}, \qquad V_iU_i^{-1}\lesseqgtr V_i^{\circ}U^{\circ-1}\lesseqgtr\tilde{V}_i\tilde{U}_i^{-1},\\
&|\tilde{V}_i\tilde{U}_i^{-1}-V_i^{\circ}U_i^{\circ-1}|<\tfrac{1}{2}(\tilde{U}_i^2)^{-1}.
\end{aligned}$$

As we have $P^{\circ}\geqq U_1^{\circ}>\exp(Q^{700})$, $P^{100}>\tilde{U}_i$ and $U_i^{\circ}>(\log P^{\circ})^{1/2}$, we have certainly that $U_i^{\circ}>\tilde{U}_i$. We have, then, $\tilde{V}_i\tilde{U}_i^{-1}$ and $V_iU_i^{-1}$ as consecutive covergents to $V_i^{\circ}U_i^{\circ-1}$. Continuing in this way we obtain real numbers $|\lambda_3|$ and $|\lambda_4|$ satisfying (ii) and (iii) in relation with $|\lambda_1|$. The members $(\lambda_1, \cdots, \lambda_4)$ belonging to $\Lambda(\eta_1, \cdots, \eta_4)$ are wider than those constructed as above.

As for the existence of p_1 in (i), suppose that there are no such p's. Then we have $U_1=XY^2$, where X divides $\Delta^1(U_1; \log P)^{0.26})$. This means that

$$X\leqq e^{(\log Q)^{0.27}}\ll U_1^{o(1)},$$

and then that U_1 is "nearly" a square of an integer. So the assumption about p_1 is not so exceptional one.

0.2.2. Let $\chi(\cdots)$ be the characteristic function with respect to $(\cdots)$, temporarily. We have easily, for given $\eta_1, \cdots, \eta_4$ ($\eta_i=\pm 1$, not in the same signature), that

$$\int\cdots\int_{E_{100}}^{2E_{100}}\sum_{x_1,\ldots,x_4;\; c_i'|\lambda_i|^{-1/2}P<x_i<c_i''|\lambda_i|^{-1/2}P}\times\chi(|\eta_1\lambda_1x_1^2+\cdots+\eta_4\lambda_4x_4^2|<2)d\lambda_1\cdots d\lambda_4$$

is

$$\gg \ll E_{100}^{4} \times (E_{100}^{-1/2})^{4} P^{2}.$$

Therefore, the assumptions on the coefficients in the statements of the Theorem 0.2 is too restrictive to be satisfactory. We have not, therefore, tried to obtain best possible results along the method shown in this note.

0.3. Contents

List of Proposition: 1.1.1, 1.1.2, 1.1.3; 2.1.4; 2.2.11, 2.2.12, 2.2.14, 2.2.15; 2.3.7, 2.3.9, 2.3.11.5; 3.1.4; 3.2.4; 4.4.19.

Also important lemmas are 4.2.7, 4.3.9, 4.4.12 and 4.4.14.

The author expresses his thanks to Professor Tomio Kubota for giving him an oppotunity to publish this lengthy note.

Chapter 1. Three Metrical Propositions on Simultaneouus Diophantine Approximations

1.1. Statements of the Propositions

1.1.1. Proposition. $^\forall c_1$ (>1), $^\forall e_1$ (>0), $^\exists h_1$ $(\gg 1)$, $^\exists h_1'$ $(\gg 1)$, $^\forall E_{100}$ (>100), $^\exists P_1$ $(\gg 1)$ *for which the following statements hold: Let λ_1 and λ_2 be positive real numbers such that*

$$\tfrac{1}{2}E_{100} \leqq \lambda_i \leqq 2E_{100} \qquad (i=1, 2)$$

and that

$$\lambda_1\lambda_2^{-1} \text{ is irrational.}$$

Let RQ^{-1} $(R \in N, Q \in N, (R, Q)=1)$ be one of the convergents of $\lambda_1\lambda_2^{-1}$ obtained from the regular continued fraction expansion of $\lambda_1\lambda_2^{-1}$ such that $Q>P_1$. Let P be any real number such that

$$Q^{1/2}(\log Q)^{h_1'} \leqq P \leqq Q^{c_1}.$$

Let H and G be any real numbers such that

$$P \gg H \geqq G > H(\log P)^{-c_1} \geqq (\log P)^{h_1}.$$

We have, then, the number of such quadruples (A_1, B_1, A_2, B_2), satisfying the conditions stated below, is

$$\leqq P^2H^{-2}(\log P)^{-e_1}.$$

The conditions imposed on (A_1, B_1, A_2, B_2) are; that

$$\begin{aligned} &A_i \in N,\ B_i \in Z, && (A_i, B_i)=1, \\ &PG^{-1} \geqq A_i \geqq PH^{-1} && (i=1, 2), \\ &\tau(X) \leqq (\log P)^{c_1} && \text{for } X=A_1, B_1, A_2, B_2. \end{aligned}$$

and there should exist a real number α such that

$$(\log H)^{c_1} \geqq \alpha \geqq (\log H)^{-c_1}$$

and

$$|\lambda_i\alpha A_i - B_i| \leqq A_i(\log H)^{c_1}P^{-2} \qquad (i=1, 2).$$

The choices of h_1 and h_1' are independent of each other.

1.1.2. Proposition. $^\forall c_2\,(>1)$, $^\forall h_2\,(>0)$, $^\forall e_2\,(>0)$, $^\exists h_2'\,(>0)$, $^\forall E_{100}$ (>100), $^\exists P_2\,(>0)$, $^\exists H_2\,(>1)$ *for which the following statements hold: Let λ_1 and λ_2 be positive real numbers such that*

$$\tfrac{1}{2}E_{100} \leqq \lambda_i \leqq 2E_{100} \qquad (i=1, 2)$$

and that

$$\lambda_1\lambda_2^{-1} \text{ is irrational.}$$

Let RQ^{-1} ($R \in \boldsymbol{N}$, $Q \in \boldsymbol{N}$, $(Q, R)=1$) be one of the convergents of $\lambda_1\lambda_2^{-1}$ obtained from the regular continued fraction expansion such that $Q>P_2$. Let H be any real number such that

$$(\log Q)^{h_2} \geqq H \geqq H_2.$$

Let P be any real number such that

$$Q^{1/2}(\log Q)^{h_2'} \leqq P \leqq Q^{c_2}.$$

We have, then, number of such quadruples (A_1, B_1, A_2, B_2), satisfying the conditions stated below, is

$$\leqq P^2H^{-2}(\log H)^{-e_2}.$$

The conditions imposed on (A_1, B_1, A_2, B_2) are; that

$$\begin{aligned} &A_i \in \boldsymbol{N},\ B_i \in \boldsymbol{Z}, \qquad (A_i, B_i)=1,\\ &PH^{-1}(\log H)^{-c_2} \leqq A_i \leqq PH^{-1} \qquad (i=1, 2), \end{aligned}$$

and that there should exist a real number α such that

$$(\log H)^{c_2} \geqq \alpha \geqq (\log H)^{-c_2}$$

and

$$|\lambda_i\alpha A_i - B_i| \leqq (\log H)^{c_2}(HP)^{-1} \qquad (i=1, 2).$$

1.1.3. Proposition. $^\forall c_3\,(\geqq 1)$, $^\forall g_0\,(>1)$, $^\forall E_{100}\,(\geqq 100)$, $^\forall g\,(g \geqq E_{100}^c$ *with* $c \geqq 1)$, $^\exists P_3$, $^\exists g'$, $^\exists g''$, $^\exists K$, $^\exists z$, $^\exists G_0$, $^\exists G$ *(all $\geqq 1$ and $K \geqq g''$), for which the following statements hold: Let λ_1 be a real number, such that*

$$\tfrac{1}{2}E_{100} \leqq \lambda_1 \leqq 2E_{100},$$

and that λ_1 has a convergent $V_1U_1^{-1}$ ($V_1 \in \boldsymbol{N}$, $U_1 \in \boldsymbol{N}$, $(V_1, U_1)=1$), obtained from the continued fraction expansion of λ_1, which satisfies

$$|\lambda_1 - V_1 U_1^{-1}| < gP^{-2}$$

and

$$G_0 \leqq U_1 \leqq P^2(\log P)^{-11}.$$

We have, then, the number of quadruples (A, B, A_1, B_1), *satisfying the conditions stated below, but that all of* A, B, A_1 *and* B_1 *are not* $[K, K^z)$*-regular, (the definition being given in* 1.1.3.2 *below), or that* $(A, A_1)\times(B, B_1)\geqq G^2$, *is*

$$\leqq g_0^{-1}P^2.$$

The conditions imposed on (A, B, A_1, B_1) *are;*

$$A,\ A_1 \in N, \qquad B,\ B_1 \in Z,$$
$$(A, B)=1, \qquad (A_1, B_1)=1,$$
$$g^{-1}P \leqq X \leqq gP \qquad \textit{for } X=A,\ A_1$$

and that there exists real α *such that*

$$g^{-1} \leqq \alpha \leqq g,$$
$$|\alpha A - B| \leqq gP^{-1}$$

and

$$|\lambda_1 \alpha A_1 - B_1| \leqq gP^{-1}.$$

1.1.3.1. Definition. Let an interval $[K, K^z)$ be given. We define $\nu_{[K,K^z)}(X)$, for a positive integer X, to be the number of different prime divisors of X, lying in $[K, K^z)$.

1.1.3.2. Definition. Let positive constants $K\,(>1)$, $z\,(>1)$, $g'\,(\in N)$ and g'' be given $(K>g''\geqq 1)$. A positive integer X is called to be $[K, K^z)$-regular when

(i) if $p<g''$, then $p^{g'}\nmid X$,

(ii) if $p\geqq g''$, then $p^2\nmid X$ (p being a prime),

and

(iii) $1\leqq \nu_{[K,K^z)}(X)\leqq 10\log z$.

(Strictly speaking, g' and g'' should explicitly appear in the terminology.)

1.1.4. The rest of this chapter is devoted to the proofs of these propositions. In the followings c, c_1, c_2, $\cdots$ are positive constants depending on foregoing c's, and e_{21}, $\cdots$ are positive constants, which may depend on e's, moreover.

1.2.

We quote some known facts in this section.

1.2.1. Lemma. *Let* ξ, ζ *and* X *be real numbers with* $\xi \neq 0$, $\frac{1}{2} > \zeta > 0$ *and* $X > 1$. *We have, then, one of the following two alternatives;*

(*Case* 1). *If there exists a pair* (u, v) *such that*

$$u \in \boldsymbol{Z}, \quad v \in \boldsymbol{Z}, \quad (u, v) = 1,$$
$$1 \leqq u \leqq (2\zeta)^{-1}, \quad \text{and} \quad |\xi u - v| < (2X)^{-1},$$

then all solutions (x, y) *such that*

$$x \in \boldsymbol{Z}, \quad y \in \boldsymbol{Z},$$
$$1 \leqq x < X, \quad \text{and} \quad |\xi x - y| < \zeta,$$

have the ratio

$$y : x = v : u.$$

(*Case* 2). *If there exists a pair* (u, v) *such that*

$$u \in \boldsymbol{Z}, \quad v \in \boldsymbol{Z}, \quad (u, v) = 1,$$
$$(2\zeta)^{-1} < u \leqq 2X, \quad \text{and} \quad |\xi u - v| < (2X)^{-1},$$

then the number of solutions (x, y) *such that*

$$x \in \boldsymbol{Z}, \quad y \in \boldsymbol{Z},$$
$$1 \leqq x < X, \quad \text{and} \quad |\xi x - y| < \zeta,$$

is

$$< 24\zeta X.$$

Proof. This is contained in the proof of Lemma 14 in [2].

1.2.2. Lemma. *Let* K, z *and* X *be real numbers with* $K > 2$, $z > 0$, *and* $X > 1$. *Let* $\prod$ *be, temporarily, the product of all prime numbers lying in the interval* $[K, K^z)$. *We have, then, the number of such integer* n *that* $1 \leqq n \leqq X$ *and that* $(n, \prod) = 1$ *is*

$$\ll_{\text{abso.}} Xz^{-1} + K^{2z}.$$

Proof. This is a simple corollary of the upper bound result of the sieve of A. Selberg. For later applications, the sieve of Brun, for instance, is insufficient. See [14], [20] or [21].

1.2.3. We quote here well-known results in the theory of regular continued fraction expansions of real numbers. See [16] or [22], for instance. Notations in this section are independent of those in later sections.

1.2.3.1. (i) Let a real number α be expanded into a regular continued fraction;

$$\alpha=[a_0; a_1, a_2, a_3, \cdots]$$
$$=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{\ddots}}}}$$

with

$$a_0 \in \boldsymbol{Z}, \quad \text{and} \quad a_i \in \boldsymbol{N} \quad (i=1, 2, 3, \cdots).$$

(ii) If α is irrational, then the expansion in (i) is unique.

(iii) If α is rational, then the expansion in (i) has only finitely many terms. We have always two choices of the last term: one with 1 and the other larger than 1. The length of the expansion varies by one, according to the choices.

(iv) We put, using (i),

$$p_k=a_k p_{k-1}+p_{k-2}, \quad p_0=a_0, \quad p_{-1}=1,$$
$$q_k=a_k q_{k-1}+q_{k-2}, \quad q_0=1, \quad q_{-1}=0,$$
$$(k=1, 2, 3, \cdots)$$

inductively. We have, then,

(iv-i) $(p_k, q_k)=1$.

(iv-ii) q_k increases strictly monotonely with k.

(iv-iii) $q_k p_{k-1}-p_k q_{k-1}=(-1)^k \quad (k\geqq 0)$.

(iv-iv) $(q_k\cdot(q_{k+1}+q_k))^{-1}\leqq(-1)^k(\alpha-p_k q_k^{-1})\leqq(q_k q_{k+1})^{-1} \quad (k\geqq 0)$.

(iv-v) $q_k\geqq(\sqrt{2})^{k-1} \quad (k\geqq 2)$.

(iv-vi) Let $h\geqq k-1\geqq 0$. We have, then,

$$p_h=x_{hk}p_{k-1}+y_{hk}p_{k-2},$$
$$q_h=x_{hk}q_{k-1}+y_{hk}q_{k-2}$$

with some pair (x_{hk}, y_{hk}) of integers such that

$$x_{hk} \in \boldsymbol{N}, \quad y_{hk} \in \boldsymbol{N} \cup \{0\},$$
$$(x_{hk}, y_{hk}) = 1,$$
$$x_{hk} \geqq y_{hk} \geqq 0$$

and that,
if $y_{hk}=0$, then $x_{hk}=1$.

The irreducible fraction $p_k q_k^{-1}$ is called as the k-th convergent to α.

(v) If an irreducible fraction ab^{-1} ($a \in \boldsymbol{Z}$, $b \in \boldsymbol{N}$, $(a, b)=1$) appears as one of the convergents of a real number α, we denote

$$\alpha(\longrightarrow ab^{-1}$$

in this note. If two irreducible fractions ab^{-1} and $a'b'^{-1}$ ($a \in \boldsymbol{Z}$, $a' \in \boldsymbol{Z}$, $b \in \boldsymbol{N}$, $b' \in \boldsymbol{N}$, $(a, b)=1$, $(a', b')=1$), with $b > b' \geqq 1$, appear as two consecutive convergents of a real number α, we denote, then,

$$\alpha(\longrightarrow ab^{-1}(\Longrightarrow a'b'^{-1}.$$

These notations are not optimal, owing to the ambiguities for rational α stated in (iii).

1.2.3.2. (i) Let a real number α and an irreducible fraction ab^{-1} ($a \in \boldsymbol{Z}$, $b \in \boldsymbol{N}$, $(a, b)=1$) satisfy

$$|\alpha - ab^{-1}| < (2b^2)^{-1},$$

then

$$\alpha(\longrightarrow ab^{-1}.$$

(ii) Let a real number α and a fraction ab^{-1} ($a \in \boldsymbol{Z}$, $b \in \boldsymbol{N}$) satisfy the condition that, for each pair (c, d) of integers with $1 \leqq d \leqq b$ and $cd^{-1} \neq ab^{-1}$, we have

$$|d\alpha - c| > |b\alpha - a|.$$

Then ab^{-1} is called as a best approximation (of the second kind) to α, [16]. The fraction ab^{-1} is irreducible.

(ii-i) If a fraction is a best approximation to α, then it is one of the convergents to α.

(ii-ii) Let α be a number which is not an integer. Then every convergent to α is a best approximation to α.

1.3. Proof of Proposition 1.1.1

1.3.1. We eliminate α in the condition as follows.

1.3.1.1. Lemma. *The quadruples* (A_1, B_1, A_2, B_2), *to be counted in* 1.1.1, *satisfy*

$$|\lambda_1\lambda_2^{-1}A_1B_2 - A_2B_1| < (\log H)^{c_{20}}G^{-2},$$
$$1 \leqq A_1B_2 \leqq (PG^{-1})^2(\log H)^{c_{20}},$$
$$A_2B_1 \neq 0$$

and

$$\tau(X) \leqq (\log P)^{c_1} \qquad \textit{for } X = A_1, B_1, A_2, B_2.$$

Proof. Suppose that $B_1 = 0$. Then, we have

$$|\lambda_1\alpha A_1| \leqq A_1(\log H)^{c_1}P^{-2},$$

which contradicts the fact that

$$|\lambda_1\alpha A_1| \geqq \lambda_1A_1(\log H)^{-c_1},$$

if P is sufficiently large. Therefore, we have $B_1 \neq 0$. Then, $B_1 \in \boldsymbol{N}$. Similarly for B_2. We have

$$1 \leqq B_i < PG^{-1}(\log H)^{c_1+1} \qquad (i=1, 2).$$

Therefore

$$|\alpha(\lambda_1A_1B_2 - \lambda_2A_2B_1)| < G^{-2}(\log H)^{2c_1+2}.$$

Dividing by $\lambda_2\alpha$, we obtain the result.

1.3.1.2. We apply 1.2.1. We have one of the following two alternatives;

EITHER (*Case* 1). There exists a pair (u, v) such that $u \in \boldsymbol{N}$, $v \in \boldsymbol{N}$, $(u, v) = 1$, and that we have

$$(A_2B_1)(A_1B_2)^{-1} = vu^{-1}$$

for all quadruples (A_1, B_1, A_2, B_2) appearing in 1.1.1.

OR (*Case* 2). The number of pairs (x, y) such that

$$x \in \boldsymbol{N}, \quad y \in \boldsymbol{Z},$$
$$1 \leqq x \leqq (PG^{-1})^2(\log H)^{c_{20}}$$

and

$$|(\lambda_1\lambda_2^{-1})x-y|<G^{-2}(\log H)^{c_{20}}$$

is

$$\leqq P^2G^{-4}(\log H)^{c_{21}}.$$

1.3.2. We treat (Case 1) in 1.3.1.2. We have

$$A_1B_2=uw \quad \text{and} \quad A_2B_1=vw$$

with

$$w=(A_1B_2,\ A_2B_1).$$

1.3.2.1. Lemma. *Suppose, in* (*Case* 1) *of* 1.3.1.2, *that* $1\leqq u<Q$, *Then, we have, by a suitable choice of* h_1', *the conclusion of* 1.1.1.

Proof. Suppose that $|(\lambda_1\lambda_2^{-1})u-v|<(2Q)^{-1}$. Then, this is $<(2u)^{-1}$. This shows, according to 1.2.3.2 (i), that $\lambda_1\lambda_2^{-1}$ $(\longrightarrow vu^{-1}$ with $1\leqq u<Q$. Then, the convergent v^*u^{*-1} of $\lambda_1\lambda_2^{-1}$ next to vu^{-1} satisfies that $u<u^*\leqq Q$. We have, then, according to 1.2.3.1 (iv-iv), that

$$|(\lambda_1\lambda_2^{-1})u-v|\geqq(u^*+u)^{-1}>(2Q)^{-1},$$

which is a contradiction. We have, therefore, that

$$|(\lambda_1\lambda_2^{-1})u-v|>(2Q)^{-1}.$$

Now we have that

$$\begin{aligned} w(2Q)^{-1} &\leqq |(\lambda_1\lambda_2^{-1})uw-vw| \\ &=|(\lambda_1\lambda_2^{-1})A_1B_2-A_2B_1| \\ &<G^{-2}(\log H)^{c_{20}}, \end{aligned}$$

that is

$$1\leqq w<(2Q)G^{-2}(\log H)^{c_{20}}.$$

The number of quadruples (A_1, B_1, A_2, B_2) to be counted is

$$\begin{aligned} &\leqq \sum_w \tau(uw)\times\tau(vw)\leqq\sum_w 1\times(\log P)^{4c_1} \\ &<(2Q)G^{-2}\,(\log H)^{c_{20}}(\log P)^{4c_1} \\ &<P^2H^{-2}(\log P)^{-(e_1+1)}QP^{-2}(\log P)^{e_{21}} \end{aligned}$$

with $e_{21}=6c_1+c_{20}+e_1+2$. If we choose h_1' so that $2h_1'>e_{21}+1$ and P_1 sufficiently large, then, we have the conclusion of 1.1.1.

1.3.2.2. Lemma. *Suppose, in* (*Case* 1) *of* 1.3.1.2, *that* $u \geqq Q$. *Then, we have the conclusion of* 1.1.1, *again.*

Proof. We have

$$w = u^{-1} A_1 B_2 < Q^{-1}(PG^{-1})^2 (\log H)^{c_{20}}.$$

Taking into consideration the fact that admissible w's satisfy

$$\tau(wu) \quad \text{and} \quad \tau(wv) \leqq (\log P)^{2c_1},$$

we have that the number of the quadruples in this case is

$$\begin{aligned}
&\leqq \sum_w \tau(uw) \times \tau(vw) \leqq \sum_w 1 \times (\log P)^{4c_1} \\
&\leqq Q^{-1}(PG^{-1})^2 (\log H)^{c_{20}} (\log P)^{4c_1} \\
&\leqq P^2 H^{-2} (\log P)^{-(e_1+1)} \times Q^{-1}(HG^{-1})^2 (\log H)^{c_{20}} (\log P)^{4c_1+e_1+1}.
\end{aligned}$$

We have

$$\begin{aligned}
&Q^{-1}(HG^{-1})^2 (\log H)^{c_{20}} (\log P)^{4c_1+e_1+1} \\
&\qquad < Q^{-1} (\log P)^{6c_1+c_{20}+e_1+1} = o(1)
\end{aligned}$$

by the assumptions, hence the conclusion of 1.1.1 in this case, again.

1.3.3. We treat (Case 2) in 1.3.1.2. We have the following

Lemma. *We have, by a suitable choice of* h_1, *the conclusion of* 1.1.1 *in this case, also.*

Proof. We have, using the condition that $\tau(X) \leqq (\log P)^{4c_1}$ for $X = A_1, \cdots, B_2$, that the number of quadruples in this case is

$$\begin{aligned}
&\leqq (\log P)^{4c_1} \times \#\{(x, y);\ \text{in (Case 2) of 1.3.1.2}\} \\
&\leqq (\log P)^{4c_1} P^2 G^{-4} (\log P)^{c_{21}} \\
&\leqq P^2 H^{-2} (\log P)^{-(e_1+1)} \times (HG^{-2})^2 (\log P)^{4c_1+c_{21}+e_1+1}.
\end{aligned}$$

By the assumptions that $G > H(\log P)^{-c_1}$, the last estimate is

$$\leqq P^2 H^{-2} (\log H)^{-(e_1+1)} \times H^{-2} (\log P)^{8c_1+c_{21}+e_1+1}.$$

We have the conclusion of 1.1.1, by choosing $2h_1 = 8c_1 + c_{21} + e_1 + 1$.

1.3.4. We have proved 1.1.1.

1.4. Proof of Proposition 1.1.2

1.4.1. Lemma. *The quadruples* (A_1, B_1, A_2, B_2) *to be counted in* 1.1.2 *satisfy*

$$|\lambda_1\lambda_2^{-1}A_1B_2 - A_2B_1| < (\log H)^{c_{30}}H^{-2},$$

and

$$PH^{-1}(\log H)^{-c_{31}} < B_i < PH^{-1}(\log H)^{c_{31}} \qquad (i=1, 2).$$

Proof. It is obtained similarly as 1.3.1.1.

1.4.1.1. Corollary. *The number of quadruples* (A_1, B_1, A_2, B_2), *to be counted in* 1.1.2 *with fixed* A_1 *and* B_1, *is at most*

$$1 + H^{-2}(\log\ H)^{c_{32}} \qquad (\ll 1).$$

Similarly, with fixed A_2 *and* B_2.

Proof. We have that

$$|B_2A_2^{-1} - (\lambda_2\lambda_1^{-1})B_1A_1^{-1}| \leqq P^{-2}(\log H)^c,$$

and that the Farey fractions of order PH^{-1} are mutually distant by $P^{-2}H^2$, at least, from each other. The rest is easy.

1.4.1.2. Corollary. *We may add the restriction that*

$$\tau(X) \leqq (\log P)^{e_2+c_2+1} \qquad \textit{for } X = A_1, \cdots, B_2$$

on A_1, B_1, A_2, *and* B_2 *in* 1.1.2.

Proof. We have

$$\sum_{X \leqq PH^{-1}} \tau(X) \ll PH^{-1}(\log P).$$

Then, we have that the number of such A_1 that satisfies $\tau(A_1) > (\log P)^{e_2+c_2+1}$ is not larger than $PH^{-1}(\log P)^{-(e_2+c_2)}$. Then, using 1.4.1.1, we have that such quadruples $(A_1, \cdots, B_2)$ that $\tau(A_1) > (\log P)^{e_2+c_2+1}$ is not larger than $P^2H^{-2}(\log P)^{-(e_2+1)}$.

1.4.1.3. Such trick of proof, as in 1.4.1.2, that, if one of $A_1, \cdots, B_2$ satisfies a certain restriction, then, the number of quadruples $(A_1, \cdots, B_2)$ is smaller than the order to be obtained, will be used very often, later,

1.4.1.4. We have $(A_1B_2, A_2B_1) = (A_1, A_2) \times (B_1, B_2)$, in 1.1.2. Therefore, we put, as standard notations in 1.4,

$$a=(A_1, A_2), \quad A_1=a\tilde{A}_1, \quad A_2=a\tilde{A}_2, \quad (\tilde{A}_1, \tilde{A}_2)=1$$

and

$$b=(B_1, B_2), \quad B_1=b\tilde{B}_1, \quad B_2=b\tilde{B}_2, \quad (\tilde{B}_1, \tilde{B}_2)=1.$$

We have $(\tilde{A}_1\tilde{B}_2, \tilde{A}_2\tilde{B}_1)=1$, We use $\tilde{X}$ to represent any one of $\tilde{A}_1, \cdots, \tilde{B}_2$.

1.4.2. We first treat those $(A_1, \cdots, B_2)$, in 1.1.2, for which

$$ab=(A_1B_2, A_2B_1)>(\log P)^{30},$$

up to 1.4.2.6. Actually, this condition is not needed until 1.4.2.5.

1.4.2.1. Lemma. *We have, using* 1.4.1.4, *that the quadruples* $(A_1, \cdots, B_2)$ *to be counted in* 1.1.2 *satisfy*

$$|\lambda_1\lambda_2^{-1}\tilde{A}_1\tilde{B}_2-\tilde{A}_2\tilde{B}_1|<(abH^2)^{-1}(\log H)^{c_{30}}$$

and

$$0<\tilde{A}_1\tilde{B}_2<(abH^2)^{-1}P^2(\log H)^{c_{31}}.$$

Proof. It is easy.

1.4.2.2. Lemma. *We have*

$$1\leqq ab\leqq \max(P, Q, P^2Q^{-1})\times H^{-2}(\log H)^{c_{33}},$$

in 1.4.1.4.

Proof. Suppose, first, that

$$|\tilde{A}_1\tilde{B}_2|<\tfrac{1}{2}abH^2(\log H)^{-c_{30}},$$

then, we have, with the notation in 1.2.3.1(v),

$$\lambda_1\lambda_2^{-1}(\longrightarrow\frac{\tilde{A}_2\tilde{B}_1}{\tilde{A}_1\tilde{B}_2} \qquad \text{(irreducible fraction)}.$$

Suppose, moreover, that

$$|\tilde{A}_1\tilde{B}_2|<Q,$$

then, we have

$$\begin{aligned}(2Q)^{-1}&<|\lambda_1\lambda_2^{-1}\tilde{A}_1\tilde{B}_2-\tilde{A}_2\tilde{B}_1|\\ &<(abH^2)^{-1}(\log H)^{c_{30}}.\end{aligned}$$

We have, therefore, one at least, of the three alternatives:

that $\tilde{A}_1\tilde{B}_2 > \frac{1}{2}abH^2(\log H)^{-c_{30}}$,

that $\tilde{A}_1\tilde{B}_2 \geqq Q$,

or that $(2Q)^{-1} < (abH^2)^{-1}(\log H)^{c_{30}}$.

Any one of them gives the conclusion, with a suitable c_{33}.

1.4.2.3. We apply 1.2.1. We have, for each time we fix a pair (a, b) appearing in 1.4.1.4 with 1.4.2.2, one at least, of the following two alternatives:

EITHER (*Case* 1). There exists a pair (u, v) such that

$$u \in \boldsymbol{N}, \quad v \in \boldsymbol{Z}, \quad (u, v) = 1,$$
$$1 \leqq u \leqq \tfrac{1}{2}abH^2(\log H)^{-c_{30}}$$

and that

$$|\lambda_1\lambda_2^{-1}u - v| < \tfrac{1}{2}P^{-2}abH^2(\log H)^{-c_{31}}.$$

OR (*Case* 2). There exists a pair (u, v) such that

$$u \in \boldsymbol{N}, \quad v \in \boldsymbol{Z}, \quad (u, v) = 1,$$
$$\tfrac{1}{2}abH^2(\log H)^{-c_{30}} < u < 2P^2(abH^2)^{-1}(\log H)^{c_{31}}$$

and that

$$|\lambda_1\lambda_2^{-1}u - v| < \tfrac{1}{2}P^{-2}abH^2(\log H)^{-c_{31}}.$$

1.4.2.4. Lemma. *Choosing a sufficiently large h_2', we have that the number of such quadruples (A_1, B_1, A_2, B_2) appearing in* 1.1.2, *that the pairs (a, b) in* 1.4.1.4 *fall into* (*Case* 1) *in* 1.4.2.3, *is*

$$< P^2H^{-2}(\log P)^{-10}.$$

Proof. As we have (Case 1) in 1.4.2.3, we have, with the notations in 1.4.1.4,

$$\frac{\tilde{A}_2\tilde{B}_1}{\tilde{A}_1\tilde{B}_2} = \frac{v}{u}.$$

As $(\tilde{A}_2\tilde{B}_1, \tilde{A}_1\tilde{B}_2) = 1$, it follows that

$$\tilde{A}_2\tilde{B}_1 = v \quad \text{and} \quad \tilde{A}_1\tilde{B}_2 = u.$$

We can suppose, using 1.4.1.2, that

$$\tau(\tilde{X})<(\log P)^{e_2+c_2+2} \qquad \text{for } \tilde{X}=\tilde{A}_1, \cdots, \tilde{B}_2.$$

Therefore, we have, for a fixed pair (a, b), that the number of quadruples $(\tilde{A}_1, \cdots, \tilde{B}_2)$ is

$$\leqq(\log P)^{4(e_2+c_2+2)}.$$

We have, using 1.4.2.2, that the number of possible pairs (a, b) is, with obvious notations,

$$\leq \sum_{(a,b);\ \text{under } 1.4.2.2} 1\leqq\sum\nolimits_n \tau(n)\leqq\max(P, Q, P^2Q^{-1})H^{-2}(\log P).$$

We have, then, the number of such quadruples $(A_1, \cdots, B_2)$ to be counted is

$$\leqq\max(P, Q, P^2Q^{-1})\cdot H^{-2}(\log P)\times(\log P)^{4(e_2+c_2)}.$$

Choosing h_2' sufficiently large, we have the conclusion.

1.4.2.5. Lemma. *We have that the number of such quadruples* (A_1, B_1, A_2, B_2) *appearing in* 1.1.2, *that the pairs* (a, b) *in* 1.4.1.4 *fall into* (*Case* 2) *in* 1.4.2.3 *and, moreover, the product ab are* $>(\log P)^{30}$, *is*

$$<P^2H^{-2}(\log P)^{-5}.$$

Proof. We have that the number or quadruples $(\tilde{A}_1, \cdots, \tilde{B}_2)$ with a fixed (a, b) under (Case 2) of 1.4.1.4 is

$$\leqq\sum\nolimits_{(x,y)}\tau(x)\cdot\tau(y),$$

where pairs (x, y) satisfy that

$$x\in N, \quad y\in Z,$$
$$1\leqq x\leqq P^2(ab\,H^2)^{-1}\log H)^{c_{32}},$$
$$|\lambda_1\lambda_2^{-1}x-y|<(abH^2)^{-1}(\log H)^{c_{32}}$$

and

$$\tau(x) \quad \text{and} \quad \tau(y)<(\log P)^{2(e_2+c_2+1)}.$$

We have one, at most, of y if x is given, and vice versa. Therefore, we have that the number of $(\tilde{A}_1, \cdots, \tilde{B}_2)$, with a fixed (a, b), is

$$\leqq\{\sum_{(x,y)}1\}^{1/2}\cdot\{\sum_x\tau(x)^4\}^{1/4}\cdot\{\sum_y\tau(y)^4\}^{1/4},$$

which is, owing to 1.2.1 (Case 2) and taking into consideration that $(\log Q)^{h_2'}\geqq H\geqq H_2$,

$$\leqq \{P^2(ab\,H^2)^{-2}(\log H)^{c_{33}}\}^{1/2} \times \{P^2(ab\,H^2)^{-1}(\log H)^{c_{32}}(\log P)^{15}\}^{1/2}$$
$$< P^2H^{-3}(ab)^{-3/2}(\log H)^{c_{34}}(\log P)^{15/2}$$
$$< P^2H^{-2}(\log H)^{-(e_2+1)}\cdot(ab)^{-3/2}(\log P)^{15/2}.$$

Now, we take into consideration that $ab > (\log P)^{30}$. We have

$$\sum_{(a,b);\ ab \geqq (\log P)^{30}} (a, b)^{-3/2} < \sum_{n;\ n \geqq (\log P)^{30}} n^{-3/2}\tau(n)$$
$$\ll (\log P)^{-14}.$$

Therefore, we have the conclusion.

1.4.2.6. Lemma. *We have that the number of such quadruples* (A_1, B_1, A_2, B_2) *appearing in* 1.1.2, *that the pairs* (a, b) *in* 1.4.1.4 *fall into* (*Case* 1) *in* 1.4.2.3, *or into* (*Case* 2) *with* $ab > (\log P)^{30}$ *in* 1.4.2.3, *is*

$$\ll P^2H^{-2}(\log P)^{-5}.$$

Proof. 1.4.2.4 and 1.4.2.5.

1.4.3. We consider (Case 2) in 1.4.2.3, with

$$1 \leqq ab \leqq (\log P)^{30}.$$

Hereafter until the end of 1.4, we do not need the assumptions that

$$\lambda_1\lambda_2^{-1}\ (\longrightarrow RQ^{-1}$$

and that

$$Q^{1/2}(\log Q)^{h_2'} \leqq P \leqq Q^{c_2}.$$

What we need is only that $(\log P) \gg \ll (\log Q)$. We prepare constants K, z, L and M, each $\geqq 1$ and to be chosen later depending on H. The pairs (a, b) are considered to lie in

$$L \geqq ab \geqq M.$$

We may regard L and M to be

$$(\log P)^{30} \geqq L \geqq M \geqq 1.$$

The constant M will appear at 1.4.5.

1.4.3.1. Lemma. *Let L be fixed. Then, the number of such quadruples* (A_1, B_1, A_2, B_2) *appearing in* 1.1.2, *that, with the notations in* 1.4.1.4,

$$L \geqq \quad a \text{ and } b \quad \geqq 1$$

and that one $\tilde{X}$, at least, of $\tilde{A}_1, \cdots, \tilde{B}_2$ has a prime divisor p with $p \geqq K$ and $p^2 | \tilde{X}$, is

$$<P^2H^{-2}(\log H)^{-(e_2+1)} \times K^{-1}(\log P)^{e_{21}}(\log L)^2$$

where $e_{21}=e_2+c_{31}+3$.

Proof. Let the pair (a, b) be fixed. Then, for instance, the number of possible $(\tilde{A}_1, \cdots, \tilde{B}_2)$, where A_1 has a prime divisor p with $p \geqq K$ and $p^2 | \tilde{A}_1$, is owing to 1.4.1.1,

$$\ll \sum_{(A_1, B_1)} 1 < \sum p^{-2}P(aH)^{-1} \times P(bH)^{-1}(\log H)^{c_{31}+1}$$
$$\ll P^2(abH^2)^{-1}K^{-1}(\log H)^{c_{31}+1}.$$

We have that

$$\sum_{a,b;\ L \geqq a,b \geqq 1} (ab)^{-1} \ll (\log L)^2,$$

therefore, the conclusion follows.

Corollary. *We may add the restriction, hereafter, that each $\tilde{X}$ of $\tilde{A}_1, \cdots, \tilde{B}_2$ in* 1.4.1.4 *is not divisible by p^2, if p is a prime not smaller than K.*

The value of K will be chosen at 1.4.8.

1.4.3.2. (i) The constants K and z are restricted to lie in

$$(\log P)^c > K > (\log H)^{e_{29}}$$

and

$$(\log P)^{1/2} > z > (\log H)^{e_{29}},$$

where c and e_{29} are constants. The values will be chosen in 1.4.8.

(ii) Let $\nu_{[K,K^z)}(X)$ be that of 1.1.3.1 for a positive integer X, i.e., the number of different prime divisors of X lying in the interval $[K, K^z)$. Let $\tau_{[K,K^z)}(X)$ be the number of such positive divisors d of X, that every prime divisor of d lies in $[K, K^z)$, or that d may be 1.

(iii) We have that

$$\nu_{[K,K^z)}(X) \leqq \nu(X),$$

where $\nu(X)$ is the number of all different prime divisors of X.

(iv) We have, for $\tilde{X}$ in 1.4.1.4, that

$$\tau_{[K,K^z)}(\tilde{X}) = 2^{\nu_{[K,K^z)}(\tilde{X})}.$$

1.4.3.3. Lemma. *Let K, z and L be fixed, under* 1.4.3 *and* 1.4.3.2 (i).

We have, then, the number of such quadruples (A_1, B_1, A_2, B_2) *appearing in* 1.1.2, *that, with the notations* 1.4.1.4 *and* 1.4.3.2 (ii),

$$L \geqq \quad a \text{ and } b \quad \geqq 1$$

and that one $\tilde{X}$, *at least, of* $\tilde{A}_1, \cdots, \tilde{B}_2$ *satisfies that*

$$\nu_{[K,K^z)}(\tilde{X})=0,$$

is

$$\leqq P^2H^{-2}(\log H)^{-(e_2+1)} \times z^{-1}(\log H)^{e_{22}}(\log L)^2.$$

Proof. This is the only step in our proof, where we need the result of the sieve of A. Selberg. Let (a, b) be fixed. We have that the number of such $(\tilde{A}_1, \cdots, \tilde{B}_2)$ that $\nu_{[K,K^z)}(\tilde{A}_1)=0$, for instance, is owing to 1.4.1.1,

$$<\sum_{\tilde{A}_1} 1 \times P(bH)^{-1}(\log H)^{c_2+1}.$$

We use 1.2.2 to obtain that the right-hand side is

$$<(P(aH)^{-1}z^{-1}+K^{2z})P(bH)^{-1}(\log H)^{c_2+1}$$
$$<P^2(abH^2)^{-1}z^{-1}(\log H)^{c_2+2},$$

taking into consideration 1.4.3.2 (i). The rest is similar to that of 1.4.3.1.

1.4.3.4. Lemma. *Let K, z and L be fixed, under* 1.4.3 *and* 1.4.3.2 (i). *We have, then, that the number of such quadruples* (A_1, B_1, A_2, B_2) *appearing in* 1.1.2, *that, with the notations in* 1.4.1.4 *and* 1.4.3.2 (ii),

$$L \geqq \quad a \text{ and } b \quad \geqq 1$$

and that one $\tilde{X}$, *at least, of* $\tilde{A}_1, \cdots, \tilde{B}_2$ *satisfies that*

$$\nu(\tilde{X}) \geqq (\log\log P)^{e_2+6},$$

is

$$\leqq P^2H^{-2}(\log\log P)^{-(e_2+3)}.$$

Proof. We have, as a well-known result,

$$\sum_{X \leqq \xi} \nu(X) \ll \xi \log\log(\xi+2).$$

Let (a, b) be fixed. We have, then, that the number of such $(\tilde{A}_1, \cdots, \tilde{B}_2)$ that $\nu(\tilde{A}_1) \geqq (\log\log P)^{e_2+6}$, for instance, is owing to 1.4.1.1,

$$\ll P^2(abH^2)^{-1}(\log\log P)^{-(e_2+5)}.$$

The rest is similar to that of 1.4.3.1.

1.4.3.5. Lemma. *Let K, z and L be fixed, under* 1.4.3 *and* 1.4.3.2 (i). *We have, then, that the number of such quadruples* (A_1, B_1, A_2, B_2) *appearing in* 1.1.2, *that, with the notations in* 1.4.1.4 *and* 1.4.3.2 (ii),

$$L \geqq \quad a \text{ and } b \quad \geqq 1$$

and that one $\tilde{X}$, at least, of $\tilde{A}_1, \cdots, \tilde{B}_2$ satisfies that

$$(\log\log P)^{e_2+7} \geqq \nu_{[K,K^z)}(\tilde{X}) \geqq 10 \log z,$$

is

$$< P^2 H^{-2}(\log H)^{-(e_2+1)} \times z^{-2}(\log H)^{e_{24}}(\log L)^2.$$

The proof of this lemma will end at 1.4.3.5.4.

1.4.3.5.1. Sublemma. *Let ν be a positive integer, and ξ_1, ξ_2 be real numbers such that $\xi_2 > \xi_1 \geqq 2$. We have, then, that*

$$\sum_{(p_1,\cdots,p_\nu)} (p_1 \cdots p_\nu)^{-1} \leqq (\nu!)^{-1}(\log\log\xi_2 - \log\log\xi_1 + c(\log\xi_1)^{-1})^\nu,$$

where $p_1, \cdots, p_\nu$ are mutually different prime numbers such that

$$\xi_1 \leqq p_1 < p_2 < \cdots < p_\nu \leqq \xi_2.$$

Proof. This is a simple corollary of Mertens' theorem.

1.4.3.5.2. Sublemma. *Let ξ be a real number with $\xi \geqq 10$, and ν be a positive integer with $\nu \geqq 10\xi$. We have, then, that*

$$(\nu!)^{-1}\xi^\nu < ce^{-3\xi},$$

with a numerical positive constant c.

Proof. We have, by Stirling's formula

$$\Gamma(x) \approx \sqrt{2\pi}\, e^{-x} x^{x-1/2} \qquad \text{as } x \longrightarrow +\infty,$$

that

$$(\nu!)\xi^\nu = \Gamma(\nu+1)^{-1}\xi^\nu \ll e^{-f(\nu)},$$

where

$$f(\nu) = \nu + \nu\log\xi - (\nu + \tfrac{1}{2})\log(\nu+1).$$

We have, if $\nu \geqq \max(\xi, 1)$,

$$\begin{aligned}\frac{df}{d\nu} &= 1+\log\xi-\log(\nu+1)-\frac{\nu+1/2}{\nu+1}\\ &=\log\xi-(2(\nu+1))^{-1}-\log(\nu+1)\\ &<\log\xi+(2\nu)^{-1}(1-\nu^{-1}+\nu^{-2})-(\log\nu+\nu^{-1}-(2\nu^2)^{-1})\\ &=\log\xi-\log\nu-(2\nu)^{-1}+(2\nu^3)^{-1}\\ &\leqq\log\xi-\log\nu\leqq 0.\end{aligned}$$

Putting $\nu=\mu\xi$ with $\mu\geqq 1$, we have, then,

$$\begin{aligned}f(\nu)&=\mu\xi+\mu\xi\log\xi-(\mu\xi+\tfrac{1}{2})\log(\mu\xi+1)\\ &<\mu\xi+\mu\xi\log\xi-\mu\xi\log\mu\xi\\ &=\mu\xi-\mu\xi\log\mu=\mu\xi(1-\log\mu),\end{aligned}$$

which is $<-3\xi$, if $\mu\geqq 10$.

1.4.3.5.3. Sublemma. *Let K and z be fixed, under* 1.4.3.2 (i). *We have, then, that the number of such integers X, that*

$$1\leqq X\leqq PH^{-1}(\log H)^{c_2+1},$$

that, if p is a prime lying in $[K, K^z)$, then, $p^2\nmid X$, and that

$$(\log\log P)^{c_2+7}\geqq\nu_{[K,K^z)}(X)\geqq 10\log z,$$

is

$$\leqq PH^{-1}z^{-2}(\log H)^{c_2+2}.$$

Proof. Let ν_0 be $=\inf.\{\nu\in N;\ \nu\geqq 10\log z\}$. Let ν run through the set of integers between ν_0 and $(\log\log P)^{c_2+7}$. We have that the number of such X to be counted is

$$\begin{aligned}&\leqq\sum_{\nu}\#\begin{pmatrix}X\in N;\ 1\leqq X\leqq PH^{-1}(\log H)^{c_2+1}, & \\ \qquad p^2\nmid X \text{ if } p \text{ is a prime lying in } [K, K^z), \\ \qquad \text{and}\quad \nu_{[K,K^z)}(X)=\nu\end{pmatrix}\\ &<\sum_{\nu}\sum_{(p_1,\cdots,p_\nu)}\#\begin{pmatrix}X\in N;\ 1\leqq X\leqq PH^{-1}(\log H)^{c_2+1}, \\ \qquad\text{and}\quad X\equiv 0 \bmod p_1\cdots p_\nu\end{pmatrix},\end{aligned}$$

where $p_1, \cdots, p_\nu$ are mutually different prime numbers such that

$$K\leqq p_1<p_2<\cdots<p_\nu<K^z.$$

Using 1.4.3.2 (i), we have that the number above to be counted is

$$\sum_{\nu}\sum_{(p_1,\cdots,p_\nu)} P(p_1\cdots p_\nu H)^{-1}(\log H)^{c_2+1}+(\log\log P)^{e_2+7}(K^z)^{(\log\log P)^{e_2+5}},$$

which is, through 1.4.3.5.1,

$$\leqq \sum_{\nu=\nu_0}^{\infty}(\nu!)^{-1}(\log z+c(\log K)^{-1})^\nu PH^{-1}(\log H)^{c_2+1}+e^{(\log P)^{2/3}}.$$

We have that, for a real $\xi>10$,

$$\sum_{\nu=\nu_0}^{\infty}(\nu!)^{-1}\xi^\nu=\sum_{\nu=0}^{\infty}((\nu+\nu_0)!)^{-1}\xi^{\nu+\nu_0}$$
$$\leqq\sum_{\nu=0}^{\infty}(\nu_0!\cdot\nu!)^{-1}\xi^{\nu+\nu_0}=(\nu_0!)^{-1}\xi^{\nu_0}e^{\xi},$$

which is

$$\ll e^{-2\xi},$$

through 1.4.3.5.2. Letting ξ be equal to $\log z+c(\log K)^{-1}$, we easily obtain the conclusion.

1.4.3.5.4. Sublemma. *We have the conclusion of* 1.4.3.5.

Proof. We use, for a fixed pair (a, b), 1.4.3.5.3 to count the number of $(\tilde{A}_1, \cdots, \tilde{B}_2)$, then the method of the proof in 1.4.3.1 to let (a, b) run freely.

1.4.3.6. Let us call a positive integer X to be $[K, K^z)$-good, up to 1.4.6, if $p^2\nmid X$ for every prime p with $K\leqq p<K^z$, and if $10\log z\geqq\nu_{[K,K^z)}(X)\geqq 1$.

1.4.3.6.1. Corollary. *We may add the restriction, hereafter, that each of $\tilde{A}_1, \cdots, \tilde{B}_2$ in* 1.4.1.4 *is $[K, K^z)$-good.*

Proof. 1.4.3.1, 1.4.3.3, 1.4.3.4 and 1.4.3.5.

1.4.3.7. (i) We decompose $\tilde{A}_1$, $\tilde{B}_1$, $\tilde{A}_2$, $\tilde{B}_2$ of 1.4.1.4, which are $[K, K^z)$-good, as follows:

$$\tilde{A}_1=a_1'\hat{A}_1,\qquad \tilde{A}_2=a_2'\hat{A}_2,$$
$$\tilde{B}_1=b_1'\hat{B}_1,\qquad \tilde{B}_2=b_2'\hat{B}_2,$$

where a_1', b_1', a_2' and b_2' consist of all prime divisors lying in the interval $[K, K^z)$, and $\hat{A}_1$, $\hat{B}_1$, $\hat{A}_2$ and $\hat{B}_2$ have no prime divisors in $[K, K^z)$.

(ii) We have, under (i),

$$K \leqq a_1', \ a_2', \ b_1' \text{ and } b_2' \leqq K^{10z \log z},$$
$$P(a_i'aH)^{-1}(\log H)^{-c_2} \leqq \hat{A}_i \leqq P(a_i'aH)^{-1},$$
$$P(b_i'bH)^{-1}(\log H)^{-c_{31}} \leqq \hat{B}_i \leqq P(b_i'bH)^{-1}(\log H)^{c_{31}}$$

and

$$(a_i', \hat{A}_i) = 1, \quad (b_i', \hat{B}_i) = 1 \quad (i = 1, 2).$$

We have, also, for $X = a_1'$, a_2', b_1' and b_2', that

$$\tau_{[K,K^z)}(X) = 2^{\nu_{[K,K^z)}(X)} \leqq z^c$$

where $c = 10 \log 2$.

1.4.4. Lemma. *We have, using* 1.4.1.4 *and* 1.4.3.7, *that the quadruples* $(A_1, \cdots, B_2)$ *to be counted in* 1.1.2 *satisfy*

$$|(\lambda_1\lambda_2^{-1}\hat{A}_1\hat{B}_1^{-1})(a_1'b_2'\hat{B}_2) - (b_1'a_2'\hat{A}_2)| < (ab\hat{B}_1H^2)^{-1}(\log H)^{c_{30}}.$$

Proof. A corollary to 1.4.2.1.

1.4.4.1. We have, each time we fix a, b, $\hat{A}_1$ and $\hat{B}_1$, one, at least, of the following two alternatives;

EITHER (*Case* 1). There exists a pair (u_1, v_1) such that

$$u_1 \in \boldsymbol{N}, \quad v_1 \in \boldsymbol{Z}, \quad (u_1, v_1) = 1,$$
$$1 \leqq u_1 \leqq \tfrac{1}{2} abB_1H^2(\log H)^{-c_{30}}$$

and

$$|(\lambda_1\lambda_2^{-1}\hat{A}_1\hat{B}_1^{-1})u_1 - v_1| < (ab\hat{A}_1H^2)(2P^2(\log H)^{c_2+1})^{-1}.$$

OR (*Case* 2). There exists a pair (u_1, v_1) such that

$$u_1 \in \boldsymbol{N}, \quad v_1 \in \boldsymbol{Z}, \quad (u_1, v_1) = 1.$$
$$\tfrac{1}{2} ab\hat{B}_1H^2(\log H)^{-c_{30}} < u_1 \leqq 2P^2(\log H)^{c_2+2}(ab\hat{A}_1H^2)^{-1}$$

and

$$|(\lambda_1\lambda_2^{-1}\hat{A}_1\hat{B}_1^{-1})u_1 - v_1| < (ab\hat{A}_1H^2)(2P^2(\log H)^{c_2+1})^{-1}.$$

1.4.4.2. Lemma. *We have, if integers* X *and* Y *are fixed, that the number of such sextuples* $(a_1', b_1', a_2', b_2', \hat{A}_2, \hat{B}_2)$, *that they appear as in* 1.1.2, 1.4.1.4 *and* 1.4.3.7,

$$a_1'b_2'\hat{B}_2 = X \quad \textit{and} \quad b_1'a_2'\hat{A}_2 = Y,$$

is $\leqq z^{c_{35}}$.

Proof. We have, for example,

$$\tau(a_1'b_1') = \tau_{[K,K^z)}(a_1'b_1') \leqq 3^{\nu_{[K,K^z)}(a_1'b_1')}$$
$$\leqq 3^{20 \log z} = z^c.$$

Also, we have a unique decomposition $(a_1'b_2') \times \hat{B}_2 = X$. The rest is easy.

1.4.5. Lemma. *Let K, z and M be fixed, under* 1.4.3 *and* 1.4.3.2 (i). *We have, then, that the number of such quadruples* (A_1, B_1, A_2, B_2) *appearing in* 1.1.2, *with the notations* 1.4.1.4 *and* 1.4.3.7 (i),
that they satisfy 1.4.3.6.1 *and* 1.4.3.7 (ii),
that we have (*Case* 2) *in* 1.4.4.1,
and that $ab \geqq M$, *is*

$$\leqq P^2 H^{-2} (\log H)^{-(e_2+1)} \times (H^2 M)^{-1} z^{c_{36}} (\log M)(\log K)^2 (\log H)^{e_{25}},$$

where e_{25} *is a suitable positive constant depending on* e_2.

Proof. Let a, b, $\hat{A}_1$ and $\hat{B}_1$ be fixed. We have, through 1.2.1 (*Case* 2), that the number of such pairs (X, Y) that

$$X \in \boldsymbol{N}, \quad Y \in \boldsymbol{Z},$$
$$1 \leqq X \leqq (ab\hat{A}_1 H^2)^{-1} P^2 (\log H)^{c_2+1},$$
$$|(\lambda_1 \lambda_2^{-1} \hat{A}_1 \hat{B}_1^{-1}) X - Y| < (ab\hat{B}_1 H^2)^{-1} (\log H)^{c_{30}}$$

is

$$< ((ab)^2 (\hat{A}_1 \hat{B}_1 H^4))^{-1} P^2 (\log H)^{c_{37}}.$$

We have, therefore, through 1.4.4 and 1.4.4.2, that the number of such sextuples $(a_1', b_1', a_2', b_2', \hat{A}_2, \hat{B}_2)$, which combined with a, b, $\hat{A}_1$ and $\hat{B}_1$, give quadruples (A_1, B_1, A_2, B_2) to be counted, is

$$\leqq z^{c_{35}} ((ab)^2 (\hat{A}_1 \hat{B}_1 H^4))^{-1} P^2 (\log H)^{c_{37}}.$$

Now, let a pair (a, b) be fixed. We have, then,

$$P(aH)^{-1} (\log H)^{-c} K^{-10z \log z} < \hat{A}_1 < P(aH)^{-1} (\log H)^c K^{-1}$$

and similarly for $\hat{B}_1$ with b in place of a. We have, therefore, taking 1.4.3.2 (i) into consideration, that

$$\sum_{\hat{A}_1, \hat{B}_1} (\hat{A}_1 \hat{B}_1)^{-1} \ll (\log (K^{10z \log z} (\log H)^c))^2$$
$$\ll z^3 (\log K)^2.$$

We have

$$\sum_{a,b;\, ab \geqq M} (ab)^{-2} \leqq \sum_{x \geqq M} x^{-2} \tau(x) \ll M^{-1} \log M.$$

We have the conclusion by combining the estimates obtained.

1.4.6. Lemma. *Let K, z and L be fixed, under* 1.4.3 *and* 1.4.3.2 (i). *We have, then, that the number of such quadruples* (A_1, B_1, A_2, B_2) *appearing in* 1.1.2, *with the notations* 1.4.1.4 *and* 1.4.3.7 (i),

that they satisfy 1.4.3.6.1. *and* 1.4.3.7 (ii),

that we have (*Case* 1) *in* 1.4.4.1,

and that $L \geqq ab$ $(\geqq 1)$, *is*

$$\leqq P^2 H^{-2} (\log H)^{-(e_2+1)} \times K^{-2} z^{c_{35}} (\log H)^{e_{26}} (\log L)^2,$$

where e_{26} *is a suitable positive constant depending on* e_2.

Proof. Let $a, b, \hat{A}_1$ and $\hat{B}_1$ be fixed. We have, through 1.2.1 (Case 1), that

$$\frac{b_1' a_2' \hat{A}_2}{a_1' b_2' \hat{B}_2} = \frac{v_1}{u_1}.$$

Both of these fractions are irreducible. We have

$$b_1' a_2' \hat{A}_2 = v_1 \quad \text{and} \quad a_1' b_2' \hat{B}_2 = u_1.$$

We have, then, through 1.4.4.2, that the number of such $(a_1', b_1', a_2', b_2', \hat{A}_2, \hat{B}_2)$, which, combined with a, b, $\hat{A}_1$ and $\hat{B}_1$, give quadruples (A_1, B_1, A_2, B_2) to be counted, is $\leqq z^{c_{35}}$. We have

$$\#\{(a, b, \hat{A}_1, \hat{B}_1)\} \leqq \sum_{a,b} P(aKH)^{-1} \times P(bKH)^{-1} (\log H)^{c_2+1}$$
$$\leqq P^2 (KH)^{-2} (\log L)^2 (\log H)^{c_2+2}.$$

We have, then, the conclusion by combining the estimates obtained.

1.4.7. Lemma. *Let K, z, L and M be given, under* 1.4.3 *and* 1.4.3.2 (i). *We have, then, the number of such quadruples* (A_1, B_1, A_2, B_2) *appearing in* 1.1.2, *with the notation* 1.4.1.4, *that*

$$L \geqq ab \geqq M,$$

is

$$\leqq P^2H^{-2}(\log H)^{-(e_2+1)}$$

$$\times\left\{\begin{array}{l} K^{-1}(\log H)^{e_{21}}(\log L)^2+z^{-1}(\log H)^{e_{22}}(\log L)^2 \\ \qquad +(\log\log P)^{-2}+z^{-2}(\log H)^{e_{24}}(\log L)^2 \\ \qquad +(H^2M)^{-1}z^{c_{36}}(\log K)^2(\log M)(\log H)^{e_{25}} \\ \qquad +K^{-2}z^{c_{35}}(\log L)^2(\log H)^{c_{26}} \end{array}\right\}.$$

Proof. 1.4.3.1, 1.4.3.3, 1.4.3.4, 1.4.3.5, 1.4.5 and 1.4.6.

1.4.8. Lemma. *Let H be* $(\log Q)^{h_2}>H$ $(\geqq H_2)$, *and L, M* $(\geqq 1)$ *be*

$$(\log P)^{30}\geqq L>M\geqq H^{-2}(\log H)^{e_{27}}(\log L)^{c_{38}},$$

with constants e_{27} and c_{38}. We have, then, the number of such (A_1, B_1, A_2, B_2) appearing in 1.1.2, *that*

$$L\geqq(A_1B_2,\ A_2B_1)\geqq M,$$

is

$$\leqq P^2H^{-2}(\log H)^{-(e_2+1)}(\log L)^{-2}.$$

Proof. We take z and K as

$$z=\max\{(\log H)^{e_{22}}(\log L)^4,\ \sqrt{(\log H)^{e_{24}}}\times(\log L)\},$$

and

$$K=\max\{(\log H)^{e_{21}}(\log L)^4,\ \sqrt{z^{c_{35}}(\log H)^{e_{26}}}\times(\log L)\}.$$

We restrict M to

$$(L>)M\geqq H^{-2}(\log H)^{e_{25}}z^{c_{36}}(\log K)^2(\log L)^3.$$

This restriction gives the same one of the lemma, with suitable e_{27} and c_{38}. It is non-void if H_2 is suitably large. We apply 1.4.7. which gives the conclusion.

1.4.8.1. Lemma. *Let e_{28} and H_2 be positive constants such that $H_2^{e_{28}}$ is sufficiently large with respect to e_{27}. We have, then, that the number of such quadruples (A_1, B_1, A_2, B_2) appearing in* 1.1.2, *that*

$$(\log P)^{30}\geqq(A_1B_2,\ A_2B_1)\geqq H^{e_{28}},$$

is

$$\leqq P^2H^{-2}(\log H)^{-(e_2+1)}\times(1+H_2^{e_{28}})^{-1}.$$

Proof. Let us define M_j and L_j inductively, as follows;

$$M_0 = H^{e_{28}}, \qquad L_0 = \exp\left((M_0 H^{3/2})^{c'_{38}}\right),$$

and

$$L_j = M_{j+1} = \exp\left((L_{j-1} H^{3/2})^{c'_{38}}\right) \qquad (j = 1, 2, 3, \cdots),$$

where $c'_{38} = (2c_{38})^{-1}$. (This L_0 has no relation with that in Theorem 0.2.) We have, easily, that

$$\log L_j \geqq M_0(j+1) \qquad (j = 0, 1, 2, \cdots),$$

and that

$$L_j > M_j \geqq H^{-2}(\log H)^{e_{27}}(\log L_j)^{c_{38}},$$

by choosing $H_2^{e_{28}}$ suitably large. We apply 1.4.8 for each pair (M_j, L_j), giving an upper bound

$$P^2 H^{-2}(\log H)^{-(e_2+1)}(M_0(j+1))^{-2}$$

on the number of such $(A_1, \cdots, B_2)$ that

$$L_j \geqq (A_1B_2, A_2B_1) \geqq M_j.$$

Summing over $j = 0, 1, 2, \cdots$, we have the conclusion.

1.4.8.2. Lemma. Let e_{28} *be an arbitrarily given positive constant. We have, by choosing* H_2 *suitably large, that the number of such quadruples* (A_1, B_1, A_2, B_2) *appearing in* 1.1.2, *that*

$$H^{e_{28}} \geqq (A_1B_2, A_2B_1) \geqq 1,$$

is

$$\leqq P^2 H^{-2}(\log H)^{-(e_2+1)}.$$

Proof. We can put $L = H^{e_{28}}$ and $M = 1$ in 1.4.8.

1.4.9. We have proved 1.1.2.

Proof. 1.4.2.6, 1.4.8.1 and 1.4.8.2.

1.5. Proof of Proposition 1.1.3

The proof of 1.1.3 is, as easily guessed, almost a corollary of that of 1.1.2. We simply list up the corresponding lemmas.

1.5.1. we put, in 1.1.3,

$$a_1=(A, A_1), \quad A=a_1\tilde{A}, \quad A_1=a_1\tilde{A}_1$$

and

$$b_1=(B, B_1), \quad B=b_1\tilde{B}, \quad B_1=b_1\tilde{B}_1.$$

1.5.2. Lemma. *We have, in* 1.1.3, *that*

$$|\lambda_1 A_1 B - AB_1| \leqq g^3.$$

Proof. Similar with 1.4.1.

1.5.2.1. Corollary. *We have, in* 1.1.3, *that the number of quadruples* (A, B, A_1, B_1) *is at most* $O(g^4)$, *if the pair* (A, B) *or* (A_1, B_1) *is fixed.*

Proof. Similar with 1.4.1.1.

1.5.2.2. Corollary. *We may, in* 1.1.3, *suppose that*

$$\tau(X) \leqq G_0(\log P)^2$$

and

$$\nu(X) \leqq 1.1(\log\log P) \qquad \textit{for } X=A, B, A_1, B_1.$$

Proof. Because

$$\sum_{X\leqq\xi} \tau(X) \ll \xi \log \xi \quad \text{and} \quad \sum_{X\leq P} (\nu(X) - \log\log P)^2 \ll P \log\log P.$$

1.5.3. Lemma. *We have, in* 1.1.3 *with* 1.5.1, *that the number of such quadruples* (A, B, A_1, B_1) *to be counted, that satisfy*

$$a_1 b_1 \geqq (\log P)^{10},$$

is

$$\leqq g_0^{-1} P^2,$$

where G_0 *in* 1.1.3. *is chosen sufficiently large.*

Proof. We have that

$$|\lambda_1 \tilde{A}_1 \tilde{B} - \tilde{A}\tilde{B}_1| < g^3(a_1 b_1)^{-1}$$

and

$$g^{-4}P^2(a_1 b_1)^{-1} \leqq \tilde{A}_1 \tilde{B} \leqq g^4 P^2 (a_1 b_1)^{-1},$$

from 1.5.2. In the followings, we take 1.5.2.2 into consideration, where G_i are suitable large positive constants.

(I) Suppose that $G_1^{-1}P^2(\log P)^{-10} \geqq a_1 b_1 \geqq G_1 P$.
We have that

$$\tilde{A}_1 \tilde{B} \leqq g^4 P^2 (a_1 b_1)^{-1} < \tfrac{1}{2} g^{-3} (a_1 b_1),$$

therefore, through 1.2.3.2 (i), that

$$\lambda_1 (\longrightarrow (\tilde{A}\tilde{B}_1)(\tilde{A}_1\tilde{B})^{-1},$$

the fraction being irreducible. We have at most $O(\log g)$ of $X = \tilde{A}_1 \tilde{B}$ and $Y = \tilde{A}\tilde{B}_1$, because of 1.2.3.1 (iv-vi), therefore at most $O((\log g)(\log P)^8)$ of $(\tilde{A}, \tilde{B}, \tilde{A}_1, \tilde{B}_1)$, if a_1 and b_1 are fixed. We have $O(G_1^{-1}P^2(\log P)^{-9})$ of (a_1, b_1), therefore the conclusion in this case.

(II) Suppose that $U_1 \leqq \frac{1}{4} g^{-3} a_1 b_1 \leqq G_1 P$ and that $|\lambda_1 U_1 - V_1| < (2g^4 P^2)^{-1} a_1 b_1$. We have, through 1.2.1 (Case 1), that

$$\tilde{A}\tilde{B}_1 = V_1 \quad \text{and} \quad \tilde{A}_1 \tilde{B} = U_1.$$

We have, therefore, that the number of quadruples $(\tilde{A}, \tilde{B}, \tilde{A}_1, \tilde{B}_1)$ is $\leqq G_2 (\log P)^8$. We have $O(G_1 P \log P)$ of pairs (a_1, b_1), therefore $O(G_3 P (\log P)^9)$ of quadruples (A, B, A_1, B_1).

(III) Suppose that $U_1 \leqq \frac{1}{4} g^{-3} a_1 b_1 \leqq G_1 P$ and that $|\lambda_1 U_1 - V_1| \geqq (2g^4 P^2)^{-1} a_1 b_1$. We have $|\lambda_1 U_1 - V_1| \leqq g_0 U_1 P^{-2}$, by the assumption of 1.1.3. We put

$$|\lambda_1 U_1 - V_1| = P^{-2} \Omega_1.$$

We have, then, from the assumptions, that

$$g U_1 \geqq \Omega_1 \geqq (2g^4)^{-1} a_1 b_1,$$

therefore, that

$$2g^5 U_1 \geqq a_1 b_1 \geqq 2g^3 U_1.$$

As $a_1 b_1$ is $\geqq (\log P)^{10}$, we can suppose that

$$U_1 \gg g^{-5} (\log P)^{10} \quad \text{and} \quad \Omega_1 \gg (\log P)^9.$$

Let a pair (a_1, b_1) and a real ξ be fixed. We count such $(\tilde{A}, \tilde{B}, \tilde{A}_1, \tilde{B}_1)$ that

$$\xi < \tilde{A}_1 \tilde{B} \leqq \xi + \tfrac{1}{2} \Omega_1^{-1} P^2$$

and that $\lambda_1 \tilde{A}_1 \tilde{B} - \tilde{A}\tilde{B}_1$ is of a fixed signature. If this is non-void, we can put, through 1.2.1 (Case 1), as

$$\tilde{A}_1\tilde{B}=X_\xi+U_1W,$$
$$\tilde{A}\tilde{B}_1=Y_\xi+V_1W$$

with

$$|W|\leqq\tfrac{1}{2}(U_1\Omega_1)^{-1}P^2,$$

where (X_ξ, Y_ξ) is a particular solution of

$$|\lambda_1X_\xi-Y_\xi|<g^3(a_1b_1)^{-1},$$
$$\lambda_1X_\xi-Y_\xi \text{ is of the fixed signature}$$

and

$$\xi<X_\xi\leqq\xi+\tfrac{1}{2}\Omega_1^{-1}P^2.$$

We have, therefore, at most

$$(2\times\tfrac{1}{2}(U_1\Omega_1)^{-1}P^2+1)G_4(\log P)^8\ll(U_1\Omega_1)^{-1}P^2(\log P)^8$$

of $(\tilde{A}, \tilde{B}, \tilde{A}_1, \tilde{B}_1)$, if a_1, b_1 and ξ are fixed. We, then, have at most

$$G_5\times g^4(a_1b_1)^{-1}\Omega_1\times(U_1\Omega_1)^{-1}P^2(\log P)^8\leqq G_6U_1^{-2}P^2(\log P)^8$$

of $(\tilde{A}, \tilde{B}, \tilde{A}_1, \tilde{B}_1)$, if a_1 and b_1 are fixed. We have, therefore, at most

$$G_7U_1^{-1}P^2(\log P)^9$$

of (A, B, A_1, B_1) with $a_1b_1\geqq(\log P)^{10}$. As U_1 is $\gg g^{-5}(\log P)^{10}$, we have done with this case also.

Now we can suppose that $U_1>\frac{1}{4}g^{-3}a_1b_1$ if $a_1b_1\ll P$.

(IV) Suppose that $U_1>\frac{1}{4}g^{-3}a_1b_1$, that $g^4P\geqq a_1b_1\geqq(\log P)^{10}$ and that $|\lambda_1U_1-V_1|<(2g^4P^2)^{-1}a_1b_1$. We have, through 1.2.1 (Case 2), that the number of pairs $(\tilde{X}, \tilde{Y})$, for which there exist such $(\tilde{A}, \tilde{B}, \tilde{A}_1, \tilde{B}_1)$ that $\tilde{X}=\tilde{A}_1\tilde{B}$ and $\tilde{Y}=\tilde{A}\tilde{B}_1$, is

$$\ll g^7P^2(a_1b_1)^{-2},$$

if a_1 and b_1 are fixed, We have, then, that the number of $(\tilde{A}, \tilde{B}, \tilde{A}_1, \tilde{B}_1)$ is

$$\ll G_8P^2(\log P)^8(a_1b_1)^{-2}.$$

We have that

$$\sum_{a_1,b_1;\ a_1b_1\geqq(\log P)^{10}}(a_1b_1)^{-2}\ll(\log P)^{-9},$$

therefore we have done in this case also.

(V) Suppose that $U_1 > \frac{1}{4} g^{-3} a_1 b_1$, that $g^4 P \geqq a_1 b_1 \geqq (\log P)^{10}$ and that $|\lambda_1 U_1 - V_1| \geqq (2g^4 P^2)^{-1} a_1 b_1$. We put, as in (III),

$$|\lambda_1 U_1 - V_1| = P^{-2} \Omega_1.$$

We have

$$g U_1 \geqq \Omega_1 \geqq (2g^4)^{-1} a_1 b_1.$$

Let a pair (a_1, b_1) and a real ξ be fixed. We count such pairs $(\tilde{X}, \tilde{Y})$ that

$$|\lambda_1 \tilde{X} - \tilde{Y}| < g^3 (a_1 b_1)^{-1},$$
$$\xi < \tilde{X} \leqq \xi + \tfrac{1}{2} \Omega_1^{-1} P^2$$

and that $\lambda_1 \tilde{X} - \tilde{Y}$ is of a fixed signature.

We have, through 1.2.1 (Case 2), that the number of pairs $(\tilde{X}, \tilde{Y})$ is

$$\leqq G_9 P^2 (\Omega_1 a_1 b_1)^{-1},$$

therefore, that the number of $(\tilde{A}, \tilde{B}, \tilde{A}_1, \tilde{B}_1)$ is

$$< G_{10} P^2 (\log P)^8 (\Omega_1 a_1 b_1)^{-1},$$

for fixed a_1, b_1 and ξ. We have, then, the number of $(\tilde{A}, \tilde{B}, \tilde{A}_1, \tilde{B}_1)$ is

$$\leqq G_{11} P^2 (\log P)^8 (\Omega_1 a_1 b_1)^{-1} \times \Omega_1 (a_1 b_1)^{-1}$$
$$\leqq G_{12} P^2 (\log P)^8 (a_1 b_1)^{-2}.$$

The rest is similar as in (IV).

(VI) We are left with the case that $a_1 b_1 > G^{-1} P^2 (\log P)^{-10}$. We suppose, first, that $\tilde{A}_1 \tilde{B} < U_1$. We have, then,

$$|\lambda_1 \tilde{A}_1 \tilde{B} - \tilde{A} \tilde{B}_1| < g^3 (a_1 b_1)^{-1} < (2 \tilde{A}_1 \tilde{B})^{-1},$$
$$1 \leqq \tilde{A}_1 \tilde{B} < U_1$$

and

$$(\tilde{A}_1 \tilde{B}, \tilde{A} \tilde{B}_1) = 1.$$

This implies, through 1.2.3.2 (i) and 1.2.3.1 (iv), that

$$(2U_1)^{-1} \leqq |\lambda_1 \tilde{A}_1 \tilde{B} - \tilde{A} \tilde{B}_1| < g^3 (a_1 b_1)^{-1},$$

therefore, that

$$a_1 b_1 < 2 g^3 U_1 \leqq 2 g^3 P^2 (\log P)^{-11}.$$

This is impossible by $a_1b_1 > G^{-1}P^2(\log P)^{-10}$.

We suppose, next, that $\tilde{A}_1\tilde{B} > U_1$. We have, in this case, that

$$(2\tilde{A}_1\tilde{B})^{-1} \leqq |\lambda_1 U_1 - V_1| \leqq g U_1 P^{-2},$$

therefore that

$$U_1 \geqq (2g)^{-1}P^2(\tilde{A}_1\tilde{B})^{-1}.$$

This implies that

$$U_1 \geqq G_{14}^{-1}P^2(\log P)^{-10},$$

which is impossible again, owing to the assumption. We have, therefore, that

$$U_1 = \tilde{A}_1\tilde{B} \quad \text{and} \quad V_1 = \tilde{A}\tilde{B}_1,$$

and, then, that the number of $(\tilde{A}, \tilde{B}, \tilde{A}_1, \tilde{B}_1)$ is

$$\leqq \tau(U_1)\tau(V_1).$$

Suppose that $\tilde{A}_1 \leqq U_1^{1/2}$. Then we have $\tilde{B} \geqq U_1^{1/2}$. This means that $b_1 \ll gPU_1^{-1/2}$ (and $a_1 \ll gP$). Suppose that $\tilde{A}_1 > U_1^{1/2}$. Then $a_1 \ll gPU_1^{-1/2}$ (and $b_1 \ll gP$). In any case, we have that the number of $(a_1, b_1, \tilde{A}, \tilde{B}, \tilde{A}_1, \tilde{B}_1)$ is

$$\ll g^2P^2U_1^{-1/2} \times \tau(U_1)\tau(V_1)$$
$$\ll g^2P^2U_1^{-1/2} \times (E_{100}U_1^2)^{0.1},$$

which is

$$\leqq g_0^{-1}P^2,$$

if U_1 is sufficiently large.

We have the conclusion of 1.5.3.

1.5.4. Lemma. *We have, in* 1.1.3 *with* 1.5.1, *that the number of such quadruples* (A, B, A_1, B_1) *to be counted, that satisfy*

$$(\log P)^{10} \geqq a_1b_1 \geqq G_0,$$

is

$$\leqq g_0^{-1}P^2,$$

for a sufficiently large constant G_0.

Proof. The proof from 1.4.3 to 1.4.8.1 applies in this case also,

1.5.5. Lemma. *We have, in* 1.1.3 *with* 1.5.1, *that the number of such quadruples* (A, B, A_1, B_1) *to be counted, that satisfy* $a_1b_1 \leqq G_0$ *and that all of* A, B, A_1 *and* B_1 *are not* $[K, K^z)$*-regular, is*

$$\leqq g_0^{-1}P^2.$$

Proof. The proof from 1.4.3 to 1.4.3.5.4 applies here.

1.5.6. We have proved 1.1.3.

Chapter 2. Gaussian Sums and Theta-Weyl Sums

2.1. Gaussian sums

We summarise classical results.

2.1.1. Definitions. (i) Let A, B and ν be integers such that $(A, B)=1$, $A>0$ and $B\neq 0$. We use the notations as follows;

$$S\left(\frac{B}{A};\nu\right):=\sum_{l;\,1\leqq l\leqq 2A} e\left(\frac{B}{2A}l^2+\frac{\nu}{2A}l\right),$$

$$S\left(\frac{B}{A}\right):=S\left(\frac{B}{A};0\right)=\sum_{l;\,1\leqq l\leqq 2A} e\left(\frac{B}{2A}l^2\right),$$

$$S^*\left(\frac{B}{A}\right):=S\left(\frac{B}{A};A\right)=S\left(\frac{B+A}{A}\right)=\sum_{l;\,1\leqq l\leqq 2A}(-1)^l e\left(\frac{B}{2A}l^2\right),$$

$$S^{**}\left(\frac{B}{A}\right):=\sum_{l;\,1\leqq l\leqq 2A} e\left(\frac{B}{2A}\left(l-\frac{1}{2}\right)^2\right),$$

$$S'\left(\frac{B}{A}\right):=\sum_{l;\,1\leqq l\leqq A} e\left(\frac{B}{A}l^2\right)$$

and

$$S'\left(\frac{B}{A};\nu\right):=\sum_{l;\,1\leqq l\leqq A} e\left(\frac{B}{A}l^2+\frac{\nu}{A}l\right).$$

(ii) Let X be an odd positive integer and Y be a non-zero integer such that $(X, Y)=1$. We use the notation

$$J\left(\frac{Y}{p}\right)$$

to denote the Legendre's symbol with respect to the solubility of $x^2\equiv Y \bmod p$, where p is an odd prime, and correspondingly,

$$J\left(\frac{Y}{X}\right)$$

to denote the Jacobi's symbol.

(iii) Let X and Y be non-zero integers such that $X>0$ and $(X, Y)=1$. Let $X=2^x\hat{X}$ with $2\nmid\hat{X}$. We put, then,

$$J_0\left(\frac{Y}{X}\right):=J\left(\frac{Y}{\hat{X}}\right).$$

(iv) Let X and Y be non-zero integers such that $X>0$. We put, then,

$$J_1(Y/X):=J_0\left(\frac{(X, Y)^{-1}Y}{(X, Y)^{-1}X}\right).$$

It is clear that

$$J_1(ZY/ZX)=(-1)^{(1/8)(\hat{X}^2-1)z}J_1(Y/X)$$

for $X=2^x\hat{X}$ and $Z=2^z\hat{Z}$ with $2\nmid\hat{X}\hat{Z}$.

2.1.2. Lemma. *Let A and B be positive integers with $(A, B)=1$. We put, temporarily,*

$$A=2^\alpha\hat{A}, \qquad B=2^\beta\hat{B},$$

where $\hat{A}$ (resp. $\hat{B}$) is the odd part of A (resp. B).

(i) *If both of A and B are odd, then,*

$$S\left(\frac{B}{A}\right)=0.$$

(ii) *If A is odd ($\alpha=0$, $A=\hat{A}$) and B is even ($\beta\geqq1$), then*

$$S\left(\frac{B}{A}\right)=(\sqrt{-1})^{((1/2)(A-1))^2}J\left(\frac{2^{\beta-1}B}{A}\right)2\sqrt{A}.$$

(iii) *If A is even with an even α ($\geqq2$) and B is odd ($\beta=0$, $B=\hat{B}$), then,*

$$S\left(\frac{B}{A}\right)=(\sqrt{-1})^{((1/2)(\hat{A}-1))^2}J\left(\frac{2B}{\hat{A}}\right)e\left(\frac{AB}{8}\right)2\sqrt{A}.$$

(iv) *If A is even with an odd α ($\geqq1$) and B is odd ($\beta=0$, $B=\hat{B}$), then,*

$$S\left(\frac{B}{A}\right)=(\sqrt{-1})^{((1/2)(\hat{A}-1))^2}J\left(\frac{B}{\hat{A}}\right)e\left(\frac{(-1)^{AB}}{8}\right)2\sqrt{A}.$$

(v) *If both of A and B are odd, then,*

$$S^*\left(\frac{B}{A}\right)=(\sqrt{-1})^{((1/2)(A-1))^2}J\left(\frac{2B}{A}\right)2\sqrt{A}.$$

(vi) *If A is odd and B is even, then,*

$$S^*\left(\frac{B}{A}\right)=0.$$

(vii) *If A is even with an even α ($\geqq 2$) and B is odd, then,*

$$S^*\left(\frac{B}{A}\right)=(\sqrt{-1})^{((1/2)(\hat{A}-1))^2}(-1)^{(1/8)(\hat{A}^2-1)(\alpha-1)}$$
$$\times J\left(\frac{B}{\hat{A}}\right)e\left(\frac{\hat{A}(A+B)}{8}\right)2\sqrt{A}.$$

(viii) *If A is even with an odd α ($\geqq 1$) and B is odd, then,*

$$S^*\left(\frac{B}{A}\right)=(\sqrt{-1})^{((1/2)(\hat{A}-1))^2}(-1)^{(1/8)(\hat{A}^2-1)(\alpha-1)}$$
$$\times J\left(\frac{B}{\hat{A}}\right)e\left(\frac{(-1)^{(1/2)(\hat{A}(A+B)-1)}}{8}\right)2\sqrt{A}.$$

(ix) *If both of A and B are odd, then,*

$$S^{**}\left(\frac{B}{A}\right)=(\sqrt{-1})^{((1/2)(A-1))^2}e\left(\frac{AB}{8}\right)J\left(\frac{2B}{A}\right)2\sqrt{A}.$$

(x) *If A is even and B is odd, then,*

$$S^{**}\left(\frac{B}{A}\right)=0.$$

(xi) *If A is odd and B is even with $\beta=1$ ($B=2\hat{B}$), then,*

$$S^{**}\left(\frac{B}{A}\right)=(\sqrt{-1})^{((1/2)(A-1))^2}c\left(\frac{A\hat{B}}{8}\right)J\left(\frac{\hat{B}}{A}\right)2\sqrt{A}.$$

(xii *If A is odd and B is even with $\beta=2$ ($B=4\hat{B}$), then,*

$$S^{**}\left(\frac{B}{A}\right)=\sqrt{-1})^{((1/2)(A-1))^2}(-1)^{(1/8)(A^2-1)+1}J\left(\frac{\hat{B}}{A}\right)2\sqrt{A}.$$

(xiii) *If A is odd and B is even with $\beta\geqq 3$, then,*

$$S^{**}\left(\frac{B}{A}\right)=(\sqrt{-1})^{((1/2)(A-1))^2}(-1)^{(1/8)(A^2-1)}J\left(\frac{B}{A}\right)2\sqrt{A}.$$

Proof. Classical results in "Kapitel II, Siebenter Abschnitt" in [1].

2.1.2.1. Corollary. *Let A, B, $\hat{A}$, $\hat{B}$, α and β be as in* 2.1.2. *We have, then, $S\left(\frac{B}{A}\right)$, $S^*\left(\frac{B}{A}\right)$, $S^{**}\left(\frac{B}{A}\right)$ are of the form $\rho J_0\left(\frac{B}{A}\right)2\sqrt{A}$, where $\rho=0$ or $\rho^8=1$. The values of ρ are determined if the residues of A, B, $\hat{A}$* mod 8, *and the residues of α and β* mod 2 *for larger values of α and β, or the values of α and β for smaller values of α and β are given.*

2.1.3. Lemma. *Let A and B be positive integers with $(A, B)=1$, and ν be an integer. We have the followings;*

(i) *If ν* is *odd and AB is even, then,*

$$S\left(\frac{B}{A}; \nu\right)=0.$$

(ii) *If ν is odd and AB is odd, then,*

$$S\left(\frac{B}{A}; \nu\right)=(-1)^{\mu}e\left(-\frac{\hat{D}\mu^2}{2A}\right)S^*\left(\frac{B}{A}\right),$$

where

$$\mu\equiv\tfrac{1}{2}(\nu-A) \qquad \bmod A$$

and

$$B\hat{D}\equiv 1 \qquad \bmod 2A.$$

(iii) *If $\nu=4\tilde{\nu}$, then,*

$$S\left(\frac{B}{A}; \nu\right)=e\left(-\frac{2D\tilde{\nu}^2}{A}\right)S\left(\frac{B}{A}\right),$$

where

$$BD\equiv 1 \qquad \bmod A.$$

(iv) *If $\nu=2\tilde{\nu}$ with $2\nmid\tilde{\nu}$ and B is odd, then,*

$$S\left(\frac{B}{A}; \nu\right)=e\left(-\frac{\hat{D}\tilde{\nu}^2}{2A}\right)S\left(\frac{B}{A}\right),$$

where

$$B\hat{D}\equiv 1 \qquad \bmod 2A.$$

(v) *If* $\nu=2\tilde{\nu}$ *with* $2\nmid\tilde{\nu}$ *and* $B=2\tilde{B}$, (*A being odd*), *then,*

$$S\left(\frac{B}{A};\nu\right)=e\left(-\frac{\tilde{D}\mu^2}{A}\right)S\left(\frac{B}{A}\right),$$

where

$$2\mu\equiv\tilde{\nu} \bmod A \quad \textit{and} \quad \tilde{B}\tilde{D}\equiv 1 \bmod A.$$

(vi) $$S\left(\frac{B}{A};-\nu\right)=S\left(\frac{B}{A};\nu\right).$$

Proof. It is easy.

2.1.4. Proposition. *Let* A, B, A' *and* B' *be positive integers and* ε *be one of* ± 1, *such that*

$$AB'-BA'=\varepsilon.$$

We have, then,

$$S\left(\frac{B}{A};\nu\right)=\tilde{\rho}\tilde{S}\left(\frac{B}{A}\right)e\left(\frac{1}{8}\varepsilon\nu^2A'A^{-1}\right)$$
$$=\rho J_0\left(\frac{B}{A}\right)e\left(\frac{\varepsilon\nu^2}{8}A'A^{-1}\right)2\sqrt{A},$$

where $\tilde{S}\left(\frac{B}{A}\right)$ *is one of* $S\left(\frac{B}{A}\right)$ *and* $S^*\left(\frac{B}{A}\right)$, $\tilde{\rho}=0$ *or* $\tilde{\rho}^8=1$, $\rho=0$ *or* $\rho^8=1$. *Their choices are determined by the residues of* $X \bmod 8$ *for* $X=A$, B, A', B' *and* $\hat{A}$ *with* $A=2^\alpha\hat{A}$ $(2\nmid\hat{A})$, *and the residue of* $\nu \bmod 4$. *Especially,* ρ *is the same for* ν *and* $-\nu$.

Proof. This is classical, and an easy corollary to 2.1.2.1 and 2.1.3.

2.2. Theta-weyl sums

2.2.0. Lemma. *Let* $F(x)$ *and* $G(x)$ *be real functions such that* $G(x)$ *is monotonic and that* $F'(x)$ *exists, is monotonic and* $F'(x)>m$ (>0). *We have, then,*

$$\left|\int_a^b G(x)e(F(x))\,dx\right|\ll m^{-1}\sup_{x;\ a\leqq x\leqq b}|G(x)|.$$

Proof. This is obtained from Lemma 4.2 in [23].

2.2.1. Lemma (van der Corput). *Let* $f(x)$ *be a real-valued function* *with a continuous and monotonely increasing derivative $f'(x)$, on $a \leqq x \leqq b$. Let $\frac{1}{10} > \eta > 0$ and $b > a$. We have, then,*

$$\sum_{k;\, a \leqq k \leqq b} e(f(k)) = \sum_{m;\, f'(a)-\eta \leqq m \leqq f'(b)+\eta} \int_a^b e(f(x)-mx)dx$$
$$+O(\log(2+f'(b)-f'(a)))+O(1+\eta^{-1}).$$

Proof. This is Lemma 4.7 in [23].

2.2.2. Lemma (van der Corput). *Let $f(x)$ be a real-valued function with a continuous and monotonely increasing derivative satisfying $|f'(x)| \leqq 1-\eta$ on $a \leqq x \leqq b$. We have, then,*

$$\sum_{k;\, a \leqq k \leqq b} e(f(k)) = \int_a^b e(f(x))dx + O(1+\eta^{-1}).$$

Proof. This is Lemma 4.8 in [23].

2.2.3. Lemma (van der Corput). *Let $f(x)$ be a real-valued function with a continuous and monotonely increasing derivative $f'(x)$ and $g(x)$ be a realvalued function with a continuous, monotone and positive-valued derivative $g'(x)$, on $a \leqq x \leqq b$. Let $\frac{1}{10} > \eta > 0$ and $a < b$. We have, then,*

$$\sum_{k;\, a \leqq k \leqq b} g(k)e(f(k)) = \sum_{m;\, f'(a)-\eta \leqq m \leqq f'(b)+\eta} \int_a^b g(x)e(f(x)-mx)dx$$
$$+O((|g(a)|+|g(b)|)(\eta^{-1}+\log(2+f'(b)-f'(a))))$$
$$+O((|g'(a)|+|g'(b)|)(\eta^{-1}+1)).$$

Proof. This is Lemma 4.10 in [23].

2.2.4. Definition. Let ε be one of ± 1 and ξ be real with $\xi \geqq 0$. We put

$$\Psi_\varepsilon(\xi) = e\left(-\frac{1}{2}\varepsilon\xi^2\right)\int_\xi^\infty e\left(\frac{1}{2}\varepsilon t^2\right)dt.$$

This is one of the so-called parabolic cylinder functions. It can be expressed using Fresnel's integrals.

2.2.5. Lemma. *We have*

(i) $$\frac{d}{d\xi}\Psi_\varepsilon(\xi) = -1 - 2\pi\sqrt{-1}\varepsilon\xi\Psi_\varepsilon(\xi).$$

(ii) $$\Psi_\varepsilon(0) = \frac{1}{2}e\left(\frac{\varepsilon}{8}\right).$$

(iii) *If ξ is a non-zero real, then,*

$$(\operatorname{sgn}\xi)\Psi_\varepsilon(|\xi|)=\frac{1}{2\pi}e\Big(-\frac{\varepsilon}{8}\Big)\int_{-\infty}^{\infty}e^{-\pi t^2}\Big(t-e\Big(\frac{\varepsilon}{8}\Big)\xi\Big)^{-1}dt.$$

(iv) $$\Psi_\varepsilon(\xi)=\frac{1}{2}e\Big(-\frac{1}{2}\varepsilon\Big(\xi^2-\frac{1}{4}\Big)\Big)-\sum_{h=1}^{k}((2h-1)!!)^{-1}$$
$$\times(-2\pi\sqrt{-1}\varepsilon)^{h-1}\xi^{2h-1}+O(((2k-1)!!)^{-1}(2\pi)^k\xi^{2k-1}).$$

(v) $$\Psi_\varepsilon(\xi)=-\sum_{h=1}^{k}((2h-3)!!)(2\pi\sqrt{-1}\varepsilon)^{-h}\xi^{-(2h-1)}$$
$$+O(((2k-1)!!)(2\pi)^{-k}\xi^{-(2k-1)}).\qquad ((-1)!!=-1).$$

(vi) *The real part of $\Psi_\varepsilon(\xi)$ decreases monotonely from $(2\sqrt{2})^{-1}$ to $0+$ as ξ varies from 0 to $+\infty$, and similarly for the imaginary part of $\varepsilon\Psi_\varepsilon(\xi)$. We have*

$$\operatorname{Re}\Psi_\varepsilon(\xi)\approx(4\pi^2\xi^3)^{-1},$$
$$\operatorname{Im}\varepsilon\Psi_\varepsilon(\xi)\approx(4\pi\xi)^{-1},\qquad \textit{as } \xi\to+\infty.$$

Proof. It is easy, or can be shown by the properties of Fresnel's integrals.

2.2.6. Lemma. *Let $\beta\,(\neq 0)$, ξ_0, ξ_1, ξ_2 be real numbers with $\xi_2>\xi_1$. We put $\varepsilon=\operatorname{sgn}\beta$. We have, then,*

$$\int_{\xi_1}^{\xi_2}e\Big(\frac{1}{2}\beta(t-\xi_0)^2\Big)dt=\chi|\beta|^{-1/2}e\Big(\frac{\varepsilon}{8}\Big)$$
$$-\Big[\operatorname{sgn}(\xi-\xi_0)|\beta|^{-1/2}e\Big(\frac{1}{2}\beta(\xi-\xi_0)^2\Big)\Psi_\varepsilon(|\beta|^{1/2}|\xi-\xi_0|)\Big]_{\xi=\xi_1}^{\xi=\xi_2},$$

where

$$\chi=\begin{cases}1 \textit{ if } \xi_2>\xi_0>\xi_1,\\ \dfrac{1}{2} \textit{ if } \xi_0=\xi_2 \quad \textit{or} \quad \textit{if } \xi_0=\xi_1,\\ 0 \textit{ otherwise,}\end{cases}$$

and sgn 0 *is* 0 *by definition.*

Proof. It is easy.

2.2.7. Definition. Let $\beta\,(\neq 0)$, γ, ξ', ξ'' be real numbers with $\xi''>\xi'+2$. We put

$$\theta(\beta^{-1}, \gamma; \xi', \xi''):=\sum_{x;\, x\in \boldsymbol{Z},\, \xi'\leqq x\leqq \xi''} e((2\beta)^{-1}(x+\gamma)^2),$$

which we call a theta-Weyl sum. ([19] with a minor change in the notations. Cf. [11]). Especially, we put

$$\theta(\beta^{-1}; \xi', \xi''):=\theta(\beta^{-1}, 0; \xi', \xi'')$$
$$=\sum_{x;\, x\in \boldsymbol{Z},\, \xi'\leqq x\leqq \xi''} e((2\beta)^{-1}x^2).$$

2.2.8. Lemma. *Let β_0 ($\notin \boldsymbol{Z}$), γ_0, ξ_0', ξ_0'', β_1, γ_1, ξ_1', ξ_1'' be real numbers, b_0 be an integer and $\varepsilon=\pm 1$, such that*

$$\beta_0\geqq \tfrac{1}{2}, \qquad \xi_0''-\xi_0'\geqq 2\beta_0,$$
$$\beta_0=b_0+\beta_1^{-1}, \qquad \beta_1^{-1}\gamma_1\equiv \tfrac{1}{2}b_0-\gamma_0 \quad \bmod 1,$$
$$\xi_1'=\beta_0^{-1}(\xi_0'+\gamma_0), \qquad \xi_1''=\beta_0^{-1}(\xi_0''+\gamma_0).$$

We have, then,

$$\theta(\varepsilon\beta_0^{-1}, \gamma_0; \xi_0', \xi_0'')=e(\varepsilon(\tfrac{1}{8}+(2\beta_1)^{-1}\gamma_1^2))\beta_0^{1/2}\theta(-\varepsilon\beta_1^{-1}, \gamma_1; \xi_1', \xi_1'')$$
$$+O(1+\beta_0^{1/2}).$$

Proof. This is Lemma 3 in [19], or a corollary to Theorem in [26].

2.2.9. Lemma. *Let β ($\neq 0$) has two consecutive convergents $A_kB_k^{-1}$ and $A_{k+1}B_{k+1}^{-1}$ such that*

$$(1\leqq)\, A_k\ll N\ll A_{k+1}\, (\leqq\infty).$$

(*That $A_{k+1}=\infty$ means $\beta=A_kB_k^{-1}$*). (*See* 1.2.3.) *We have, then,*

$$|\theta(\beta^{-1}, \gamma; \xi, \xi+N)|\ll \min(NA_k^{-1/2}, A_{k+1}^{-1/2}),$$

for any γ and ξ.

Proof. This is a corollary to Theorems 1 and 2 in [19].

2.2.10. Definition. Let A and A' be positive integers such that $(A, A')=1$. Let AA'^{-1} has a regular continued fraction expansion

$$AA'^{-1}=[b_0; b_1, b_2, \cdots, b_{k_0}].$$

(The ambiguity of b_{k_0} in 1.2.3.1 (iii) does not affect the followings.) Let $r_kq_k^{-1}$ be the convergents of AA'^{-1} corresponding to b_k's. Let g be a large positive constant and let us have that $A>g^{100}$. Suppose we have that

$$b_{k+1}<r_k^8 \text{ for every } k \text{ with } g^{10}<r_k<A^{0.3}$$

and that

$$b_{k+1} < g^8 \text{ for every } k \text{ with } 1 < r_k < g^{10}.$$

We call, then, the fraction AA'^{-1} has "good partial fractions with respect to g".

2.2.11. Proposition. *Let $g \gg 1$ and $P \geqq P_0$. We have, then, that the number of such pairs (A, A'), that*

$$g^{-1/4}P < \quad A \text{ and } A' \quad < g^{1/4}P,$$
$$(A, A') = 1$$

and that the fractions AA'^{-1} have NOT good partial fractions with respect to g, is

$$\ll g^{-6}P^2.$$

Proof. We use the notations in 2.2.10. First, suppose that $r_{k+1} = A$ (and $q_{k+1} = A'$) and $b_{k+1} \geqq r_k^8$. This means $r_k < (gP)^{1/9}$ and $b_{k+1} \ll gPr_k^{-1}$, as $r_{k+1} \gg \ll b_{k+1} r_k$. We have, then, that the number of those pairs (A, A') is

$$\leqq \sum_{(r_k, q_k)} \sum_{b_{k+1}} 2 \ll \sum_{r_k} g^{1/2} r_k \times gPr_k^{-1} \ll g^2 P^{1+1/9} = o(P^2).$$

Next, suppose that AA'^{-1} has a k with

$$g^{10} < r_k < A^{0.3} \quad \text{and} \quad b_{k+1} \geqq r_k^8.$$

We can express, through 1.2.3 (iv–vi),

$$A = r_{k+1}x + x_k y,$$
$$A' = q_{k+1}x + q_k y$$

where

$$x, y \in \boldsymbol{N}, \quad (x, y) = 1, \quad x \geqq y \geqq 1.$$

We have

$$x < r_{k+1}^{-1} A < r_{k+1}^{-1} g^{1/4} P.$$

We have that the number of those pairs (A, A') is

$$\leqq \sum_{(r_k, q_k);\ r_k > g^{10}} \ \sum_{b_{k+1};\ b_{k+1} > r_k^9} \ \sum_{(x,y)} 1$$
$$\ll \sum_{(r_k, q_k)} \sum_{b_{k+1}} ((b_{k+1} r_k)^{-1} g^{1/4} P)^2$$
$$< \sum_{(r_k, q_k)} r_k^{-11} g^{1/2} P^2$$
$$< \sum_{r_k} g^{1/2} r_k r_k^{-11} g^{1/2} P^2 < g^{-10} P^2.$$

Next, suppose that AA'^{-1} has a k with

$$r_k < g^{10} \quad \text{and} \quad b_{k+1} > g^8.$$

We proceed as in the last case. We have that the number of those pairs (A, A') is

$$\begin{aligned}
&\leqq \sum_{(r_k, q_k);\ r_k < g^{10}} \ \sum_{b_{k+1};\ b_{k+1} > g^8} \ \sum_{(x,z)} 1 \\
&\ll \sum_{(r_k, q_k)} \sum_{b_{k+1}} ((b_{k+1} r_k)^{-1} g^{1/4} P)^2 \\
&< \sum_{(r_k, q_k)} g^{-8} \times r_k^{-2} g^{1/2} P^2 \\
&< \sum_{r_k} g^{1/2} r_k \times g^{-8} \times r_k^{-2} g^{1/2} P^2 \\
&< g^{-7} \log g \times P^2 < g^{-6} P^2.
\end{aligned}$$

We have proved the proposition.

2.2.12. Proposition. *Let ξ', ξ'' be integers with $\xi'' > \xi' + 2$. Let β be an irrational real number, $B'A'^{-1}$ and BA^{-1} be irreducible fractions such that, putting as*

$$\beta = BA^{-1} + \omega \quad \textit{and} \quad \varepsilon = \operatorname{sgn} \omega,$$

we have

$$\begin{aligned}
&|A\omega\xi| \leqq 0.4 \qquad \textit{for } \xi = \xi', \xi'', \\
&|A^2\omega| \ll 1, \\
&AB' - BA' = \varepsilon, \\
&1 \leqq A' < A,
\end{aligned}$$

and that the fraction $A(2A')^{-1}$ has "good partial fractions with respect to g", 2.2.9, where g is a given constant with $A > g^{100}$ and $g \geqq 0.9$. We have, then,

$$\begin{aligned}
\theta(\beta; \xi', \xi'') = \frac{1}{2A} S\left(\frac{B}{A}\right) \times \int_{\xi'}^{\xi''} e\left(\frac{1}{2}\omega x^2\right) dx \\
+ \sum_{\nu'}^{(g^{10})} \frac{1}{2A} S\left(\frac{B}{A}; \nu\right) e\left(\frac{1}{2}\omega\xi^2 - \frac{1}{2A}\xi\nu\right) \\
\times \operatorname{sgn}\left(\xi - \frac{\nu}{2A\omega}\right) |\omega|^{-1/2} \Psi_\varepsilon\left(|\omega|^{1/2} \left|\xi - \frac{\nu}{2A\omega}\right|\right)\Bigg|_{\xi=\xi'}^{\xi=\xi''} \\
+ O(g^{-1}\sqrt{A}),
\end{aligned}$$

where ν runs through the set of integers such that

$$\nu\neq 0, \qquad |\nu|\leqq g^{10}.$$

Note that we need not restrict ourselves that $A\ll\xi''-\xi'$. Cf., [11].

Proof. We proceed as follows; Let $L\to+\infty$. Then,

$$\theta(\beta;\xi',\xi'')$$
$$=\sum_{y,l;\ \xi'\leqq 2Ay+l\leqq\xi'',0\leqq l<2A}\sum e\left(\frac{B}{2A}l^2\right)e\left(\frac{\omega}{2}2(Ay+l)^2\right)$$
$$=\sum_{0\leqq l<2A}e^2\left(\frac{B}{2A}l^2\right)\left[\begin{array}{l}\frac{1}{2}\sum_{y;\ 2Ay+l=\xi'\text{ or }\xi''}e\left(\frac{\omega}{2}(2Ay+l)^2\right)\\ \quad+\sum_{|\nu|\leq L}\int_{y;\ \xi'<2Ay+l<\xi''}e\left(\frac{\omega}{2}(2Ay+l)^2-\nu y\right)dy\\ \quad+o(1)\end{array}\right]$$
$$=O(1)+\frac{1}{2A}S\left(\frac{B}{A}\right)\times\int_{\xi'<x<\xi''}e\left(\frac{\omega}{2}x^2\right)dx$$
$$+\sum_{\nu;\ L\geqq|\nu|\neq 0}\frac{1}{2A}S\left(\frac{B}{A};\nu\right)e\left(\frac{\omega}{2}\xi^2-\frac{\xi\nu}{2A}\right)$$
$$\times\operatorname{sgn}\left(\xi-\frac{\nu}{2A\omega}\right)|\omega|^{-1/2}\Psi_\varepsilon\left(|\omega|^{1/2}\left|\xi-\frac{\nu}{2A\omega}\right|\right)\Bigg|_{\xi=\xi'}^{\xi=\xi''}.$$

We have, by 2.1.4, that

$$(2A)^{-1}\left|S\left(\frac{B}{A};\nu\right)\right|\leqq A^{1/2}$$

and that

$$S\left(\frac{B}{A};\nu\right)=-\rho e\left(-\frac{\varepsilon A'}{8A}\nu^2\right)2\sqrt{A},$$

where $\rho=0$ or $\rho^8=1$, and ρ is the same one if A, B, A', B', ε and the residue class of ν mod 4 are fixed. We use the approximation 2.2.5 (v) with $k=1$ to treat the sum over ν. First, we have

$$\sum_{\nu;L\geqq|\nu|\geqq g}(2A)^{-1}S\left(\frac{B}{A};\nu\right)\times O(|\omega|^{-1/2}(|\omega|^{1/2}|\xi-(2A\omega)^{-1}\nu|)^{-3})$$
$$\ll\sum_{\nu}|\omega|A^{5/2}|\nu-2A\omega\xi|^{-2}$$
$$\ll|\omega|A^{5/2}g^{-2}\ll g^{-2}A^{1/2}.$$

Next, we have

$$\sum_{\nu;\ L\geqq|\nu|\geqq g}(2A)^{-1}S\Big(\frac{B}{A};\nu\Big)\times(2A)(2A\omega\xi-\nu)^{-1}\times e\Big(-\frac{\xi\nu}{2A}\Big)$$
$$=\sum_{\nu}S\Big(\frac{B}{A};\nu\Big)e\Big(-\frac{\xi\nu}{2A}\Big)\times\Big(\frac{-1}{\nu}+O\Big(\frac{|2A\omega\xi|}{\nu^2}\Big)\Big)$$
$$=-\sum_{\nu;\ L\geqq|\nu|\geqq g}\nu^{-1}S\Big(\frac{B}{A};\nu\Big)e\Big(-\frac{\xi\nu}{2A}\Big)+O(g^{-1}A^{1/2})$$
$$=2\sqrt{A}\sum_{\tilde\nu=0,\cdots,3}\rho_{\tilde\nu}\sum_{\nu;\ \nu\equiv\tilde\nu \bmod 4,L\geqq|\nu|\geqq g}\nu^{-1}e\Big(-\frac{\varepsilon A'}{8A}\nu^2-\frac{\xi\nu}{2A}\Big).$$

We have that, with $L'=[(4A)^{-1}L]+1$,

$$\sum_{\nu_0;\ |\nu_0|\leqq 2A}\Big|\sum_{l;\ L'\geqq l\geqq g}((\nu_0+4Al)^{-1})+(\nu_0-4Al)^{-1})\Big|$$
$$\ll\sum_{\nu_0}\sum_{l}(4Al)^{-2}|\nu_0|\ll g^{-1},$$

therefore, that

$$\sum_{\nu;\ L\geqq|\nu|\geqq gA}(2A)^{-1}S\Big(\frac{B}{A};\nu\Big)\times(2A)(2A\omega\xi-\nu)^{-1}e\Big(-\frac{\xi\nu}{2A}\Big)\ll g^{-1}\sqrt{A}.$$

Now we treat the summation over ν with $gA\geqq|\nu|\geqq g^{10}$. We have, fixing $\tilde\nu=0,\cdots,3$ and treating gA and g^{10} as if they are integers,

$$\sum_{\nu;\ gA\geqq\nu>g^{10},\ \nu\equiv\tilde\nu \bmod 4}\nu^{-1}e\Big(-\frac{\varepsilon A'}{8A}\nu^2-\frac{\xi\nu}{2A}\Big)$$
$$=\sum_{N;\ gA\geqq N>g^{10}}N^{-1}\Big(\sum_{\nu;\ 1\leqq\nu\leqq N,\ \nu\equiv\tilde\nu \bmod 4}e(\cdots)-\sum_{\nu;\ 1\leqq\nu\leqq N-1,\nu\equiv\tilde\nu \bmod 4}e(\cdots)\Big)$$
$$\ll\sum_{N=g^{10},gA}N^{-1}\Big|\sum_{\nu;\ 1\leqq\nu\leqq N,\ \nu\equiv\tilde\nu \bmod 4}e(\cdots)\Big|$$
$$+\sum_{N;\ gA\geqq Ng^{10}}N^{-2}\Big|\sum_{\nu;\ 1\leqq\nu\leqq N,\ \nu\equiv\tilde\nu \bmod 4}e(\cdots)\Big|$$
$$\ll\sum_{N=g^{10},gA}N^{-1}\Big|\sum_{\nu;\ 1\leqq\nu\leqq 4^{-1}N}e\Big(-\frac{\varepsilon 2A'}{2A}\nu^2-\frac{\tilde\xi}{2A}\nu\Big)\Big|$$
$$+\sum_{N;\ gA\geqq N\geqq g^{10}}N^{-2}\Big|\sum_{\nu;\ 1\leqq\nu\leqq 4^{-1}N}e\Big(-\frac{\varepsilon 2A'}{2A}\nu^2-\frac{\tilde\xi}{2A}\nu\Big)\Big|$$
$$+O(g^{-10}),$$

where $\tilde\xi=4\xi+4\varepsilon A'\tilde\nu$. Such N's, that

$$\Big|\sum_{\nu;\ 1\leqq\nu\leqq 4^{-1}N}e\Big(-\frac{\varepsilon 2A'}{2A}\nu^2-\frac{\tilde\xi}{2A}\nu\Big)\Big|\ll N^{0.9},$$

contribute

$$\ll \max_{N;\, gA \geqq N \geqq g^{10}} N^{-0.1} \ll g^{-1}.$$

Suppose that we have, for a certain N, that

$$\left| \sum_{\nu;\, 1 \leqq \nu \leqq 4^{-1}N} e\left(-\frac{\varepsilon 2A'}{2A} \nu^2 - \frac{\tilde{\xi}}{2A} \nu \right) \right| > N^{0.9},$$

then, the consecutive convergents $r_k q_k^{-1}$ and $r_{k+1} q_{k+1}^{-1}$ of $A(2A')^{-1}$ with $r_k \ll N \ll r_{k+1}$ satisfy, through 2.2.9, that

$$(*) \qquad N r_k^{-1/2} \gg N^{0.9} \quad \text{and} \quad r_{k+1}^{1/2} \gg N^{0.9},$$

i.e.,

$$r_k \ll N^{0.2} \quad \text{and} \quad r_{k+1} \gg N^{1.8}.$$

If $(A \geqq) r_k \geqq \frac{1}{2}A$, then $A \ll N \ll gA$, therefore,

$$\left| \sum_{\nu} e\left(-\frac{\varepsilon A'}{2A} \nu^2 - \frac{\tilde{\xi}}{2A} \nu \right) \right| \ll N A^{-1/2} \ll g N^{1/2}.$$

This gives an admissible error $O(g^{-4} A^{1/2})$. If $r_k < \frac{1}{2}A$, therefore, if $r_{k+1} \leqq A$, then, using notations in 2.2.10,

$$b_{k+1} \gg \ll r_{k+1} r_k^{-1} \gg N^{1.6} \gg r_k^8.$$

We may suppose that this conclusion is

$$b_{k+1} > r_k^8$$

by adjusting the implied constants. We have also

$$r_k < A^{0.3}.$$

Suppose especially that $g^{10} \geqq r_k \geqq 1$. This implies

$$b_{k+1} > (g^{10})^{1.8} g^{-10} = g^8,$$

as $r_{k+1} \geqq N \geqq g^{10}$. As $A(2A')^{-1}$ has good partial fractions with respect to g, the existence of such N satisfying $(*)$ with $r_k < \frac{1}{2}A$ is denied. We have proved the proposition.

2.2.13. Lemma. *There exists a positive numerical constant c (small) such that, if c_0 is a fixed constant with $c > c_0 > 0$, then we have*

$$\left| \int_U^{\kappa U} e\left(\frac{1}{2} u^2 \right) du \right| \gg \ll (U + U^{-1})^{-1}$$

for any κ and U with

$$1+2c_0>\kappa>1+c_0 \quad (>1)$$

and

$$\infty>U>0.$$

The implied constant in "$\ll$" is absolute, and that in "$\gg$" depends only on c_0.

Proof. (i) Suppose that U is $>cc_0^{-1/2}$ with a large positive constant c. We have

$$\begin{aligned}\int_{U^2}^{(\kappa U)^2} e(u)u^{-1/2}du &= (2\pi i)^{-1}((\kappa U)^{-1}e(\kappa^2U^2)-U^{-1}e(U^2))+O(U^{-3})\\ &=(2\pi i)^{-1}e(U^2)(\kappa^{-1}-1)U^{-1}\\ &\quad+(2\pi i)(e(\kappa^2U^2)-e(U^2))(\kappa U)^{-1}\\ &\quad+O(U^{-3}),\end{aligned}$$

by 2.2.5 (v).

(i-i) If we have

$$|\kappa^{-1}(e(\kappa^2U^2)-e(U^2))|\geqq 2|\kappa^{-1}-1| \qquad (\gg\ll c_0),$$

then, we have

$$\begin{aligned}\left|\int_{U^2}^{(\kappa U)^2} e(u)u^{-1/2}du\right| &\gg\ll |e(\kappa^2U^2)-e(U^2)|\times(\kappa U)^{-1}\\ &\gg U^{-1}|\kappa^{-1}-1|\gg U^{-1}.\end{aligned}$$

We have also, by the mean-value theorem on integrals,

$$\left|\int_{U^2}^{(\kappa U)^2} e(u)u^{-1/2}du\right|\ll U^{-1}.$$

(i-ii) If we have

$$|\kappa^{-1}(e(\kappa^2U^2)-e(U^2))|\leqq \tfrac{1}{2}|\kappa^{-1}-1|,$$

then, we have

$$\left|\int_{U^2}^{(\kappa U)^2} e(u)u^{-1/2}du\right|\gg\ll|\kappa^{-1}-1|\cdot U^{-1}\gg\ll U^{-1}.$$

(i-iii) If we have

$$\tfrac{1}{2}|\kappa^{-1}-1|\leqq|\kappa^{-1}(e(\kappa^2U^2)-e(U^2))|\leqq 2|\kappa^{-1}-1|,$$

i.e., if we have

$$|(\kappa^2-1)U^2 \bmod 1| \gg \ll |\kappa-1| \gg \ll c_0,$$

we have, then,

$$\begin{aligned}|e(\kappa^2U^2)-\kappa^{-1}e(U^2)| &= |(e((\kappa^2-1)U^2)-1)+(1-\kappa^{-1})| \\ &= |2\pi i((\kappa^2-1)U^2 \bmod 1)+(1-\kappa^{-1})|+O(c_0^2) \\ &\gg \ll |\kappa-1| \gg \ll c_0,\end{aligned}$$

for suitably chosen small c_0. Therefore we have the conclusion again.

(ii) Suppose that U is $< cc_0^{1/2}$ with a small positive constant c. We have

$$\begin{aligned}\int_U^{\kappa U} e(\tfrac{1}{2}u^2)du &= \int_U^{\kappa U} (1+O(u^2))du = (\kappa-1)U+O(U^3) \\ &\gg \ll c_0 U.\end{aligned}$$

(iii) Suppose that $c_0^{1/2} \ll U \ll c_0^{-1/2}$. Using numerical tables of values of Fresnel's integrals, or the graph of "Cornu's spiral", we have

$$\left|\int_U^{\kappa U} e(\tfrac{1}{2}u^2)\,du\right| \gg \ll 1$$

if U is $\gg \ll 1$, the upper bound being absolute.

We have proved the lemma. See, also, Lemma 7 of [11].

2.2.14. Proposition. *Let c_0 be as in* 2.2.13, *Let ξ' and ξ'' be positive integers such that*

$$\xi'' > \xi'+2 \quad (\geqq 3)$$

and that

$$(1+2c_0)\xi' > \xi'' > (1+c_0)\xi'.$$

Let β be a non-zero irrational real number. Let $A_{k_0}B_{k_0}^{-1}$ and $A_{k_0+1}B_{k_0+1}^{-1}$ be two consecutive convergents to β such that

$$A_{k_0}^{-1}\xi'' > c_0^{-1}$$

and that

$$A_{k_0+1}^{-1}\xi'' \leqq c_0^{-1}.$$

We have, then,

EITHER

$$|\theta(\beta^{-1};\, \xi',\, \xi'')| \ll A_{k_0}^{1/2},$$

OR

$$|\theta(\beta^{-1};\xi',\xi'')| \gg \ll A_{k_0+1}^{1/2} \times \min(\Xi, \Xi^{-1}),$$

where $\Xi := (A_{k_0}A_{k_0+1})^{-1/2}\xi''$. *The implied constants depend only on* c_0.

Proof. Theorems 1 and 2 in [19] with the formula for X_{k+2} in "Corrigendum", Lemma 2.2.13 and the fact that

$$|\beta^{-1} - B_{k_0}A_{k_0}^{-1}| \gg \ll (A_{k_0}A_{k_0+1})^{-1}$$

through 1.2.3.1 (iv-iv).

2.2.15. Proposition. *Let* ξ' *and* ξ'' *be as in* 2.2.14, *We have, then,*

$$\int_0^2 \left| \sum_{x;\ \xi'<x\leqq\xi'',\ x\in N} e\left(\frac{\alpha}{2}x^2\right)\right|^4 d\alpha \ll \xi''(\log \xi'')^3.$$

Proof. Let $r(n)$ denote the number of integer solutions (x, y) of $x^2 + y^2 = n$. We have, then,

$$\int_0^2 \left|\sum_x e\left(\frac{\alpha}{2}x^2\right)\right|^4 d\alpha \leqq \sum_{n;\ n\leqq\xi''^2,\ n\in N} (r(n))^2 \leqq \sum_n (\tau(n))^2 \ll \xi^2(\log \xi'')^3.$$

A slightly better estimate can be obtained, if we use 2.2.14.

2.3. Jacobi's symbol.

2.3.1. Lemma. *Let* A, B, A', B' *be positive integers and* ε *be* ± 1, *such that*

$$AB' - BA' = \varepsilon.$$

Let p *and* q *be positive integers with* $(p, q) = 1$. *We have, then,*

(0) $$(Ap + A'q,\ Bp + B'q) = 1.$$

(i) $$S\left(\frac{Bp+B'q}{Ap+A'q}\right)\cdot((Ap+A'q)(4qA)^{-1})^{-1/2}$$
$$= \begin{cases} \rho S^{**}\left(\frac{-\varepsilon p}{q}\right)S^{*}\left(\frac{B}{A}\right) & \textit{if } AB \textit{ is odd,} \\ \rho S\left(\frac{-\varepsilon p}{q}\right)S\left(\frac{B}{A}\right) & \textit{if } AB \textit{ is even and } AA' + B(B'+1) \textit{ is even,} \\ \rho S^{*}\left(\frac{-\varepsilon p}{q}\right)S\left(\frac{B}{A}\right) & \textit{if } AB \textit{ is even and } AA' + B(B'+1) \textit{ is odd.} \end{cases}$$

(ii) $S^*\left(\dfrac{Bp+B'q}{Ap+A'q}\right)\cdot((Ap+A'q)(4qA)^{-1})^{-1/2}$

$$=\begin{cases}\rho S^{**}\left(\dfrac{-\varepsilon p}{q}\right)S\left(\dfrac{B}{A}\right) & \text{if } A \text{ is odd and } B \text{ is even,}\\ \rho S\left(\dfrac{-\varepsilon p}{q}\right)S^*\left(\dfrac{B}{A}\right) & \text{if } (A, B)\not\equiv(1, 0) \bmod 2 \text{ and } AA' \\ & \quad +(A+B)(A'+B'+1) \text{ is even,}\\ \rho S^*\left(\dfrac{-\varepsilon p}{q}\right)S^*\left(\dfrac{B}{A}\right) & \text{if } (A, B)\not\equiv(1, 0) \bmod 2 \text{ and } AA' \\ & \quad +(A+B)(A'+B'+1) \text{ is odd.}\end{cases}$$

In (i) *and* (ii), $\rho^8=1$ *and* ρ *is determined if the residues of* A, B, A', B' mod 8 (*and* $\varepsilon=\pm 1$) *are given.*

Proof. This lemma is implicitly well-known in the classical theory of theta-series. So, we do not write down its proof. The explicit values of ρ's are, following the order of their appearances in the lemma,

(i) $(-1)^{(A+1)/2}e(\frac{1}{8}\varepsilon(1+A'(A+2)+(A+1)^2B'))$,

$e(\frac{1}{8}\varepsilon)$,

$e(\frac{1}{8}\varepsilon)$,

(ii) $(-1)^{(A+1)/2}e(\frac{1}{8}\varepsilon(1+A'(A+2)+(A+1)^2(A'+B'))$,

$e(\frac{1}{8}\varepsilon)$,

$e(\frac{1}{8}\varepsilon)$,

respectively.

2.3.2. Lemma. *Let* A *and* B *be positive integers with* $(A, B)=1$ *and* $A=2^\alpha\hat{A}$, $2\nmid\hat{A}$. *Let* S_i *be one of* $S\left(\dfrac{B}{A}\right)$ *and* $S^*\left(\dfrac{B}{A}\right)$, $i=1, \cdots, 4$. *We have, then,*

$$S_1\times\cdots\times S_4=16\rho A^2,$$

where $\rho=0$ *or* $\rho^8=1$ *and* ρ *is determined if the residues of* A, $\hat{A}$, B mod 8, *the residue of* α mod 2, ε *and the choices of* S_i *are given.*

Proof. 2.1.2.

2.3.3. Lemma. *Let* A, B, A', B' *be positive integers and* ε *be* ± 1 *such that*

$$AB'-BA'=\varepsilon$$

with $A=2^\alpha\hat{A}$, $2\nmid\hat{A}$. *Let* p_i *and* q_i *be positive integers with* $(p_i, q_i)=1$, $i=1, \cdots, 4$. *Let* S_i *be one of*

$$S\left(\frac{Bp_i+B'q_i}{Ap_i+A'q_i}\right) \quad and \quad S^*\left(\frac{Bp_i+B'q_i}{Ap_i+A'q_i}\right)$$

for each $i=1, \cdots, 4$. *We have, then,*

$$S_1\times\cdots\times S_4=16\rho((Ap_1+A'q_1)\cdots(Ap_4+A'q_4))^{1/2},$$

where $\rho=0$ *or* $\rho^8=1$, *and* ρ *is determined if the residues of* A, $\hat{A}$, B, A', B' mod 8, (ε), *the residue of* α mod 2, *the values of* p_i, q_i, *and the choice of* S_i *are given.*

Proof. This is a corollary of 2.3.1, 2.3.2 and 2.1.2.1.

2.3.4. Lemma. *Let* X, Y *and* Z *be positive integers, such that* $X=2^x\hat{X}$, $Y=2^y\hat{Y}$ *and* $Z=2^z\hat{Z}$ *with* $2\nmid\hat{X}\hat{Y}\hat{Z}$. *We have, then,*

$$J_1(X/YZ)=J\times J_1(X/Y)J_1(X/Z),$$

where

$$J=J\left(\frac{\ddot{d}d_Z}{\ddot{Y}}\right)J\left(\frac{\ddot{d}d_Y}{\ddot{Z}}\right)J_1\left(\frac{\ddot{X}}{d}\right)J\left(\frac{d_Yd_Z}{d'}\right).$$

Here we have put as

$$\hat{X}=dd_Yd_Z\ddot{X}, \quad \hat{Y}=dd'd_Y\ddot{Y}, \quad \hat{Z}=dd'd_Z\ddot{Z},$$

$$d=(\hat{X}, \hat{Y}, \hat{Z}), \quad d'=(d^{-1}\hat{Y}, d^{-1}\hat{Z}), \quad \ddot{d}=(d, \ddot{X}),$$
$$d_Y=(d^{-1}\hat{X}, (dd')^{-1}\hat{Y}), \qquad d_Z=(d^{-1}\hat{X}, (dd'^{-1}\hat{Z})).$$

We have, also, a corresponding formula for $J_1(YZ/X)$, *in which* $J_1(Y/X)$ *appears in place of* $J_1(X/Y)$, *etc. We have*

$$(d_Yd_Z\ddot{X}, d')=1, \quad (d_Y\ddot{Y}, d_Z\ddot{Z})=1, \quad (\ddot{X}, d_Yd_Z\ddot{Y}\ddot{Z})=1.$$

If $(X, YZ)=1$, *we have* $J=1$.

Proof. We have

$$J_1(X/YZ)=e\left(\frac{1}{16}x((\hat{Y}\hat{Z})^2-1)\right)J_1(\hat{X}/\hat{Y}\hat{Z})$$
$$=e('')J_1(\ddot{X}/dd'^2\ddot{Y}\ddot{Z}),$$

which is

$$=e('')J\left(\frac{(\ddot{d}^{-1}\ddot{X})}{(\ddot{d}^{-1}d)\ddot{Y}\ddot{Z}}\right),$$

as $(\ddot{X}, d_1')=1$ and $(\ddot{d}^{-1}\ddot{X}, \ddot{d}^{-1}d\ddot{Y}\ddot{X})=1$. We have, then, this is

$$=e('')J\left(\frac{(\ddot{d}^{-1}\ddot{X})}{(\ddot{d}^{-1}d)}\right)d\left(\frac{\ddot{d}}{\ddot{Y}}\right)J\left(\frac{\ddot{X}}{\ddot{Y}}\right)J\left(\frac{\ddot{X}}{\ddot{Z}}\right)$$
$$=e('')J_1\left(\frac{\ddot{X}}{d}\right)J\left(\frac{\ddot{d}}{\ddot{Y}}\right)J\left(\frac{\ddot{d}}{\ddot{Z}}\right)J\left(\frac{\ddot{X}}{\ddot{Y}}\right)J\left(\frac{\ddot{X}}{\ddot{Z}}\right),$$

as $J\left(\frac{\ddot{d}^{-1}\ddot{X}}{\ddot{d}^{-1}d}\right)=J_1\left(\frac{\ddot{X}}{d}\right)$ owing to $2\nmid\ddot{X}$. On the other hand, we have

$$J_1(X/Y)=e\left(\frac{1}{16}x(\hat{Y}^2-1)\right)J_1(\hat{X}/\hat{Y})$$
$$=e('')J\left(\frac{d_Z\ddot{X}}{d'\ddot{Y}}\right)$$
$$=e('')J\left(\frac{d_Z}{d'}\right)J\left(\frac{d_Z}{\ddot{Y}}\right)J\left(\frac{\ddot{X}}{d'}\right)J\left(\frac{\ddot{X}}{\ddot{Y}}\right),$$

and similarly for $J_1(X/Z)$. We have

$$e\left(\frac{1}{16}((\hat{Y}\hat{Z})^2-1)\right)=e\left(\frac{1}{16}((\hat{Y}^2-1)+(\hat{Z}^2-1))\right),$$

as $\hat{Y}$ and $\hat{Z}$ are odd. Combining these, we obtain the result for $J_1(X/YZ)$. As for $J_1(YZ/X)$, we have

$$J_1(YZ/X)=e\left(\frac{1}{16}(y+z)(\hat{X}^2-1)\right)J_1(\hat{Y}\hat{Z}/\hat{X}),$$
$$J_1(Y/Z)=e\left(\frac{1}{16}y(\hat{X}^2-1)\right)J_1(\hat{Y}/\hat{Z})$$

and similarly for $J_1(Z/X)$. Then, the rest is similar as above.

2.3.5. Lemma. *Let* $\ddot{A}$, $\ddot{B}$, $\ddot{A}^{\Delta}$, $\ddot{B}^{\Delta}$, A_1, B_1, A_1', B_1', p_1, q_1, t_1, U_1, V_1, a, b *be positive integers and* $\ddot{\varepsilon}=\pm1$, $\varepsilon_1=\pm1$ *such that*

$$U_1aB_1=t_1((p_1+q_1)\ddot{B}-q_1\ddot{B}^{\Delta}),$$
$$V_1bA_1=t_1((p_1+q_1)\ddot{A}-q_1\ddot{A}^{\Delta}),$$
$$\ddot{A}\ddot{B}^{\Delta}-\ddot{B}\ddot{A}^{\Delta}=\ddot{\varepsilon},$$
$$A_1B_1'-B_1A_1'=\varepsilon_1,$$
$$(a,b)=1,\quad (U_1,V_1)=1,\quad (p_1,q_1)=1,$$
$$t_1=(U_1aB_1, V_1bA_1).$$

We divide t_1 *as*

$$t_1 = t_1^{(ab)} t_1^{(A)} t_1^{(B)},$$

so that

$$t_1^{(ab)} = (U_1 a,\ V_1 b), \quad t_1^{(A)} \mid (U_1 a,\ A_1), \quad t_1^{(B)} \mid (V_1 b,\ B_1).$$

We have, then,

$$J_0\left(\frac{B_1}{A_1}\right) = \rho_1 J^* \times J_0\left(\frac{(p_1+q_1)\ddot{B} - q_1 \ddot{B}^{\Delta}}{(p_1+q_1)\ddot{A} - q_1 \ddot{A}^{\Delta}}\right) J_0\left(\frac{(p_1+q_1)\ddot{A} - q_1 \ddot{A}^{\Delta}}{(t_1^{(B)} t_1^{(ab)})^{-1} U_1 a}\right).$$

Here we have put as

$$J^* = J_0\left(\frac{\ddot{A}^{\Delta}}{(t_1^{(B)} t_1^{(ab)})^{-1} V_1 b}\right) J_0\left(\frac{A_1}{t_1^{(B)}}\right) J_0\left(\frac{A_1'}{t_1^{(A)}}\right) J_0\left(\frac{-\ddot{\varepsilon}(p_1+q_1) t_1^{(ab)-1} U_1 a}{(t_1^{(B)} t_1^{(ab)})^{-1} V_1 b}\right)$$
$$\times J_0\left(\frac{t_1^{(B)}}{t_1^{(A)}}\right) J_0\left(\frac{(t_1^{(ab)} t_1^{(A)})^{-1} U_1 a}{t_1^{(B)}}\right) J_0\left(\frac{-\varepsilon_1}{t_1^{(A)}}\right),$$

and $\rho_1^8 = 1$. *This* ρ_1 *is determined if the residues of* x mod 2, *the residues of* X, $\hat{X}$ mod 8 *with* $X = 2^x \hat{X}$ $(2 \nmid \hat{X})$, *for* $X = t_1^{-1} U_1 a$, $t_1^{-1} V^1 b$, $t_1^{(B)}$, A_1 *and* B_1, *are fixed.*

Proof. In this proof, ρ_1 has the same properties as those in the statement of the lemma, but it may not be the same one, as it occurs. We have, putting $u_1 = t_1^{(A)} t_1^{(ab)}$ and $v_1 = t_1^{(B)} t_1^{(ab)}$,

$$\begin{aligned}
J_1(U_1 a B_1 / V_1 b A_1) &= J_0\left(\frac{t_1^{-1} U_1 a B_1}{t_1^{-1} V_1 b A_1}\right) \\
&= J_0\left(\frac{(u_1^{-1} U_1 a)(t_1^{(B)-1} B_1)}{(v_1^{-1} V_1 b)(t_1^{(A)-1} A_1)}\right) \\
&= J_0\left(\frac{u_1^{-1} U_1 a}{v_1^{-1} V_1 b}\right) J_0\left(\frac{t_1^{(B)-1} B_1}{v_1^{-1} V_1 b}\right) J_0\left(\frac{u_1^{-1} U_1 a}{t_1^{(A)-1} A_1}\right) J_0\left(\frac{t_1^{(B)-1} B_1}{t_1^{(A)-1} A_1}\right) \\
&= J_0\left(\frac{t_1^{-1} U_1 a B_1}{v_1^{-1} V_1 b}\right) J_0\left(\frac{u_1^{-1} U_1 a}{t_1^{-1} V_1 b A_1}\right) J_0\left(\frac{t_1^{(B)-1} B_1}{t_1^{(A)-1} A_1}\right) J_0\left(\frac{t_1^{(A)}}{v_1^{-1} V_1 b}\right) \\
&\quad \times J_0\left(\frac{u_1^{-1} U_1 a}{t_1^{(B)}}\right) J_0\left(\frac{u_1^{-1} U_1 a}{v_1^{-1} V_1 b}\right).
\end{aligned}$$

We have, first

$$J_0\left(\frac{t_1^{-1}U_1aB_1}{v_1^{-1}V_1b}\right)=J_0\left(\frac{(p_1+q_1)\ddot{B}-q_1\ddot{B}^\Delta}{(v_1^{-1}V_1b)}\right)$$
$$=J_0\left(\frac{\ddot{A}^\Delta}{(v_1^{-1}V_1b)}\right)J_0\left(\frac{(p_1+q_1)\ddot{A}^\Delta\ddot{B}-q_1\ddot{A}^\Delta\ddot{B}^\Delta}{(v_1^{-1}V_1b)}\right).$$

We have

$$(p_1+q_1)\ddot{A}^\Delta\ddot{B}-q_1\ddot{A}^\Delta\ddot{B}^\Delta=(p_1+q_1)(\ddot{A}\ddot{B}^\Delta-\ddot{\varepsilon})-q_1{}^\Delta\ddot{B}^\Delta$$
$$=\ddot{B}^\Delta((p_1+q_1)\ddot{A}-q_1\ddot{A}^\Delta)-\ddot{\varepsilon}(p_1+q_1)$$
$$\equiv-\ddot{\varepsilon}(p_1+q_1)\qquad \bmod (v_1^{-1}V_1b).$$

This gives us that

$$J_0\left(\frac{t_1^{-1}U_1aB_1}{v_1^{-1}V_1b}\right)=J_0\left(\frac{-\ddot{\varepsilon}(p_1+q_1)}{(v_1^{-1}V_1b)}\right)J_0\left(\frac{\ddot{A}^\Delta}{(v_1^{-1}V_1b)}\right).$$

We have, next, that

$$J_0\left(\frac{u_1^{-1}U_1a}{t_1^{-1}V_1bA_1}\right)=\rho_1J_0\left(\frac{t_1^{-1}V_1bA}{u_1^{-1}U_1a}\right)=\rho_1J_0\left(\frac{(p_1+q_1)\ddot{A}-a_1\ddot{A}^\Delta}{u_1^{-1}U_1a}\right).$$

We have also

$$J_0\left(\frac{t_1^{(B)-1}B_1}{t_1^{(A)-1}A_1}\right)=J_0\left(\frac{t_1^{(B)}}{t_1^{(A)}}\right)J_0\left(\frac{B_1}{t_1^{(A)}}\right)J_0\left(\frac{t_1^{(B)}}{A_1}\right)J_0\left(\frac{B_1}{A_1}\right)$$
$$=J_0\left(\frac{t_1^{(B)}}{t_1^{(A)}}\right)J_0\left(\frac{-\varepsilon_1A_1'}{t_1^{(A)}}\right)J_0\left(\frac{t_1^{(B)}}{A_1}\right)J_0\left(\frac{B_1}{A_1}\right).$$

Combining all the above, we have the result of the lemma.

2.3.6. Lemma. *Let X, Y, Y_0 and D $(D>0)$ be integers such that*

$$(X,\ Y)\,|\,D,\qquad (X,\ Y_0)\,|\,D$$

and that

$$Y\equiv Y_0 \mod D.$$

We have, then,

$$(X,\ Y)=(X,\ Y_0).$$

Proof. It is sufficient to prove the equivalence of the solubility in integers x and y of

$$Xx+Yy=k$$

and that in x_0 and y_0 of

$$Xx_0+Y_0y_0=k,$$

for every integer k. Suppose that we have integers x_0 and y_0 satisfying the second equation. We must solve the first one. It is sufficient to solve

$$X(x-x_0)+Y(y-y_0)=(Y_0-Y)y_0.$$

Putting $Z=(X, Y)$, we obtain the equation

$$(Z^{-1}X)(x-x_0)+(Z^{-1}Y)(y-y_0)=(Z^{-1}(Y_0-Y))y_0,$$

where $Z^{-1}(Y_0-Y)$ is an integer. This equation is clearly soluble in x and y. The converse argument holds also. Therefore, we obtain the lemma.

2.3.7. Proposition. *Let U_i, V_i, t_i, t'_i, p_i, q_i, p'_i, q'_i, $(i=1, \cdots, 4)$, a and b be positive integers and $\varepsilon=\pm 1$, such that*

$$t_it'_i((p_i+q_i)q'_i+q_ip'_i)=U_iV_iab,$$
$$(a, b)=1, \quad (U_i, V_i)=1, \quad (p_i, q_i)=1, \quad (p'_i, q'_i)=1.$$

Let W, $\ddot{A}$, $\ddot{B}$, $\ddot{A}^{\Delta}$, $\ddot{B}^{\Delta}$, $\ddot{B}_0$, $\ddot{A}_0^{\Delta}$, $\ddot{B}_0^{\Delta}$, A_i, B_i, A'_i, B'_i, $A_i^{(0)}$, $B_i^{(0)}$, $A_i'^{(0)}$, $B_i'^{(0)}$ $(i=1, \cdots, 4)$ be positive integers and $\ddot{\varepsilon}=\pm 1$, such that

$$\ddot{A}\ddot{B}^{\Delta}-\ddot{B}\ddot{A}^{\Delta}=\ddot{A}\ddot{B}_0^{\Delta}-\ddot{B}_0\ddot{A}_0^{\Delta}=\ddot{\varepsilon},$$
$$A_iB'_i-B_iA'_i=A_i^{(0)}B_i'^{(0)}-B_i^{(0)}A_i'^{(0)}=\ddot{\varepsilon},$$
$$\begin{pmatrix} U_ia & 0 \\ 0 & V_ib \end{pmatrix}\begin{pmatrix} B_i & B'_i \\ A_i & A'_i \end{pmatrix}=\begin{pmatrix} \ddot{B} & \ddot{B}^{\Delta} \\ \ddot{A} & \ddot{A}^{\Delta} \end{pmatrix}\begin{pmatrix} t_i(p_i+q_i) & t'_ip'_i \\ -t_iq_i & t'_iq'_i \end{pmatrix},$$
$$\begin{pmatrix} U_ia & 0 \\ 0 & V_ib \end{pmatrix}\begin{pmatrix} B_i^{(0)} & B_i'^{(0)} \\ A_i^{(0)} & A_i'^{(0)} \end{pmatrix}=\begin{pmatrix} \ddot{B}_0 & \ddot{B}_0^{\Delta} \\ \ddot{A} & \ddot{A}_0^{\Delta} \end{pmatrix}\begin{pmatrix} t_i(p_i+q_i) & t'_ip'_i \\ -t_iq_i & t'_iq'_i \end{pmatrix},$$
$$\ddot{A}^{\Delta}\equiv\ddot{A}_0^{\Delta} \qquad \mathrm{mod}\, bW[V_1, \cdots, V_4]$$

and

$$t_i^{(A)}t_i^{(ab)} \mid W$$

for $i=1, \cdots, 4$, where $t_i^{(A)}$ and $t_i^{(ab)}$ are defined from $U_iaB_i^{(0)}$ and $V_ibA_i^{(0)}$ similarly as in 2.3.5. We have, then,

$$J_0\left(\frac{B_1}{A_1}\right)\times\cdots\times J_0\left(\frac{B_4}{A_4}\right)=\rho\prod_{i=1,\cdots,4} J_0\left(\frac{(p_i+q_i)\ddot{A}-q_i\ddot{A}^{\Delta}}{(t_i^{(A)}t_i^{(ab)})^{-1}U_ia}\right),$$

where $\rho^8=1$ and ρ is determined if U_i, V_i, t_i, t'_i, p_i, q_i, p'_i, q'_i, a, b, $\ddot{\varepsilon}$, $\ddot{A}$, $\ddot{A}_0^{\Delta}$, *the residues of* x mod 2, *the residues of* X, $\hat{X}$ mod 8, *with* $X=2^x\hat{X}\,(2\nmid\hat{X})$, *for* $X=\ddot{A}$, $\ddot{B}$, $\ddot{A}^{\Delta}$, $\ddot{B}^{\Delta}$, A_i, B_i, *are given.*

Proof. We apply 2.3.6 to $J_0(B_1/A_1)$, for instance, and 2.3.3 to the product

$$\prod_{i=1,\cdots,4} J_0((p_i+q_i)\ddot{B}-q_i\ddot{B}^{\Delta}/(p_i+q_i)\ddot{A}-q_i\ddot{A}^{\Delta}).$$

We must check that the corresponding J^* has the properties of ρ in the proposition. We have the same $t_i^{(ab)}$ and $t_i^{(A)}$ for $\ddot{A}^{\Delta}$ and $\ddot{A}_0^{\Delta}$, owing to the assumption that $t_i \mid W$ and by 2.3.6. We have, therefore, the same $t_i^{(B)}$ for $\ddot{A}^{\Delta}$ and $\ddot{A}_0^{\Delta}$, because t_i is fixed. It is clear, then, that

$$A_i\equiv A_i^{(0)}, \quad A_i'\equiv A_i'^{(0)}, \quad \bmod t_i^{(A)}t_i^{(B)},$$

therefore, that we have the same J^* for $\ddot{A}^{\Delta}$ and $\ddot{A}_0^{\Delta}$.

2.3.8. Lemma. *Suppose the assumptions of* 2.3.7 *hold, and, moreover, that we have*

$$\ddot{A}^{\Delta}\equiv\ddot{A}_0^{\Delta} \quad \bmod abW[U_1V_1, \cdots, U_4V_4].$$

We have, then,

$$J_0\left(\frac{B_1}{A_1}\right)\times\cdots\times J_0\left(\frac{B_4}{A_4}\right)=\rho,$$

where ρ *has similar properties as in* 2.3.7.

Proof. This is a corollary to 2.3.7.

2.3.9. Proposition. *Let* U_i, V_i, t_i, p_i, q_i $(i=1, \cdots, 4)$ *be positive integers such that*

$$(U_i, V_i)=1, \qquad (p_i, q_i)=1.$$

Let a, b, $\ddot{A}$, $\ddot{A}_0^{\Delta}$, $A_i^{(0)}$ *be positive integers such that*

$$(\ddot{A}, \ddot{A}_0^{\Delta})=1, \qquad (a, b)=1,$$
$$t_i((p_i+q_i)\ddot{A}-\ddot{A}_0^{\Delta})=V_ibA_i^{(0)}, \qquad (i=1, \cdots, 4).$$

We put $t_i^{(ab)}$, $t_i^{(A)}$ *and* $t_i^{(B)}$ *as*

$$t_i^{(ab)}=(U_ia, V_ib)$$

and

$$t_i = t_i^{(ab)} t_i^{(A)} t_i^{(B)}$$

with

$$t_i^{(A)} \mid (U_i a, A_i^{(0)}) \quad \text{and} \quad (t_i^{(B)}, A_i^{(0)}) = 1,$$

and suppose that all prime divisors of $t_i^{(A)} t_i^{(ab)}$ *are smaller than* T. *Let* $U_i^{\#}$ *be the square-free kernel of the odd divisor part of* $U_i a$. *Let* W *and* Z ($\geqq T \geqq 3$), *be positive integers. Let us suppose that they have the following properties;*

We suppose that all prime divisors of $a \times t_1 \cdots t_4 \times (U_i^{\#}, q_i W[V_1, \cdots, V_4])$ *are smaller than* Z, *and that there exist one* i_0 ($1 \leqq i_0 \leqq 4$) *and a prime divisor* p_0 *of* $U_{i_0}^{\#}$ *satisfying*

$$\begin{aligned} &p_0 \geqq Z, && p_0 \mid U_{i_0}^{\#}, \\ &p_0 \nmid U_i^{\#} && \text{for all } i \neq i_0,\ i = 1, \cdots, 4. \end{aligned}$$

We let $\ddot{A}^{\Delta}$ *vary so that*

$$\begin{aligned} &\ddot{A}^{\Delta} \notin \mathcal{S}, \\ &\ddot{A}^{\Delta} \equiv \ddot{A}_0^{\Delta} \mod 8abW[V_1, \cdots, V_4], \\ &(\ddot{A}^{\Delta}, \ddot{A}) = 1 \end{aligned}$$

and

$$X < \ddot{A}^{\Delta} \leqq X + 8abWW'[V_1, \cdots, V_4],$$

with fixed positive integers X *and* W', *where* $\mathcal{S}$ *is a set of integers to allow exceptions for* $\ddot{A}^{\Delta}$. *Putting*

$$V_i b A_i = t_i((p_i + q_i)\ddot{A} - q_i \ddot{A}^{\Delta}),$$

where A_i *become integers, we suppose, moreover, that* $\mathcal{S}$ *contains all* $\ddot{A}^{\Delta}$, *for which* $\prod_{i=1,\dots,4} (U_i a, A_i)$ *have prime divisors larger than* T. *We suppose that, with* $\bar{t}_i^{(A)}$ *and* $\bar{t}_i^{(ab)}$ *for* $\ddot{A}^{\Delta}$ *similar as* $t_i^{(A)}$ *and* $t_i^{(ab)}$ *as are in* 2.3.7,

$$W \equiv 0 \mod \bar{t}_i^{(A)} \bar{t}_i^{(ab)} \quad i = 1, \cdots, 4$$

for all $\ddot{A}^{\Delta}$ *to vary stated above, including* $t_i^{(A)} t_i^{(ab)}$ *of* $\ddot{A}_0^{\Delta}$;

Under these situations we have

$$\begin{aligned} &\sum_{\ddot{A}^{\Delta}} \prod_{i=1,\dots,4} J_0\left(\frac{(p_i+q_i)\ddot{A} - q_i\ddot{A}^{\Delta}}{(\bar{t}_i^{(A)} \bar{t}_i^{(ab)})^{-1} U_i a}\right) \\ &\quad \ll (U_1^{\#} \cdots U_4^{\#})^{-1/2} \cdot [U_1^{\#}, \cdots, U_4^{\#}] \cdot \Delta_U^{\#} \cdot \log (U_1 \cdots U_4) \\ &\quad \times (ab)^4 \cdot \prod_i (U_i a, A_i^{(0)}) \cdot \tau(\Delta^1(\ddot{A}; Z)) \end{aligned}$$

$$
\begin{aligned}
&+Z^{-1}\nu(\ddot{A})\ddot{A}\phi(\ddot{A})^{-1}W' \\
&+\nu(U_1\cdots U_4a)(T^{-1}W'+1) \\
&+2^{\nu(\ddot{A})}+\sum_{\ddot{A}^{\Delta}}{}'' 1.
\end{aligned}
$$

Here, in the left-hand side, $\ddot{A}^{\Delta}$ varies as is stated above. Also $\sum''_{\ddot{A}^{\Delta}}$ is taken over such $\ddot{A}^{\Delta}$ that

$$
\begin{aligned}
&\ddot{A}^{\Delta}\in\mathscr{S}, \\
&\ddot{A}^{\Delta}\equiv\ddot{A}_0^{\Delta} \mod 8abW[V_1,\cdots,V_4], \\
&(\ddot{A}^{\Delta},\ddot{A})=1, \\
&X<\ddot{A}^{\Delta}\leqq X+8abWW'[V_1,\cdots,V_4],
\end{aligned}
$$

and that all prime divisors of $\prod_{i=1,\dots,4}(U_ia, A_i)$ are smaller than T. We put as

$$\Delta^1(X;Z)=\prod_{p;\ \mathrm{prime},p|X,p<Z} p$$

and

$$\Delta_U^{\#}=\prod_{(1,2,3,4)}(U_{i_1}^{\#}, U_{i_2}^{\#}U_{i_3}^{\#}U_{i_4}^{\#}\cdot\Delta^1(U_{i_1}^{\#};Z))$$

where (i_1, i_2, i_3, i_4) is taken over possible permutations of $1,\cdots,4$.

Remark. We can use, in $J_0(\cdots)$, $t_i^{(A)}$ in place of $\bar{t}_i^{(A)}$, that corresponds to $\ddot{A}_0^{\Delta}$, as we have seen in the proof of 2.3.7.

2.3.10. The proof of 2.3.9 will end at 2.3.10.7.

2.3.10.1. Lemma. *The number of $\ddot{A}^{\Delta}$, for which $\prod_i(U_ia, A_i)$ have prime divisors larger than T, is*

$$\ll\nu(U_1\cdots U_4a)(T^{-1}W'+1).$$

Proof. It is easy.

2.3.10.1.1. Corollary. *We may suppose, in the left-hand side $\sum_{\ddot{A}^{\Delta}}\prod_i J_0(\cdots)$, the summation is taken over*

$$
\begin{aligned}
&\ddot{A}^{\Delta}\equiv\ddot{A}_0^{\Delta} \mod 8abW[V_1,\cdots,V_4], \\
&(\ddot{A}^{\Delta},\ddot{A})=1, \\
&X<\ddot{A}^{\Delta}\leqq X+8abWW'[V_1,\cdots,V_4],
\end{aligned}
$$

and may use $t_i^{(A)}$ given by $\ddot{A}_0^{\Delta}$ instead of $\bar{t}_i^{(A)}$.

2.3.10.2. We use the Möbius function $\mu(d)$ with $d\,|\,\ddot{A}$, to get rid of the condition that $(\ddot{A}^{\Delta}, \ddot{A})=1$. The conditions, that

$$\ddot{A}^{\Delta}\equiv\ddot{A}_0^{\Delta} \mod 8abW[V_1, \cdots, V_4]$$

and

$$\ddot{A}^{\Delta}\equiv 0 \mod d,$$

give us that

$$\ddot{A}_0^{\Delta}\equiv 0 \mod (d, 8abW[V_1, \cdots, V_4]).$$

This, with

$$d\,|\,\ddot{A} \quad\text{and}\quad (\ddot{A}, \ddot{A}_0^{\Delta})=1,$$

gives us that d must satisfy

$$(d, 8abW[V_1, \cdots, V_4])=1.$$

The last condition is also sufficient for the existence of $\ddot{A}^{\Delta}$ as a solution of the first congruences. We put, temporarily, $\tilde{U}_i^{\#}$ as the square-free kernel of $(t_i^{(A)}t_i^{(ab)})^{-1}U_i a$. Then $\tilde{U}_i^{\#}$ is a divisor of $U_i^{\#}$, and p_0 and i_0 have the same properties for $\tilde{U}_i^{\#}$'s, as all prime divisors of $t_i^{(A)}t_i^{(ab)}$ are smaller than Z. Taking into account 2.3.10.1.1 and the fact that

$$J_0\left(\frac{(p_i+q_i)\ddot{A}-q_i\ddot{A}^{\Delta}}{(t_i^{(A)}t_i^{(ab)})^{-1}U_i a}\right)=J\left(\frac{(p_i+q_i)\ddot{A}-q_i\ddot{A}^{\Delta}}{\tilde{U}_i^{\#}}\right),$$

we must estimate the sum

$$\sum_{d;\ d|\ddot{A},(d,8abW[V_1,\cdots,V_4])=1} \mu(d)\times\sum_{\ddot{A}^{\Delta}}\prod_{i=1,\cdots,4} J\left(\frac{(p_i+q_i)\ddot{A}-q_i\ddot{A}^{\Delta}}{\tilde{U}_i^{\#}}\right),$$

where $\ddot{A}^{\Delta}$ runs over such integers that

$$\ddot{A}^{\Delta}\equiv\ddot{A}_0^{\Delta} \mod d8abW[V_1, \cdots, V_4],$$

and

$$X<\ddot{A}^{\Delta}\leqq X+8abWW'[V_1, \cdots, V_4].$$

2.3.10.3. Lemma. *The contribution of such d's to the sum in* 2.3.10.2, *which have prime divisors not smaller than Z, is*

$$\ll Z^{-1}\nu(\ddot{A})\ddot{A}\phi(\ddot{A})^{-1}W'+2^{\nu(\ddot{A})}.$$

Proof. Letting d run over as stated above, we have

$$\sum_d \mu(d)^2(d^{-1}W'+1)$$
$$\leqq \sum_{p,d;\ pd|\ddot{A},p\geqq Z,p \text{ is a prime}} (pd)^{-1}\mu(d)^2 W'+2^{\nu(\ddot{A})}$$
$$\leqq (\sum_{p;\ \text{prime},p|\ddot{A},p\geqq Z} p^{-1})\cdot(\sum_{d;\ d|\ddot{A}} d^{-1}\mu(d)^2)W'+2^{\nu(\ddot{A})}$$
$$\ll Z^{-1}\nu(\ddot{A})\ddot{A}\phi(\ddot{A})^{-1}W'+2^{\nu(\ddot{A})},$$

by the formula (1.5.24) in [20].

2.3.10.4. We consider a Gaussian sum

$$\sum_{l;\ 1\leqq l\leqq U} J\left(\frac{l}{U}\right)e\left(\frac{l}{U}\right),$$

where $U(>1)$ is a square-free odd positive integer. Here we put $J(l/U)=0$ if $(l, U)>1$. It is well-known that this sum has the absolute value $U^{1/2}$, and that

$$J\left(\frac{X}{U}\right)\times\sum_l J\left(\frac{l}{U}\right)e\left(\frac{l}{U}\right)=\sum_{l;\ 1\leqq l\leqq U} J\left(\frac{l}{U}\right)e\left(\frac{lX}{U}\right)$$

for integers X. We let d's run, in 2.3.10.2, over such integers that d are square-free divisors of $\ddot{A}$ whose prime divisors are all smaller than Z, i.e., $d\,|\,\Delta^1(\ddot{A}; Z)$. It is sufficient to estimate the sum

$$(\tilde{U}_1^\#\cdots\tilde{U}_4^\#)^{-1/2}\sum_d\ \sum_{l_1,\cdots,l_4}\Big|\sum_{\ddot{A}^\Delta} e\Big(\sum_{i=1,\cdots,4}\tilde{U}_i^{\#-1}l_iq_i\ddot{A}^\Delta\Big)\Big|,$$

where d runs as is stated above, $\ddot{A}^\Delta$ runs as is stated in the last of 2.3.10.2 and l_i runs so that $1\leqq l_i\leqq\tilde{U}_i^\#$ and $(l_i, \tilde{U}_i^\#)=1$.

2.3.10.5. All prime divisors of $(\tilde{U}_i^\#, 8l_iq_idW[V_1, \cdots, V_4])$ are smaller than Z, but there exist a prime divisor p_0 and i_0 such that

$$p_0\geqq Z,\quad p_0\,|\,\tilde{U}_{i_0}^\#,\quad p_0\nmid\tilde{U}_i^\#\quad\text{for } i\neq i_0,$$

by the assumption on Z. These mean that

$$\sum_i \tilde{U}_i^{\#-1}l_iq_i\times 8dW[V_1, \cdots, V_4]$$

is not an integer. Therefore the summation over $\ddot{A}^\Delta$ in 2.3.10.4 is

$$\ll\Big\|\sum_i \tilde{U}_i^{\#-1}l_iq_idW[V_1, \cdots, V_4]\Big\|^{-1},$$

where $\|\xi\|$ is the distance of ξ from the nearest integer.

2.3.10.6. Sublemma. *Let* D_i $(i=1, \cdots, 4)$ *and* D *be*

$$D_i=\tilde{U}_i^{\#}(\tilde{U}_i^{\#}, q_i dW[V_1, \cdots, V_4])^{-1}$$

and

$$D=[D_1, \cdots, D_4],$$

temporarily. Then, the number of quadruples $(l_1, \cdots, l_4)$ *in* 2.3.10.4 *for given* l*, such that*

$$\sum_i \tilde{U}_i^{\#-1}\cdot l_i q_i dW[V_1, \cdots, V_4]\equiv D^{-1}l \quad \bmod 1,$$

is

$$\ll \prod_{(1,2,3,4)} (U_{i_1}^{\#}, U_{i_2}^{\#}U_{i_3}^{\#}U_{i_4}^{\#}\Delta^1(U_{i_1}^{\#}; Z)),$$

where $\prod_{(1,2,3,4)}$ *is taken over possible permutations of* $(1, 2, 3, 4)$.

Proof. As $(l_i, \tilde{U}_i^{\#})=1$, the reduced denominator of the left-hand side of the congruence is D. Putting X_i as

$$D_i^{-1}\times X_i=\tilde{U}_i^{\#-1}\times q_i dW[V_1, \cdots, V_4], \qquad (D_i, X_i)=1,$$

temporarily, we must solve the congruence

$$\sum_i (DD_i^{-1}X_i)l_i\equiv l \quad \bmod D,$$

under

$$0\leqq l_i<D_i(\tilde{U}_i^{\#}, q_i dW[V_1, \cdots, V_4]),$$
$$(l_i, \tilde{U}_i^{\#})=1.$$

We have

$$(DD_1^{-1}X_1, D_1)=1$$

as D is square-free. We have also

$$(DD_1^{-1}X_1)l_1\equiv l \quad \bmod D_1(D_1, D_2D_3D_4)^{-1}.$$

Therefore l_1 is determined mod $D_1(D_1, D_2D_3D_4)^{-1}$. Then, the number of possible l_1 is

$$\ll(D_1, D_2D_3D_4)\times(\tilde{U}_1^{\#}, q_1 dW[V_1, \cdots, V_4])$$
$$\ll(U_1^{\#}, U_2^{\#}U_3^{\#}U_4^{\#}\Delta^1(U_1^{\#}; Z)).$$

Similarly for other l_i's, and we have the conclusion.

2.3.10.7. We return to 2.3.10.5. We have, putting the estimate obtained as $\varDelta_U^{\#}$ temporarily, that

$$\begin{aligned}
&\sum_{l_1,\cdots,l_4} \Big\|\sum_i \tilde{U}_i^{\#} l_i q_i d W[V_1, \cdots, V_4]\Big\|^{-1} \\
&\quad \ll \sum_{l;\ 1\leq l\leq (1/2)D} l^{-1} D \times \varDelta_U^{\#} \\
&\quad \ll D \log D \times \varDelta_U^{\#} \\
&\quad \ll [U_1^{\#}, \cdots, U_4^{\#}] \times \varDelta_U^{\#} \log (U_1 \cdots U_4).
\end{aligned}$$

We have, therefore, the sum in 2.3.10.4 to be estimated is

$$\begin{aligned}
&\ll (\tilde{U}_1^{\#} \times \cdots \times \tilde{U}_4^{\#})^{-1/2} \cdot [U_1^{\#}, \cdots, U_4^{\#}] \cdot \varDelta_U^{\#} \cdot \log (U_1 \cdots U_4) \times \sum_{d;\ d|\varDelta^1(\ddot{A};\ Z)} 1 \\
&\ll (U_1^{\#} \cdots U_4^{\#})^{-1/2} \cdot \Big(\prod_i t_i^{(A)} t_i^{(ab)}\Big)^{1/2} \cdot [U_1^{\#}, \cdots, U_4^{\#}] \cdot \varDelta_U^{\#} \\
&\qquad\qquad \cdot \log (U_1 \cdots U_4) \cdot \tau(\varDelta^1(\ddot{A};\ Z)).
\end{aligned}$$

As $t_i^{(A)} t_i^{(ab)} \mid ab(U_i a, A_i^{(0)})$, we have the conclusion of 2.3.9.

2.3.11. To apply 2.3.9 effectively, let us consider the following problem; Let g', K be (large) positive integers which are regarded as constants. Let $\ddot{A}$, a, b, U_i, V_i, u_i, v_i, p_i, q_i, p_i', q_i', t_i, t_i' $(i=1, \cdots, 4)$ be fixed positive integers such that

$$\begin{aligned}
&U_i V_i ab = t_i t_i' ((p_i+q_i) q_i' + p_i' q_i), \\
&2^{u_i} \| U_i, \quad 2^{v_i} \| V_i \quad (u_i \times v_i = 0), \\
&(a\ddot{A}, b) = 1, \qquad (U_i, V_i) = 1, \\
&2 \nmid ab, \qquad 2^{g'+1} \nmid \ddot{A}, \\
&(p_i, q_i) = 1, \qquad (p_i', q_i') = 1.
\end{aligned}$$

We put $u_0 = \max(u_1, \cdots, u_4)$ and $v_0 = \max(v_1, \cdots, v_4)$. Let $\ddot{\varepsilon}$ be ± 1. Let $\ddot{A}_0^{\varDelta}$, $\ddot{B}_0^{\varDelta}$ and $\ddot{B}_0^{\varDelta}$ be positive integers such that

$$\begin{aligned}
&\ddot{A}\ddot{B}_0^{\varDelta} - \ddot{B}_0 \ddot{A}_0^{\varDelta} = \ddot{\varepsilon}, \\
&2^{g'+1} \nmid X \qquad \text{for } X = \ddot{A}_0^{\varDelta}, \ddot{B}_0^{\varDelta}, \ddot{B}_0,
\end{aligned}$$

and that

$$\begin{pmatrix} U_i a & 0 \\ 0 & V_i b \end{pmatrix} \begin{pmatrix} B_i^{(0)} & B_i'^{(0)} \\ A_i^{(0)} & A_i'^{(0)} \end{pmatrix} = \begin{pmatrix} \ddot{B}_0 & \ddot{B}_0 \\ \ddot{A}_0 & \ddot{A}_0 \end{pmatrix} \begin{pmatrix} t_i(p_i+q_i) & t_i' p_i' \\ -t_i q_i & t_i' q_i' \end{pmatrix}$$

have integer solutions $A_i^{(0)}$, $A_i'^{(0)}$, $B_i^{(0)}$, $B_i'^{(0)}$ for $i=1, \cdots, 4$. We have, then,

$$A_i^{(0)}B_i'^{(0)}-B_i^{(0)}A_i'^{(0)}=\ddot{\varepsilon},$$
$$(U_iaB_i^{(0)},\ V_ibA_i^{(0)})=t_i$$

and

$$(U_iaB_i'^{(0)},\ V_ibA_i'^{(0)})=t_i'.$$

The problem is: Let T, W and $\mathscr{S}$ be as in 2.3.9. Find a positive (small) integer W'' for which, for every integer $\ddot{A}^{\Delta}$ with

$$(\ddot{A},\ \ddot{A}^{\Delta})=1,$$
$$\ddot{A}^{\Delta}\equiv\ddot{A}_0^{\Delta} \mod 2^{u_0+g'+1}\times abWW''[V_1,\cdots,V_4]$$

and

$$\ddot{A}^{\Delta}\notin\mathscr{S}$$

we can find $\ddot{B}$ and $\ddot{B}^{\Delta}$ such that

$$\ddot{A}\ddot{B}^{\Delta}-\ddot{B}\ddot{A}^{\Delta}=\ddot{\varepsilon},$$

and that

$$\begin{pmatrix}U_ia & 0\\ 0 & V_ib\end{pmatrix}\begin{pmatrix}B_i & B_i'\\ A_i & A_i'\end{pmatrix}=\begin{pmatrix}\ddot{B} & \ddot{B}^{\Delta}\\ \ddot{A} & \ddot{A}^{\Delta}\end{pmatrix}\begin{pmatrix}t_i(p_i+q_i) & t_i'q_i'\\ -t_iq_i & t_i'q_i'\end{pmatrix}$$

have integer solutions A_i, A_i', B_i, B_i' with

$$\ddot{B}\equiv\ddot{B}_0 \mod 2^{g'+u_0+1}a.$$

(It follows from the assumptions that A_i and A_i' are integers.)

The set $\mathscr{S}$ can be restricted in connection with applications in 4.5.3. Note that we have

$$(U_iaB_i,\ V_ibA_i)=t_i,$$
$$(U_iaB_i',\ V_ibA_i')=t_i'$$

and

$$A_iB_i'-B_iA_i'=\ddot{\varepsilon}.$$

2.3.11.1. Let $\overline{W}$ be $bWW''[V_1,\cdots,V_4]$, temporarily. Let us put

$$\ddot{B}=\ddot{B}_0+2^{g'+u_0+1}a\beta,\quad \ddot{B}^{\Delta}=\ddot{B}_0^{\Delta}+2^{g'+u_0+1}a\beta^{\Delta},$$
$$\ddot{A}^{\Delta}=\ddot{A}_0^{\Delta}+2^{g'+u_0+1}a\overline{W}\alpha^{\Delta},$$

with integers β, β^{Δ}, α^{Δ}. As

$$\ddot{A}\ddot{B}^{\Delta}-\ddot{B}\ddot{A}^{\Delta}=\ddot{A}\ddot{B}_0^{\Delta}-\ddot{B}_0\ddot{A}_0^{\Delta},$$

we must have

$$\ddot{A}2^{g'+u_0+1}a\beta^{\Delta}=\ddot{A}^{\Delta}2^{g'+u_0+1}\alpha\beta+\ddot{B}_0 2^{g'+u_0+1}\overline{W}\alpha^{\Delta},$$

i.e.,

$$\ddot{A}\beta^{\Delta}=\ddot{A}^{\Delta}\beta+\ddot{B}_0\overline{W}\alpha^{\Delta}.$$

Preparing $\ddot{C}$ and $\ddot{C}^{\Delta}$, which depend on $\ddot{A}$ and $\ddot{A}^{\Delta}$, so that

$$\ddot{A}\ddot{C}^{\Delta}-\ddot{A}^{\Delta}\ddot{C}=1,$$

we have

$$\begin{aligned}\ddot{C}\ddot{A}\beta^{\Delta}&=\ddot{C}(\ddot{A}^{\Delta}\beta+\ddot{B}_0\overline{W}\alpha^{\Delta})\\&=\ddot{C}\ddot{B}_0\overline{W}\alpha^{\Delta}+(\ddot{A}\ddot{C}^{\Delta}-1)\beta.\end{aligned}$$

Therefore

$$\beta\equiv\ddot{C}\ddot{B}_0\overline{W}\alpha^{\Delta} \bmod \ddot{A}.$$

Then, we can put

$$\beta=\ddot{C}\ddot{B}_0\overline{W}\alpha^{\Delta}+\ddot{A}z$$

with an integer z. We have, then, that

$$\begin{aligned}\ddot{A}\beta^{\Delta}&=\ddot{A}^{\Delta}(\ddot{C}\ddot{B}_0\overline{W}\alpha^{\Delta}+\ddot{A}z)+\ddot{B}_0\overline{W}\alpha\\&=(\ddot{A}^{\Delta}\ddot{C}+1)\ddot{B}_0\overline{W}\alpha^{\Delta}+\ddot{A}\ddot{A}^{\Delta}z\\&=\ddot{A}\ddot{C}^{\Delta}\ddot{B}_0\overline{W}\alpha^{\Delta}+\ddot{A}\ddot{A}^{\Delta}z,\end{aligned}$$

therefore that

$$\beta^{\Delta}=\ddot{C}^{\Delta}\ddot{B}_0\overline{W}\alpha^{\Delta}+\ddot{A}^{\Delta}z.$$

Conversely, if

$$\beta=\ddot{C}\ddot{B}_0\overline{W}\alpha^{\Delta}+\ddot{A}z$$

and

$$\beta^{\Delta}=\ddot{C}^{\Delta}\ddot{B}_0\overline{W}\alpha^{\Delta}+\ddot{A}^{\Delta}z$$

with an integer z, we have

$$\ddot{A}\beta^{\Delta}=\ddot{A}^{\Delta}\beta+\ddot{B}_0\overline{W}\alpha^{\Delta}$$

and, then,

$$\ddot{A}\ddot{B}^{\Delta}-\ddot{B}\ddot{A}^{\Delta}=\ddot{\varepsilon}.$$

2.3.11.2. Substituting these, we have

$$\begin{pmatrix}\ddot{B} & \ddot{B}^{\Delta}\\ \ddot{A} & \ddot{A}^{\Delta}\end{pmatrix}\times\begin{pmatrix}t_i(p_i+q_i) & t'_ip'_i\\ -t_iq_i & t'_iq'_i\end{pmatrix}$$
$$=\begin{pmatrix}U_ia & 0\\ 0 & V_ib\end{pmatrix}\times\begin{pmatrix}B_i^{(0)} & B_i'^{(0)}\\ A_i^{(0)} & A_i'^{(0)}\end{pmatrix}+2^{g'+u_0+1}az\begin{pmatrix}V_ibA_i & V_ibA'_i\\ 0 & 0\end{pmatrix}$$
$$+2^{g'+u_0+1}a\overline{W}\begin{pmatrix}X_iB_0 & Y_iB_0\\ t'_ip'_i & t'_iq'_i\end{pmatrix},$$

where

$$X_i=t_i((p_i+q_i)\ddot{C}-q_i\ddot{C}^{\Delta}),$$

and

$$Y_i=t'_i(p'_i\ddot{C}+q'_i\ddot{C}^{\Delta}).$$

Here A_i, A'_i, $\ddot{C}$, $\ddot{C}^{\Delta}$ are considered to be known as they are obtained from $\ddot{A}^{\Delta}$ and $\ddot{A}$. The problem is reduced to show the following; Find W'', so that, for every α^{Δ} to be taken into account, we can find z satisfying

$$zV_ibA_i+\overline{W}\alpha^{\Delta}X_i\ddot{B}_0\equiv 0 \mod U_i$$

and

$$zV_ibA'_i+\overline{W}\alpha^{\Delta}Y_i\ddot{B}_0\equiv 0 \mod U_i,$$

for $i=1,\cdots,4$.

2.3.11.3. Sublemma. *The system of congruences*

$$c_i^{(1)}x\equiv a_i^{(1)} \mod b_i$$

and

$$c_i^{(2)}x\equiv a_i^{(2)} \mod b_i \quad (i=1,\cdots,4),$$

is soluble in x, *if and only if*

$$a_i^{(h)}\equiv 0 \mod (c_i^{(h)}, b_i) \quad (h=1,2;\ i=1,\cdots,4),$$
$$c_i^{(1)}a_i^{(2)}\equiv c_i^{(2)}a_i^{(1)} \mod b_i(c_i^{(1)}, c_i^{(2)}, b_i) \quad (i=1,\cdots,4),$$
$$c_i^{(h)}a_j^{(h)}\equiv c_j^{(h)}a_i^{(h)} \mod (b_ib_j,\ b_ic_j^{(h)},\ b_jc_i^{(h)})$$
$$(h=1,2;\ i,j=1,\cdots,4;\ i\neq j)$$

and

$$c_i^{(1)}a_j^{(2)} \equiv c_j^{(2)}a_i^{(1)} \quad \bmod (b_ib_j,\ b_ic_j^{(1)},\ b_jc_i^{(2)}) \quad (i, j=1, \cdots, 4;\ i\neq j).$$

Proof. It is elementary, Section 8.1 in [12], for instance.

2.3.11.4. Returning to the end of 2.3.11.2, it is sufficient for the solubility in z (for every α^{Δ}), if

$$(U_i,\ V_ibA_i)\mid \overline{W}, \qquad (U_i,\ V_ibA_i')\mid \overline{W},$$
$$V_ib\overline{W}\ddot{B}_0(A_iY_i-A_i'X_i)\equiv 0 \quad \bmod U_i(U_i,\ V_ibA_i,\ V_ibA_i'),$$
$$b\overline{W}\ddot{B}_0(V_iA_iX_j-V_jA_jX_i)\equiv 0 \quad \bmod (U_iU_j,\ U_iU_jbA_j,\ U_jV_ibA_i),$$
$$b\overline{W}\ddot{B}_0(V_iA_i'Y_j-V_jA_j'Y_i)\equiv 0 \quad \bmod (U_iU_j,\ U_iV_jbA_j',\ U_jV_ibA_i')$$

and

$$b\overline{W}\ddot{B}_0(V_iA_iY_j-V_jA_j'X_i)\equiv 0 \quad \bmod (U_iU_j,\ U_iV_jbA_j,\ U_jV_ibA_i'),$$

for $i, j=1, \cdots, 4$ and $i\neq j$. We have

$$\begin{aligned}
\ddot{A}(A_iY_i-A_i'X_i) &= \ddot{A}(t_i'A_i(p_i'\ddot{C}+q_i'\ddot{C}^{\Delta})-t_iA_i'((p_i+q_i)\ddot{C}-q_i\ddot{C}^{\Delta}))\\
&= t_i'A_i(p_i'\ddot{A}\ddot{C}+q_i'(\ddot{A}^{\Delta}\ddot{C}+1))-t_iA_i'((p_i+q_i)\ddot{A}\ddot{C}-q_i(\ddot{A}^{\Delta}\ddot{C}+1))\\
&= \ddot{C}(A_i\times t_i'(p_i'\ddot{A}+q_i'\ddot{A}^{\Delta})-A_i'\times t_i((p_i+q_i)\ddot{A}-q_i\ddot{A}^{\Delta}))\\
&\quad +t_i'q_i'A_i+t_iq_iA_i'\\
&= \ddot{C}(A_i\times V_ibA_i'-A_i'\times V_ibA_i)\\
&\quad +(V_ib)^{-1}(t_i'q_i'\times t_i((p_i+q_i)\ddot{A}-q_i\ddot{A}^{\Delta})+t_iq_i\times t_i'(p_i'\ddot{A}+q_i'\ddot{A}^{\Delta}))\\
&= (V_ib)^{-1}t_it_i'((p_i+q_i)q_i'+p_i'q_i)\ddot{A}\\
&= \ddot{A}U_ia,
\end{aligned}$$

therefore

$$V_ib\overline{W}\ddot{B}_0(A_iY_i-A_i'X_i)=U_iV_iab\overline{W}\ddot{B}_0.$$

We have

$$(U_i,\ V_ibA_i,\ V_ibA_i')\mid V_ib(U_i,\ A_i,\ A_i').$$

Putting $U_i=U_i'(U_i,\ U_j)$ and $U_j=U_i'(U_i,\ U_j)$ temporarily, we have

$$\begin{aligned}
(U_iU_j,\ U_iV_jbA_j,\ U_jV_ibA_i) &= (U_i,\ U_j)\times((U_i,\ U_j)U_i'U_j',\ U_i'V_jbA_j,\ U_i'V_ibA_i)\\
&\mid (U_i,\ U_j)\times b(U_i,\ V_iA_i)(U_j,\ V_jA_j)\mid b(U_i,\ U_j)\cdot(U_i,\ A_i)\cdot(U_j,\ A_j),
\end{aligned}$$

the last owing to $(U_i,\ V_i)=1$ and $(U_j,\ V_j)=1$. Similarly

$$(U_iU_j,\ U_iV_jbA_j',\ U_jV_ibA_i')\mid b(U_i,\ U_j)(U_i,\ A_i')(U_j,\ A_j'),$$

and

$$(U_iU_j,\ U_iV_jbA_j,\ U_jV_ibA'_i)\mid b(U_i,\ U_j)(U_i,\ A'_i)(U_j,\ A_j).$$

It is sufficient, therefore, to suppose that

$$\overline{W}\equiv 0 \quad \mathrm{mod}\,(U_i,\ A_i,\ A'_i) \quad (i=1,\ \cdots,4)$$
$$\overline{W}\equiv 0 \quad \mathrm{mod}\, b(U_i,\ U_j)(U_i,\ X^{(i)})(U_jY^{(j)}) \quad (i,j=1,\ \cdots,4;\ i\neq j)$$

with $X^{(i)}=A_i,\ A'_i$ and $Y^{(j)}=A_j,\ A'_j$, for all $\ddot{A}^{\Delta}$ to be considered. Now let us suppose

that $\quad p^{g'+1}\nmid X$ if p is a prime, $p\mid U_1\cdots U_4,\ 2\leqq p<K$,

that $\quad p^2\nmid X$ if p is a prime, $p\mid U_1\cdots U_4,\ K\leqq p\leqq T$,

and that $\quad p\nmid X$ if p is a prime, $p\mid U_1\cdots U_4,\ p>T$,

for every $X=A_i$ and A'_i with $\ddot{A}^{\Delta}\notin\mathscr{S}$. Then, the assumption that

$$\begin{aligned} WW''\equiv 0 \ \mathrm{mod}\,(&\prod_{p;\ \mathrm{prime},2\leqq p<K} p)^{g'} \\ &\times \text{L.C.M. of } \{(U_i,\ U_j);\ i,j=1,\ \cdots,4,\ i\neq j\} \\ &\times \Delta^1(U_1\cdots U_4;\ T) \end{aligned}$$

is sufficient for the problem in 2.3.11. We have therefore the following proposition.

2.3.11.5. Proposition. *Let g', K be (large) constant positive integers. Let $\ddot{A}$, a, b, U_i, V_i, p_i, q_i, p'_i, q'_i, t_i, t'_i $(i=1,\ \cdots,4)$ be positive integers such that*

$$U_iV_iab=t_it'_i((p_i+q_i)q'_i+p'_iq_i),$$
$$(U_i,\ V_i)=1,\quad (p_i,\ q_i)=1,\quad (p'_i,\ q'_i)=1,$$
$$(a\ddot{A},\ b)=1,\quad 2\nmid ab,\quad 2^{g'+1}\nmid\ddot{A}.$$

Let $\ddot{\varepsilon}$ be ± 1. Let $\ddot{A}_0^{\Delta}$; $\ddot{B}_0^{\Delta}$ and $\ddot{B}_0$ be positive integers such that

$$\ddot{A}\ddot{B}_0^{\Delta}-\ddot{B}_0\ddot{A}_0^{\Delta}=\ddot{\varepsilon},$$
$$2^{g'+1}\nmid X \qquad \text{tor } X=\ddot{A}_0^{\Delta},\ \ddot{B}_0^{\Delta},\ \ddot{B}_0,$$

and that

$$\begin{pmatrix} U_ia & 0 \\ 0 & V_ib \end{pmatrix}\begin{pmatrix} B_i^{(0)} & B_i'^{(0)} \\ A_i^{(0)} & A_i'^{(0)} \end{pmatrix}=\begin{pmatrix} \ddot{B}_0 & \ddot{B}_0^{\Delta} \\ \ddot{A} & \ddot{A}_0^{\Delta} \end{pmatrix}\begin{pmatrix} t_i(p_i+q_i) & t'_ip'_i \\ -t_iq_i & t'_iq'_i \end{pmatrix}$$

have integer solutions $A_i^{(0)}$, $A_i'^{(0)}$, $B_i^{(0)}$, $B_i'^{(0)}$ for $i=1,\ \cdots,4$. Let T, W, $\mathscr{S}$ be as in 2.3.9. Let W'' be such that

$$WW'' \equiv 0 \mod \Big(\prod_{p;\ \mathrm{prime}, 2\leqq p<K} p\Big)^{g'}$$
$$\times \text{L.C.M. of } \{(U_i, U_j);\ i, j=1, \cdots, 4,\ i\neq j\}$$
$$\times \Delta^1(U_1\cdots U_4;\ T)$$

with the notation $\Delta^1(\cdots)$ *in* 2.3.9. *Let* u_0 *be such that* $2^{u_0} \| [U_1, \cdots, U_4]$. *Let* $\ddot{A}^\Delta$ *be an integer such that*

$$\ddot{A}^\Delta \equiv \ddot{A}_0^\Delta \mod 2^{g'+u_0+1}abWW''[V_1, \cdots, V_4],$$
$$(\ddot{A}^\Delta, \ddot{A})=1$$

and

$$\ddot{A}^\Delta \notin \mathscr{S}.$$

Putting A_i *and* A_i' *as*

$$V_i b A_i = t_i((p_i+q_i)\ddot{A} - q_i\ddot{A}^\Delta)$$

and

$$V_i b A_i' = t_i'(p_i'\ddot{A} + q_i'\ddot{A}^\Delta),$$

which become integers, we suppose, in addition, for $X=A_i$ *and* A_i',

that $p^{g'+1}\nmid X$ *if* p *is a prime*, $p\,|\,U_1\cdots U_4$ *and* $2\leqq p<K$,

that $p^2\nmid X$ *if* p *is a prime*, $p\,|\,U_1\cdots U_4$ *and* $K\leqq p\leqq T$,

and that $p\nmid X$ *if* p *is a prime*, $p\,|\,U_1\cdots U_4$ *and* $p>T$.

We can find, then, $\ddot{B}$ *and* $\ddot{B}^\Delta$ *such that*

$$\ddot{A}\ddot{B}^\Delta - \ddot{B}\ddot{A}^\Delta = \ddot{\varepsilon},$$
$$\ddot{B} \equiv \ddot{B}_0 \mod 2^{g'+u_0+1}\alpha,$$

and that, with above A_i *and* A_i',

$$\begin{pmatrix} U_i a & 0 \\ 0 & V_i b\end{pmatrix}\begin{pmatrix} B_i & B_i' \\ A_i & A_i'\end{pmatrix} = \begin{pmatrix} \ddot{B} & \ddot{B}^\Delta \\ \ddot{A} & \ddot{A}^\Delta\end{pmatrix}\begin{pmatrix} t_i(p_i+q_i) & t_i'p_i' \\ -t_iq_i & t_i'q_i'\end{pmatrix}$$

have integer solutions B_i *and* B_i'.

Chapter 3. Kloosterman's Sum

3.1.

As for the so-called Kloosterman's sum, it would be desirable to consult the result of A. Weil, ((70), p. 35 [13], on its history). But we will not pursue here the best possible result along our method, and so we consult Kloosterman's original result, (2.43-Lemma 4c, p. 423, [17]).

3.1.1. Lemma. *Let p be an odd prime and u be an integer with $p \nmid u$. We have, then,*

$$\sum_v \left| \sum_x{}' e\left(\frac{1}{p}(ux+v\bar{x})\right)\right|^4 < 5p^3,$$

where the summation are taken over so that

$$1 \leqq v \leqq p, \quad 1 \leqq x \leqq p, \quad p \nmid x$$

and

$$x\bar{x} \equiv 1 \bmod p.$$

Proof. This is a corollary of what was proved originally on the line 9, p. 426, [17]. The notation σ_3 was used in the proof of Lemma 4c.

3.1.1.1. Corollary. *Let p be an odd prime, u and v be integers with $p \nmid u$ or $p \nmid v$. We have, then,*

$$\left| \sum_x{}' e\left(\frac{1}{p}(ux+v\bar{x})\right)\right| < 5^{1/4}p^{3/4},$$

where the summation is taken over so that

$$1 \leqq x \leqq p, \quad p \nmid x \quad \text{and} \quad x\bar{x} \equiv 1 \bmod p.$$

3.1.2. Lemma. *Let $A(>0)$, u and v be integers. Let A has the standard decomposition $A = p_1^{e_1} \cdots p_l^{e_l}$, where p_j's are mutually different primes and $e_j \in \boldsymbol{N}$. We have, then,*

$$\sum_x{}' e\left(\frac{1}{A}(ux+v\bar{x})\right) = \prod_{j=1}^{l} \left\{ \sum_{x_j}{}' e\left(\frac{1}{p_j^{e_j}}(u_j x_j + v_j \bar{x}_j)\right)\right\},$$

where the summation is taken over

$$1 \leqq x \leqq A, \quad (x, A) = 1, \quad x\bar{x} \equiv 1 \bmod A,$$
$$1 \leqq x_j \leqq p_j^{e_j}, \quad (x_j, p_j) = 1, \quad x_j \bar{x}_j \equiv 1 \bmod p_j^{e_j},$$

and u_j's are suitably chosen so that, for every j, we have the equivalence between $p_j^t \| u$ and $p_j^t \| u_j$, and similarly for v_j's.

Proof. This follows from a standard argument, (2.42-Lemma 4b, [17]).

3.1.3. Lemma. *Let $A=a\ddot{A}$ (>0), u and v be integers such that*

$$\mu(\ddot{A})\neq 0, \quad 2\nmid\ddot{A}, \quad (a, \ddot{A})=1.$$

We have, then,

$$\left|\sum_x{}' e\left(\frac{1}{A}(ux+v\bar{x})\right)\right|\leqq\{A^3 5^{\nu(\ddot{A}_{(u,v)})}(\ddot{A}, u, v)a\}^{1/4},$$

where the summation is taken over

$$1\leqq x\leqq A, \quad (A, x)=1, \quad x\bar{x}\equiv 1 \bmod A$$

and we have put

$$\ddot{A}_X=\prod_{p;\ \mathrm{prime},\, p\mid\ddot{A},\, p\nmid X} p$$

for a square-free $\ddot{A}$.

Proof. Let $A=p_1^{e_1}\cdots p_l^{e_l}$ ($e_j=1$ if $p_j\mid\ddot{A}$) be the standard decomposition of A. We apply 3.1.2. If p_j is such that $p_j\mid\ddot{A}$ and $p_j\nmid(u, v)$, then, we use 3.1.1.1. If $p_j\mid(\ddot{A}, u, v)$, then, we use trivial estimate that the absolute value of the p_j-component is $\leqq p_j$. If $p_j\nmid\ddot{A}$, therefore if $p_j^{e_j}\| a$, we use again trivial estimate that the absolute value is $\leqq p_j^{e_j}$. These give the stated estimate.

3.1.4. Proposition. *Let $F(x)$ be a complex-valued function of $x\in\boldsymbol{Z}$. Let $A=a\ddot{A}$ be a positive integer such that $\mu(\ddot{A})\neq 0$, $2\nmid\ddot{A}$ and $(a, \ddot{A})=1$. Let ξ' and ξ'' be real such that $\frac{1}{2}<\xi''-\xi'\leqq A$. Let u, v, λ and Λ be integers such that $1\leqq\Lambda\ll\xi''-\xi'$. We have, then,*

$$\begin{aligned}&\left|\sum_{x;\ \xi'<x\leqq\xi'',\ (x,A)=1,\ x\equiv\lambda \bmod \Lambda}{}' F(x)e\left(\frac{1}{A}(ux+v\bar{x})\right)\right|\\ &\quad\ll\{A^3 5^{\nu(\ddot{A}_v)}a\}^{1/4}\\ &\quad\times\left\{\begin{aligned}&A^{-1}\cdot(A, \Lambda)\sigma_{1/4}((\ddot{A}, v))\times\Big|\sum_{x;\ \xi'<x\leqq\xi'',\ x\equiv\lambda \bmod \Lambda} F(x)e(A^{-1}ux)\Big|\\ &+\sigma_{-3/4}((\ddot{A}, v))(\log A)\times\sum_{x;\ \xi'<x\leqq\xi'',\ x\equiv\lambda \bmod \Lambda}|F(x)e(A^{-1}ux)\\ &\qquad\qquad-F(x+\Lambda)e(A^{-1}u(x+\Lambda))|\\ &+\sigma_{-3/4}((\ddot{A}, v))(\log A)\times(|F(x_\lambda')|+|F(x_\lambda'')|)\end{aligned}\right\}\end{aligned}$$

where $x\bar{x}\equiv 1 \bmod A$ and x_λ', x_λ'' are such that

$$\begin{aligned}&x_\lambda'\equiv\lambda \bmod \Lambda, \qquad x_\lambda'<\xi'\leqq x_\lambda'+\Lambda,\\ &x_\lambda\equiv\lambda \bmod \Lambda, \qquad x_\lambda''-\Lambda\leqq\xi''<x_\lambda''\end{aligned}$$

and

$$\sigma_a(X)=\sum_{d;\ d\mid X} d^a.$$

Proof. We have the left-hand side to be proved is

$$=A^{-1}\left|\begin{array}{l}\displaystyle\sum_{l;\ 1\leqq l\leqq A}\left\{\sum_{x;\ \xi'<x\leqq\xi'',x\equiv\lambda \bmod A} F(x)e\left(-\frac{1}{A}(l-u)x\right)\right\}\\ \displaystyle\qquad\times\left\{\sum_{x;\ 1\leqq x\leqq A,(x,A)=1} e\left(\frac{1}{A}(lx+v\bar{x})\right)\right\}\end{array}\right|$$

$$\leqq A^{-1}\sum_l\Big|\sum_x F(x)e(\cdots)\Big|\cdot\Big|\sum_x{}' e(\cdots)\Big|.$$

As for the terms with $l\Lambda\equiv 0 \bmod A$, we proceed as follows: we have

$$\begin{aligned}&\sum_{l;\ l\Lambda\equiv 0 \bmod A,1\leqq l\leqq A}(\ddot{A}, l, v)^{1/4}\\ &\quad\leqq\sum_{l;\ 1\leqq l\leqq(A,\Lambda)}(\ddot{A}, v, A(A, \Lambda)^{-1}l)^{1/4}\\ &\quad\leqq\sum_{d;\ d\mid(\ddot{A},v)} d^{1/4}\cdot\sum_{l;\ 1\leqq l\leqq(A,\Lambda),\ A(A,\Lambda)^{-1}l\equiv 0 \bmod d} 1\\ &\quad\leqq\sum_d d^{1/4}\{(A, \Lambda)d^{-1}(d, (A, \Lambda)^{-1}A)+1\}\\ &\quad\leqq\sigma_{1/4}((\ddot{A}, v))+(A, \Lambda)\sum_{d;\ d\mid(\ddot{A},v)} d^{-3/4}(d, (A, \Lambda)^{-1}A).\end{aligned}$$

We put, taking into account that $\mu(\ddot{A})\neq 0$, as $d=d_1d_2$, where

$$d_1\mid(\ddot{A}, v),\quad (d_1, (A, \Lambda)^{-1}A)=1,\quad d_2\mid(\ddot{A}, v, (A, \Lambda)^{-1}A).$$

We have, then,

$$\begin{aligned}&\sum_{d;\ d\mid(\ddot{A},v)} d^{-3/4}(d, (A, \Lambda)^{-1}A)\\ &\quad=\Big\{\sum_{d_1;\ d_1\mid(\ddot{A},v),(d_1,(A,\Lambda)^{-1}A)=1} d_1^{-3/4}\Big\}\times\Big\{\sum_{d_2;\ d_2\mid(\ddot{A},v,(A,\Lambda)^{-1}A)} d_2^{1/4}\Big\}\\ &\quad=\sigma_{-3/4}((\ddot{A}, v)_{((A,\Lambda)^{-1}A)})\times\sigma_{1/4}((\ddot{A}, v, (A, \Lambda)^{-1}A)).\end{aligned}$$

Consulting 3.1.3, we have the contribution of such terms with $l\Lambda\equiv 0 \bmod A$ is

$$\begin{aligned}&\leqq\{\sigma_{-3/4}((\ddot{A}, v)_{((A,\Lambda)^{-1}A)})\times\sigma_{1/4}((\ddot{A}, v, (A, \Lambda)^{-1}A))\times(A, \Lambda)+\sigma_{1/4}((\ddot{A}, v))\}\\ &\quad\times\{A^3 5^{\nu(\ddot{A}_v)}a\}^{1/4}\times\{A^{-1}\Big|\sum_{x;\ \xi'<x\leqq\xi'',x\equiv\lambda \bmod A} F(x)e(A^{-1}ux)\Big|\},\end{aligned}$$

which is a slightly stronger estimation than the corresponding one in the conclusion of the proposition.

As for the terms with $l\Lambda\not\equiv 0 \bmod A$, we proceed as follows: we have, using a partial summation,

$$\Big|\sum_{x;\ \xi'<x\leqq\xi'',\ x\equiv\lambda \bmod \Lambda} F(x)e(-A^{-1}(l-u)x)\Big|$$
$$\ll\{|F(x'_\lambda)|+|F(x''_\lambda)|+\sum_{x;\ \xi'<x\leqq\xi'',\ x\equiv\lambda \bmod \Lambda}|F(x)e(A^{-1}ux)$$
$$-F(x+\Lambda)e(A^{-1}u(x+\Lambda))|\}\cdot\|A^{-1}\Lambda l\|^{-1}.$$

where $\|\xi\|=$ the distance of ξ and the nearest integer for a real ξ. We have

$$\sum_{l;\ 1\leqq l\leqq A,\ \Lambda l\not\equiv 0 \bmod A}(\ddot{A}, l, v)^{1/4}\|A^{-1}\Lambda l\|^{-1}$$
$$=\sum_{d;\ d|(\ddot{A},v)} d^{1/4}\sum_{l;\ 1\leqq l\leqq d^{-1}A,\ \Lambda l\not\equiv 0 \bmod d^{-1}A}\|A^{-1}d\Lambda l\|^{-1}$$
$$\ll\sum_{d;\ d|(\ddot{A},v)} d^{1/4}\times d^{-1}A\log A$$
$$\leqq\sigma_{-3/4}((\ddot{A}, v))A\log A.$$

Consluting 3.1.3, we have the contribution of such terms with $l\Lambda\not\equiv 0 \bmod A$, is

$$\leqq\sigma_{-3/4}((\ddot{A}, v))\times A\log A\times A^{-1}\{A^3 5^{\nu(A_v)}a\}^{1/4}$$
$$\times\{|F(x'_\lambda)|+|F(x''_\lambda)|+\sum_{x;\ \xi'<x\leqq\xi'',\ x\equiv\lambda \bmod \Lambda}|F(x)e(A^{-1}ux)$$
$$-F(x+\Lambda)e(A^{-1}u(x+\Lambda))|\}.$$

These give the conclusion of the proposition.

3.2. In this section, we prove a proposition, which will be used in 4.5.6, together with 2.3.8. We suppose similar situations as in 2.3.11.5, but here, the moduli of congruences are chosen to fit 2.3.8.

3.2.1. Let K, z, g' be large constant positive integers. Let a, b, U_i, V_i, p_i, q_i, p'_i, q'_i, t_i, t'_i, $(i=1, \cdots, 4)$, $\ddot{\varepsilon}(=\pm 1)$, $\ddot{A}$, $\ddot{A}_0^\Delta$, $\ddot{B}_0^\Delta$, $\ddot{B}_0$, $A_i^{(0)}$, $A_i'^{(0)}$, $B_i^{(0)}$, $B_i'^{(0)}$, T and $\mathscr{S}$ be as in 2.3.11.5. We suppose that $p^2\nmid\ddot{A}$ if $p\geqq K$, that $p^{g'+1}\nmid\ddot{A}$ if $p<K$ and that $(a, \ddot{A})=1$, Let a and b satisfy

$$1\leqq ab\leqq g.$$

Let t' be a positive integer such that

$$1\leqq t'<ab.$$

Let us suppose that

$$\ddot{A}_0^\Delta+t'\ddot{A}\equiv 0 \mod b$$

and

$$\ddot{B}_0^\Delta+t'\ddot{B}_0\equiv 0 \mod a.$$

Let a_0 be

$$a_0 = \prod_{p;\ \mathrm{prime}, K \leqq p < K^z} p,$$

and let us suppose that $ab \mid a_0$. Let W_0° be such that

$$W_0^\circ \equiv 0 \bmod \begin{pmatrix} 2^{g'+1} a_0 \cdot \Bigl(\prod_{p;\ \mathrm{prime},\ p \mid U_1 V_1 \cdots U_4 V_4,\ p<K} p \Bigr)^{g'} \\ \times \mathrm{L.C.M.\ of\ } \{(U_{i_1}, U_{i_2});\ i_1, i_2 = 1, \cdots, 4,\ i_1 \neq i_2\} \\ \times \Delta^1(U_1 V_1 \cdots U_4 V_4;\ T) \end{pmatrix}.$$

We put W_0 as

$$W_0 = W_0^\circ [U_1 V_1, \cdots, U_4 V_4].$$

Let $\ddot{A}^\Delta$ vary so that

$$\ddot{A}^\Delta \equiv \ddot{A}_0^\Delta \bmod W_0,$$
$$(\ddot{A}^\Delta, \ddot{A}) = 1$$

and

$$\ddot{A}^\Delta \notin \mathscr{S},$$

where the set $\mathscr{S}$ is supposed to include the $\mathscr{S}$ in 2.3.9. We have, by 2.3.11.5, integers $\ddot{B}$, $\ddot{B}^\Delta$, A_i, A_i', B_i, B_i', such that

$$\ddot{A}\ddot{B}^\Delta - \ddot{B}\ddot{A}^\Delta = \ddot{\varepsilon},$$
$$\ddot{B} \equiv \ddot{B}_0 \bmod 2^{g'+u_0+1} a_0,$$
$$\ddot{A} > \ddot{A}^\Delta > g^{-1} \ddot{A},$$

and

$$\begin{pmatrix} U_i a & 0 \\ 0 & V_i b \end{pmatrix} \begin{pmatrix} B_i & B_i' \\ A_i & A_i' \end{pmatrix} = \begin{pmatrix} \ddot{B} & \ddot{B}^\Delta \\ \ddot{A} & \ddot{A}^\Delta \end{pmatrix} \begin{pmatrix} t_i(p_i + q_i) & t_i' p_i' \\ -t_i q_i & t_i' q_i' \end{pmatrix}.$$

3.2.2. Let E_{100} and g be large constant positive integers. Let P be a positive real number which is supposed sufficiently large in relation with E_{100} and g. Let $\eta_0, \eta_1, \cdots, \eta_4$ be ± 1. Let α^* and Ω_i $(i = 1, \cdots, 4)$ be real numbers such that

$$g > \alpha^* > 1,$$
$$|\Omega_i| \leqq (g^3 P^2)^{-1}.$$

We put as

$$\lambda_i = \eta_i(V_i U_i^{-1} + \Omega_i)$$

and suppose that

$$\tfrac{1}{2}E_{100} \leqq |\lambda_i| \leqq 2E_{100},$$
$$1 \leqq U_i (\ll P)$$

and that

$$g^{-1}U_i \leqq t_i q_i \leqq g U_i \quad (i=1, \cdots, 4).$$

Later U_i's will be more restricted. Let t be an integer such that $0 \leqq t \leqq g$. We choose $\ddot{B}$ in 3.2.1, if possible, so that it satisfies

$$t\ddot{A} < \ddot{B} < (t+1)\ddot{A}.$$

Therefore $\ddot{B}$ and $\ddot{B}^{\Delta}$ are at most one for $\ddot{A}^{\Delta}$ (and $\ddot{A}$), if they exist in the stated interval. We put as

$$A = a\ddot{A}, \qquad B = b\ddot{B},$$
$$A' = b^{-1}(\ddot{A}^{\Delta} + t\ddot{A}), \qquad B' = a^{-1}(\ddot{B}^{\Delta} + t'\ddot{B}),$$

which are positive integers and satisfy

$$AB' - BA' = \ddot{\varepsilon}.$$

We suppose that

$$gP \geqq A > A' \geqq g^{-1}P,$$
$$gP \geqq B > B' \geqq g^{-1}P,$$
$$gP \geqq X \geqq g^{-1}P$$

for $X = \ddot{A}$, $\ddot{A}_0^{\Delta}$, $\ddot{B}_0$, $\ddot{B}_0^{\Delta}$, $\ddot{A}^{\Delta}$, $\ddot{B}$, $\ddot{B}^{\Delta}$, $A_i^{(0)}$, $A_i'^{(0)}$, $B_i^{(0)}$, $B_i'^{(0)}$, B_i and B_i', and that

$$g\xi_i'' > A_i > A_i' > g^{-1}\xi_i''.$$

We put

$$F_i = \Omega_i B A^{-1}$$

and

$$M_i = \ddot{\varepsilon}\{|\lambda_i| A^{-1}(A\alpha^* + A')^{-1} + t_i q_i (U_i A A_i)^{-1}\},$$

for $i=1, \cdots, 4$. We have, then,

$$(g^2P)^{-2} < |M_i| < (g^{-2}P)^{-2}.$$

Let μ be one of 0 and ± 1. Let ν_i $(i=1, \cdots, 4)$ vary in the set of integers such that their absolute values are $\leqq g$. Let h_i and l_i $(i=1, \cdots, 4)$ vary in the set of non-negative integers such that their values are $0 \leqq l_i \ll h_i \ll g$. Let ξ_i $(i=1, \cdots, 4)$ be one of the positive integers ξ'_i and ξ''_i such that $g^{-1}P<\xi'_i, \xi''_i \leqq gP$.

3.2.3. Now we prepare pairs of functions of $\ddot{A}^{\Delta}$; Let I' and I'' be such that $I' \cup I'' = \{1, 2, 3, 4\}$ and $I' \cap I'' = \emptyset$, one of which can be empty. We put

$$f(B)=(AB^{-1})^2 \times e(\tfrac{1}{2}\eta_0 \sum_i{}' \eta_i \xi_i^2 F_i)$$

and

$$g(A')=(A\alpha^* + A')^{-2} \times \prod_i{}' \begin{pmatrix} A_i^{1/2+l_i-2h_i} \cdot |M_i|^{l_i-h_i} \cdot \xi_i^{l_i} \\ \times e(\tfrac{1}{8}\varepsilon \nu_i^2 A'_i A_i^{-1} + \tfrac{1}{2}\eta_0\eta_i\xi_i^2 M_i - \eta_0\eta_i\xi_i\nu_i(2A_i)^{-1}) \end{pmatrix}$$
$$\times \prod_i{}'' \tfrac{1}{2} A_i^{-1/2} |M_i|^{-1/2},$$

where $\sum'_i$ and $\prod'_i$ are taken over I', and $\prod''_i$ over I''. As these definitions show, in $g(A')$, $\ddot{A}^{\Delta}$ can be regarded to vary as a real variable lying in an interval, and similarly for $\ddot{B}$ in $f(B)$. We prepare constants c'_* for each choice of μ, $(\nu)=(\nu_1, \cdots, \nu_4)$, $(h)=(h_1, \cdots, h_4)$, $(l)=(l_1, \cdots, l_4)$, I' and I''. We put

$$\tilde{S}(\ddot{A}^{\Delta}) = \sum_{\mu,(\nu),(h),(l),(\xi)} \sum_{I',I''} c'_* f(B) g(A') e(A^{-1}\mu B),$$

where μ, (ν), (h), (l), (ξ) vary as were described in 3.2.2. This is, in fact, the main term without the factors of Jacobi's symbol of 4.5.2, where the product on the right-hand side is expanded. What we must consider is the sum

$$\sum_{\ddot{A}^{\Delta}} \tilde{S}(\ddot{A}^{\Delta}),$$

in which $\ddot{A}^{\Delta}$ varies as was described in 3.2.1 and 3.2.2, and others are supposed to be fixed.

3.2.4. Proposition. *Under the situations from* 3.2.1 *to* 3.2.3, *we have, if* $W_0=o(P)$,

$$\sum_{\ddot{A}^{\Delta}} \tilde{S}(\ddot{A}^{\Delta}) = \sum_{\mu,(\nu),(h),(l),(\xi)} \sum_{I',I''} c'_* \ddot{A}^{-1} \times \ddot{A}_{\tilde{W}_0}^{-1} \phi(\ddot{A}_{\tilde{W}_0}^{\Delta}) \times (2^{g'+u_0+1} \cdot a_0)^{-1}$$
$$\cdot (2^{g'+u_0+1} \cdot a_0, \ddot{A}, W_0) \times W_0^{-1} \int_{\ddot{A}^{\Delta}} g(A') d\ddot{A}^{\Delta} \times \sum_{\ddot{B}} f(B) e(A^{-1}\mu B)$$
$$+ O(\sum_{\ddot{A}^{\Delta}}{}'' |\tilde{S}(\ddot{A}^{\Delta})|) + O(G\{\ddot{A}, W_0\} + 2^{\nu(\ddot{A})}\})$$

$$+O(GP^{3/4}(\ddot{A}, W_0)\times 5^{(1/4)\nu(\ddot{A})}\sigma_{-3/4}(\ddot{A}_{W_0})\ddot{A}_{W_0}\phi(A_{W_0})^{-1}(\log P)^2).$$

Here G is a sufficiently large positive constant depending on E_{100}, K, z, g *and* c_**'s. The notation* $\ddot{A}_{\tilde{W}_0}$ *is that in* 3.1.3 *with* $\tilde{W}_0=$L.C.M. *of* $2^{g+u_0+1}a_0$ *and* $(\ddot{A}, W_0)$. *The sum over* $\ddot{A}^\Delta$ *on the left-hand side is taken over as was described in* 3.2.1 *and* 3.2.2. *The integral over* $\ddot{A}^\Delta$ *on the right-hand side is taken over the corresponding interval which is obtained from* 3.2.1 *and* 3.2.2. *This is independent of* $\ddot{A}_0^\Delta$, $\ddot{B}_0^\Delta$ *and* $\ddot{B}_0$. *Its precise choice will be seen in* 3.2.5.2, 3.2.5.3 (i) *and* 3.2.5.7. (*See also* 4.5.3.1.) *The sum over* $\ddot{B}$ *is taken over*

$$t\ddot{A}<\ddot{B}<(t+1)\ddot{A}.$$

Also $\sum''_{\ddot{A}^\Delta}$ *is taken over*

$$\ddot{A}^\Delta\in\mathscr{S},\quad (\ddot{A}^\Delta, \ddot{A})=1 \quad \text{and} \quad \ddot{A}^\Delta\equiv\ddot{A}_0^\Delta \bmod W_0.$$

Remark. In application of this proposition in 4.5, we must take $\tilde{W}_0=o(P^{1/4})$.

3.2.5. The proof of 3.2.4 will end at 3.2.5.9. In the followings, G is a suitably large positive constant which may not be the same one as it appears. We will concern ourselves mainly in $f\cdot g\cdot e(\cdots)$ with $I'=\{1,\cdots, 4\}$ and $I''=\emptyset$, as the calculations for other I' and I'''s go similarly and as it is sufficient to keep in mind to take the summation over μ, (ν), $\cdots$, (ξ), I' and I'' before-hand. So the error terms, corresponding to $O(\sum''_{\ddot{A}^\Delta}|\tilde{S}(\ddot{A}^\Delta)|)$, will be suggested by $O(\sum^{('')}_{\ddot{A}^\Delta}|\tilde{S}(\ddot{A}^\Delta)|)$.

3.2.5.1. Lemma. *We have*

$$|g(A')|\leqq G.$$

Proof. It is easy.

3.2.5.2. The conditions on $\ddot{A}^\Delta$, that

$$\ddot{A}>\ddot{A}^\Delta>g^{-1}\ddot{A},$$
$$gP\geqq A>A'\geqq g^{-1}P,$$

and

$$g\xi''_i>A_i>A'_i>g^{-1}\xi''_i,$$

imply that $\ddot{A}^\Delta$ may be considered to vary so that

$$\ddot{A}^\Delta\equiv\ddot{A}_0^\Delta \bmod W_0,$$
$$(\ddot{A}^\Delta, \ddot{A})=1,$$
$$\ddot{A}^\Delta\notin\mathscr{S}$$

and that $\ddot{A}^{\Delta}$ lies in a certain interval which is determined by a, b, U_i, V_i, p_i, q_i, p_i', q_i', t_i, t_i', $\ddot{\varepsilon}$, $\ddot{A}$, t, t', α^*, g, P and ξ_i'', and which is independent of $\ddot{A}_0^{\Delta}$, $\ddot{B}_0$, $\ddot{B}_0^{\Delta}$. We denote this interval as $J((\ddot{A}))$, temporarily. We have

$$\sum_{\ddot{A}^{\Delta};\text{ as above}} f(B)g(A')e(A^{-1}\mu B)$$
$$=\sum_{l;\,-(1/2)\ddot{A}<l\leqq(1/2)\ddot{A}} \ddot{A}^{-1}\times\{\sum_{\substack{\ddot{A}^{\Delta};\,\ddot{A}\\ \in J((\ddot{A})),\ \ddot{A}^{\Delta}\equiv\ddot{A}_0^{\Delta}\bmod W_0}} g(A')e(\ddot{A}^{-1}\ddot{A}^{\Delta}l)\}$$
$$\times\{\sum_{\ddot{B};(\ddot{B},\ddot{A})=1,\ t\ddot{A}<\ddot{B}<(t+1)\ddot{A},\ \ddot{B}\equiv\ddot{B}_0\bmod 2^{g'+u_0+1}a_0} f(B)e(A^{-1}\mu B+\ddot{A}^{-1}l\ddot{\varepsilon}\bar{\ddot{B}})\}$$
$$+O(\sum_{\ddot{A}^{\Delta}}^{('')}|\tilde{S}(\ddot{A}^{\Delta})|),$$

for which we put as

$$=\sum_l \ddot{A}^{-1}\times\sum_{\ddot{A}^{\Delta}}^{(l)}\times\sum_{\ddot{B}}^{(l)}+O(\sum_{\ddot{A}^{\Delta}}^{('')}|\tilde{S}(\ddot{A}^{\Delta})|),\quad\text{say.}$$

Here $\ddot{B}\bar{\ddot{B}}\equiv 1 \bmod \ddot{A}$. Also on the right-hand side, the condition that $(\ddot{A}^{\Delta}, \ddot{A})=1$ is dropped off. This is because we have, on performing the summation over l first, that

$$\ddot{A}^{\Delta}\equiv(-\ddot{\varepsilon})\bar{\ddot{B}} \bmod \ddot{A},$$

which means that $(\ddot{A}^{\Delta}, \ddot{A})=1$. The term $O(\sum_{\ddot{A}^{\Delta}}^{('')}|\tilde{S}(\ddot{A}^{\Delta})|)$ arises from the possibilities that $\ddot{B}^{\Delta}$, with $\bar{\ddot{B}}\equiv-\ddot{\varepsilon}\ddot{A}^{\Delta} \bmod \ddot{A}$, may give that $\ddot{A}^{\Delta}\in\mathscr{S}$.

3.2.5.3. Let us first treat the sum $\sum_{\ddot{A}^{\Delta}}^{(l)}$.

(i) As a rational function of $\ddot{A}^{\Delta}$, the numerator and the denominator of

$$\{A_1^{1/2+l_1-2h_1}\cdots\}\times|M_1|^{l_1-h_1}\cdots\times(A\alpha^*+A')^{-2}$$

are of degree $O(|h_1|+\cdots+|h_4|+1)$. Therefore, by dissecting $J((\ddot{A}))$ into $O((|h_1|+\cdots+|h_4|+1)^3)$ ($\ll g^3$) subintervals, we may suppose that this rational function is monotone in each subinterval. On the other hand, the argument part

$$\tilde{g}_l(\ddot{A}^{\Delta})=\tfrac{1}{8}\ddot{\varepsilon}\sum_i \nu_i^2A_i'A_i^{-1}+\tfrac{1}{2}\ddot{\varepsilon}\eta_0\sum_i \eta_i\xi_i^2(|\lambda_i|A^{-1}(A\alpha^*+A')^{-1}+t_iq_i(U_iAA_i)^{-1})$$
$$-\eta_0\sum_i \eta_i\xi_i\nu_i(2A_i)^{-1}+l\ddot{A}^{\Delta}\ddot{A}^{-1}$$

of $g(A')e(\ddot{A}^{-1}l\ddot{A}^{\Delta})$ satisfies that

$$\frac{d}{d\ddot{A}^{\Delta}}\tilde{g}_l(\ddot{A}^{\Delta})=l\ddot{A}^{-1}+O(GP^{-1}),$$

therefore that

$$\left|\frac{d}{d\ddot{A}^{\Delta}}\tilde{g}_l(\ddot{A}^{\Delta})\right|<\frac{3}{4}.$$

As a rational function of $\ddot{A}^{\Delta}$, $\dfrac{d}{d\ddot{A}^{\Delta}}\tilde{g}_l(\ddot{A}^{\Delta})$ may be regarded to be monotone in each subinterval stated above, by adding suitable $O(G)$ points of dissection. We put, temporarily, as

$$\ddot{A}^{\Delta}=\ddot{A}_0^{\Delta}+yW_0 \in J((\ddot{A})),$$

where y is considered as a real variable. We have, then,

$$\frac{d}{dy}\tilde{g}_l(\ddot{A}^{\Delta})=W_0(\ddot{A}^{-1}l+O(GP^{-1})).$$

We have at most one integer m_l such that

$$|m_l-W_0((A^{-1}l)+O(GP^{-1}))|<\tfrac{3}{4},$$

as $W_0=o(P)$. We apply 2.2.3, (in each subinterval and, then, by adding the results), to obtain

$$\sum_{\ddot{A}\Delta}^{(l)}=\int_{\ddot{A}\Delta\in J((\ddot{A}))}|g(A')|\cdot e(\tilde{g}_l(\ddot{A}^{\Delta})-m_l y)dy$$
$$+O\left(\sup|g(A')|+\sup G\left|\frac{d}{dy}g(A')\right|\right).$$

The $O(\cdots)$-terms contribute $\ll G$.

(ii) Suppose that l is such that

$$\|\ddot{A}^{-1}W_0 l\|\geqq G^2W_0P^{-1},$$

where $\|\xi\|$ is the distance of ξ from the nearest integer. Then 2.2.0 tells us that

$$\left|\int_{\ddot{A}\Delta\in J((\ddot{A}))}|g(A')|e(\tilde{g}_l(\ddot{A}^{\Delta})-m_l y)\,dy\right|$$
$$\ll\sup|g(A')|\times\|\ddot{A}^{-1}W_0l\|^{-1}\ll G\cdot\|\ddot{A}^{-1}W_0l\|^{-1}.$$

Thus we obtain

$$\Big|\sum_{\ddot{A}\Delta}^{(l)}\Big|\ll G\|\ddot{A}^{-1}W_0l\|^{-1},$$

if $\|\ddot{A}^{-1}W_0l\|\geqq G^2W_0P^{-1}$.

(iii) Suppose that l is such that

$$\|\ddot{A}^{-1}W_0l\|\leqq G^2W_0P^{-1}.$$

We take a trivial estimate to obtain

$$|\sum_{\ddot{A}\Delta}^{(l)}| \ll G\ddot{A}W_0^{-1}.$$

The number of such l, that

$$-\tfrac{1}{2}\ddot{A} < l \leqq \tfrac{1}{2}\ddot{A}$$

and

$$\|\ddot{A}^{-1}W_0 l\| \leqq G^2 W_0 P^{-1},$$

is

$$\ll G^2 W_0,$$

because

$$|l - W_0^{-1}\ddot{A}m_l| = \|\ddot{A}^{-1}W_0 l\| \times W_0^{-1}\ddot{A} \ll G^2$$

with $|m_l| \ll W_0$.

3.2.5.4. We treat the sum $\sum_{\ddot{B}}^{(l)}$.

(i) We apply 3.1.4 to this sum, substituting

$$\left.\begin{array}{l} \ddot{A} = \prod_{p;\, p<K,\, p^e \| A} p^e \times \prod_{p;\, p \geqq K,\, p | \ddot{A}} p, \\ \ddot{B},\ 0,\ \ddot{\varepsilon}l, \\ \ddot{B} \equiv \ddot{B}_0 \bmod 2^{g'+u_0+1}a_0 \\ t\ddot{A},\ (t+1)\ddot{A}, \\ f(B)e(A^{-1}\mu B), \end{array}\right\} \quad \text{for} \quad \left\{\begin{array}{l} A = a \times \ddot{A}, \\ x,\ u,\ v, \\ x \equiv a \bmod \Lambda, \\ \xi',\ \xi'', \\ F(x), \end{array}\right.$$

respectively. We have that

$$|\sum_{\ddot{B}}^{(l)}| \ll \{A^3 5^{\nu(\ddot{A})}\}^{1/4}$$
$$\times \left\{\begin{array}{l} \ddot{A}^{-1}G\sigma_{1/4}((\ddot{A}, l)) \times |\sum_{\ddot{B}} f(B)\, e(A^{-1}\mu B)| \\ \quad + \sigma_{-3/4}((\ddot{A}, l)) \log \ddot{A} \times \sum_{\ddot{B}} |f(B)e(A^{-1}\mu B) - f(B^\circ)e(A^{-1}\mu B^\circ)| \\ \quad + \sigma_{-3/4}((\ddot{A}, l))(\log \ddot{A}) \times g^2 \end{array}\right\},$$

where B° is $b(\ddot{B} + 2^{g'+u_0+1}a_0)$. We have trivially

$$|\sum_{\ddot{B}} f(B)e(A^{-1}\mu B)| \ll \sum_{\ddot{B}} g^2 \ll G\ddot{A}.$$

We have that

$$|f(B)e(A^{-1}\mu B)-f(B^{\circ})e(A^{-1}\mu B^{\circ})| \ll G\ddot{A}^{-1},$$

therefore that

$$\sum_{\ddot{B}} |f(B)e(A^{-1}\mu B)-f(B^{\circ})e(A^{-1}\mu B^{\circ})| \ll G.$$

We have, then,

$$|\sum_{\ddot{B}}{}^{(l)}| \ll G\{\ddot{A}^{3}5^{\nu}(\ddot{A})\}^{1/4}\sigma_{1/4}((\ddot{A}, l)) \log P.$$

(ii) A trivial estimate is

$$|\sum_{\ddot{B}}{}^{(l)}| \ll g^{2}\ddot{A}.$$

3.2.5.5. We have, from 3.2.5.3 (ii) and 3.2.5.4 (i), that

$$\sum_{l;\ -(1/2)\ddot{A}<l\leqq(1/2)\ddot{A},\ \|\ddot{A}^{-1}W_0 l\|\geqq G^2W_0P^{-1}} \ddot{A}^{-1}\times\sum_{\ddot{A}}{}^{(l)}\sum_{\ddot{B}}{}^{(l)}$$
$$\ll G\sum_{l;\ \text{as above}} \ddot{A}^{-1}\times\|\ddot{A}^{-1}W_0 l\|^{-1}\times\{A^{3}5^{\nu(\ddot{A})}\}^{1/4}\sigma_{1/4}((\ddot{A}, l)) \log P.$$

We have

$$\sum_{l;\ \text{as above}} \sigma_{1/4}((\ddot{A}, l))\,\|\ddot{A}^{-1}W_0 l\|^{-1}$$
$$\leqq \sum_{d;\ d|\ddot{A}} \sigma_{1/4}(d) \sum_{l;\ |l|\leqq(1/2)Ad^{-1},\ \|(\cdots)\|\geqq G^2W_0P^{-1}} \|(\ddot{A}d^{-1})^{-1}W_0 l\|^{-1}$$
$$\ll \sum_{d;\ d|\ddot{A}} \sigma_{1/4}(d)\times \sum_{l;\ (1/2)\ddot{A}d^{-1}\geqq l\geqq G^2W_0P^{-1}\ddot{A}d^{-1}} \ddot{A}d^{-1}l^{-1}$$
$$\times \sum_{l';\ 0<l'\leqq(1/2)\ddot{A}d^{-1},\ W_0l'\equiv l \bmod \ddot{A}d^{-1}} 1$$
$$\ll \sum_{d} \sigma_{1/4}(d)\ddot{A}d^{-1}(W_0\,\ddot{A}d^{-1}) \log \ddot{A}.$$

As $p^2 \nmid \ddot{A}$ if $p>K$, this is

$$\ll G\sum_{d;\ d|(W_0,\ddot{A})^{-1}\ddot{A}} \mu(d)^2\sigma_{1/4}(d)d^{-1}\times\sum_{d;\ d|(W_0,A)} \mu(d)^2\sigma_{1/4}(d)d^{-2}\times\ddot{A}(\ddot{A}, W_0) \log \ddot{A}$$
$$\ll G\sigma_{-3/4}(\ddot{A}_{W_0})\ddot{A}_{W_0}\phi(\ddot{A}_{W_0})^{-1}\times\ddot{A}(\ddot{A}, W_0) \log P.$$

We have, therefore, that

$$\sum_{l;\ -(1/2)\ddot{A}<l\leqq(1/2)\ddot{A},\ \|\ddot{A}^{-1}W_0 l\|\geqq G^2W_0P^{-1}} \ddot{A}^{-1}\sum_{\ddot{A}}{}^{(l)}\sum_{\ddot{B}}{}^{(l)}$$
$$\ll G\ddot{A}^{3/4}\cdot(\ddot{A}, W_0)5^{1/4(\ddot{A})}\sigma_{-3/4}(\ddot{A}_{W_0})\ddot{A}_{W_0}\phi(\ddot{A}_{W_0})^{-1}(\log P)^2.$$

3.2.5.6. We have, from 3.2.5.3 (iii) and 3.2.5.4 (i), that

$$\sum_{\substack{l;\,-(1/2)\ddot{A}<l\leqq(1/2)\ddot{A},\\ \|\ddot{A}^{-1}W_0 l\|\leqq G^2W_0P^{-1},(l,\ddot{A})\leqq 10G^2W_0}} \ddot{A}^{-1}\sum_{\ddot{A}}^{(l)}\sum_{\ddot{B}}^{(l)} \ll \sum_{l;\text{ as above}} \ddot{A}^{-1}G\ddot{A}W_0^{-1}\{\ddot{A}^3 5^{\nu(\ddot{A})}\}^{1/4}\sigma_{1/4}((\ddot{A},l))\log P.$$

We have

$$\begin{aligned}\sum_{l;\text{ as above}} \sigma_{1/4}((\ddot{A},l)) &\leqq \sum_{d;\,d|\ddot{A},d\leqq 10G^2W_0} \sigma_{1/4}(d)\cdot \sum_{\substack{l;\,|l|\leqq(2d)^{-1}\ddot{A},\\ \|(\ddot{A}d^{-1})^{-1}W_0l\|\leqq G^2W_0P^{-1}}} 1\\ &\ll \sum_{d;\,d|\ddot{A},d\leqq 10G^2W_0} \sigma_{1/4}(d)\cdot(W_0,\ddot{A}d^{-1})d^{-1}G^2W_0\\ &\ll G\sigma_{-3/4}(\ddot{A}_{W_0})\ddot{A}_{W_0}\phi(\ddot{A}_{W_0})^{-1}\cdot(\ddot{A},W_0)W_0.\end{aligned}$$

Substituting this, we have

$$\sum_{\substack{l;\,-(1/2)\ddot{A}<l\leqq(1/2)\ddot{A},\\ \|\ddot{A}^{-1}W_0 l\|\leqq G^2W_0P^{-1},(l,\ddot{A})\leqq 10G^2W_0}} \ddot{A}^{-1}\sum_{\ddot{A}}^{(l)}\sum_{\ddot{B}}^{(l)} \ll G\ddot{A}^{3/4}\cdot(\ddot{A},W_0)\times 5^{(1/5)\nu(\ddot{A})}\sigma_{-3/4}(\ddot{A}_{W_0})\ddot{A}_{W_0}\phi(\ddot{A}_{W_0})^{-1}\log P.$$

3.2.5.7. Suppose that l is such that

$$(l,\ddot{A})\geqq 10G^2W_0 \quad\text{and}\quad \|\ddot{A}^{-1}W_0l\|\leqq G^2W_0P^{-1}.$$

The irreducible denominator of $\ddot{A}^{-1}W_0l$ is, then,

$$\leqq(10G^2W_0)^{-1}\ddot{A}<\tfrac{1}{3}(G^2W_0)^{-1}P.$$

Therefore $A^{-1}W_0l$ is an integer. The m_l in 3.2.5.3 (i) is

$$m_l=\ddot{A}^{-1}W_0l,$$

and, there,

$$\ddot{A}^{-1}l\ddot{A}^{\Delta}-m_ly=\ddot{A}^{-1}l(\ddot{A}_0^{\Delta}+yW_0)-\ddot{A}^{-1}W_0ly=\ddot{A}^{-1}l\ddot{A}_0^{\Delta}.$$

We have, from 3.2.5.3 (i), that

$$\begin{aligned}\sum_{\ddot{A}\Delta}^{(l)} &= e(\ddot{A}^{-1}l\ddot{A}_0^{\Delta})\int_{\ddot{A}\Delta\in J((\ddot{A}))} g(A')dy+O(G)\\ &= e(\ddot{A}^{-1}l\ddot{A}_0^{\Delta})\cdot W_0^{-1}\int_{\ddot{A}\Delta\in J((A))} g(A')d\ddot{A}^{\Delta}+O(G).\end{aligned}$$

We have, therefore,

$$\sum_{\substack{l;\,-(1/2)\ddot{A}<l\leqq(1/2)\ddot{A},\\ \|\ddot{A}^{-1}W_0 l\|\leqq G^2W_0P^{-1},\,(l,\ddot{A})\geqq 10G^2W_0}} \ddot{A}^{-1}\sum_{\ddot{A}}^{(l)}\sum_{\ddot{B}}^{(l)}$$

$$=\sum_{l;\,-(1/2)\ddot{A}<l\leqq(1/2)\ddot{A},\,\ddot{A}^{-1}W_0 l\in Z} \ddot{A}^{-1}$$

$$\times\left\{e(\ddot{A}^{-1}l\ddot{A}_0^{\Delta})W_0^{-1}\int_{\ddot{A}^{\Delta}} g(A')d\ddot{A}^{\Delta}+O(G)\right\}$$

$$\times\Big\{\sum_{\substack{\ddot{B};\,(\ddot{B},\ddot{A})=1,\,t\ddot{A}<\ddot{B}<(t+1)\ddot{A},\\ \ddot{B}\equiv\ddot{B}_0 \bmod 2^{g'+u_0+1}a_0}} f(B)e(A^{-1}\mu B+\ddot{A}^{-1}l\ddot{\varepsilon}\ddot{B})\Big\}$$

$$=\sum_{l;\text{ as above}} \ddot{A}^{-1}\times\left\{W_0^{-1}\int_{\ddot{A}^{\Delta}} g(A')d\ddot{A}^{\Delta}\right\}$$

$$\times\Big\{\sum_{\ddot{B};\text{ as above}} f(B)e(A^{-1}\mu B+\ddot{A}^{-1}l(\ddot{A}_0^{\Delta}+\ddot{\varepsilon}\bar{\ddot{B}}))\Big\}$$

$$+O(G(\ddot{A},W_0)).$$

We have that

$$\sum_{l;\,-(1/2)\ddot{A}<l\leqq(1/2)\ddot{A},\,\ddot{A}^{-1}W_0 l\in Z} e(\ddot{A}^{-1}l(\ddot{A}_0^{\Delta}+\ddot{\varepsilon}\ddot{B}))$$

$$=\sum_{l;\,-(1/2)(\ddot{A},W_0)<l\leqq(1/2)(\ddot{A},W_0)} e((\ddot{A},W_0)^{-1}(\ddot{A}_0^{\Delta}+\ddot{\varepsilon}\bar{\ddot{B}})l)$$

$$=\begin{cases}(\ddot{A},W_0) & \text{if } \ddot{A}_0^{\Delta}+\ddot{\varepsilon}\bar{\ddot{B}}\equiv 0 \bmod (\ddot{A},W_0),\\ 0 & \text{otherwise.}\end{cases}$$

The condition that $\ddot{A}_0^{\Delta}+\ddot{\varepsilon}\bar{\ddot{B}}\equiv 0 \bmod (\ddot{A},W_0)$ is equivalent to that $\ddot{B}\equiv \ddot{B}_0 \bmod (\ddot{A},W_0)$. Thus we have

$$\sum_{l;\text{ as above}} \ddot{A}^{-1}\sum_{\ddot{A}}^{(l)}\sum_{\ddot{B}}^{(l)}$$

$$=\{\ddot{A}W_0\}^{-1}\cdot(\ddot{A},W_0)\cdot\int_{\ddot{A}^{\Delta}\in J((\ddot{A}))} g(A')d\ddot{A}^{\Delta}$$

$$\times\sum_{\ddot{B}} f(B)e(A^{-1}\mu B)+O(G(\ddot{A},W_0)),$$

where $\ddot{B}$ is taken over

$$t\ddot{A}<\ddot{B}<(t+1)\ddot{A},$$
$$(\ddot{B},\ddot{A})=1$$

and

$$\ddot{B}\equiv\ddot{B}_0 \bmod 2^{g'+u_0+1}a_0, \text{ and also mod } (\ddot{A},W_0).$$

3.2.5.8. We want to get rid of the last two conditions on $\ddot{B}$. We have, putting $\tilde{W}_0=$ L.C.M. of $2^{g'+u_0+1}a_0$ and $(\ddot{A},\tilde{W}_0)$,

$$\sum_{\substack{\ddot{B};\ t\ddot{A}<\ddot{B}<(t+1)\ddot{A},\ (\ddot{B},\ddot{A})=1,\\ \ddot{B}\equiv\ddot{B}_0 \bmod \tilde{W}_0}} f(B)e(A^{-1}\mu B)$$

$$=\sum_{d;\ d|\ddot{A}}\mu(d)\sum_{\substack{\ddot{B};\ t\ddot{A}<\ddot{B}<(t+1)\ddot{A},\\ \ddot{B}\equiv\ddot{B}_0 \bmod \tilde{W}_0,\ \ddot{B}\equiv 0 \bmod d}} f(B)e(A^{-1}\mu B)$$

Here d must satisfy that $(d, \tilde{W}_0)\,|\,\ddot{B}_0$. As $(\ddot{A}, \ddot{B}_0)=1$, this means that $(d, \tilde{W}_0)=1$. Therefore, putting $\bar{d}$ as

$$d\bar{d}\equiv 1 \bmod \tilde{W}_0,$$

$\ddot{B}$ satisfies

$$\ddot{B}\equiv d\bar{d}\ddot{B}_0 \bmod d\tilde{W}.$$

We have, then, the above sum is

$$=\sum_{d;\ d|\ddot{A},(d,\tilde{W}_0)=1}\mu(d)\sum_{\substack{\ddot{B};\ t\ddot{A}<\ddot{B}<(t+1)\ddot{A},\\ \ddot{B}\equiv(d\bar{d})\ddot{B}_0 \bmod d\tilde{W}_0}} f(B)e(A^{-1}\mu B).$$

Suppose that $d\tilde{W}_0\ll P$. We have, if $|\ddot{B}-\ddot{B}^\circ|\ll d\tilde{W}_0$, then

$$f(B)e(A^{-1}\mu B)=f(B^\circ)e(A^{-1}\mu B^\circ)+O(GP^{-1}d\tilde{W}_0),$$

where B° corresponds to $\ddot{B}^\circ$. Therefore we have that

$$f(B^\circ)e(A^{-1}\mu B^\circ)$$
$$=(d\tilde{W}_0)^{-1}\sum_{\ddot{B};\ -(1/2)d\tilde{W}_0<\ddot{B}-\ddot{B}_0<(1/2)d\tilde{W}_0}\{f(B)e(A^{-1}\mu B)+O(GP^{-1}d\tilde{W}_0)\}$$

then, that

$$\sum_{\ddot{B};t\ddot{A}<\ddot{B}<(t+1)\ddot{A},\ddot{B}\equiv(d\bar{d})\ddot{B}_0 \bmod d\tilde{W}_0} f(B)e(A^{-1}\mu B)$$
$$=O(G)+(d\tilde{W}_0)^{-1}\sum_{\ddot{B};t\ddot{A}<\ddot{B}<(t+1)\ddot{A}} f(B)e(A^{-1}\mu B).$$

This last estimate holds good also if $d\tilde{W}_0\gg P$.

We have, therefore, that

$$\sum_{\ddot{B};t\ddot{A}<\ddot{B}<(t+1)\ddot{A},(\ddot{B},\ddot{A})=1,\ddot{B}\equiv\ddot{B}_0 \bmod \tilde{W}_0} f(B)e(A^{-1}\mu B)$$
$$=\sum_{d;d|\ddot{A},(d,\tilde{W}_0)=1}\mu(d)\ \{(d\tilde{W}_0)^{-1}\sum_{\ddot{B},t\ddot{A}<\ddot{B}<(t+1)\ddot{A}} f(B)e(A^{-1}\mu B)+O(G)\}$$
$$=\tilde{W}_0^{-1}\cdot\ddot{A}_{\tilde{W}_0}^{-1}\phi(\ddot{A}_{\tilde{W}_0})\times\{\sum_{\ddot{B};t\ddot{A}<\ddot{B}<(t+1)\ddot{A}} f(B)e(A^{-1}\mu B)\}+O(G2^{\nu(\ddot{A})}).$$

We know that $(\ddot{A}, W_0)\tilde{W}_0^{-1}=(2^{g'+u_0+1}a_0)^{-1}(2^{g'+u_0+1}a_0, \ddot{A}, W_0)$.

3.2.5.9. Combining all, we have the estimates in 3.2.4. The importance of 3.2.4 lies in the fact that the "main" term in the right-hand side does not depend on the choice of $\ddot{A}_0^{\Delta}$, $\ddot{B}_0^{\Delta}$ and $\ddot{B}_0^{\Delta}$. What we needed for the proof was the existence of them for the given $\ddot{A}$, W_0, U_i, V_i, p_i, q_i, etc.

Chapter 4. Proof of the Theorem

We proceed to the proof of Theorem 0.2. The proof is, in principle, traditional, a combination of [5] (=Chapter 20 of [8], see also Chapter 11 of [25],) and [17]. Up to the end of 4.3, we cut off such values of the variable α of integration that can be treated metrically, using propositions in Chapter 1 and 2.2.14. Then, we seek for relations of convergents of α left and those of $\lambda_i\alpha$ $(i=1, \cdots, 4)$. They can be found in 4.4.14. We apply 3.2.4, and 2.3.9, with 2.2.12 and 2.3.11.5. Then we will be done with the proof. We will not need the assumptions (iv) and (v) of the Theorem until the end of 4.4.

4.1. Formulation and easier parts of the proof

4.1.1. Lemma. *For real ξ, we have*

$$\int_{-\infty}^{\infty} e(\xi\alpha)((\pi\alpha)^{-1}(\sin \pi\alpha))^2 d\alpha = \max\{0, 1-|\xi|\}.$$

Proof. It is easy (Lemma 50 in [8]).

4.1.2. Lemma. *Let $f_1, f_2, \cdots, f_n$ be real. We put*

$$F(\alpha) = \sum_{i=1,\cdots,n} e(\alpha f_i) \qquad (\alpha \in \boldsymbol{R}).$$

We have, then, for $U>4$,

$$\int_{\alpha;|\alpha|>U} |F(\alpha)|^2((\pi\alpha)^{-1}(\sin\pi\alpha))^2 d\alpha$$
$$\leqq 16U^{-1}\int_{-\infty}^{\infty} |F(\alpha)|^2\{(\pi\alpha)^{-1}(\sin\pi\alpha)\}^2 d\alpha.$$

Proof. This is Lemma 2 in [6]. In fact, we can proceed without this lemma.

4.1.3. Let $\eta_1, \cdots, \eta_4$ be ± 1, which are not in the same signature. Let c_0 be as in 2.2.14. Let c_i', c_i'' $(i=1, \cdots, 4)$ be positive small real numbers, satisfying $1+2c_0 > c_i'' c_i'^{-1} > 1+c_0$, which will be determined in

4.1.7. Let E_{100} be a large positive integer. Let h_0 be a positive constant, which will be determined in 4.3.2. Let L_0 be a positive constant, which will be determined in 4.4.5, 4.4.11, 4.4.19, 4.4.21 and 4.5.2. Let $\lambda_1, \cdots, \lambda_4$ ($\lambda_2 = \pm E_{100}$) and P be real numbers satisfying the conditions (i) ~ (v) of the Theorem. Let us put

$$\kappa = E_{100}^{-1/2}, \qquad \kappa_i' = c_i' |\lambda_i|^{-1/2} + O(P^{-1}),$$
$$\kappa_i'' = c_i'' |\lambda_i|^{-1/2} + O(P^{-1}),$$
$$\xi_i' = \kappa_i' P, \quad \text{and} \quad \xi_i'' = \kappa_i'' P \quad (i = 1, \cdots, 4),$$

so that ξ_i' and ξ_i'' shall be positive integers.

In this chapter the positive constants g, g', G are suitably chosen, depending on $c_0, c_i', c_i'', h_0, L_0, E_{100}$. They may be different as they appear.

4.1.4. Definition. Let α be real, to be used as a variable of integration. We put

$$S_i(\alpha) = \sum_{x_i; x_i \in N, \xi_i' < x_i \leqq \xi_i''} e(\tfrac{1}{2}\lambda_i \alpha x_i^2),$$

for $i = 1, \cdots, 4$, which are one of $\theta(\beta^{-1}; \xi', \xi'')$ in 2.2.7. When we want to treat one of $S_i(\alpha)$, we write sometimes

$$S_0(\alpha) = \sum_{x; x \in N, \xi' < x \leqq \xi''} e(\tfrac{1}{2}\lambda \alpha x^2),$$

or

$$S(\beta) = \sum_{x; x \in N, \xi' < x \leqq \xi''} e(\tfrac{1}{2}\beta x^2).$$

In the followings, the suffix i will be used correspondingly to $S_i(\alpha)$.

This notation might cause slight confusions, as we have already used $S_i(BA^{-1})$ in Chapter 2 to mean Gaussian sum.

4.1.5. Formulation. We have the number of such quadruples $(x_1, \cdots, x_4)$, that

$$x_1, \cdots, x_4 \in N, \quad \xi_i < x_i \leqq \xi_i'' \quad (i = 1, \cdots, 4)$$

and

$$|\lambda_1 x_1^2 + \cdots + \lambda_4 x_4^2| < 2,$$

is

$$\geqq \int_{-\infty}^{\infty} ((\pi\alpha)^{-1} (\sin \pi\alpha))^2 S_1(\alpha) \cdots S_4(\alpha) d\alpha.$$

Proof. 4.1.1 and 4.1.4.

4.1.5.1. Our aim is to show; that the contribution of such α, that $|\alpha|\ll P^{-1}$, is $\gg\ll|\lambda_1\cdots\lambda_4|^{-1/2}P^2$, and that the contribution of other α is a "minor" one i.e. $o(|\lambda_1\cdots\lambda_4|^{-1/2}P^2)$, if P satisfies the conditions in the Theorem and P is sufficiently large. We estimate, for a certain portion of $|\alpha|\gg P^{-1}$,

$$\int(1+\alpha^2)^{-1}|S_1(\alpha)\cdots S_4(\alpha)|\,d\alpha$$

from above in 4.1, 4.2 and 4.3. For the rest of α ($|\alpha|\gg P^{-1}$), we must treat

$$\int((\pi\alpha)^{-1}(\sin\pi\alpha))^2S_1(\alpha)\cdots S_4(\alpha)d\alpha$$

itself, in 4.4 and 4.5. At each step of the proof, P_0 is supposed to be taken sufficiently large, the number of steps being bounded.

4.1.6. Lemma. *Such α, that $|\alpha|>(\log P)^4$, gives a minor contribution to* 4.1.5.

Proof. We have, from 4.1.2 and 2.2.15, that

$$\int_{\alpha;|\alpha|>(\log P)^4}((\pi\alpha)^{-1}(\sin\pi\alpha))^2|S_0(\alpha)|^4\,d\alpha$$
$$\ll(\log P)^{-4}(E_{100}^{-1/2}P)^2(\log P)^3\ll P^2(\log P)^{-1}.$$

We have, then,

$$\int_{\alpha;|\alpha|>(\log P)^4}((\pi\alpha)^{-1}(\sin\pi\alpha))^2|S_1(\alpha)\cdots S_4(\alpha)|\,d\alpha$$
$$\ll\sum_{i=1,\cdots,4}\int_{\alpha;|\alpha|>(\log P)^4}((\pi\alpha)^{-1}(\sin\pi\alpha))^2|S_i(\alpha)|^4\,d\alpha\ll P^2(\log P)^{-1}.$$

4.1.7. Lemma. *There exist small positive constants c_i', c_i'' ($c_i''>c_i'>0$, $1+2c_0>c_i''c_i'^{-1}>1+c_0$, $i=1,\cdots,4$) and c_{200}'' (>0) depending on η_i's, such that*

$$\int_{\alpha;|\alpha|<\min_{i=1,\cdots,4}\{(2c_i'P)^{-1}\}}((\pi\alpha)^{-1}(\sin\pi\alpha))^2S_1(\alpha)\cdots S_4(\alpha)d\alpha$$
$$=\tilde{\theta}|\lambda_1\cdots\lambda_4|^{-1/2}P^2,$$

where $\tilde{\theta}\in C$ and $(1\gg)\ \mathrm{Re}\,\tilde{\theta}>3c_{200}''$.

Proof. It goes along a classical line, (Lemma 53 of [8]). The assumption, that $\eta_1,\cdots,\eta_4$ are not in the same signature, is needed only here.

(i) If $|\alpha\lambda\xi''|\leqq\frac{1}{2}$, we have, through 2.2.2, that

$$S_0(\alpha)=\int_{x;\,\xi'<x<\xi''} e(\tfrac{1}{2}\lambda\alpha x^2)dx+O(1),$$

therefore especially

$$\ll 1+\min(\xi'', |\alpha\lambda|^{-1/2}).$$

Let U be such that $1\ll\max_i c_iU\leqq\frac{1}{2}$, temporarily. We have

$$\int_{\alpha|\alpha|<UP^{-1}}((\pi\alpha)^{-1}(\sin\pi\alpha))^2 S_1(\alpha)\cdots S_4(\alpha)d\alpha$$

$$=\int_{\alpha;\,|\alpha|<UP^{-1}} d\alpha((\pi\alpha)^{-1}(\sin\pi\alpha))^2\prod_{i=1,\cdots,4}\left(\int_{x_i;\,\xi_i'<x_i<\xi_i''} e(\tfrac{1}{2}\alpha\lambda_i x_i^2)dx_i+O(1)\right)$$

$$=\int_{\alpha;\,|\alpha|<UP^{-1}} d\alpha((\pi\alpha)^{-1}(\sin\pi\alpha))^2\prod_{i=1,\cdots,4}\left(\int_{x_i;\,\xi_i'<x_i<\xi_i''} e(\tfrac{1}{2}\alpha\lambda_i x_i^2)dx_i\right)$$

$$+O\left(\int_{\alpha;\,|\alpha|<UP^{-1}} d\alpha(1+\alpha^2)^{-1}(1+\min\{E_{100}^{-1/2}P,\,E_{100}^{-1/2}|\alpha|^{-1/2}\})^3\right).$$

We have, easily, that the $O(\cdots)$-term is

$$\ll E_{100}^{-3/2}P.$$

(ii) We have

$$\int_{\alpha;\,|\alpha|<UP^{-1}} d\alpha((\pi\alpha)^{-1}(\sin\pi\alpha))^2\prod_{i=1,\cdots,4}\left(\int_{x_i;\,\xi_i'<x_i<\xi_i''} e(\tfrac{1}{2}\alpha\lambda_i x_i^2)dx_i\right)$$

$$=|\lambda_1\cdots\lambda_4|^{-1/2}P^4\int_{\alpha;\,|\alpha|<UP^{-1}} d\alpha((\pi\alpha)^{-1}(\sin\pi\alpha))^2$$

$$\times\prod_i\left(\int_{u_i} e(\tfrac{1}{2}(\alpha P^2\eta_i)u_i^2)du_i\right),$$

where

$$\xi_i'|\lambda_i|^{1/2}P^{-1}<u_i<\xi_i''|\lambda_i|^{1/2}P^{-1}.$$

We have, easily,

$$\left|\int_{u_i} e(\tfrac{1}{2}(\alpha P^2\eta_i)u_i^2)du_i\right|\ll|\alpha P^2|^{-1/2},$$

(or as a corollary to 2.2.13). The integral to be evaluated is, then,

$$=|\lambda_1\cdots\lambda_4|^{-1/2}P^4\int_{\alpha;\,-\infty<\alpha<\infty} d\alpha((\pi\alpha)^{-1}(\sin\pi\alpha))^2\prod_i\left(\int_{u_i} e(\tfrac{1}{2}(\alpha P^2\eta_i)u_i^2)du_i\right)$$

$$+O(E_{100}^{-2}U^{-1}P)$$

$$=|\lambda_1\cdots\lambda_4|^{-1/2}P^4\int\cdots\int_{(u_1,\cdots,u_4)}\max\{1-\tfrac{1}{2}P^2|\eta_1u_1^2+\cdots+\eta_4u_4^2|,0\}$$
$$\times du_1\cdots du_4+O(P),$$

where u_i's vary in the same intervals as before.

(iii) As η_i's are not in the same signature, they are one of $(1, 1, 1, -1)$ and $(1, 1, -1, -1)$ in some re-arrangement. We have

$$\int\cdots\int_{(u_1,\cdots,u_4);c_i'+O(E_{100}^{1/2}P^{-1})<u_i<c_i''-O(E_{100}^{1/2}P^{-1})}$$
$$\max\{1-\tfrac{1}{2}P^2|\eta_1u_1^2+\cdots+\eta_4u_4^2|,0\}du_1\cdots du_4\geqq 4c_{200}P^{-2},$$

by choosing, for instance, as

$$c_1'=c_2'=c_3'=\sqrt{3}\,c_4'\qquad \text{if } (\eta_i)_i=(1,1,1,-1)$$

and

$$c_1'=\cdots=c_4'\qquad \text{if } (\eta_i)_i=(1,1,-1,-1).$$

These give the result.

4.1.8. Lemma. *Such α, that*

$$\tfrac{1}{2}\min((c_i'P)^{-1})<|\alpha|<P^{-2/3}(\log P)^{-1},$$

gives a minor contribution to 4.1.5.

Proof. Let U be $P\gg U\gg 1$. Let α be

$$UP^{-1}>|\alpha|\gg P^{-1}.$$

We have

$$|S_0(\alpha)|\ll|\alpha\lambda|^{-1/2}+\xi''|\alpha\lambda|^{1/2},$$

by dividing the range of the summation into $O(1+|\alpha\lambda|\xi'')$ subintervals of length $\gg\ \ll|\alpha\lambda|^{-1}$, as, in each subinterval, we have

$$|\textstyle\sum_x^{O(|\alpha\lambda|^{-1})}e(\tfrac{1}{2}\alpha\lambda x^2)|\ll|\alpha\lambda|^{-1/2},$$

by 2.2.2, 2.2.6 and 2.2.5(iv). We have, then,

$$\int_{\alpha;P^{-1}\ll|\alpha|<UP^{-1}}((\pi\alpha)^{-1}(\sin\pi\alpha))^2|S_1\cdots S_4|d\alpha$$
$$\ll\int_\alpha d\alpha((\alpha E_{100})^{-1/2}+\xi''|\alpha E_{100}|^{-1/2})^4$$
$$\ll E_{100}^{-2}P+\xi''^4E_{100}^2(UP^{-1})^3,$$

which is

$$\ll P^2 (\log P)^{1/2},$$

if $U \ll P^{1/3} (\log P)^{-1}$.

4.1.9. We are left with such α, that

$$(\log P)^4 > |\alpha| > P^{-2/3} (\log P)^{-1}.$$

We may suppose that α and $\lambda_i \alpha$ are irrational, if needed.

(i) We put, for $j=1, \cdots, 4$,

$$\Psi_j \underset{\text{def}}{=} \begin{pmatrix} \alpha \in \boldsymbol{R}; (\log P)^4 > |\alpha| > P^{-2/3} (\log P)^{-1}, \\ |S_j(\alpha)| \geqq |S_i(\alpha)| \quad (i=1, \cdots, 4) \end{pmatrix}.$$

Up to the end of 4.3, j will be used in this meaning only. There, j will be supposed to be a fixed one, even if it is not stated explicitly.

(ii) Let the successive convergents of $|(\lambda_i \alpha)^{-1}|$, (1.2.3.1 (iv)), be

$$A_{k_i}^{(i)} / B_{k_i}^{(i)} \qquad k_i = 0, 1, 2, \cdots$$

We choose $k_i^{(0)}$ uniquely by

$$c_0 \xi_i'' > A_{k_i^{(0)}}^{(i)} \quad \text{and} \quad c_0 \xi_i'' \leqq A_{k_i^{(0)}+1}^{(i)}.$$

We will often write as

$$A_{k_i^{(0)}}^{(i)} = A_*^{(i)}, \qquad B_{k_i^{(0)}}^{(i)} = B_*^{(i)},$$
$$A_{k_i^{(0)}+1}^{(i)} = A^{(i)*}, \qquad B_{k_i^{(0)}+1}^{(i)} = B^{(i)*}.$$

It is clear that we have $A_*^{(i)} B_*^{(i)} \neq 0$.

4.2. Intermediate domain I

4.2.1. Lemma. *Such* α, *that*

$$|S_{i_1}(\alpha)| \geqq (\log P)^4 |S_{i_2}(\alpha)|$$

for some i_1 *and* i_2 *with* $i_1 \neq i_2$, *gives a minor contribution to* 4.1.5.

Proof. We have, for such α, that

$$|S_1(\alpha) \cdots S_4(\alpha)| \leqq (\log P)^{-4} |S_j(\alpha)|^4.$$

Then, 2.2.15 applied to $|S_j(\alpha)|^4$ gives the result.

4.2.2. We may suppose, therefore, that $\alpha \in \Psi_j$ satisfies that

$$(\log P)^{-4}|S_j(\alpha)| \leqq |S_i(\alpha)| \leqq |S_j(\alpha)|$$

for all $i=1, \cdots, 4$.

4.2.3. Lemma. *Let c_{300} be a (large) positive numerical constants, which will be chosen depending on* 4.2.3.1, 4.2.4, 4.2.5 *and* 4.2.11.4. *Then, such α, that*

$$\alpha \in \Psi_j, \qquad |S_j(\alpha)| \leqq c_{300}(\xi_j'')^{1/2},$$

and that

$$A_*^{(i)} \leqq c_{300}^{-10} E_{100}^{-1} \xi_i''$$

or

$$A^{(i)*} \geqq c_{300}^{10} E_{100} \xi_i''$$

for some i, gives a minor contribution to 4.1.5.

Proof. We have, for such α, that

$$|S_1(\alpha) \cdots S_4(\alpha)| \leqq c_{300}^4 \xi_i''^2 \ll c_{300}^4 E_{100}^{-1} P^2.$$

We have, for a fixed i,

$$\int_{\alpha; A^{(i)*} \geqq c_{300}^{10} E_{100} \xi_i''} (1+\alpha^2)^{-1} d\alpha \ll c_{300}^{-10} E_{100}^{-1},$$

and similarly for those α with $A_*^{(i)} \leqq c_{300}^{-10} E_{100}^{-1} \xi_i''$. The contribution is, then,

$$\ll c_{300}^{-6} |\lambda_1 \cdots \lambda_4|^{-1/2} P^2.$$

Choosing c_{300} suitably large, we have the result. The choice of c_{300} depends only on c_i''s and c_i'''s.

4.2.3.1. Lemma. *Let $|(\lambda_i \alpha)^{-1}|$ has successive convergents as*

$$|(\lambda_i \alpha)^{-1}| (\longrightarrow A^{(i)***} B^{(i)***-1} (\Longrightarrow A^{(i)**} B^{(i)**-1} (\Longrightarrow A^{(i)*} B^{(i)*-1}$$
$$(\Longrightarrow A_*^{(i)} B_*^{(i)-1},$$

with

$$A^{(i)***} > A^{(i)**} > A^{(i)*} > c_0 \xi_i'' \geqq A_*^{(i)} \qquad (>1)$$

and

$$c_{300}^{10}E_{100}\xi_i'' \geqq A^{(i)*}.$$

Then, such α, *that*

$$|S_j(\alpha)| \leqq c_{300}(\xi_j'')^{1/2}$$

and that

$$A^{(i)**} \geqq c_{300}^{20}E_{100}^{2}\xi_i'' \quad or \quad A^{(i)***} \geqq c_{300}^{30}E_{100}^{3}\xi_i''$$

for some i, gives a minor contribution to 4.1.5.

Proof. If $A^{(i)**} \geqq (\cdots)$, then, α gives a minor contribution, similarly as in the proof of 4.2.3. If $A^{(i)**} \leqq (\cdots)$ and $A^{(i)***} \geqq (\cdots)$, then, similar calculation gives the result again.

4.2.4. Such α, that

$$\alpha \in \Psi_j \quad \text{and} \quad |S_j(\alpha)| \leqq c_{300}(\xi_j'')^{1/2}$$

and that α does not fit to satisfy neither 4.2.3 nor 4.2.3.1, will be treated in 4.4 and 4.5.

4.2.5. We treat, up to the end of 4.3, such α, that $\alpha \in \Psi_j$,

$$|S_j(\alpha)| \geqq c_{300}(\xi_j'')^{1/2}$$

and that

$$(\log P)^{-4}|S_j(\alpha)| \leqq |S_i(\alpha)| \leqq |S_j(\alpha)|$$

for all $i=1, \cdots, 4$. Choosing c_{300} suitably large depending on c_i' and c_i'', we do not have the first alternative in 2.2.14, for $S_j(\alpha)$. Therefore, we have

$$\begin{aligned} |S_j(\alpha)| &\gg \ll (A^{(j)*})^{1/2} \times \min\{\xi_j''(A_*^{(j)}A^{(j)*})^{-1/2}, (\xi_j''(A_*^{(j)}A^{(j)*})^{-1/2})^{-1}\} \\ &\gg c_{300}(\xi_j'')^{1/2}, \end{aligned}$$

for such α.

4.2.6. Subdivision of the domain of integration

Consulting 1.2.3, we can divide the half-line of positive real numbers, with a fixed j, into the disjoint union of subintervals of the form

$$(-1)^{k+1}[(bB+B')(bA+A')^{-1}, ((b+1)B+B')((b+1)A+A')^{-1}),$$

where

$$B \in \boldsymbol{Z},\ b \text{ and } A \in \boldsymbol{N},\quad (A, B)=1,\quad 1 \leqq A \leqq [c_0\xi_j''] < bA+A',$$
$$B/A\ (\Longrightarrow B'/A' \quad \text{with} \quad AB'-BA'=(-1)^k \text{ and } k=0, 1.$$

Similarly for the half-line of negative real numbers. Here the orientation of the interval [,) is supposed to be adjusted by $(-1)^{k+1}$ to fit in increasing order. As k is taken into consideration, there occurs no ambiguity concerning 1.2.3.1 (iii). Note that, p_k and q_k in 1.2.3 (iv) are B and A here, but that they are $A_k^{(i)}$ and $B_k^{(i)}$ of $|\lambda_i\alpha|^{-1}$, when they are used in connection with $S_i(\alpha)$. The length of the subinterval is

$$\{(bA+A')((b+1)A+A')\}^{-1} \qquad (\leqq (bA)^{-2}).$$

4.2.7. Lemma. *Let $V_1, V_2, U_0, H_1, H_2, H_3$ and H_4 be*

$$(\log P)^{1000} > V_1, V_2 \gg 1,$$
$$(PH_1^{-1})(\log P)^{-10000} > U_0 \gg 1,$$
$$(\log P)^{-5000}\big(\min_{i=1,\cdots,4} \xi_i''\big) \gg H_1 \gg 1$$
$$(\log P)^{50} > H_2, H_3, H_4 \gg 1.$$

Let j be fixed. Suppose that $A_*^{(j)}, B_*^{(j)}, A^{(j)*}$ of $\lambda_j\alpha$, for α in 4.2.5, *satisfy that*

$$(\log P)^4 > |\alpha| > U_0 \quad or \quad P^{-2/3}(\log P)^{-1} < |\alpha| < U_0^{-1},$$

that

$$\tau(A_*^{(j)}) \geqq V_1, \qquad \tau(B_*^{(j)}) \geqq V_2$$
$$(c_0\xi_j'' >) A_*^{(j)} \geqq H_1^{-1} c_0 \xi_j'' \qquad (\gg 1),$$

that

$$A_*^{(j)}A^{(j)*} \leqq H_2^{-1}(c_0\xi_j'')^2 \quad or \quad A_*^{(j)}A^{(j)*} \geqq H_3(c_0\xi_j'')^2$$

and that

$$|S_{i_0}(\alpha)| \leqq H_4^{-1}|S_j(\alpha)|$$

for some i_0. We have, then, the contribution to 4.1.5 *of such α, is*

$$\ll (\xi_j'')^2 (H_2^{-3} \wedge H_3^{-1}) H_4^{-1} (1+U_0)^{-1}(1+\log H_1)$$
$$\times (1+V_1(\log P)^{-1})^{-1}(1+V_2(\log P)^{-1})^{-1}.$$

Here $(\wedge)$ is to suggest that the corresponding condition on $A_*^{(j)}A^{(j)*}$ is chosen.

The proof of this lemma will end at 4.2.7.6.

4.2.7.1. Sublemma. *It is sufficient for* 4.2.7, *to estimtate*

$$H_4^{-1}\sum_{A,B,b}\{1+(BA^{-1})^2\}^{-1}\times\min\{(A^2b)^2\xi_j''^{-4},\ \{(A^2b)^2\xi_j''^{-4}\}^{-1}\},$$

where A, B, b *run such that*

$$\begin{aligned}&A,B,b\in N,\qquad bA>\tfrac{1}{3}c_0\xi_j'',\\&bA^2<H_2^{-1}(c_0\xi_j'')^2\quad\text{or}\quad bA^2>H_3(\tfrac{1}{3}c_0\xi_j'')^2,\\&c_0\xi_j''\geqq A\geqq H_1^{-1}\tfrac{1}{3}c_0\xi_j''\qquad(\gg 1),\\&\tau(A)\geqq V_1,\qquad \tau(B)\geqq V_2,\\&U_0\leqq BA^{-1}\leqq(\log P)^4\quad\text{or}\quad P^{-2/3}(\log P)^{-1}\leqq BA^{-1}\leqq U_0^{-1}.\end{aligned}$$

Proof. We apply the subdivision of 4.2.6 to $\lambda_j\alpha$ and use 4.2.5. The b, A, B are b, $A_*^{(j)}$, $B_*^{(j)}$, respectively. We have $2bA_*^{(j)}\geqq A^{(j)*}\geqq bA_*^{(j)}$. In each subinterval, we have

$$\begin{aligned}|S_1(\alpha)\cdots S_4(\alpha)|&\leqq H_4^{-1}|S_j(\alpha)|^4\\&\ll H_4^{-1}(bA_*^{(j)})^2\times(\min\{\xi_j''(bA_*^{(j)2})^{-1/2},\{\xi_j''(bA_*^{(j)2})^{-1/2}\}^{-1}\})^{-4}.\end{aligned}$$

The length of each subinterval is

$$\ll(bA_*^{(j)})^{-2}.$$

The rest, then, is easy.

4.2.7.2. Sublemma. *We have, for fixed* A *and* b, *that*

$$\sum_B\{1+(BA^{-1})^2\}^{-1}\ll A(1+V_2(\log P)^{-1})^{-1}(1+U_0)^{-1}.$$

Proof. We have

$$\sum_{X;\,\xi\leqq X\leqq\xi+\xi_1}\tau(X)\ll(\xi_1+1)\log(\xi+2)+\xi^{1/2}$$

for $1\ll\xi_1\ll\xi$, because

$$\sum_{X;\,1\leqq X\leqq\xi}\tau(X)=\xi\log\xi+(2\gamma-1)\xi+O(\xi^{1/2}).$$

We have, then, for an integer m with $0\leqq mA<P^2$, that the sum of such B, that

$$(m-1)A<B\leqq(m+1)A\quad\text{and}\quad\tau(B)\geqq V_2,$$

is

$$\ll A(1+V_2(\log P)^{-1})^{-1}+(mA)^{1/2}.$$

We have

$$\begin{aligned}&\sum_{B;\tau(B)\geqq V_2,(\log P)^4>BA^{-1}\geqq U_0}(1+(BA^{-1})^2)^{-1}\\&\ll\sum_{m;(\log P)^4\gg m>U_0}(1+m^2)^{-1}\times\{A(1+V_2(\log P)^{-1})^{-1}+(mA)^{1/2}\}\\&\ll A(1+V_2(\log P)^{-1})^{-1}(1+U_0)^{-1}+A^{1/2}(1+U_0)^{-1/2}\\&\ll A(1+V_2(\log P)^{-1})^{-1}(1+U_0)^{-1}.\end{aligned}$$

We have, also,

$$\begin{aligned}\sum_{B;\tau(B)\geqq V_2,U_0^{-1}>BA^{-1}>0}1&\ll(U_0^{-1}A+1)(1+V_2(\log P)^{-1})^{-1}+(U_0^{-1}A)^{1/2}\\&\ll A(1+V_2(\log P)^{-1})^{-1}(1+U_0)^{-1}.\end{aligned}$$

These give the result.

4.2.7.3. Sublemma. *We have*

$$\sum_{A;\xi_j''\gg A\gg H_1^{-1}\xi_j'',\tau(A)\geqq V_1}A^{-1}\ll(1+\log H_1)(1+V_1(\log P)^{-1})^{-1}.$$

Proof. If $V_1\ll\log P$, then, the estimate is trivial. If $V_1\gg\log P$ then, we have, by a partial summation, that

$$\begin{aligned}\sum_A A^{-1}\ll&\sum_{m;\xi_j''\gg m\gg H_1^{-1}\xi_j''}m^{-2}\Big(\sum_{A;m\geqq A\gg H_1^{-1}\xi_j,\tau(A)\geqq V_1}1\Big)\\&+\xi_j''^{-1}\Big(\sum_{A;\xi_j''\gg A\gg H_1^{-1}\xi_j'',\tau(A)\geqq V_1}1\Big)\\&\ll\sum_m m^{-2}\{m(1+V_1(\log P)^{-1})^{-1}+m^{1/2}\}\\&+\xi''^{-1}_j\{\xi_j''(1+V_1\log P)^{-1})^{-1}+\xi_j''^{1/2}\},\end{aligned}$$

which is

$$\ll(1+\log H_1)(1+V_1(\log P)^{-1})^{-1}+(H_1^{-1}\xi_j'')^{-1/2}.$$

This is

$$\ll(1+\log H_1)(1+V_1(\log P)^{-1})^{-1},$$

as

$$H_1\ll(\log P)^{-5000}\xi_j''.$$

4.2.7.4. Sublemma. *The contribution to* 4.2.7.1, *of such triple* (A, B, b) *that* $bA^2>H_3(2\xi_j')^2$, *is*

$$\ll (\xi_j'')^2 H_3^{-1} H_4^{-1}(1+\log H_1)(1+V_1(\log P)^{-1})^{-1}$$
$$\times(1+V_2(\log P)^{-1})^{-1}(1+U_0)^{-1}.$$

Proof. We choose $(A^2b)^{-2}\xi_j''^4$ in $\min\{\cdots,\cdots\}$ in 4.2.7.1. Applying 4.2.7.2 on the summation over B, then, after summing over $b \gg A^{-2}H_3\xi_j''^2$, we apply 4.2.7.3 on the summation over A.

4.2.7.5. Sublemma. *The contribution to* 4.2.7.1, *of such triple* (A, B, b) *that* $bA^2 < H_2^{-1}(2c_0\xi_j'')^2$, *is*

$$\ll (\xi_j'')^2 H_2^{-3} H_4^{-1}(1+\log H_1)(1+V_1(\log P)^{-1})^{-1}$$
$$\times(1+V_2(\log P)^{-1})^{-1}(1+U_0)^{-1}.$$

Proof. We choose $(A^2b)^2\xi_j''^{-4}$ in $\min\{\cdots,\cdots\}$ in 4.2.7.1. We proceed as in the proof of 4.2.7.4. Here we sum over $1 \leqq b \ll A^{-2}H_2^{-1}\xi_j''^2$, instead.

4.2.7.6. We have proved 4.2.7.

4.2.8. Let H be

$$1 \ll H \ll \min_i \{\xi_i''\}(\log P)^{-5000}.$$

Let j be fixed and we consider α's as in 4.2.5 and satisfying

$$(c_0\xi_j'') \geqq A_*^{(j)} \geqq H^{-1}(c_0\xi_j'').$$

We impose the following conditions "$(*)_j$" on α such that

$$(*)_j \qquad \begin{aligned} &(\log H)^3 \geqq |\alpha| \geqq (\log H)^{-3}, \\ &\tau(A_*^{(j)}) \leqq (\log P)^4, \qquad \tau(B_*^{(j)}) \leqq (\log P)^5, \\ &(\log H)^3(c_0\xi_j'')^2 \geqq A_*^{(j)}A^{(j)*} \geqq (\log H)^{-1}(c_0\xi_j'')^2, \end{aligned}$$

and that, for $i=1,\cdots,4$,

$$(\log H)^{-3}|S_j(\alpha)| \leqq |S_i(\alpha)| \qquad (\leqq |S_j(\alpha)|).$$

Here and in the followings, $\log H$ is considered to mean $\gg\ll 1$, if H is $\gg\ll 1$.

4.2.9. Lemma. *Let H and j be fixed as in* 4.2.8. *Then, such α as in* 4.2.5 *and satisfying*

$$c_0\xi_j'' \geqq A_*^{(j)} \geqq H^{-1}(c_0\xi_j''),$$

but that does not satisfy conditions in $(*)_j$ *in* 4.2.8, *gives a "minor" contribution*

$$\ll (\xi_j'')^2 (\log H)^{-2}$$

to 4.1.5.

Proof. Let us have

$$A_*^{(j)} A^{(j)*} \geqq (\log H)^3 (c_0 \xi_j'')^2$$

for instance. We substitute $(\log H)^3$ for H_3, and 1 for the rest, in 4.2.7. The other cases are similar.

4.2.10. Lemma. *Suppose that* α *satisfies the conditions in* 4.2.8. *We have, then,*

$$A_*^{(i)} \ll A_*^{(j)} (\log H)^{10}$$

for $i=1, \cdots, 4$.

Proof. We have, by 2.2.9, that

$$|S_i(\alpha)| \ll \xi_i'' (A_*^{(i)})^{-1/2}.$$

We have, from 4.2.5 and the last but one condition in 4.2.8, that

$$\begin{aligned} |S_j(\alpha)| &\gg (A^{(j)*})^{1/2} (\log H)^{-3/2} \\ &\gg ((\xi_j'')^2 (A_*^{(j)} \log H)^{-1})^{1/2} (\log H)^{-3/2} \\ &\gg \xi_j'' \{A_*^{(j)} (\log H)^4\}^{-1/2}. \end{aligned}$$

The last condition in 4.2.8 implies

$$\xi_i'' (A_*^{(i)})^{-1/2} \gg \xi_j'' \{A_*^{(j)} (\log H)^4\}^{-1/2} (\log H)^{-3}.$$

This implies the conclusion.

4.2.11. We have corresponding results about $\lambda_i \alpha$, as are shown in the followings.

4.2.11.1. Let H and H' be

$$1 \ll H \quad \text{and} \quad H' \ll \min_i \{\xi_i''\} (\log P)^{-5000}.$$

Let j be fixed and we consider α's as in 4.2.5 and satisfying that

$$H^{-1/3} \xi_j'' \geqq A_*^{(j)} \geqq H^{-1} (c_0 \xi_j'')$$

and

$$(\log H)^{-3}|S_j(\alpha)|\leqq|S_i(\alpha)| \qquad (\leqq|S_j(\alpha)|)$$

for $i=1,\cdots,4$. We impose the following conditions "$(*)_i$" on α such that

$$(*)_i \qquad \begin{aligned} &H'^{-1/3}(c_0\xi_i'')>A_*^{(i)}\geqq H'^{-1}(c_0\xi_i''),\\ &(\log H')^{-1}(\log H)^{-11}\leqq|\alpha|\leqq(\log H')(\log H)^{11},\\ &\tau(A_*^{(i)})\leqq(\log P)^{12},\\ &\tau(B_*^{(i)})\leqq(\log P)^{12}, \end{aligned}$$

and

$$(\log H')^{-1}(\log H)^{-12}(c_0\xi_i'')^2\leqq A_*^{(i)}A^{(i)*}\leqq(\log H')(\log H)^{12}(c_0\xi_i'')^2,$$

for $i=1,\cdots,4$. Here $\log H'$ is considered to mean $\gg\ll 1$, if H' is $\gg\ll 1$.

4.2.11.2. Sublemma. *Let H, j and i be fixed as in* 4.2.11.1. *Then, such α as in* 4.2.5 *and satisfying*

$$H^{1/3}\xi_j''\geqq A_*^{(j)}\geqq H^{-1}(c_0\xi_j'')$$

and

$$(\log H)^{-3}|S_j(\alpha)|\leqq|S_i(\alpha)| \qquad (\leqq|S_j(\alpha)|),$$

but that does not satisfy the conditions $(*)_i$ *in* 4.2.11.1 *for this i, gives a "minor" contribution* $\ll(\xi_j'')^2(\log H)^{-2}$ *to* 4.1.5.

Proof. We have

$$|S_1(\alpha)\cdots S_4(\alpha)|\leqq|S_i||S_j|^3\leqq(\log H)^9|S_i(\alpha)|^4.$$

We proceed, then, similarly for i and H' as in j and H of 4.2.7 and 4.2.8.

4.2.11.3. Sublemma. *Suppose that α satisfies the conditions in* 4.2.11.1 *for $i=1,\cdots,4$ and the conditions in* 4.2.8 *for j. We have, then,*

$$A_*^{(j)}(\log H)^7\geqq A_*^{(i)}\geqq A_*^{(j)}(\log H')^{-6}(\log H)^{-25},$$

if c_{300} in 4.2.3 *is chosen sufficently large.*

Proof. The left-hand side to be shown is the same as 4.2.10. Suppose that

$$|S_i(\alpha)|\geqq c_{300}(\xi_i'')^{1/2}.$$

Then, we have similarly as in 4.2.5 and as in the proof of 4.2.10, that

$$|S_i(\alpha)| \gg (\log H')^{-3} (\log H)^{-12} \xi_i''(A_*^{(i)})^{-1/2}.$$

Combining with

$$|S_i(\alpha)| \leqq |S_j(\alpha)| \ll \xi_j''(A_*^{(j)})^{-1/2} \ll \xi_i''(A_*^{(j)})^{-1/2},$$

we have the right-hand side of the conclusion in this case. Suppose that

$$|S_i(\alpha)| \leqq c_{300}(\xi_i'')^{1/2}.$$

Then,

$$|S_j(\alpha)| \leqq |S_i(\alpha)| (\log H)^3 \ll (\xi_i'')^{1/2} (\log H)^3 \ll (\xi_j'')^{1/2} (\log H)^3.$$

Therefore

$$A_*^{(j)} \gg \xi_j'' (\log H)^{-8},$$

as

$$|S_j(\alpha)| \gg \xi_j''(A_*^{(j)})^{-1/2} (\log H)^{-2}.$$

This is impossible if H is $\gg 1$ because

$$A_*^{(j)} \ll \xi_j'' H^{-1/3}.$$

If H is $\gg \ll 1$, then $A_*^{(j)}$ is $\gg \ll \xi_j''$, therefore $|S_j(\alpha)|$ is $\gg \ll (\xi_j'')^{1/2}$, by 2.2.9 and 4.2.5. We have $|S_i(\alpha)|$ is $\gg \ll (\xi_j'')^{1/2}$, which is $\gg \ll (\xi_i'')^{1/2}$. Therefore, $A_*^{(i)}$ is $\gg \ll \xi_i''$, by 2.2.9. Then, we have $A_*^{(i)} \gg \ll A_*^{(j)}$.

4.2.11.4. Sublemma. *We may suppose, in* 4.2.11.1, *that* $\log H'$ *is* $\gg \ll \log H$.

Proof. We have, from 4.2.11.1 and 4.2.11.3, that

$$H^{-1/3}\xi_j'' (\log H)^7 \gg H'^{-1}\xi_i''$$

and

$$H'^{-1/3}\xi_i'' \gg H^{-1}\xi_j'' (\log H')^{-6} (\log H)^{-25}.$$

These give the conclusion.

4.2.12. Let H be such that

$$1 \ll H \ll P (\log P)^{-5000}.$$

Let j be fixed and we consider α's as in 4.2.5 and satisfying that

$$(\log H)^3 \geqq |\alpha| \geqq (\log H)^{-3},$$
$$H^{-1/2}(c_0\xi_j'') \geqq A_*^{(j)} \geqq H^{-1}(c_0\xi_j''),$$

and that, for $i=1, \cdots, 4$,

$$(\log H)^{-3}|S_j(\alpha)| \leqq |S_i(\alpha)| \qquad (\leqq |S_j(\alpha)|).$$

We impose the following conditions "$((*))_i$" on α such that

$$((*))_i \qquad \begin{aligned} &A_*^{(j)}(\log H)^7 \geqq A_*^{(i)} \geqq A_*^{(j)}(\log H)^{-31}, \\ &\tau(A_*^{(i)}) \leqq (\log P)^{12}, \qquad \tau(B_*^{(i)}) \leqq (\log P)^{12}, \end{aligned}$$

and

$$(\log H)^{-13}(c_0\xi_i'')^2 \leqq A_*^{(i)}A^{(i)*} \leqq (\log H)^{13}(c_0\xi_i'')^2,$$

for $i=1, \cdots, 4$.

4.2.13. Lemma. *Such α as in* 4.2.5 *and satisfying*

$$(\log H)^3 \geqq |\alpha| \geqq (\log H)^{-3},$$
$$H^{-1/2}(c_0\xi_j'') \geqq A_*^{(j)} \geqq H^{-1}(c_0\xi_j''),$$

and, for $i=1, \cdots, 4$,

$$(\log H)^{-3}|S_j(\alpha)| \leqq |S_i(\alpha)| \qquad (\leqq |S_j(\alpha)|),$$

but that does not satisfy the condition $((*))_i$ in 4.2.12 *for at least one i, gives a "minor" contribution*

$$\ll (\log H)^{-2}(\xi_j'')^2,$$

to 4.1.5.

Proof. By 4.2.11 and 4.2.12.

4.3. Intermediate domain II

4.3.1. Hereafter we will make use of the choices of Q, etc. Here h_0 is the constant in the statement of the Theorem.

4.3.2. Lemma. *There exist positive (large) numerical constants h_0 and h_1, h_0 being independent of h_1, such that, for arbitrarily fixed H with*

$$P(\log P)^{-5000} \geqq H \geqq (\log P)^{h_1},$$

we have

$$\int_\alpha |S_1(\alpha)\cdots S_4(\alpha)|\times(1+\alpha^2)^{-1}d\alpha \ll P^2(\log P)^{-2},$$

where α varies under the conditions that

$$\alpha\in\Psi_j,\qquad (\log H)^{-3}\leqq|\alpha|\leqq(\log H)^3,$$
$$H^{-1}\xi_i''(\log P)^{11}\geqq A_*^{(i)}\geqq H^{-1}\xi_i''(\log P)^{-32},$$
$$(\log H)^{-14}\xi_i^2\leqq A_*^{(i)}A^{(i)*}\leqq(\log H)^{14}\xi_i''^2$$

and

$$\tau(A_*^{(i)})\leqq(\log P)^{12},\qquad \tau(B_*^{(i)})\leqq(\log P)^{12}$$

for every $i=1,\cdots,4$.

Proof. We have

$$|S_1(\alpha)\cdots S_4(\alpha)|\leqq|S_j(\alpha)|^4\ll(\xi_j''A_*^{(j)-1/2})^4$$
$$\ll(H^{1/2}(E_{100}^{-1/2}P)^{1/2}(\log P)^{16})^4\ll H^2E_{100}^{-1}P^2(\log P)^{64}.$$

We can apply Proposition 1.1.1, substituting

$$\begin{array}{ll} A_*^{(i)},\, B_*^{(i)} & A_i,\, B_i,\quad (i=1,2)\\ H(\log P)^{23} & H,\\ & \text{for}\\ H(\log P)^{45} & G,\\ \xi_j'' & P, \end{array}$$

as $\log H\gg\ll\log(H(\log P)^c)$ owing to the assumption that $H\gg(\log P)^{h_1}$. We obtain, for a certain h_0 that

$$\int_\alpha(1+\alpha^2)^{-1}d\alpha\leqq\int d\alpha\ll(\xi_j''^2(\log H)^{-14})^{-1}\times\#\{(A_*^{(1)},B_*^{(1)},A_*^{(2)},B_*^{(2)})\}$$
$$\ll(\xi_j''^2(\log H)^{-14})^{-1}\times\xi_j''^2H^{-2}(\log P)^{-80}\ll H^{-2}(\log H)^{14}(\log P)^{-80}.$$

We, then, obtain

$$\int_\alpha|S_1(\alpha)\cdots S_4(\alpha)|\cdot(1+\alpha^2)^{-1}d\alpha$$
$$\ll H^2E_{100}^{-1}P^2\cdot(\log P)^{64}\cdot H^{-2}(\log H)^{14}(\log P)^{-80}$$
$$\ll E_{100}^{-1}P^2(\log P)^{-2}\ll P^2(\log P)^{-2}.$$

4.3.3. Lemma. *Using h_0 and h_1 of 4.3.2, we have*

$$\sum_{j=1}^{4}\int_{\alpha}(1+\alpha^2)^{-1}|S_1(\alpha)\cdots S_4(\alpha)|\,d\alpha\ll E_{100}^{-1}P^2(\log\log P)^{-2},$$

where α varies as $\alpha\in\Psi_j$, $\xi_j''(\log P)^{-h_1}\geqq A_*^{(j)}\geqq(\log P)^{5000}$, $|S_j(\alpha)|\geqq c_{300}\xi_j''^{1/2}$.

Proof. Let us put H_l as

$$H_l=(\log P)^{h_1+l}\qquad(l=0,1,2,\cdots).$$

We have, with an l_0 $(=(\log P)(\log\log P)^{-1}+O(1))$, that

$$H_{l_0-1}\leqq P(\log P)^{-5000}\leqq H_{l_0}.$$

We estimate, for $l=0,1,2,\cdots,l_0$, as

$$\int_{\alpha;\alpha\in\Psi_j,\xi_j''H_l^{-1}\geqq A_*^{(j)}\geqq\xi_j''H_{l+1}^{-1},|S_j(\alpha)|\geqq c_{300}\xi_j''^{1/2}}\times(1+\alpha^2)^{-1}|S_1(\alpha)\cdots S_4(\alpha)|\,d\alpha\ll\xi_j''^2(\log H_l)^{-2},$$

using 4.2.13, 4.3.2 or 4.2.9 (and 4.2.1, if needed). We sum over $l=0,1,\cdots,$ to obtain

$$\sum_{l=0,1,\cdots}(\log H_l)^{-2}\ll\sum_{l}(l+h_1)^{-2}(\log\log P)^{-2}\ll(\log\log P)^{-2}.$$

4.3.4. Lemma. *There exist positive (large) constants h_0, h_2 and H_2' (depending on h_1), such that, for arbitrarily fixed H with*

$$(\log P)^{h_1}\geqq H\geqq H_2',$$

we have

$$\int_{\alpha}|S_1(\alpha)\cdots S_4(\alpha)|\cdot(1+\alpha^2)^{-1}d\alpha\ll\xi_j''^2(\log H)^{-2},$$

where α varies under the conditions that

$$\alpha\in\Psi_j,\qquad(\log H)^{-3}\leqq|\alpha|\leqq(\log H)^3,$$
$$H^{-1}\xi_i''(\log H)^{11}\geqq A_*^{(i)}\geqq H^{-1}\xi_i''(\log H)^{-32},$$
$$(\log H)^{-14}\xi_i''^2\leqq A_*^{(i)}A^{(i)*}\leqq(\log H)^{14}\xi_i''^2,$$

and

$$\tau(A_*^{(i)})\leqq(\log P)^{12},\qquad\tau(B_*^{(i)})\leqq(\log P)^{12}$$

for $i=1,\cdots,4$.

Proof. We have

$$\begin{aligned}|S_1(\alpha)\cdots S_4(\alpha)|&\leqq|S_j(\alpha)|^4\ll(\xi_j''A_*^{(j)-1/2})^4\\&\ll(H^{1/2}(E_{100}^{-1/2}P)^{1/2}(\log H)^{11})^4\\&\ll H^2(\log H)^{44}E_{100}^{-1}P^2.\end{aligned}$$

We apply Proposition 1.1.2, substituting

$$\begin{array}{lcl} A_*^{(i)},\ B_*^{(i)} & & A_i,\ B_i\quad (i=1,2),\\ H & \text{for} & H,\\ \xi_j''(\log H)^{12} & & P,\ \text{respectively.}\end{array}$$

We obtain, for certain h_2 and H_2', that

$$\begin{aligned}\int_\alpha(1+\alpha^2)^{-1}d\alpha&\ll(\xi_j''^2(\log H)^{-14})^{-1}\#\{(A_*^{(1)},B_*^{(1)},A_*^{(2)},B_*^{(2)})\}\\&\ll\xi_j''^{-2}(\log H)^{14}(\xi_j''(\log H)^{12})^2(\log H)^{-e_2},\end{aligned}$$

which is

$$\ll(\log H)^{-50}$$

for $e_2=90$.

4.3.5. Lemma. *Using h_0 and h_1 of* 4.3.2, *h_2 of* 4.3.4 *and a suitable H_2, we have*

$$\sum_{j=1}^{4}\int_{\alpha;\alpha\in\Psi_j,H_2^{-1}\xi_j''\geqq A_*^{(j)}\geqq(\log P)^{-h_1}\xi_j'',|S_j(\alpha)|\geqq c_{300}\xi_j''^{1/2}}\times(1+\alpha^2)^{-1}|S_1(\alpha)\cdots S_4(\alpha)|\,d\alpha\ll\delta E_{100}^{-2}P^2,$$

for an arbitrarily given small positive constant δ.

Proof. We put H_l inductively as

$$H_{l+1}=H_l(\log H_l)^2\qquad l=2,3,\cdots$$

beginning with $H_2=e_3(\delta^{-1}E_{100})\cdot H_2'(\log H_2')^{-11}$, where H_2' is that of 4.3.4 and $e_3(*)=\exp\exp\exp(*)$. We have, obviously

$$H_l\geqq(\log\log H_2)^l\qquad\text{for } l=2,3,\cdots.$$

We have

$$\int_{\alpha;\alpha\in\Psi_j,H_l^{-1}\xi_j''\geqq A_*^{(j)}\geqq H_{l+1}^{-1}\xi_j''}\times(1+\alpha^2)^{-1}|S_1(\alpha)\cdots S_4(\alpha)|\,d\alpha\ll\xi_j''^2(\log H_l)^{-2},$$

by 4.2.13, 4.3.4, or 4.2.9. We sum over l, to obtain

$$\sum_{l=2}^{\infty}(\log H_l)^{-2} \ll (\log\log\log H_2)^{-2}\sum_{l=2}^{\infty} l^{-2} \ll_{(H_2')} (\delta E_{100}^{-1})^2.$$

These give the result.

4.3.6. Lemma. *We have, with h_0 and H_2 of* 4.3.5, *that*

$$\sum_{j=1}^{4}\int_{\alpha;\alpha\in\Psi_j, H_2^{-1}\xi_j''\geqq A_*^{(j)}\geqq(\log P)^{5000}, |S_i(\alpha)|\geqq c_{300}\xi_j''^{1/2}} \times(1+\alpha^2)^{-1}|S_1(\alpha)\cdots S_4(\alpha)|\,d\alpha \ll \delta E_{100}^{-2}P^2,$$

for an arbitrarily given small positive constant δ. Here H_2 may depend on E_{100} and δ.

Proof. 4.3.3 and 4.3.5.

4.3.7. Lemma. *There exists a positive (large) constant H_3, for arbitrarily given small constant δ, such that*

$$\int_{\alpha}(1+\alpha^2)^{-1}|S_1(\alpha)\cdots S_4(\alpha)|\,d\alpha \ll \delta E_{100}^{-2}P^2,$$

where α varies under the conditions that

$$H_3 \geqq |\alpha| \geqq H_3^{-1},$$
$$c_0\xi_j'' \geqq A_*^{(j)} \geqq H_3^{-1}(c_0\xi_j''),$$

but that does not satisfy one, at least, of the following

$$H_3^{-1}|S_j(\alpha)| \leqq |S_i(\alpha)| \qquad (\leqq |S_j(\alpha)|),$$
$$H_3 A_*^{(j)} \geqq A_*^{(i)} \geqq H_3^{-1}A_*^{(j)},$$
$$H_3^{-1}(c_0\xi_i'')^2 \geqq A_*^{(i)}A^{(i)*} \geqq H_3(c_0\xi_i)^2,$$
$$\tau(A_*^{(i)}) \leqq (\log P)^{12}, \qquad \tau(B_*^{(i)}) \leqq (\log P)^{12}$$

for $i=1,\cdots,4$.

Proof. 4.2.7 and by calculations similar to those in 4.2.9, 4.2.11 and 4.2.13.

4.3.8. We are left with such α, that

(i) Those in 4.2.4.

(ii) Those that have been left in 4.3.7, i.e., that satisfy all of

$$H_3 \geqq |\alpha| \geqq H_3^{-1},$$
$$H_3^{-1}|S_j(\alpha)| \leqq |S_i(\alpha)| \qquad (\leqq |S_j(\alpha)|),$$
$$c_0\xi_i'' \geqq A_*^{(i)} \geqq H_3^{-1}(c_0\xi_i),$$
$$H_3^{-1}(c_0\xi_i'')^2 \leqq A_*^{(i)}A^{(i)*} \leqq H_3(c_0\xi_i'')^2,$$
$$\tau(A_*^{(i)}) \leqq (\log P)^{12}, \qquad \tau(B_*^{(i)}) \leqq (\log P)^{12},$$

for $i=1, \cdots, 4$.

(iii) Those satisfying, for $i=1, \cdots, 4$,

$$(\log P)^{6000} > A_*^{(i)} \qquad (\geqq 1),$$
$$P^2(\log P)^{-20} < A_*^{(i)}A^{(i)*} < P^2(\log P)^{20},$$

and

$$(\log P)^4 > |\alpha| > P^{-3/4}.$$

The case (i) and (ii) will be treated in 4.4 and 4.5.

4.3.9. Lemma. *Those α in* (iii) *of* 4.3.8 *do not exist, if h_0 is* $\geqq 8000$.

Proof. This is classical, [5]. We have, for α in (iii) of 4.3.8, that

$$|\lambda_i\alpha| A_*^{(i)} = B_*^{(i)} + O(P^{-2}(\log P)^{20})$$

for $i=1, 2$. We have, as in 1.3.1.1, that

$$\| |\lambda_1\lambda_2^{-1}|(A_*^{(1)}B_*^{(2)}) - (A_*^{(2)}B_*^{(1)})| \ll P^{-2}(\log P)^{7000}.$$

If $B_*^{(2)}=0$, then, $B_*^{(1)}=0$. This contradicts that

$$|\alpha| > P^{-3/4}.$$

Therefore $B_*^{(2)} \neq 0$ and $B_*^{(2)} \geqq 1$. Similarly $B_*^{(1)} \geqq 1$. As $1 \leqq A_*^{(1)}B_*^{(2)} \leqq (\log P)^{1500} = o(P^2(\log P)^{-7000})$, the fraction, (or its reduced one),

$$(A_*^{(2)}B_*^{(1)})(A_*^{(1)}B_*^{(2)})^{-1}$$

is a convergent to $|\lambda_1\lambda_2^{-1}|$ with a smaller denominator than that of $R'Q'^{-1}$, by 1.2.3.2 (i). Then

$$\| |\lambda_1\lambda_2^{-1}|(A_*^{(1)}B_*^{(2)}) - (A_*^{(2)}B_*^{(1)})| \times ((A_*^{(1)}B_*^{(2)}), (A_*^{(2)}B_*^{(1)}))^{-1}$$
$$\geqq (2Q')^{-1} \geqq Q^{-1} \geqq (P^2(\log P)^{-h_0})^{-1}.$$

This contradicts that the left-hand side is $O(P^{-2}(\log P)^{7000})$, if $h_0 \geqq 8000$. The case (iii) of 4.3.8 is impossible.

4.3.10. Lemma. *We may add to* (ii) *of* 4.3.8, *the conditions that* α *satisfies, moreover, by taking* H_3 *larger if needed,*

$$H_3(c_0\xi_i'')\geqq A^{(i)***},\ A^{(i)**}\qquad (i=1,\cdots,4),$$

where $A^{(i)***}$, $A^{(i)**}$ *are those in* 4.2.3.1.

Proof. Similar to 4.2.3.1.

4.4. Kloosterman's domain I

We treat such α as is left in (i) and (ii) of 4.3.8. We will rely, in 4.5, on 2.3.7, 2.3.9, 2.3.11.5 and 3.2.4, where we need the essences of [17]. In this section, the key steps are 4.4.14 and 4.4.19.

4.4.1. (i) From (i) and (ii) of 4.3.8, we have to treat such α, with a large positive constant H_0 depending on E_{100} and an arbitrarily small positive constant δ of 4.3.5 and 4.3.7, that

$$H_0^{-1}<|\alpha|<H_0$$

and

$$H_0\xi_i''>A^{(i)***}>A^{(i)**}>A^{(i)*}>c_0\xi_i''\geqq A_*^{(i)}>H_0^{-1}\xi_i'',$$

where

$$|\lambda_i\alpha|^{-1}\ (\longrightarrow\frac{A^{(i)***}}{B^{(i)***}}\ (\Longrightarrow\frac{A^{(i)**}}{B^{(i)**}}\ (\Longrightarrow\frac{A^{(i)*}}{B^{(i)*}}\ (\Longrightarrow\frac{A_*^{(i)}}{B_*^{(i)}},$$

and α and $\lambda_i\alpha$ may be considered as irrational if needed. As we have obtained upper bounds of "L^1-norm" of parts of 4.1.5, in the preceedings, we can widen the range of α which is left as above, if needed. We will take into consideration 1.1.3, 2.2.11 and lemmas in 1.4 up to 1.4.7.

(ii) With a large positive constant g_0, which will be explained in 4.4.4, we prepare the convergents of $|\alpha|$ as

$$|\alpha|\ (\longrightarrow B^*/A^*\ (\Longrightarrow B_*/A_*\ (\Longrightarrow B_{**}/A_{**}$$

with

$$A^*>g_0P\geqq A_*>A_{**}\qquad(\geqq 1)$$

As we have

$$|S_i|\ll(H_0\xi_i'')^{1/2}$$

from (i), we may suppose that

$$g_0^2P > A^* > g_0P \geqq A_* > A_{**} \geqq g_0^{1/2}P,$$

by taking g_0 large in connection with H_0. See 4.2.3 and 4.2.3.1. The notation will be modified later, in 4.4.3.1.

4.4.2. Lemma. *Such* α, *as is stated in* 4.4.1 *and satisfying one of the following conditions, gives a minor contribution* $\ll \delta|\lambda_1\cdots\lambda_4|^{-1/2}P^2$ *to* 4.1.5. *The conditions are*;

(i) *that* $X(2X')^{-1}$ *has not "good partial fractions with respect to* g_0', 2.2.10", *for some choice of*

$$(X, X') = (A^{(i)***}, A^{(i)**}), (A^{(i)**}, A^{(i)*}), (A^{(i)*}, A_*^{(i)}).$$

or (ii) *that* X *is not* $[K, K^z)$-regular, 1.1.3.2, *for some choice of* $X = A^*, B^*, A_*, B_*, A_{**}, B_{**}, A^{(i)***}, B^{(i)***}, A^{(i)**}, B^{(i)**}, A^{(i)*}, B^{(i)*}, A_*^{(i)}, B_*^{(i)}$

or (iii) *that, if* λ_i *is irrational,* $(Y, Y') > G$ *for some choice of*

$$Y = A^*, A_*, A_{**}$$

and

$$Y' = A^{(i)***}, A^{(i)**}, A^{(i)*}, A_*^{(i)},$$

or

$$Y = B^*, B_*, B_{**}$$

and

$$Y' = B^{(i)***}, B^{(i)**}, B^{(i)*}, B_*^{(i)}.$$

Here, the positive constants g_0', G, K *and* z *are supposed to be sufficiently large depending on on* E_{100}, (H_0 *and* g_0). *Also,* G_0' *in the assumption* (iii) *of the Theorem is chosen sufficiently large. Their definite choices will be explained in* 4.4.4.

Proof. Apply 2.2.11 to (i), 1.1.3 to (iii), and then, lemmas from 1.4.3 to 1.4.3.6.1, to (iii). Here, G_0' used in (iii) of the Theorem must be chosen sufficiently large to fit G_0 of 1.1.3.

4.4.3. Now α is supposed to satisfy all of the followings;

(0) That α is as is stated in 4.4.1.

(i) That $X(2X')^{-1}$ has "good partial fractions" for each choice of X and X' as in 4.4.2 (i).

(ii) That X is $[K, K^z)$-regular for each choice of X as in 4.4.2 (ii). and (iii) that, if λ_i is irrational, then $(Y, Y')\leqq G$ for each choice of Y and Y' as in 4.4.2 (iii).

4.4.3.1. In the followings, we often put as

$$A_*=A=a\ddot{A}, \qquad B_*=B=b\ddot{B},$$
$$A_{**}=A', \qquad B_{**}=B',$$

where all prime divisors of a and b lie in $[K, K^z)$ and no prime divisors of $\ddot{A}$ and $\ddot{B}$ lie in $[K, K^z)$. We have

$$1\leqq\nu_{[K,K^z)}(x)\leqq 10\log z$$

for $x=a$ and b, by 4.4.3 (ii).

4.4.4. As we use so many positive constants, we list them up here, according to their order of being fixed.

(i) $\eta_1\cdots\eta_4=\pm 1$; given first in the Theorem.

(ii) $\delta\,(>0)$; a sufficiently small positive numerical constant, prepared to state 4.3.5, 4.3.7 and 4.4.2.

(iii) $c_i', c_i'', c_0', c_{200}''$; given in 4.1.7.

(iii′) h_0; given in 4.3.6.

(iv) H_0; given in 4.4.1.

(iv′) g_0; given in 4.4.1, and will be fixed in 4.4.5 and 4.4.7.

(v) g_0', G_0', G, K, z; given in 4.4.2 and

(v-0) G as in 4.4.2,

(v-i) G_0' as in 4.4.2, and will be fixed in 4.4.7,

(v-ii) z is chosen as in 4.4.2, according to 1.4.3.3 and 1.4.3.5,

(v-iii) g_0' will be fixed in 4.5.1,

(v-iv) K will be fixed in 4.4.5, 4.4.7, 4.4.9 and 4.4.15.

(vi) L_0; appearing in (ii) and (iii) of the assumptions of the Theorem will be fixed in 4.4.5, 4.4.7, 4.4.11, 4.4.19, 4.4.21 and 4.5.2.

(vii) G', g'; will be fixed in 4.4.15.

(viii) G_0; appearing in (iii), (vi) and (v) of the Theorem will be fixed in the final step in 4.5.7.6. Also T and Z in 2.3.11 will be fixed in connection with G_0.

In the followings, these constants are supposed to be taken sufficiently large.

4.4.5. Lemma. *Each of* $(U_i aY)(V_i bX)^{-1}$, *where* $(X, Y)=(A^{(i)**}, B^{(i)**})$, $(A^{(i)*}, B^{(i)*})$, $(A_*^{(i)}, B_*^{(i)})$ $(i=1, \cdots, 4)$, *has* $\ddot{B}\ddot{A}^{-1}$ *as one of its convergents.*

Proof. Suppose $(X, Y)=(A^{(i)**}, B^{(i)**})$, for instance. We have that

$$||\lambda_i\alpha|-B^{(i)**}A^{(i)**-1}|\ll H_0(\xi_i'')^{-2},$$

therefore that

$$|(V_iB)(U_iA)^{-1}-B^{(i)**}A^{(i)**-1}|\ll H_0(\xi_i'')^{-2},$$

hence that

$$|(V_ib\ddot{B})(U_ia\ddot{A})^{-1}-B^{(i)**}A^{(i)**-1}|\ll H_0(\xi_i'')^{-2}.$$

This means that

$$|\ddot{B}\ddot{A}^{-1}-(U_iaB^{(i)**})(V_ibA^{(i)**})^{-1}|\ll H_0ab^{-1}(\xi_i'')^{-2}.$$

We have

$$\ddot{A}=a^{-1}A\ll a^{-1}g_0\xi_i''$$

and

$$ab\geqq K^2.$$

If K is $\geqq cH_0g_0^2$, where the positive numerical constant c is taken sufficiently large, then,

$$\ddot{A}^2<c^{-1}H_0^{-1}a^{-1}b\xi_i''^2.$$

This means, by 1.2.3.2 (i), that

$$(U_iaB^{(i)**})(V_ibA^{(i)**})^{-1}\quad (\longrightarrow \ddot{B}\ddot{A}^{-1}.$$

4.4.6. Lemma. *Let the preceding convergents of* $\ddot{B}\ddot{A}^{-1}$ *be* $\ddot{B}^{\Delta}\ddot{A}^{\Delta-1}$ *and* $\ddot{B}^{\nabla}\ddot{A}^{\nabla-1}$, *one lying to the right and the other to the left of* $\ddot{B}\ddot{A}^{-1}$, 1.2.3.1 (iii). *We have, then, for each choice of i, one of the following three alternatives*;

$$(\mathrm{i})_i\quad \begin{cases}(U_iaB^{(i)**})(V_ibA^{(i)**})^{-1} & (\longrightarrow \ddot{B}\ddot{A}^{-1}\ (\Longrightarrow \ddot{B}'\ddot{A}'^{-1},\\ (U_iaB^{(i)*})(V_ibA^{(i)*})^{-1} & (\longrightarrow \ddot{B}\ddot{A}^{-1}\ (\Longrightarrow \ddot{B}'\ddot{A}'^{-1},\end{cases}$$

or

$$(\mathrm{ii})_i\quad \begin{cases}(U_iaB^{(i)*})(V_ibA^{(i)*})^{-1} & (\longrightarrow \ddot{B}\ddot{A}^{-1}\ (\Longrightarrow \ddot{B}'\ddot{A}'^{-1},\\ (U_iaB_*^{(i)})(V_ibA_*^{(i)})^{-1} & (\longrightarrow \ddot{B}\ddot{A}^{-1}\ (\Longrightarrow \ddot{B}'\ddot{A}'^{-1},\end{cases}$$

or

$$(\mathrm{iii})_i\quad \begin{cases}(U_iaB^{(i)**})(V_ibA^{(i)**})^{-1} & (\longrightarrow \ddot{B}\ddot{A}^{-1}\ (\Longrightarrow \ddot{B}^{\Delta_i}\ddot{A}^{\Delta_i-1},\\ (U_iaB^{(i)*})(V_ibA^{(i)*})^{-1} & (\longrightarrow \ddot{B}\ddot{A}^{-1}\ (\Longrightarrow \ddot{B}^{\nabla_i}\ddot{A}^{\nabla_i-1},\\ (U_iaB_*^{(i)})(V_ibA_*^{(i)})^{-1} & (\longrightarrow \ddot{B}\ddot{A}^{-1}\ (\Longrightarrow \ddot{B}^{\Delta_i}\ddot{A}^{\Delta_i-1},\end{cases}$$

where

$$\ddot{B}'\ddot{A}'^{-1} \text{ is one of } \ddot{B}^{\varDelta}\ddot{A}^{\varDelta-1} \quad \text{and} \quad \ddot{B}^{\nabla}\ddot{A}^{\nabla-1},$$

and

$$(\varDelta_i, \nabla_i) \text{ is one of } (\varDelta, \nabla) \quad \text{and} \quad (\nabla, \varDelta).$$

Remark. We have

$$\ddot{A}^{\varDelta}+\ddot{A}^{\nabla}=\ddot{A},$$
$$\ddot{B}^{\varDelta}+\ddot{B}^{\nabla}=\ddot{B}.$$

Proof. 4.4.5.

4.4.7. Lemma. *Such* α, *that we have* (i)$_i$ *or* (ii)$_i$ *of* 4.4.6 *for some* i, *gives a minor contribution* $\ll\delta|\lambda_1\cdots\lambda_4|^{-1/2}P^2$ *to* 4.1.5.

Proof. Suppose we have the case (i)$_i$, for instance. We have

$$\begin{aligned}|\lambda_i\alpha| &= (V_iU_i^{-1}+O((L_0P^2)^{-1})(BA^{-1}+O((g_0P^2)^{-1})\\ &= (V_iB)(U_iA)^{-1}+O((H_0L_0^{-1}+E_{100}g_0^{-1})P^{-2}).\end{aligned}$$

We have, on the other hand, by 1.2.3.1 (iv-iv), that

$$||\lambda_i\alpha|-B^{(i)**}A^{(i)**-1}|\geqq(2A^{(i)***}A^{(i)**})^{-1}\gg(H_0\xi_i'')^{-2}$$

and

$$||\lambda_i\alpha|-B^{(i)*}A^{(i)*-1}|\geqq(2A^{(i)**}A^{(i)*})^{-1}\gg(H_0\xi_i'')^{-2}.$$

By choosing L_0 and g_0 sufficiently large, these mean that, if both of $B^{(i)**}A^{(i)**-1}$ and $B^{(i)*}A^{(i)*-1}$ lie on the same side to $(U_iB)(V_iA)^{-1}$, they must lie on the same side to $|\lambda_i\alpha|$. This is impossible as $A^{(i)**}B^{(i)**-1}$ and $A^{(i)*}B^{(i)*-1}$ are consecutive convergents to $|\lambda_i\alpha|^{-1}$. Therefore

$$B^{(i)**}A^{(i)**-1}\gtreqqless(V_iA)(U_iA)^{-1}\gtreqqless B^{(i)*}A^{(i)*-1}.$$

Suppose that λ_i is irrational, then $U_i>G_0'$, owing to the assumption (iii) in the Theorem. Suppose, moreover, that

$$B^{(i)**}A^{(i)**-1}=(V_iB)(U_iA)^{-1},$$

then

$$(V_iB,\ U_iA)\gg(U_iP)(H_0P)^{-1}\gg U_i^{3/4}.$$

Then

$$(U_i,\ B)\gg U_i^{1/3} \quad \text{or} \quad (V_i,\ A)\gg U_i^{1/3+0.01}\gg V_i^{1/3}.$$

Then the number of possible pairs (A, B) is

$$\begin{aligned}&\ll \sum_{d;d\mid U_i, d\gg U_i^{1/3}} d^{-1}H_0(g_0P)^2 + \sum_{d;d\mid V_i, d\gg V_i^{1/3}} d^{-1}H_0(g_0P)^2 \\ &\ll (U_i^{-1/3}\tau(U_i) + V_i^{-1/3}\tau(V_i))H_0(g_0P)^2 \\ &\ll U_i^{-0.1}P^2.\end{aligned}$$

This gives a minor contribution to 4.1.5.

Suppose λ_i is irrational and $B^{(i)*}A^{(i)*-1} = (V_iB)(U_iA)^{-1}$. We have a similar conclusion as before.

Suppose that λ_i is rational and $B^{(i)**}A^{(i)**-1} = (V_iB)(U_iA)^{-1}$. ($U_i = 1$ by the assumption of the Theorem.) We have

$$\begin{aligned}E_{100} \gg V_i &\geqq (V_i, A) = (V_iB, U_iA) = U_iA\cdot A^{(i)**-1} \\ &= AA^{(i)**-1} > g_0^{1/2}H_0^{-1},\end{aligned}$$

which is impossible if g_0 is sufficiently large.

Suppose that λ_i is rational and $B^{(i)*}A^{(i)*-1} = (V_iB)(U_iA)^{-1}$. We have a similar conclusion as before.

Suppose that we have no equality. But this case does not occur, because $(U_iaB^{(i)**})^{-1}(V_ibA^{(i)**})^{-1}$ and $(U_iaB^{(i)*})(V_ibA^{(i)*})^{-1}$ must lie on the same side to $\ddot B\ddot A^{-1}$, owing to (i)$_i$.

4.4.8. Lemma. *We have, from* (iii)$_i$ $(i=1, \cdots, 4)$ *of* 4.4.6, *that*

$$\begin{pmatrix} U_ia & 0 \\ 0 & V_ib \end{pmatrix}\begin{pmatrix} B^{(i)**} & B^{(i)*} & B^{(i)}_* \\ A^{(i)**} & A^{(i)*} & A^{(i)}_* \end{pmatrix}$$
$$= \begin{pmatrix} \ddot B^{\Delta_i} & \ddot B^{\nabla_i} \\ \ddot A^{\Delta_i} & \ddot A^{\nabla_i} \end{pmatrix}\begin{pmatrix} t_i^{**}(p_i^{**}+q_i^{**}) & t_i^*p_i^* & t_{i*}(p_{i*}+q_{i*}) \\ t_i^{**}p_i^{**} & t_i^*(p_i^*+q_i^*) & t_{i*}q_{i*} \end{pmatrix},$$

where

$$\begin{aligned}&t_i^{**}, p_i^{**}, q_i^{**}, \cdots, q_{i*} \in N, \\ &p_i^{**} \geqq q_i^{**} \geqq 1, \qquad (p_i^{**}, q_i^{**}) = 1, \\ &p_i^* \geqq q_i^* \geqq 1, \qquad (p_i^*, q_i^*) = 1, \\ &p_i{}^* \geqq q_{i*} \geqq 1, \qquad (p_{i*}, q_{i*}) = 1, \\ &t_i^{**} = (U_iaB^{(i)**}, V_ibA^{(i)**}), \\ &t_i^* = (U_iaB^{(i)*}, V_ibA^{(i)*}), \\ &t_{i*} = (U_iaB^{(i)}_*, V_ibA^{(i)}_*).\end{aligned}$$

Proof. (iii)$_i$ of 4.4.6 and 1.2.3.1 (iv-vi). If $q_i^* = 0$, for instance, then $(U_iaB^{(i)*})(V_iaB^{(i)*})^{-1} = (\ddot B^{\Delta_i} + \ddot B^{\nabla_i})(\ddot A^{\Delta_i} + A^{\nabla_i})^{-1} = \ddot B\ddot A^{-1}$. This was discussed in the proof of 4.4.7. Therefore $q_i^* \geqq 1$.

4.4.9. Lemma.
(i) *Putting**)

$$\ddot{A}\ddot{B}^{\Delta}-\ddot{B}\ddot{A}^{\Delta}=\ddot{\varepsilon}^{\nabla} \quad (=\pm 1),$$
$$\ddot{A}\ddot{B}^{\nabla}-\ddot{B}\ddot{A}^{\nabla}=\ddot{\varepsilon}^{\Delta} \quad (=\pm 1), \qquad ((4.4.6)),$$

we have

$$\ddot{\varepsilon}^{\nabla}=-\ddot{\varepsilon}^{\Delta}.$$

(ii) *Putting*

$$AB'-BA'=\varepsilon \quad (=\pm 1), \qquad ((4.4.3.1)),$$
$$\ddot{A}\ddot{B}'-\ddot{B}\ddot{A}'=\ddot{\varepsilon}' \quad (=\pm 1), \qquad ((4.4.6)),$$

where $(\ddot{A}', \ddot{B}')$ *is one of* $(\ddot{A}^{\Delta}, \ddot{B}^{\Delta})$ *and* $(\ddot{A}^{\nabla}, \ddot{B}^{\nabla})$, *we have*

$$\ddot{A}'=(\varepsilon\ddot{\varepsilon}')(bA'-t'\ddot{A}),$$
$$\ddot{B}'=(\varepsilon\ddot{\varepsilon}')(aB'-t'\ddot{B}),$$

with

$$g_0^{-1/2}ab \ll t' \leqq ab.$$

If $\varepsilon\ddot{\varepsilon}'=1$, *then* $t'<ab$.

Proof. (i) is well-known; Theorems 158 and 164 in [12], for instance. As for (ii), it is easy except the order of t'. We have, by choosing K sufficiently large,

$$t'\ddot{A}=bA'-\ddot{\varepsilon}'\varepsilon\ddot{A}' \gg \ll bA',$$

therefore

$$t' \gg bA'\ddot{A}^{-1}=abA'A^{-1} \gg g_0^{-1/2}ab.$$

We have

$$t'\ddot{A} \leqq bA'+\ddot{A}'<bA+\ddot{A}=(ab+1)\ddot{A},$$

therefore $t' \leqq ab$. If $\varepsilon\ddot{\varepsilon}'=1$, then the same argument gives us that $t'<ab$.

4.4.9.1. As a converse to 4.4.9, we have

Lemma. *Let* $a, b, \ddot{A}, \ddot{B}, \ddot{A}', \ddot{B}', \ddot{\varepsilon}', \varepsilon$ *be given, so that* $(a\ddot{A}, b\ddot{B})=1$, $\ddot{A}\ddot{B}'-\ddot{B}\ddot{A}'=\ddot{\varepsilon}'$ *and* $\ddot{\varepsilon}', \varepsilon$ *are* ± 1. *Let* t' *be such that*

*) These notations may cause of some irritations. See the end of 4.5.3.1.

$$\ddot{A}'+(\varepsilon\ddot{\varepsilon}')t'\ddot{A}\equiv 0 \bmod b,$$
$$\ddot{B}'+(\varepsilon\ddot{\varepsilon}')t'\ddot{B}\equiv 0 \bmod a,$$
$$0<t'\leqq ab,$$

($t'<ab$ if $\varepsilon\ddot{\varepsilon}'=1$). Let us put

$$A=a\ddot{A},\qquad B=b\ddot{B},$$
$$A'=b^{-1}(\ddot{A}'+(\varepsilon\ddot{\varepsilon}')t'\ddot{A}),\qquad B'=a^{-1}(\ddot{B}'+(\varepsilon\ddot{\varepsilon}')t'\ddot{B}).$$

Then, they are integers and

$$AB'-BA'=\varepsilon.$$

Proof. It is easy.

4.4.10. Lemma. *We have, from* 4.4.8, *that*

$$(V_iB)(U_iA)^{-1}-B^{(i)**}A^{(i)**-1}=(\ddot{\varepsilon}^{\Delta_i}t_i^{**}q_i^{**})(U_iAA^{(i)**})^{-1},$$
$$(V_iB)(U_iA)^{-1}-B^{(i)*}A^{(i)*-1}=-(\ddot{\varepsilon}^{\Delta_i}t_i^{*}q_i^{*})(U_iAA^{(i)*})^{-1},$$
$$(V_iB)(U_iA)^{-1}-B_*^{(i)}A_*^{(i)-1}=(\ddot{\varepsilon}^{\Delta_i}t_{i*}q_{i*})(U_iAA_*^{(i)})^{-1}.$$

Here we have put as

$$\ddot{A}^{\Delta_i}\ddot{B}^{\nabla_i}-\ddot{B}^{\Delta_i}\ddot{A}^{\nabla_i}=\ddot{\varepsilon}^{\Delta_i}\qquad(=-\ddot{\varepsilon}^{\nabla_i}=\pm 1).$$

We have also

$$A^{(i)**}B^{(i)*}-B^{(i)**}A^{(i)*}=-A^{(i)*}B_*^{(i)}+B^{(i)*}A_*^{(i)}=\ddot{\varepsilon}^{\Delta_i}.$$

Proof. It is easy.

4.4.11. Lemma. *We have, for*

$$|\lambda_i|=V_iU_i^{-1}+\Omega_i\qquad(\textit{Assumption (iii) of the Theorem})$$

and

$$|\alpha|=BA^{-1}+\omega\qquad((4.4.3.1))$$

with

$$|\Omega_i|\leqq(L_0P^2)^{-1}\quad\textit{and}\quad\omega=\varepsilon A^{-1}(A\alpha^*+A')^{-1},$$

that

$$\operatorname{sgn}(\Omega_iBA^{-1}+\omega|\lambda_i|)=\operatorname{sgn}\omega=\varepsilon$$

where

$$\varepsilon = AB' - BA' = \pm 1.$$

Proof. We have

$$|\Omega_i BA^{-1}| \ll H_0(L_0P^2)^{-1}$$

and

$$|\omega\lambda_i| \gg E_{100}(2g_0^3P^2)^{-1},$$

the latter owing to $A\alpha^* + A' \leqq 2A^* \ll g_0^2P$. Then the conclusion follows easily.

4.4.12. Lemma. *We have, in* 4.4.8 *and* 4.4.11, *that*

$$\text{if } \varepsilon = \ddot{\varepsilon}^{\Delta_i}, \text{ then } t_i^* q_i^* > t_i^{**} q_i^{**} \qquad (\geqq 1),$$

and

$$\text{if } \varepsilon = -\ddot{\varepsilon}^{\Delta_i}\ (= \ddot{\varepsilon}^{\Gamma_i}), \text{ then } t_{i*}q_{i*} > t_i^* q_i^* \qquad (\geqq 1).$$

Proof. We have

$$\begin{aligned}|\lambda\alpha_i| - B^{(i)**}A^{(i)**-1} &= (V_iU_i^{-1} + \Omega_i)(BA^{-1} + \omega) - B^{(i)**}A^{(i)**-1}\\ &= ((V_iB)(U_iA) - B^{(i)**}A^{(i)**-1}) + (\Omega_iBA^{-1} + \omega|\lambda_i|)\\ &= (\ddot{\varepsilon}^{\Delta_i}t_i^{**}q_i^{**})(U_iAA^{(i)**})^{-1} + (\Omega_iBA^{-1} + \omega|\lambda_i|),\end{aligned}$$

by 4.4.10. Similarly

$$|\lambda_i\alpha| - B^{(i)*}A^{(i)*-1} = -(\ddot{\varepsilon}^{\Delta_i}t_i^*q_i^*)(U_iAA^{(i)*})^{-1} + (\Omega_iBA^{-1} + \omega|\lambda_i|),$$
$$|\lambda_i\alpha| - B_*^{(i)}A_*^{(i)-1} = (\ddot{\varepsilon}^{\Delta_i}t_{i*}q_{i*})(U_iAA_*^{(i)})^{-1} + (\Omega_iBA^{-1} + \omega|\lambda_i|).$$

On the other hand, the signatures of the above three formulae vary between positive and negative values alternatively. We have also that

$$\begin{aligned}A_*^{(i)}||\lambda_i\alpha| - B_*^{(i)}A_*^{(i)-1}| &> A^{(i)*}\,||\lambda_i\alpha| - B^{(i)*}A^{(i)*-1}|\\ &> A^{(i)**}\,||\lambda_i\alpha| - B^{(i)**}A^{(i)**-1}|.\end{aligned}$$

(See 1.2.3.1 (iv-iv)).

(i) Suppose that $\varepsilon = \ddot{\varepsilon}^{\Delta_i} = 1$. We have, by 4.4.11, that

$$\Omega_iBA^{-1} + \omega|\lambda_i| > 0.$$

Therefore

$$|\lambda_i\alpha| - B^{(i)**}A^{(i)**-1} > 0.$$

Then

$$|\lambda_i\alpha|-B^{(i)*}A^{(i)*-1}<0.$$

These mean that

$$(t_i^* q_i^*)(U_i AA^{(i)*})^{-1}-(\Omega_i BA^{-1}+\omega|\lambda_i|)>0.$$

Then

$$\begin{aligned}A^{(i)*}\{&(t_i^* q_i^*)(U_i AA^{(i)*})^{-1}-(\Omega_i BA^{-1}+\omega|\lambda_i|)\}\\&=A^{(i)*}(-|\lambda_i\alpha|+B^{(i)*}A^{(i)*-1})\\&\quad>A^{(i)**}(|\lambda_i\alpha|-B^{(i)**}A^{(i)**-1})\\&=A^{(i)**}\{(t_i^{**} q_i^{**})(U_i AA^{(i)**})^{-1}+(\Omega_i BA^{-1}+\omega|\lambda_i|)\}.\end{aligned}$$

We have, then,

$$\begin{aligned}t_i^* q_i^*&>t_i^{**} q_i^{**}+(A^{(i)**}+A^{(i)*})\cdot U_i A\cdot(\Omega_i BA^{-1}+\omega|\lambda_i|)\\&>t_i^{**} q_i^{**}.\end{aligned}$$

(ii) Suppose that $\varepsilon=\ddot{\varepsilon}^{\Delta_i}=-1$. We have

$$|\lambda_i\alpha|-B^{(i)**}A^{(i)**}<0,$$

as

$$\Omega_i BA^{-1}+\omega|\lambda_i|<0.$$

Then we have

$$(t_i^* q_i^*)(U_i AA^{(i)*})^{-1}>|\Omega_i BA^{-1}+\omega|\lambda_i||,$$

as

$$|\lambda_i\alpha|-B^{(i)*}A^{(i)*-1}>0.$$

We have

$$\begin{aligned}A^{(i)*}\{&(t_i^* q_i^*)(U_i AA^{(i)*})^{-1}-|\Omega_i BA^{-1}+\omega|\lambda_i||\}\\&>A^{(i)**}\{(t_i^{**} q_i^{**})(U_i AA^{(i)**})^{-1}+|\Omega_i BA^{-1}+\omega|\lambda_i||\}.\end{aligned}$$

Then

$$\begin{aligned}t_i^* q_i^*&>t_i^{**} q_i^{**}+(A^{(i)*}+A^{(i)**})U_i A|\Omega_i BA^{-1}+\omega|\lambda||\\&>t_i^{**} q_i^{**}.\end{aligned}$$

(iii) Suppose that $\varepsilon=-\ddot{\varepsilon}^{\Delta_i}=1$. We have

$$|\lambda_i\alpha| - B^{(i)*}A^{*-1} > 0,$$

as

$$\Omega_i BA^{-1} + \omega|\lambda_i| > 0.$$

Then

$$|\lambda_i\alpha| - B_*^{(i)}A_*^{(i)-1} < 0.$$

Therefore

$$(t_{i*}q_{i*})(U_i AA_*^{(i)})^{-1} - (\Omega_i BA^{-1} + \omega|\lambda_i|) > 0.$$

Then

$$\begin{aligned} &A_*^{(i)}\{(t_{i*}q_{i*})(U_i AA_*^{(i)})^{-1} - (\Omega_i BA^{-1} + \omega|\lambda_i|)\} \\ &\qquad > A^{(i)*}\{(t_i^* q_i^*)(U_i AA^{(i)*})^{-1} + (\Omega_i BA^{-1} + \omega|\lambda_i|)\}. \end{aligned}$$

Therefore

$$\begin{aligned} t_{i*}q_{i*} &> t_i^* q_i^* + (A_*^{(i)} + A^{(i)*})U_i A(\Omega_i BA^{-1} + \omega|\lambda_i|) \\ &> t_i^* q_i^*. \end{aligned}$$

(iv) Suppose that $\varepsilon = -\ddot{\varepsilon}^{\Delta_i} = -1$. We have

$$|\lambda_i\alpha| - B^{(i)*}A^{(i)*-1} < 0,$$

as

$$\Omega_i BA^{-1} + \omega|\lambda_i| < 0.$$

Then

$$|\lambda_i\alpha| - B_*^{(i)}A_*^{(i)-1} > 0.$$

Then

$$(t_{i*}q_{i*})(U_i AA_*^{(i)})^{-1} > |\Omega_i BA^{-1} + \omega|\lambda_i||.$$

Therefore

$$\begin{aligned} &A_*^{(i)}\{(t_{i*}q_{i*})(U_i AA_*^{(i)})^{-1} - |\Omega_i BA^{-1} + \omega|\lambda_i||\} \\ &\qquad > A^{(i)*}\{(t_i^* q_i^*)(U_i AA^{(i)*})^{-1} + |\Omega_i BA^{-1} + \omega|\lambda_i||\}. \end{aligned}$$

Then

$$\begin{aligned} t_{i*}q_{i*} &> t_i^* q_i^* + (A_*^{(i)} + A^{(i)*}) \cdot U_i A \cdot |\Omega_i BA^{-1} + \omega|\lambda_i|| \\ &> t_i^* q_i^*. \end{aligned}$$

4.4.13. With the notations in 4.4.11, 4.4.10, 4.4.8 and 4.4.3.1, let us put A_i, B_i, A'_i, B'_i, ε_i as follows;

(i) If $\varepsilon=\ddot{\varepsilon}^{\Delta_i}$, then

$$\begin{aligned} &B_i=B^{(i)**}, \qquad B'_i=B^{(i)*},\\ &A_i=A^{(i)**}, \qquad A'_i=A^{(i)*},\\ &\varepsilon_i=\ddot{\varepsilon}^{\Delta_i}\ (=A_iB'_i-B_iA'_i=AB'-BA'=\varepsilon). \end{aligned}$$

OR (ii) If $\varepsilon=-\ddot{\varepsilon}^{\Delta_i}\ (=\ddot{\varepsilon}^{\nabla_i})$, then

$$\begin{aligned} &B_i=B^{(i)*}, \qquad B'_i=B^{(i)}_{*},\\ &A_i=A^{(i)*}, \qquad A'_i=A^{(i)}_{*},\\ &\varepsilon_i=\ddot{\varepsilon}^{\nabla_i}\ (=A_iB'_i-B_iA'_i=AB'-BA'=\varepsilon). \end{aligned}$$

Corresponding to the above cases, the letters p^*_i, q^*_i, t^*_i, p^{**}_i, $\cdots$ etc., will be denoted as

$$p_i,\ q_i,\ t_i \qquad \text{for } B_iA_i^{-1}$$

and

$$p'_i,\ q'_i,\ t'_i \qquad \text{for } B'_iA_i'^{-1}.$$

Also, we choose Δ, with $\ddot{\varepsilon}^{\Delta}$ in 4.4.9, to satisfy

$$\ddot{\varepsilon}^{\Delta}=\varepsilon$$

so that we have

$$\Delta=\Delta_i \qquad \text{in (i)}$$

or

$$\Delta=\nabla_i \qquad \text{in (ii)}.$$

4.4.14. Lemma. *With the notations in* 4.4.13, *we have*

$$\begin{pmatrix} U_ia & 0\\ 0 & V_ib\end{pmatrix}\begin{pmatrix} B_i & B'_i\\ A_i & A'_i\end{pmatrix}=\begin{pmatrix} \ddot{B} & \ddot{B}^{\nabla}\\ \ddot{A} & \ddot{A}^{\nabla}\end{pmatrix}\begin{pmatrix} t_i(p_i+q_i) & t'_ip'_i\\ -t_iq_i & t'_ip'_i\end{pmatrix}.$$

where

$$\begin{aligned} &A_iB'_i-B_iA'_i=\varepsilon_i=\varepsilon, \qquad H_0\xi_i>A_i>c_0\xi''_i,\\ &A=a\ddot{A}, \quad B=b\ddot{B}, \quad AB'-BA'=\varepsilon \qquad (4.4.3.1),\\ &p_i\geqq q_i\geqq 1, \qquad p'_i\geqq q'_i\geqq 1,\\ &(p_i,\ q_i)=1, \qquad (p'_i,\ q'_i)=1, \end{aligned}$$

$$t_i=(U_iaB_i,\ V_ibA_i),\qquad t'_i=(U_iaB'_i,\ V_iaA'_i),$$
$$t_it'_i(p_iq'_i+q_ip'_i+q_iq'_i)=U_iV_iab,$$
$$t'_iq'_i>t_iq_i$$

and

$$t_i(p_i+q_i)>t'_iq'_i.$$

Proof. Suppose that $\varepsilon=\ddot{\varepsilon}^{\Delta_i}$ $(=\varepsilon^{\Delta})$. We have, from 4.4.13 (i), 4.4.8, 4.4.9 (i) and 4.4.10, that

$$\begin{aligned}
\begin{pmatrix} U_ia & 0\\ 0 & V_ib\end{pmatrix}\begin{pmatrix} B_i & B'_i\\ A_i & A'_i\end{pmatrix}&=\begin{pmatrix} U_ia & 0\\ 0 & V_ib\end{pmatrix}\begin{pmatrix} B^{(i)**} & B^{(i)*}\\ A^{(i)**} & A^{(i)*}\end{pmatrix}\\
&=\begin{pmatrix} \ddot{B}^{\Delta_i} & \ddot{B}^{\Gamma_i}\\ \ddot{A}^{\Delta_i} & \ddot{A}^{\Gamma_i}\end{pmatrix}\begin{pmatrix} t_i^{**}(p_i^{**}+q_i^{**}) & t_i^*p_i^*\\ t_i^{**}p_i^{**} & t_i^*(p_i^*+q_i^*)\end{pmatrix}\\
&=\begin{pmatrix} \ddot{B} & \ddot{B}^{\Gamma}\\ \ddot{A} & \ddot{A}^{\Gamma}\end{pmatrix}\begin{pmatrix} 1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix} t_i(p_i+q_i) & t'_ip'_i\\ t_ip_i & t'_i(p'_i+q'_i)\end{pmatrix}\\
&=\begin{pmatrix} \ddot{B} & \ddot{B}^{\Gamma}\\ \ddot{A} & \ddot{A}^{\Gamma}\end{pmatrix}\begin{pmatrix} t_i(p_i+q_i) & t'_ip'_i\\ -t_iq_i & t'_iq'_i\end{pmatrix}.
\end{aligned}$$

Suppose that $\varepsilon=-\ddot{\varepsilon}^{\Delta_i}=\varepsilon^{\Gamma_i}=\varepsilon^{\Delta}$. Then

$$\begin{aligned}
\begin{pmatrix} U_ia & 0\\ 0 & V_ib\end{pmatrix}\begin{pmatrix} B_i & B'_i\\ A_i & A'_i\end{pmatrix}&=\begin{pmatrix} U_ia & 0\\ 0 & V_ib\end{pmatrix}\begin{pmatrix} B^{(i)*} & B^{(i)}_*\\ A^{(i)*} & A^{(i)}_*\end{pmatrix}\\
&=\begin{pmatrix} \ddot{B}^{\Delta_i} & \ddot{B}^{\Gamma_i}\\ \ddot{A}^{\Delta_i} & \ddot{A}^{\Gamma_i}\end{pmatrix}\begin{pmatrix} t_i^*p_i^* & t_{i*}(p_{i*}+q_{i*})\\ t_i^*(p_i^*+q_i^*) & t_{i*}p_{i*}\end{pmatrix}\\
&=\begin{pmatrix} \ddot{B} & \ddot{B}^{\Gamma}\\ \ddot{A} & \ddot{A}^{\Gamma}\end{pmatrix}\begin{pmatrix} 1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} t_ip_i & t'_i(p'_i+q'_i)\\ t_i(p_i+q_i) & t'_ip'_i\end{pmatrix}\\
&=\begin{pmatrix} \ddot{B} & \ddot{B}^{\Gamma}\\ \ddot{A} & \ddot{A}^{\Gamma}\end{pmatrix}\begin{pmatrix} t_i(p_i+q_i) & t'_ip'_i\\ -t_iq_i & t'_iq'_i\end{pmatrix}.
\end{aligned}$$

That $t'_iq'_i>t_iq_i$ follows from 4.4.12. We have

$$0<V_ib(A_i-A'_i)=\{t_i(p_i+q_i)-t'_iq'_i\}\ddot{A}-\{t_iq_i+t'_iq'_i\}\ddot{A}^{\Delta}.$$

Therefore we must have $t_i(p_i+q_i)>t'_iq'_i$.

4.4.15. Lemma. *With the notations of* 4.4.14, *such* α, *that does not satisfy all of the following conditions, gives a minor contribution to* 4.1.5. *The conditions are, with sufficiently large positive constants* G' *and* g':

(i) *That* X *is expressible as a product of a* (*square-free*) *divisor of*

$\Delta^1_{G'}(U_iV_i)$ and of integer of order $O(G')$, and also $G'\cdot 2^{\nu(U_iV_i)}\geqq X(\geqq 1)$, where $X=t_i$ and t'_i.

(ii) *That (X, Y) is expressible as a product of a (square-free) divisor of $\Delta^1_{G'}(Y)$ and of an integer of order $O(G')$, and also*

$$G'2^{\nu(Y)}\geqq(X, Y),$$

where $X=A, B, A', B', A_i, B_i, A'_i, B'_i$ and $Y=U_i, V_i$.

(iii) *That (X, Y) is a (square-free) divisor of $\Delta^1_{G'}(Y)$, where $X=p_i, q_i, p_i+q_i$ and $Y=t_i^{-1}U_iV_iab$, or $X=p'_i, q'_i, p'_i+q'_i$ and $Y=t_i'^{-1}U_iV_iab$. Also*

$$G'U_i\geqq t_ip_i>t_iq_i\geqq G'^{-1}U_i$$

and

$$G'U_i\geqq t'_ip'_i>t'_iq'_i\geqq G'^{-1}U_i.$$

(iv) *That*

$$(q_i, q'_i)t_it'_i\mid U_iV_iab,$$
$$(q_i, q'_i)\leqq G'4^{\nu(U_iV_i)}.$$

Also that $p^2\nmid(q_i, q'_i)$, if p is a prime and $p>K$.

(v) *And that*

$$2^{g'+1}\nmid X$$

for $X=A, B, A', B', A_i, B_i, A'_i, B'_i$.

Remark. Above conditions are trivial, if $\lambda_i\in \boldsymbol{Q}$.

Proof. It is easy for (v).

As for (iv), we have, from 4.4.14. that

$$t_it'_i(q_i, q'_i)\mid U_iV_iab,$$

and that

$$U_iV_iabB'_iA_i=t_it'_i(p'_i\ddot{B}+q'_i\ddot{B}^r)((p_i+q_i)\ddot{A}-q_i\ddot{A}^r).$$

Therefore we have

$$p_ip'_i\ddot{B}\ddot{A}\equiv 0 \bmod (q_i, q'_i).$$

But $(p_ip'_i, (q_i, q'_i))=1$. Therefore we have

$$\ddot{B}\ddot{A}\equiv 0 \bmod (q_i, q'_i).$$

As $(\ddot{A}, \ddot{B})=1$ and $p^2 \nmid \ddot{A}\ddot{B}$ if $p>K$, we have the latter assertion in (iv). Suppose, then, that

$$(q_i, q_i')\geqq G'4^{\nu(U_iV_i)}.$$

We can put $(q_i, q_i')=d_1d_2$, where

$$d_1d_2 \mid U_iV_iab, \quad \ddot{A}\equiv 0 \bmod d_1, \quad \ddot{B}\equiv 0 \bmod d_2.$$

We have either $d_1\geqq G'^{1/2}2^{\nu(U_iV_i)}$ or $d_2>G'^{1/2}2^{\nu(U_iV_i)}$. Also we can suppose that $p^2 \nmid d_1d_2$ if $p>K$. The number of such (A, B), that $d_1\geqq G'^{1/2}2^{\nu(U_iV_i)}$ for some i, is

$$\begin{aligned}
&\ll H_0g_0P\sum_a\sum_{d_1}(ad_1)^{-1}g_0P\\
&\ll H_0g_0P\sum_a(aG'^{1/2}2^{\nu(U_iV_i)})^{-1}\times K'2^{\nu(U_iV_i)}g_0P\\
&\ll G'^{-1/2}H_0g_0^2K''P^2,
\end{aligned}$$

where K' is $\prod_{p;p\leqq g''}p^{g'}$ and K'' is $K'\cdot z\log z\cdot\log K$. If G' is sufficiently large, the contribution of those (A, B) is of minor one. Similarly for d_2.

Let us consider (i). We have

$$t_i=(U_iaB_i, V_ibA_i)\,|\,ab(U_i, A_i)(V_i, B_i).$$

Similar consideration as in the latter half of the proof of (iv) gives us that the contribution of those (A_i, B_i) with $t_i>G'2^{\nu(U_iV_i)}$ is of minor one. We can suppose that $p^2 \nmid A_iB_i$ if $p>K$, therefore $p^2 \nmid A_iB_i$ if $p>K^z$. Then the number of such pairs (A_i, B_i), that there exists a prime common divisor of U_i and A_i with $p>G'\nu(U_i)$, is, taking $G'>K^z$,

$$\begin{aligned}
&\ll \sum_{p;\,\mathrm{prime},\,p\mid U_i,\,p>G'\nu(U_i)} p^{-1}g_0P\cdot H_0g_0P\\
&\ll (G'\nu(U_i))^{-1}\nu(U_i)H_0g_0^2P\\
&\ll G'^{-1}H_0g_0^2P.
\end{aligned}$$

This gives a minor contribution. Similarly for other cases, and we easily have (i) for t_i. Similarly for t_i'.

(ii) The proof is similar as in (i).

(iii) We have

$$t_ip_i\leqq\ddot{B}^{-1}U_iaB_i=B^{-1}U_iabB_i\ll(g_0{}^{-1/2}P)^{-1}U_iab(H_0P)\ll G'U_i.$$

Similarly we have $t_i'p_i'\ll G'U_i$. Suppose that $t_ip_i<G'^{-1/2}U_i$. We have, then,

$$U_iaB_i = t_i((p_i+q_i)\ddot{B} - q_i\ddot{B}^r) \ll t_ip_i\ddot{B} < G'^{-1/2}U_i\ddot{B}$$
$$\ll (G'a)^{-1/2}U_iaB_i.$$

This is impossible if G' is large. Now suppose that $t_ip_i > G'^{-1/2}U_i$ and $t_iq_i < G'^{-1}U_i$. Then $p_iq_i^{-1} > G'^{1/2}$. We have

$$|(U_iaB_i)(V_ibA_i)^{-1} - \ddot{B}\ddot{A}^{-1}| = q_i\ddot{A}^{-1}((p_i+q_i)\ddot{A} - q_i\ddot{A}^r)^{-1}$$
$$< q_i\ddot{A}^{-1}(p_i\ddot{A})^{-1} < G'^{-1/2}\ddot{A}^{-2}.$$

We have, then,

$$|(V_i\ddot{B})(U_i\ddot{A})^{-1} - (aB_i)(bA_i)^{-1}| \ll (E_{100}G')^{-1/2}\ddot{A}^{-2}.$$

This is

$$< \tfrac{1}{2}(bA_i)^{-2}$$

if G' is sufficiently large in connection with H_0, g_0, K and z. Then we have

$$(V_i\ddot{B})(U_i\ddot{A})^{-1} \ (\longrightarrow \text{the reduced fraction of } (aB_i)(bA_i)^{-1}.$$

We have $(a, A_i) \leqq (A, A_i) \leqq G$. Therefore, the denominator of the reduced fraction of $(aB_i)(bA_i)^{-1}$ is

$$\gg (GH_0)^{-1}P.$$

This means that the number of possible reduced fraction arising from possible $(aB_i)(bA_i)^{-1}$ is $\ll GH_0$, if $\ddot{A}$, $\ddot{B}$, a and b are fixed. Then, 1.4.4.2 tells us that the number of (a, b, A_i, B_i) is

$$\ll z^cGH_0,$$

if $\ddot{A}$ and $\ddot{B}$ are fixed. We have $\ddot{A}$ and $\ddot{B}$ are $< K^{-1}(g_0P)$. Therefore taking K sufficiently large, the contribution is of minor one, if $t_iq_i < G'^{-1}U_i$. The proof is similar for $t_i'q_i'$.

Dividing t_i as $t_i = y_iz_i$, temporarily, with

$$y_i | a(U_i, A_i), \qquad z_i | b(V_i, B_i), \qquad (y_i, z_i) = 1,$$

we see that $(t_i^{-1}U_iV_iab, X)$ is a divisor of $(y_i^{-1}U_ia, X)(z_i^{-1}V_ib, X)$. Suppose that

$$(t_i^{-1}U_iV_iab, X) \nmid \Delta^1_{G'}(t_i^{-1}U_iV_iab).$$

As $(U_i, V_i) = 1$ and $(a, b) = 1$, we have, then, either

$$(y_i^{-1}U_ia, X)\nmid \Delta_{G'}^1(y_i^{-1}U_ia),$$

or

$$(z_i^{-1}V_ib, X)\nmid \Delta_{G'}^1(z_i^{-1}V_ib).$$

Then, there exists a prime p, such that either

$$p>G'\nu(U_i) \quad \text{and} \quad p\,|\,(y_i^{-1}U_ia, X),$$

or

$$p>G'\nu(V_i) \quad \text{and} \quad p\,|\,(z_i^{-1}V_ib, Y).$$

If X is p_i, for instance, this means that either

$$p>G'\nu(U_i),\ p\,|\,U_i \quad \text{and} \quad p\,|\,(\ddot{B}-\ddot{B}^r)$$

or

$$p>G'\nu(V_i),\ p\,|\,V_i \quad \text{and} \quad p\,|\,(\ddot{B}-\ddot{B}^r),$$

because $p\nmid q_i$. Then the number of possible pairs $(\ddot{B}, \ddot{B}^r)$ is

$$\ll \sum_{p;\text{ as above}} p^{-1}H_0(g_0P)^2 \ll \{(G'\nu(U_i))^{-1}\nu(U_i)+(G'\nu(V_i))^{-1}\nu(V_i)\}\cdot H_0(g_0P)^2$$
$$\ll G^{-1}H_0g_0^2P^2.$$

There are $O(H_0^2g_0^3ab)$ at most of $(\ddot{A}, \ddot{A}^r)$ for given $(\ddot{B}\ \ddot{B}^r)$. Therefore, there are at most $O(G'^{-1}(K^{20z\log z}H_0g_0^2)^2P^2)$ of (A, B). Taking G' sufficiently large we have done with the case of $(t_i^{-1}U_iV_iab, p)$. Similarly for other cases.

4.4.15.1. Corollary. *Hereafter we can add the conditions of* 4.4.15 *on* 4.4.14.

4.4.16. Lemma. *If i, t_i, t'_i, q_i, q'_i, a, b are fixed, the number of possible pairs (p_i, p'_i), satisfying the conditions in* 4.4.15 *and*

$$t_it'_i(p_iq'_i+q_ip'_i+q_iq'_i)=U_iV_iab,$$

is

$$<G'^4(q_i, q'_i)^2.$$

Proof. We have

$$p_iq'_i+p'_iq_i=(t_it'_i)^{-1}U_iV_iab-q_iq'_i.$$

Therefore the solution $(p_i \bmod (q_i, q'_i)^{-1}q_i,\ p'_i \bmod (q_i, q'_i)^{-1}q'_i)$ is unique, if ab is fixed. Then we have

$$\ll \{t_i t'_i q_i q'_i\}^{-1} \cdot \{(q_i, q'_i) G' U_i\}^2 \ll G'^4 (q_i, q'_i)^2$$

of (p_i, p'_i).

The next two lemmas 4.4.17 and 4.4.17.1 are to suggest the nature of p_i, q_i, etc.

4.4.17. Lemma. *If i, t_i, t'_i, a, b are fixed, the number of possible quadruples (p_i, q_i, p'_i, q'_i), satisfying the conditions in* 4.4.15, *and also*

$$(p_i, q_i)=1, \quad (p'_i, q'_i)=1, \quad p_i \geqq q_i \geqq 1, \quad p'_i \geqq q'_i \geqq 1$$

and

$$t_i t'_i (p_i q'_i + q_i p'_i + q_i q'_i) = U_i V_i ab,$$

is

$$\ll G'^4 (t_i^{-2} + t_i'^{-2}) U_i^2.$$

Proof. Fixing p_i and q_i, the solution (p'_i mod (p_i+q_i), q'_i mod q_i) is unique, as $(p_i+q_i, q_i)=1$. Therefore we have

$$\ll \{(p_i+q_i)^{-1} t_i'^{-1} G' U_i + 1\} \cdot \{q_i{}^{1} t'_i G' U_i + 1\}$$

of pairs (p'_i, q'_i), if p_i and q_i are fixed. We have

$$\sum_{p_i; G' U_i > p_i > G'^{-1} U_i} (p_i+q_i)^{-1} \ll \log G'$$

and

$$\sum_{q_i; G' U_i > q_i > G'^{-1} U_i} q_i^{-1} \log G' \ll (\log G')^2.$$

We have, also,

$$\sum_{p_i, q_i} (p_i+q_i)^{-1} \ll \sum_{p_i} 1 \ll t_i^{-1} G' U_i,$$
$$\sum_{p_i, q_i} q_i^{-1} \ll \sum_{p_i} 1 \times \log G' \ll t_i^{-1} G' \log G' \cdot U_i,$$
$$\sum_{p_i, q_i} 1 \ll (t_i^{-1} G U_i)^2.$$

These gives us the number of quadruples (p_i, q_i, p'_i, q'_i) is

$$\ll (t_i'^{-2} + (t_i t'_i)^{-1} + t_i^{-2}) G'^4 U_i^2$$
$$\ll G'^4 (t_i^{-1} + t_i'^{-1})^2 U_i^2.$$

4.4.17.1. Lemma. *We have*

$$\sum_{t_i, t_i'} (t_i t_i')^{-1} \ll G'^2 \{\exp(\sum_p p^{-1})\}^2 \ll G'^3 (\log \nu(U_i V_i))^2$$

where p runs through the set of primes such that

$$p \leqq G'\nu(U_i V_i) \quad \textit{and} \quad p \mid U_i V_i.$$

Proof. Easily obtained from 4.4.15 (i).

4.4.18. Lemma. *We have, from* 4.4.14,

$$\begin{pmatrix} V_i & 0 \\ 0 & U_i \end{pmatrix} \begin{pmatrix} B & B' \\ A & A' \end{pmatrix} = \begin{pmatrix} B_i & B_i' \\ A_i & A_i' \end{pmatrix} \begin{pmatrix} t_i' q_i' & t_i'(ab)^{-1}(-p_i' + t' q_i') \\ t_i q_i & t_i(ab)^{-1}((p_i + q_i) + t' q_i) \end{pmatrix},$$

where t' is that of 4.4.9 (ii) (*and* 4.4.9.1) *with $\ddot{\varepsilon}' = \ddot{\varepsilon}^{\Delta}$ $(=\varepsilon)$.*

Proof. It is easy.

4.4.19. Proposition. *Suppose that integers A, B, A', B', t', t_i, t_i', p_i, q_i, p_i', q_i', a, b are given, so that*

$$AB' - BA' = \varepsilon = \pm 1,$$
$$A = a\ddot{A}, \qquad B = b\ddot{B},$$
$$\det \begin{pmatrix} t_i' q_i' & t_i'(ab)^{-1}(-p_i' + t' q_i') \\ t_i q_i & t_i(ab)^{-1}((p_i + q_i) + t' q_i) \end{pmatrix} = U_i V_i,$$
$$G'^2 > (t_i' q_i')(t_i q_i)^{-1} > 1.$$

Let $\ddot{A}^{\triangledown}$ and $\ddot{B}^{\triangledown}$ be chosen as $\ddot{A}'$ and $\ddot{B}'$ of 4.4.9 (ii) *with $\ddot{\varepsilon}' = \varepsilon$. Suppose that we have integers A_i, B_i, A_i', B_i' satisfying* 4.4.14 *and that $A_i > A_i' > 0$. Then, we have*

EITHER that

$$(|\lambda_i| BA^{-1})^{-1} (\longrightarrow A_i B_i^{-1} (\Longrightarrow A_i' B_i'^{-1},$$

OR that such α, that $|\alpha| (\longrightarrow BA^{-1} (\Longrightarrow B'A'^{-1}$, gives a minor contribution to 4.1.5.

We have to choose L_0 sufficiently large, depending on G'.

Proof. We have

$$\begin{aligned}(V_i B)(U_i A)^{-1} &= \{(t_i' q_i') B_i + (t_i q_i) B_i'\} \cdot \{(t_i' q_i') A + (t_i q_i) A_i'\}^{-1} \\ &= \{\zeta_i B_i + B_i'\} \cdot \{\zeta_i A_i + A_i'\}^{-1},\end{aligned}$$

where $\zeta_i = (t_i' q_i')(t_i q_i)^{-1}$. Then, through Theorem 172 in [12] and by the assumption that $\zeta_i = (t_i' q_i')(t_i q_i)^{-1} > 1$, we have $((V_i B)(U_i A)^{-1})^{-1} (\longrightarrow A_i B_i^{-1}$

$(\Longrightarrow A_i'B_i'^{-1}$. If λ_i is rational, (therefore an integer by the assumption of Theorem), then $|\lambda_i|=V_iU_i^{-1}$. Therefore, we are done with, in this case.

Suppose λ_i is irrational. We have

$$|\lambda_i|=V_iU_i^{-1}+\Omega_i$$

with $|\Omega_i|\leqq(L_0P^2)^{-1}$. Then, we have

$$(|\lambda_i|BA^{-1})^{-1}=\{(\zeta_i+\tau_i)A_i+A_i'\}\{(\zeta_i+\tau_i)B_i+B_i'\}^{-1},$$

where

$$\tau_i=(-\varepsilon_i)\{\Omega_iBA^{-1}(\zeta_iA_i+A_i')^2\}\{1+\varepsilon_i\Omega_iBA^{-1}A_i(\zeta_iA_i+A_i')\}^{-1}.$$

Here, taking L_0 sufficiently large, we can suppose that

$$|\Omega_iBA^{-1}A_i(\zeta_iA_i+A_i')|<\tfrac{1}{2}$$

and that

$$|\tau_i|<L_0^{-1/2}.$$

If $\zeta_i+\tau_i>1$, then we are done with, by the same theorem in [12]. It is impossible that $\zeta_i+\tau_i=1$, as λ_i is irrational. Suppose, then, $\zeta_i+\tau_i<1$. This means that $\varepsilon_i\Omega_i<0$ and $1>\zeta_i+\tau_i>1-L_0^{-1/2}$, as $\zeta_i>1$. Let us put as

$$\zeta_i+\tau_i=(1+\zeta_i'^{-1})^{-1}.$$

Then $\zeta_i'\gg L_0^{1/2}$. And then,

$$\begin{aligned}(|\lambda_i|BA^{-1})^{-1}&=\{(1+\zeta_i'^{-1})^{-1}A_i+A_i'\}\{(1+\zeta_i'^{-1})^{-1}B_i+B_i\}^{-1}\\&=\{\zeta_i'(A_i+A_i')+A_i'\}\{\zeta_i'(B_i+B_i')+B_i'\}^{-1}.\end{aligned}$$

Therefore the theorem in [12] tells us that

$$(|\lambda_i|BA^{-1})^{-1}(\longrightarrow(A_i+A_i')(B_i+B_i')^{-1}(\Longrightarrow A_i'B_i'^{-1}.$$

Also we have

$$(A_i+A_i')B_i'-(B_i+B_i')A_i'=\varepsilon_i.$$

We have

$$\begin{aligned}&|\lambda_i|BA^{-1}-(B_i+B_i')(A_i+A_i')^{-1}\\&\quad=\varepsilon_i\{(A_i+A_i')(\zeta_i'(A_i+A_i')+A_i')\}^{-1}\ll L_0^{-1/2}(A_i+A_i')^{-2}.\end{aligned}$$

We can suppose that A and B are $[K,K^{\varkappa})$-regular. Then, calculations, similar to the proof in the case $p_iq_i^{-1}>G'^{1/2}$ of 4.4.15 (iii), applied to

$$|\{a(B_i+B_i')\}\{b(A_i+A_i')\}^{-1}-|\lambda_i|\ddot{B}\ddot{A}^{-1}|\ll L_0^{-1/3}\{b(A_i+A_i')\}^{-2},$$

tell us that there are $O(L_0^{-1/4}P^2)$ of (A, B, A', B'). Then such α gives a minor contribution $O(L_0^{-1/5}P^2)$ to 4.1.5, by taking L_0 sufficiently large.

4.4.20. Lemma. *Suppose we have*

$$|\alpha|=BA^{-1}+\omega,$$
$$|\lambda_i|=V_iU_i^{-1}+\Omega_i$$

and

$$|\lambda_i\alpha|=B_iA_i+\omega_i,$$

so that

$$V_iB=t_i'q_i'B_i+t_iq_iB_i',$$
$$U_iA=t_i'q_i'A_i+t_iq_iA_i',$$

and

$$A_iB_i'-B_iA_i'=\varepsilon_i\,(=\pm1).$$

We have, then,

$$\omega_i=\Omega_iBA^{-1}+|\lambda_i|\omega+(\varepsilon_it_iq_i)(U_iAA_i)^{-1}.$$

Proof. It is easy.

Remark. We will use this lemma, in the form that

$$\omega=\varepsilon\{A(A\alpha^*+A')\}^{-1},$$
$$AB'-BA'=\varepsilon\,(=\pm1),$$
$$\alpha^*>1 \qquad \text{(real number)},$$

and A, B, A', B', A_i, t_i, q_i, t_i', q_i' are to satisfy 4.4.14, where $\ddot{A}^{\prime}$ and $\ddot{B}^{\prime}$ are $\ddot{A}'$ and $\ddot{B}'$ of 4.4.9 (ii) with $\ddot{\varepsilon}'=\varepsilon$.

4.4.21. Lemma. *With the notation of* 4.4.20, *we put*

$$M_i=\varepsilon|\lambda_i|\{A(A\alpha^*+A')\}^{-1}+\varepsilon_it_iq_i(U_iAA_i)^{-1}.$$

We suppose that, $\varepsilon=\varepsilon_i$, *(See* 4.4.14). *We have, then,*

$$(H_0g_0P)^{-2}\ll|M_i|\ll G'(H_0^{-1}P)^{-2}.$$

Proof. We have, from 4.4.1, that $(1<)\alpha^*\ll g_0^2$, because $A^*\gg\ll\alpha^*A$ and $A=A_*$. The rest is easy.

4.4.22. Remark. Suppose that $\varepsilon\,(=\pm 1)$ a, b, a_i, b_i $(i=1, \cdots, 4)$ are fixed. Suppose $\ddot{A}$ and $\ddot{B}$ are given so that $A=a\ddot{A}$ and $B=b\ddot{B}$ lie in the interval $[g_0^{1/2}P, g_0P]$. We have, then, by choosing K sufficiently large, at most one possibility of (A_i, B_i) such that $a_i|A_i$, $b_i|B_i$ and that A_i and B_i belong to $[H_0^{-1}P, H_0P]$. See 1.4.1.1. We have uniquely $\ddot{A}^{\nabla}$ and $\ddot{B}^{\nabla}$ such that

$$\ddot{A}\ddot{B}^{\nabla}-\ddot{B}\ddot{A}^{\nabla}=\varepsilon$$

and

$$\ddot{A}>\ddot{A}^{\nabla}\geqq 1.$$

We have, also, uniquely A_i' and B_i' such that

$$A_iB_i'-B_iA_i'=\varepsilon$$

and

$$A_i>A_i'\geqq 1.$$

These mean that, supposing that ε, a, b, a_i, b_i are fixed and K is sufficiently large, the possibility for t_i, t_i', p_i, q_i, p_i', q_i' $(i=1, \cdots, 4)$ in 4.4.14 is one at most, if $\ddot{A}$ and $\ddot{B}$ are fixed. Therefore, fixing ε, a, b, a_i, b_i, we can divide the set of $(\ddot{A}, \ddot{B})$'s disjointly, according to t_i, t_i', p_i, q_i, p_i', q_i' $(i=1, \cdots, 4)$ in 4.4.14.

These arguments go similarly when we begin with a, b, $\ddot{A}$ and $\ddot{A}^{\nabla}$, in place of a, b, $\ddot{A}$ and $\ddot{B}$.

Note that we have not used the assumptions (iv) and (v) up to now. These heavy assumptions will be needed in 4.5.

4.5. Kloosterman's domain II

4.5.1. Lemma. *Suppose that we have the conclusion of* 4.4.14. *Suppose that* $A_i(2A_i')^{-1}$ *has "good partial fractions with respect to* H_0, 2.2.10". (*We suppose that such* α, *that gives a minor contribution to* 4.1.5, *is not considered here, according to* 4.1.3, 4.4.1, 4.4.3, 4.4.3.1, 4.4.4, 4.4.15, 4.4.19 *and* 4.4.21.) *We put as*

$$\begin{aligned}&\operatorname{sgn}\alpha=\eta_0\ (=\pm 1), \qquad \operatorname{sgn}\lambda_i=\eta_i\ (=\pm 1),\\ &F_i=\Omega_iBA^{-1},\\ &M_i=\varepsilon\{|\lambda_i|A^{-1}(A\alpha^*+A')^{-1}+t_iq_i(U_iAA_i)^{-1}\}.\end{aligned}$$

Note that $\varepsilon_i=\varepsilon$ *by* 4.4.14. *We have, then, for* $S_i(\alpha)$ *in* 4.1.4, *that*

$$S_i(\alpha)=\frac{1}{2}\rho_0(2A_i)^{-1}S\left(\frac{B_i}{A_i}\right)(\varepsilon\omega_i)^{-1/2}$$
$$+\sum_{\xi_i}(\pm)\sum_{h_i}\sum_{\nu_i}\rho_{\nu_i}((2h_i-1)!!)^{-1}(-\eta_i\eta_0 2\pi\sqrt{-1})^{h_i-1}(2A_i)^{-1}S_i^{(\nu_i)}\left(\frac{B_i}{A_i}\right)$$
$$\times e\left(\frac{\varepsilon}{8}\frac{A_i'}{A_i}\nu_i^2\right)e\left(\frac{1}{2}\eta_0\eta_i\omega_i\xi_i^2-\frac{1}{2A_i}\eta_i\eta_0\xi^i\nu_i\right)$$
$$\times\omega_i^{h_i-1}\left(\xi_i-\frac{\nu_i}{2A_i\omega_i}\right)^{2h_i-1}$$
$$+O(H_0^{-20}P^{1/2}).$$

Here ξ_i is ξ_i'' or ξ_i' and $\pm$ corresponds to this choice, h_i is taken over $1\leqq h_i\leqq H_0^{500}$, ν_i is taken over $|\nu_i|\leqq H_0^{200}$. Also $S_i^{(\nu_i)}(B_i/A_i)$ is one of $S(B_i/A_i)$ and $S^*(B_i/A_i)$, and ρ_0, ρ_{ν_i} are one of such ρ that $\rho=0$ or $\rho^8=1$. Their choices are determined by the residues mod 8 *of A_i, B_i, A_i', B_i', (ε_i), and $\hat{A}_i, \hat{A}_i$ being the odd divisor part of A_i, and by residues* mod 4 *of ν_i. $\omega_i=F_i+M_i$.*

Proof. We apply 2.2.12. Putting $|\lambda_i\alpha|=B_iA_i^{-1}+\omega_i$, we have $\omega_i=F_i+M_i$, owing to 4.4.20. We have, on the other hand, by 4.4.19, that

$$(|\lambda_i|BA^{-1})^{-1})\longrightarrow A_i/B_i\,(\Longrightarrow A_i'/B_i',$$

and, therefore, with $0<\theta_+$, $\theta_+'<1$,

$$\begin{aligned}\omega_i&=|\lambda_i|(BA^{-1}+\varepsilon\theta_+A^{-2})-B_iA_i^{-1}\\&=\varepsilon\theta_+'A_i^{-2}+\varepsilon\theta_+|\lambda_i|A^{-2}\end{aligned}$$

We have, then,

$$|A_i\omega_i\xi_i|<A_i^{-1}\xi_i''+A^{-2}A_i\xi_i''2E_{100}<0.2+o(1)<0.3,$$

and

$$A_i^2|\omega_i|\ll 1.$$

We have, combining with 2.1.4, that

$$S_i(\alpha)=\sum_{\xi_i}(\pm)\sum_{\nu_i}\rho_{\nu_i}(2A_i)^{-1}S_i^{(\nu_i)}\left(\frac{B_i}{A_i}\right)e\left(\frac{1}{2}\omega_i\xi_i^2-(2A_i)^{-1}\xi_i\omega_i\right)$$
$$\times\operatorname{sgn}(\xi_i-\nu_i(2A_i\omega_i)^{-1})|\omega_i|^{-1/2}\cdot\Psi_\varepsilon(|\omega_i|^{1/2}|\xi_i-\nu_i(2A_i\omega_i)^{-1}|)$$
$$+O(H_0^{-20}P^{1/2}),$$

where, ν_i, ρ_{ν_i} and $S_i^{(\nu_i)}(\cdots)$ are as explained above. We substitute 2.2.5

(iv) with $k=H_0^{100}$ there. The term corresponding to

$$\tfrac{1}{2}e(-\tfrac{1}{2}\varepsilon(\xi^2-\tfrac{1}{4}))$$

in 2.2.5 (iv) appears as

$$\sum_{\nu_i}\rho_{\nu_i}(2A_i)^{-1}S_i^{(\nu_i)}\left(\frac{B_i}{A_i}\right)e((8A_i)^{-1}\varepsilon A_i'\nu_i^2)e\left(\frac{1}{2}\,\omega_i\xi_i^2-(2A_i)^{-1}\xi_i\nu_i\right)$$
$$\times\operatorname{sgn}(\xi_i-\nu_i(2A_i\omega_i)^{-1})|\omega_i|^{-1/2}$$
$$\times\frac{1}{2}e\left(-\frac{1}{2}\omega_i(\xi_i-\nu_i(2A_i\omega_i)^{-1})^2\right)e\left(\frac{1}{8}\,\varepsilon\right).$$

We have

$$\begin{aligned}(8A_i)^{-1}\varepsilon A_i'\nu_i^2&+\tfrac{1}{2}\omega_i\xi_i^2-(2A_i)^{-1}\xi_i\nu_i-\tfrac{1}{2}\omega_i(\xi_i-\nu_i(2A_i\omega_i)^{-1})^2\\ &=(8A_i)^{-1}\varepsilon A_i'\nu_i^2-(8A_i)^{-2}\omega_i^{-1}\nu_i^2\\ &=(8A_i)^{-1}\varepsilon A_i'\nu_i^2-(8A_i)^{-2}\varepsilon A_i(A_iB_i^*+A_i')\nu_i^2\\ &=-\tfrac{1}{8}\varepsilon\beta_i^*\nu_i^2,\end{aligned}$$

where we have put β_i^* as $|\omega_i|^{-1}=A_i(A_i\beta_i^*+A_i')$. We have

$$\begin{aligned}\sum_{\nu_i}&\text{(the above sum)}\\ &=\sum_{\nu_i}\rho_{\nu_i}(2A_i)^{-1}\cdot S_i^{(\nu_i)}\left(\frac{B_i}{A_i}\right)\cdot e\left(-\frac{1}{8}\varepsilon\beta_i^*\nu_i^2\right)\\ &\quad\times\operatorname{sgn}(\xi_i-\nu_i(2A_i\omega_i)^{-1})\cdot|\omega_i|^{-1/2}e\left(\frac{1}{8}\varepsilon\right).\end{aligned}$$

As we have $\rho_{\nu_i}=\rho_{-\nu_i}$ and $\operatorname{sgn}(\xi_i-\nu_i(2A_i\omega_i)^{-1})=-\operatorname{sgn}(\xi_i-(-\nu_i)(2A_i\omega_i)^{-1})$ if $\nu_i\neq 0$, the terms with $\nu_i\neq 0$ all cancel out each other. The term with $\nu_i=0$ gives us

$$\frac{1}{2}e\left(\frac{1}{8}\cdot\varepsilon\right)\rho_0(2A_i)^{-1}S\left(\frac{B_i}{A_i}\right)|\omega_i|^{-1/2},$$

which gives us the first term of the conclusion by adjusting ρ_0. As for the other terms of 2.2.5 (iv), we have

$$\begin{aligned}A_i^{-1/2}&|\omega_i|^{-1/2}(|\omega_i|^{1/2}|\xi_i-\nu_i(2A_i\omega_i)^{-1}|)^{2k-1}\\ &=(A_i|\omega_i|)^{-1/2}((A_i|\omega_i|^{1/2})^{-1}|\nu_i-(2A_i\omega_i\xi_i)|)^{2k-1}\\ &\leqq A_i^{1/2}(H_0H_0^{200})^{2k-1},\end{aligned}$$

and that

$$((2k-1)!!)^{-1} \leqq ((k-1)!)^{-1} \ll k^{-(k-(1/2))}.$$

The choice $k = H_0^{500}$ is sufficient for the conclusion.

4.5.2. Lemma. *Under the same assumptions as in* 4.5.1, *we have*

$$((\pi\alpha)^{-1}(\sin \pi\alpha))^2 S_1(\alpha) \cdots S_4(\alpha) = \sum_{\mu=0,\pm 1} (-1)^{\mu}(B^{-1}A)^2 e(A^{-1}\mu B)$$

$$\times \prod_{i=1,\cdots 4,} \left\{ \begin{aligned} &\frac{1}{2}\rho_0(2A_i)^{-1} S\left(\frac{B_i}{A_i}\right)|M_i|^{-1/2} \\ &+\sum_{\xi_i}\sum_{\nu_i}\sum_{h_i}\sum_{l_i} c_*^{(i)}|M_i|^{l_i-h_i} A_i^{l_i+(1/2)-2h_i}\xi_i^{l_i} \\ &\times 2A_i^{-1/2}\cdot S_i^{(\nu_i)}\left(\frac{B_i}{A_i}\right)\cdot e\left(\frac{1}{2}\eta_0\eta_i F_i \xi_i^2\right) \\ &\times e\left(\frac{1}{8}\varepsilon A_i^{-1}A_i'\nu_i^2 + \frac{1}{2}\eta_0\eta_i\xi_i^2 M_i - (2A_i)^{-1}\eta_0\eta_i\xi_i\nu_i\right) \end{aligned} \right\}$$

$$+O(\delta H_0^{-2}P^2).$$

Here

$$c_*^{(i)} = \pm(\sqrt{2}\,\pi)^{-1/2}\rho_{\nu_i}((2h_i-1)!!)^{-1}(-\eta_i\eta_0\sqrt{-1})^{h_i-1}\varepsilon^{h_i+l_i}(-1)^{l_i}$$
$$\times \binom{2h_i-1}{l_i}\left(\frac{1}{2}\nu_i\right)^{2h_i-1-l_i},$$

where $\pm$ *corresponds to* $\xi_i = \xi_i''$ *or* ξ_i'. (*If* $\nu_i = 0$, *then* $\nu_i^0 = 1$ *and* $\nu_i^l = 0$ *for* $l > 0$). *Also* l_i *is taken over* $0 \leqq l_i \leqq 2h_i - 1$. *The other notations are the same as in* 4.5.1, *and we must take* L_0 *sufficiently large as the last but one step about* 4.4.4. *We have, also, that, after expanding the product on the right-hand side, each term is*

$$\ll H_0^2 g_0 \prod_{i=1,\cdots,4} (1+|c_*^{(i)}|(H_0 g_0)^{2h_i})\cdot P^2$$

and

$$\sum_{(\xi),(\nu),(h),(l)} \prod_i (1+|c_*^{(i)}|(H_0 g_0)^{2h_i}) \ll \exp(H_0^{500} g_0^2).$$

Remark. We will often write the right-hand side of the formula of the lemma as

$$\sum_{\mu,(\xi),(\nu),(h),(l)} (\cdots),$$

when it is expanded.

Proof. We have

$$((\pi\alpha)^{-1}(\sin \pi\alpha))^2 = \sum_{\mu=0,\pm 1}(-1)^{\mu}(4\pi^2)^{-1}(B^{-1}A)^2 e(A^{-1}\mu B)+o(1).$$

We have, as for the first term in the right-hand side in 4.5.1, that

$$(\varepsilon\omega_i)^{-1/2}=(|M_i|+|F_i|)^{-1/2}=|M_i|^{-1/2}(1+O(L_0^{-1}g_0^6)).$$

Similarly

$$\omega_i^{h_i-1}=\varepsilon_i^{h_i-1}|M_i|^{h_i-1}(1+O(2^{h_i}L_0^{-1}g_0^6)).$$

Also

$$(\xi_i-(2A_i\omega_i)^{-1}\nu_i)^{2h_i-1}=\sum_{l_i}\binom{2h_i-1}{l_i}\xi_i^{l_i}(-1)^{l_i}(\varepsilon\nu_i(2A_i)^{-1}(|F_i|+|M_i|))^{2h_i-1-l_i}$$

$$=\sum_{l_i}\binom{2h_i-1}{l_i}\xi_i^{l_i}(-1)^{l_i}\varepsilon^{l_i+1}(\nu_i(2A_i|M_i|)^{-1})^{2h_i-1-l_i}(1+O(2^{2h_i}L_0^{-1}g_0^6)).$$

Taking L_0 sufficiently large with respect to δ, H_0 and g_0, we obtain the expansion to be proved. The rest is easy.

4.5.3. Let the constants in 4.4.4, except G_0, be fixed as have been explained up to now. To treat α which are left untouched in 4.4, we proceed as follows: We suppose that

$$Z \geqq T \geqq G_0\nu([U_1V_1, \cdots, U_4V_4]).$$

Then we take variables and constants, which are positive integers except ε, η_0, α^* and α, in the following order.

(0) Let a_0 be

$$a_0=\prod_{p;\,\text{prime},\,K\leqq p<K^z} p.$$

Let $\varepsilon=\pm 1$, $\eta_0=\pm 1$, a and b be fixed so that

$$ab|a_0, \qquad (a,b)=1,$$

and

$$1\leqq \nu(x)\leqq 10\log z \qquad \text{for } x=a,\, b.$$

(i) Let us put, with u_0 such that $2^{u_0}\|[U_1, \cdots, U_4]$,

$$W_*^\circ = 2^{g'+u_0+5}a_0\Big(\prod_{p;\,\text{prime},\,p<K,\,p|U_1\cdots U_4} p\Big)^{g'}\times \Delta^1(U_1\cdots U_4;T)$$
$$\times \text{L.C.M. of } \{(U_{i_1}, U_{i_2});\ i_1, i_2=1,\cdots,4 \text{ and } i_1\neq i_2\},$$

$$W_0^\circ=2^{g'+5}a_0\Big(\prod_{p;\,\text{prime},\,p<K,\,p\mid U_1V_1\cdots U_4V_4} p\Big)^{g'}\times\Delta^1(U_1V_1\cdots U_4V_4;\,T)$$
$$\times\text{L.C.M. of }\{(U_{i_1},U_{i_2});\ i_i,\ i_2=1,\cdots,4\text{ and } i_i\neq i_2\},$$
$$W_*=W_*^\circ[V_1,\cdots,V_4]$$

and

$$W_0=W_0^\circ[U_1V_1,\cdots,U_4V_4].$$

See 2.3.9, 2.3.11.5 and 3.2.1. Clearly W_* divides W_0.

(ii) $t;\ 0\leqq t\leqq G'$,

(ii′) $t';\ g_0^{-1/2}b<t'<ab$. (4.4.9 (ii)).

(iii) $t_i,\ t'_i,\ p_i,\ q_i,\ p'_i,\ q'_i\quad(i=1,\cdots,4)$;

$$t_it'_i(p_iq'_i+q_ip'_i+q_iq'_i)=U_iV_iab,$$
$$p_i\geqq q_i\geqq 1,\quad p'_i\geqq q'_i\geqq 1,\quad (p_i,q_i)=1,\quad (p'_i,q'_i)=1,$$
$$t'_iq'_i>t_iq_i>0,\quad t_i(p_i+q_i)>t'_ip'_i,$$

and

$$G'^2>(t'_iq'_i)(t_iq_i)^{-1}\quad(>1)\quad(4.4.19).$$

We impose the conditions in 4.4.15 on them.

(iv) $\ddot{A}$; has no prime divisor in $[K,K^z)$,

$$p^{g'+1}\nmid\ddot{A}\text{ if }p\text{ is a prime and }2\leqq p<K,$$
$$p^2\nmid\ddot{A}\text{ if }p\text{ is a prime and }p\geqq K,$$
$$g_0P>a\ddot{A}>g_0^{1/2}P,$$
$$\nu(\ddot{A})<1.1\log\log P.$$

The last condition is admissible by a well-known theorem of Hardy and Ramanujan, Problem 20 on p. 31 [20], for insatnce.

(iv′) $(\ddot{A}_0^\gamma,\ddot{B}_0^\gamma,\ddot{B}_0)$; Representatives enough to cover the set of $(\ddot{A}^\gamma,\ddot{B}^\gamma,\ddot{B})$, in which $\ddot{A}^\gamma$ is taken mod W_0, and $\ddot{B}^\gamma$ and $\ddot{B}$ are taken mod $2^{g'+u_0+5}a_0$, satisfying the following conditions;

$$\ddot{A}\ddot{B}^\gamma-\ddot{B}\ddot{A}^\gamma=\varepsilon,$$
$$p^{g'+1}\nmid\ddot{B}\qquad\text{if }p\text{ is a prime and }2\leqq p\leqq K,$$
$$p^2\nmid\ddot{B}\qquad\text{if }p\text{ is a prime and }p>K,$$
$$\ddot{B}\text{ has no prime divisor in }[K,K^z),$$
$$\ddot{A}^\gamma+t'\ddot{A}\equiv 0\bmod b,$$
$$\ddot{B}^\gamma+t'\ddot{B}\equiv 0\bmod a,$$

$$\begin{pmatrix}(U_ia)^{-1} & 0\\ 0 & (V_ib)^{-1}\end{pmatrix}\begin{pmatrix}\ddot{B} & \ddot{B}^{\digamma}\\ \ddot{A} & \ddot{A}^{\digamma}\end{pmatrix}\begin{pmatrix}t_i(p_i+q_i) & t'_ip'_i\\ -t_iq_i & t'_iq'_i\end{pmatrix}\in M_2(\boldsymbol{Z}),$$
$$\ddot{A}>\ddot{A}^{\digamma}>g_0^{-3}\ddot{A},$$

$$2c_0V_ib\xi''_i>t_i((p_i+q_i)\ddot{A}-q_i\ddot{A}^{\digamma})>t'_i(p'_i\ddot{A}+q'_i\ddot{A}^{\digamma})>H_0^{-2}V_ib\xi''_i,$$

$$(G'\ddot{A}\gg)\ddot{B}>G'^{-1}\ddot{A}$$

and

$$(t+1)\ddot{A}>\ddot{B}>t\ddot{A}.$$

(v) $\ddot{A}^{\digamma}$; $(\ddot{A}, \ddot{A}^{\digamma})=1$,

$$\ddot{A}^{\digamma}\equiv\ddot{A}_0^{\digamma} \bmod W_0,$$
$$\ddot{A}>\ddot{A}^{\digamma}>g_0^{-3}\ddot{A},$$

and

$$2c_0V_ib\xi''_i>t_i((p_i+q_i)\ddot{A}-q_i\ddot{A}^{\digamma})>t'_i(p'_i\ddot{A}+q'_i\ddot{A}^{\digamma})>H_0^{-2}V_ib\xi''_i.$$

(v′) $\ddot{B}, \ddot{B}^{\digamma}$; $\ddot{A}\ddot{B}^{\digamma}-\ddot{B}\ddot{A}^{\digamma}=\varepsilon$,

$$\ddot{B}\equiv\ddot{B}_0 \bmod 2^{g'+u_0+5}a_0,$$
$$(t+1)\ddot{A}>\ddot{B}>\max(t\ddot{A}, G'^{-1}\ddot{A}).$$
$$(U_ia)^{-1}(t_i(p_i+q_i)\ddot{B}-t_iq_i\ddot{B}^{\digamma})\in\boldsymbol{Z},$$
$$(U_ia)^{-1}(t'_ip'_i\ddot{B}+t'_iq'_i\ddot{B}^{\digamma})\in\boldsymbol{Z}, \qquad (2.3.11.5).$$

(We have $\ddot{B}^{\digamma}\equiv\ddot{B}_0^{\digamma} \bmod 2^{g'+u_0+5}a_0$.)

(vi) $\begin{pmatrix}B & B'\\ A & A'\end{pmatrix}$; $B=b\ddot{B}$, $B'=a^{-1}(\ddot{B}^{\digamma}+t'\ddot{B})$, $A=a\ddot{A}$, $A'=b^{-1}(\ddot{A}^{\digamma}+t'\ddot{A})$,

and A, A', B, B' are $[K, K^z)$-regular and satisfying the conditions in 4.4.15.

(vii) $\begin{pmatrix}B_i & B'_i\\ A_i & A'_i\end{pmatrix}$; $\begin{pmatrix}U_ia & 0\\ 0 & V_ib\end{pmatrix}\begin{pmatrix}B_i & B'_i\\ A_i & A'_i\end{pmatrix}=\begin{pmatrix}\ddot{B} & \ddot{B}^{\digamma}\\ \ddot{A} & \ddot{A}^{\digamma}\end{pmatrix}\begin{pmatrix}t_i(p_i+q_i) & t'_ip'_i\\ -t_iq_i & t'_iq'_i\end{pmatrix}$,

and A_i, A'_i, B_i, B'_i are supposed to satisfy the conditions in 4.4.15 and 4.5.1. (A_i, A'_i, B_i, B'_i are integers by (iv′), (v), (v′).)

(viii) α^*; $g_0>\alpha^*>1$ (real).

(viii′) $\alpha=BA^{-1}+\varepsilon A^{-1}(A\alpha^*+A')^{-1}$.

We prepare the set $\mathscr{S}$ of $\ddot{A}^{\digamma}$'s for a fixed $\ddot{A}$ of (iv), for which all of (iv′) ~ (viii) are not satisfied.

4.5.3.1. Explanation of 4.5.3. The existence of $\ddot{A}^{\digamma}, \ddot{B}, \ddot{B}_0^{\digamma}$ for given $\ddot{A}, \ddot{A}_0^{\digamma}, \ddot{B}_0, \ddot{B}_0^{\digamma}$ is assured by 2.3.11.5, even to the modulus $W_0^{\circ}[V_1, \cdots, V_4]$;

or if they do not fall into the range stated above, they give minor contributions to 4.1.5, as had been explained up to 4.4.22. We have

$$AB'-BA'=\varepsilon,\quad A>A'>0,\quad B>B'>0,$$
$$t_i=(U_iaB_i,\ V_ibA_i),\quad t_i'=(U_iaB_i',\ V_ibA_i'),$$
$$2c_0\xi_i''>A_i>A_i'>H_0^{-2}\xi_i''.$$

Especially $2^{2g'+2}\nmid t_i$. The conditions in (v) given by inequalities show that $\ddot{A}^{\Gamma}$ in (v) lies in an interval $J((\ddot{A}))$, say, determined by (0)$\sim$(iv) of 4.5.3, not depending on $\ddot{A}_0^{\Gamma}$, $\ddot{B}_0^{\Gamma}$ and $\ddot{B}_0$ in (iv′) there. Let us use the notation $((\ddot{A}))$ to suggest the choices of (0)$\sim$(iv) in 4.5.3. Therefore $\ddot{A}^{\Gamma}$ in (v) of 4.5.3 may be considered as those, for which

$$(\ddot{A}^{\Gamma},\ddot{A})=1,\qquad \ddot{A}^{\Gamma}\equiv\ddot{A}_0^{\Gamma} \bmod W_0$$

and

$$\ddot{A}^{\Gamma}\in J((\ddot{A})),$$

where $\ddot{A}_0^{\Gamma}$ is one of (iv′) in 4.5.3. In (viii′) we may have cases to obtain such α that was treated in the preceding sections. Estimates there had been in "L^1-norm", so we have no need to mind having such cases. By (viii′) we can take α^* in (viii) as a variable of integration in 4.1.5, after t, A, A' in $J((\ddot{A}))$ are fixed. Owing to the length of this note, there are confusions in the use of $^{\Delta}$ and $^{\Gamma}$. Those $\ddot{A}^{\Delta}$, etc., in 2.3 and 3.2 should correspond to $\ddot{A}^{\Gamma}$, etc., in 4.4. See the footnote of 4.4.9 (i).

4.5.4. Lemma. *We have, under* $((\ddot{A}))$ *in* 4.5.3.1 *that*

$$S_1^{(\nu_1)}\left(\frac{B_1}{A_1}\right)\cdots S_4^{(\nu_4)}\left(\frac{B_4}{A_4}\right)=16\rho(A_1\cdots A_4)^{1/2}$$

in 4.5.2, *where* $\rho=0$ *or* $\rho^8=1$,

$$\ddot{A}^{\Gamma}\equiv\ddot{A}_0^{\Gamma} \bmod W_0,\qquad \ddot{B}\equiv\ddot{B}_0 \bmod 2^{g'+u_0+5}a_0,$$

and ρ *is determined if* $\ddot{A}_0^{\Gamma}$, $\nu_1,\cdots,\nu_4$ *are fixed under* $((\ddot{A}))$.

Proof. From 4.4.15 (iv), (v) and the moduli W_0 and $2^{g'+u_0+5}a_0$, we have the same residues mod 8 of x, X and $\hat{X}$ with $X=2^x\hat{X}$, $(2\nmid\hat{X})$, for $X=\ddot{A}^{\Gamma}$ and $\ddot{A}_0^{\Gamma}$, or for $X=\ddot{B}$ and $\ddot{B}_0$, or $X=A_i$ and $A_i^{(0)}$, or $X=B_i$ and $B_i^{(0)}$. Then we can consult 2.3.8.

4.5.5. We proceed from 4.5.2 and 4.5.3 as follows;

$$\int_{\alpha \text{ in } 4.5.3} ((\pi\alpha)^{-1}(\sin \pi\alpha))^2 S_1(\alpha)\cdots S_4(\alpha)d\beta$$
$$= \sum_{(4.5.3)} \int_{\alpha^*;\, g_0>\alpha^*>1} d\alpha^*(-\varepsilon)(A\alpha^*+A')^{-2}\times\begin{Bmatrix}\text{the ``main'' terms in the}\\ \text{right-hand side of 4.5.2}\end{Bmatrix}$$
$$+O(\delta H_0^{-2}P^2),$$

where $\sum_{(4.5.3)}$ is to suggest the choices in 4.5.3 and, in the integral, we use 4.5.3 (viii′). We have, by the remark in 4.4.22, that this is

$$=\sum_{\ddot{A};\,((\ddot{A}))} \int_{\alpha^*} \sum_{(\ddot{A}_0^{\bar{r}},\ddot{B}_0^{\bar{r}},\ddot{B}_0);(\text{iv}')} \left\{\sum_{\ddot{A}^{\bar{r}};(\text{v})}(\cdots)\right\} d\alpha^* + O(\delta H_0^{-2}P^2),$$

where $\sum_{((\ddot{A}))}$ suggests that the choices of (0) ~ (iv) in 4.5.3, $g_0>\alpha^*>1$, and $\sum_{(\ddot{A}_0^{\bar{r}},B_0^{\bar{r}});(\text{iv}')}$ is taken over 4.5.3 (iv′). Also the sum $\sum_{\ddot{A}^{\bar{r}};(\text{v})}$ is taken over 4.5.3 (v) and, if there do not exist $\ddot{B}$ and $\ddot{B}^{\bar{r}}$ of 4.5.3 (v′) ~ (vii), we can regard as $\ddot{A}^{\bar{r}} \in \mathscr{S}$.

4.5.5.1. We suppose that $W_0=o(P)$.

4.5.6. We have, through 2.3.8 and 3.2.4 with the notations 3.2.3, 3.2.4 and $S_i^{(\nu_i)}(\cdots)$ in 4.5.1 with $g'+4$ in place of g' in $\tilde{W}_0$ in 3.2.4, that

$$\sum_{\ddot{A}^{\bar{r}};(\text{v})}(\cdots) = \sum_{\mu,(\nu),(h),(l),(\xi)} \sum_{I',I''} c'_* 16^{-1}(A_1\cdots A_4)^{-1/2} S_1^{(\nu_1)}\left(\frac{B_1}{A_1}\right)\cdots S_4^{(\nu_4)}\left(\frac{B_4}{A_4}\right)$$
$$\times \ddot{A}^{-1}\ddot{A}_{\tilde{W}_0}^{-1}\phi(\ddot{A}_{\tilde{W}_0})\times(2a^{g'+u_0+5}a_0)^{-1}(2^{g'},\ddot{A})$$
$$\times W_0^{-1}\int_{\ddot{A}^{\bar{r}}} g(A')d\ddot{A}^{\bar{r}}\times\sum_{\ddot{B}} f(B)e(A^{-1}\mu B)$$
$$+O(\Sigma''_{\ddot{A}^{\bar{r}}}|\tilde{S}(\ddot{A}^{\bar{r}})|)$$
$$+O(GP^{3/4}(\ddot{A},W_0)\times(\log P)^4).$$

Here we have used the fact that, from $\nu(\ddot{A})<1.1\log\log P$,

$$2^{\nu(\ddot{A})}<\log P,\quad 5^{(1/4)\nu(\ddot{A})}<(\log P)^{1/2},\quad \sigma_{-3/4}(\ddot{A})<\log P,$$

and

$$\ddot{A}_{W_0}\phi(\ddot{A}_{W_0})^{-1}\ll\log\log P.$$

Also we have used the fact that $(2^{g'+u_0+5}a_0,\ddot{A},W_0)=(2^{g'+u_0+5},\ddot{A},W_0)=(2^{g'},\ddot{A})$.

4.5.6.1. By what were explained in 4.4.22. and 4.5.3.1, we have

$$\sum_{\ddot{A};((\ddot{A}))}\int_{\alpha^*} d\alpha^* \sum_{(\ddot{A}_0^{\bar{r}},\ddot{B}_0^{\bar{r}}\ddot{B}_0);(\text{iv}')} \sum''_{\ddot{A}^{\bar{r}}} O(|\tilde{S}(\ddot{A}^{\bar{r}})|)\ll\delta|\lambda_1\cdots\lambda_4|^{-1/2}P^2.$$

We have, also, that, each time adjusting G suitably larger,

$$\sum_{\ddot{A};((\ddot{A}))}\int_{\alpha^*} d\alpha^* \sum_{(\ddot{A}_0^{\nabla},\ddot{B}_0^{\nabla},\ddot{B}_0);(\text{iv}')} O(GP^{3/4}(\ddot{A},\, W_0)\,(\log P)^4)$$
$$\ll \sum_{\ddot{A}} \sum_{\ddot{A}_0^{\nabla}} (\ddot{A},\, W_0)\times GP^{3/4}\,(\log P)^4$$
$$\ll \sum_{\ddot{A}} (\ddot{A},\, W_0)\times GW_0P^{3/4}\log P)^4,$$

which is, owing to 4.4.15 (ii) and 2.3.10.1,

$$\ll G\{T^{-1}\nu(U_1V_1\cdots U_4V_4)P+\tau(\Delta^1(W_0;\, T))P\}W_0P^{3/4}\,(\log P)^4$$
$$\ll GP^2\{T^{-1}\nu(U_1V_1\cdots U_4V_4)+\tau(\Delta^1(U_1V_1\cdots U_4V_4;\, T))\}W_0P^{-1/4}\,(\log P)^4.$$

This is

$$\ll\delta|\lambda_1\cdots\lambda_4|^{-1/2}P^2$$

by the assumption of the Theorem, as T will be chosen $=G_0\nu(U_1V_1\cdots U_4V_4)$ in 4.5.7.6.

4.5.7. Let $\ddot{A}$, under the choices $((\ddot{A}))$, be fixed. Let μ, (ξ), (ν), (h), (l) be fixed also. We consider the sum

$$\sum_{(\ddot{A}_0^{\nabla},\ddot{B}_0^{\nabla},\ddot{B}_0);(\text{iv}')} 16^{-1}(A_1\cdots A_4)^{-1/2}S_i^{(\nu_1)}\left(\frac{B_1}{A_1}\right)\cdots S_4^{(\nu_4)}\left(\frac{B_4}{A_4}\right)$$

arising from the plausible "main" term in the right-hand side in 4.5.6.

4.5.7.1. We take up the modulus W_* in 4.5.3 (i), which is a divisor of W_0. Let $(\ddot{A}_\circ^{\nabla}, \ddot{B}_\circ^{\nabla}, \ddot{B}_\circ^{\nabla})$'s be representatives enough to cover the set of $(\ddot{A}_0^{\nabla}, \ddot{B}_0^{\nabla}, \ddot{B}_0)$'s in 4.5.3 (iv′), in which $\ddot{A}_\circ^{\nabla}$ is taken mod W_* and $\ddot{B}_\circ^{\nabla}$ and $\ddot{B}_\circ$ are taken mod $2^{g'+u_0+5}a_0$. We fix $(\ddot{A}_\circ^{\nabla}, \ddot{B}_\circ^{\nabla}, \ddot{B}_\circ)$ and let $(\ddot{A}_0^{\nabla}, \ddot{B}_0^{\nabla}, \ddot{B}_0)$ run so that

$$\ddot{A}_0^{\nabla}\equiv\ddot{A}_\circ^{\nabla} \bmod W_*$$

and

$$\ddot{B}_0^{\nabla}\equiv\ddot{B}_\circ^{\nabla} \bmod 2^{g'+u_0+5}a_0.$$

As $2^{g'}\nmid X$ for $X=\ddot{A}_0^{\nabla}, \cdots, \ddot{B}_\circ$, we have the same power of 2 for $\ddot{A}_0^{\nabla}$ and $\ddot{A}_\circ^{\nabla}$, etc. We have, then, through 2.1.2.1, 2.3.11.5, 2.3.7 and 4.4.15, that

$$16^{-1}(A_1\cdots A_4)^{-1/2}S_1^{(\nu_1)}\left(\frac{B_1}{A_1}\right)\cdots S_4^{(\nu_4)}\left(\frac{B_4}{A_4}\right)=\rho\sum_{i=1,\cdots,4} J_0\left(\frac{(p_i+q_i)\ddot{A}-q_i\ddot{A}_0^{\nabla}}{(t_i^{(A)}t_i^{(ab)})^{-1}U_ia}\right),$$

where $t_i^{(A)}$ and $t_i^{(ab)}$ have the corresponding meaning to $t_1^{(A)}$ and $t_1^{(ab)}$ in 2.3.5 given with respect to $(\ddot{A}_\circ^{\nabla}, \ddot{B}_\circ^{\nabla}, \ddot{B}_\circ)$.

4.5.7.2. We prepare Z, which will be the Z_0 in the Theorem, such that $Z \geqq T\,(\geqq 3)$, that $Z \geqq a_0$, that all common prime divisors of $U_1^\sharp \cdots U_4^\sharp$ and $V_1 \cdots V_4$ are $<Z$, that all common prime divisors of U_{i_1} and U_{i_2} $(i_1 \neq i_2)$ are $<Z$, and that there exist one i_0 $(i_0 = 1, \cdots, 4)$ and a prime divisor p_0 of $U_{i_0}^\sharp$ satisfying $p_0 \geqq Z$ (and, therefore, $p_0 \nmid U_i^\sharp$ for $i \neq i_0$).

4.5.7.3. We want to apply 2.3.9 with this Z and W_*. We have

$$\sum_{(\ddot{A}_0^\nabla, \ddot{B}_0^\nabla, \ddot{B}_0);\,(\text{iv}'),\ \ddot{A}_0^\nabla \equiv \ddot{A}_0^\nabla \bmod W_*,\ \ddot{B}_0^\nabla \equiv \ddot{B}_0^\nabla \bmod 2g' + u_0 + 5a_0,} \prod_i \quad J_0(\cdots)$$

$$= \sum_{\ddot{A}_0^\nabla;\,\ddot{A}_0^\nabla \equiv \ddot{A}_0^\nabla \bmod W_*} \prod_i J_0(\cdots) + O\Big(\sum\nolimits_{\ddot{A}_0^\nabla}^{('')} 1\Big),$$

where $\sum_{\ddot{A}_0^\nabla}^{('')}$ is used to suggest that we will first perform the summation $\sum_{\mu,(\nu),(h),(\xi),(l)}$ in 4.5.6, as were explained at the end of 3.2.5. We have, then, the sum is

$$\begin{aligned} &\ll (U_1^\sharp \cdots U_4^\sharp)^{-1/2} [U_1^\sharp \cdots U_4^\sharp] \{\Delta^1(U_1; Z) \cdots \Delta^1(U_4; Z)\} \log (U_1 \cdots U_4) \\ &\quad \times a_0^4 \prod_i (U_i a, A_i^{(0)}) \cdot \tau(\Delta^1(\ddot{A}, Z)) \\ &\quad + Z^{-1} W_0 W_*^{-1} (\log \log P)^2 + G\nu(U_1 \cdots U_4) T^{-1} W_0 W_*^{-1} \\ &\quad + G\nu(U_1 \cdots U_4) + (\log P)^{0.8} + \sum\nolimits_{\ddot{A}_0^\nabla}^{('')} 1, \end{aligned}$$

because $(U_{i_1}^\sharp, U_{i_2}^\sharp U_{i_3}^\sharp U_{i_4}^\sharp \Delta^1(U_{i_1}^\sharp, Z))$ in 2.3.9 is a divisor of $\Delta^1(U_{i_1}; Z)$.

4.5.7.4. As for $\sum_{\ddot{A}_0^\nabla}^{('')} 1$, we go backwards;

$$\sum_{\ddot{A};\,((\ddot{A}))} \sum_{(\ddot{A}_0^\nabla, \ddot{B}_0^\nabla, \ddot{B}_0)} \sum\nolimits_{\ddot{A}_0^\nabla}^{('')} \qquad \text{(the "main" term in 4.5.6)}$$

$$= \sum_{\ddot{A}} \sum_{(\ddot{A}_0^\nabla, \ddot{B}_0^\nabla, \ddot{B}_0)} \sum\nolimits_{\ddot{A}_0^\nabla}^{('')} \begin{pmatrix} \sum_{\ddot{A}^\nabla;(\text{v})} ((\cdots), \text{ the left-hand side in 4.5.6}) \\ + O((\cdots) \text{ in the right-hand side in 4.5.6}) \end{pmatrix}$$

$$\ll \sum_{\ddot{A}} \sum_{\ddot{A}^\nabla \in \mathscr{S}} ((\pi\alpha)^{-1} (\sin \pi\alpha))^2 |S(\alpha) \cdots S_4(\alpha)|$$

$$+ \sum_{\ddot{A}} \sum_{A^\nabla \in \mathscr{S}} O(\cdots) + O(\delta |\lambda_1 \cdots \lambda_4|^{-1/2} P^2),$$

which is absorbed in the estimates in 4.5.6.1.

4.5.7.5. Let us combine 4.5.7.3 with the "main" term in 4.5.6. We have, by 3.2.5.1, the "main" term on the right-hand side is

$$\ll G W_0^{-1} P,$$

because

$$\ddot{A}_{\bar{W}_0}^{-1}\phi(\ddot{A}_{\bar{W}_0})\times(2^{g'+u_0+5}a_0)^{-1}(2^{g'},\ddot{A})\ll 1.$$

Therefore the combined contribution of the "main" term of 4.5.6 and 4.5.7.3 (without $O(\sum_{\ddot{A}^{\nabla}}^{(\prime\prime)}1)$) to 4.1.5 is

$$\ll \sum_{\varepsilon,a,b,t,t'}\ \sum_{t_i,t_i',p_i,q_i,p_i',q_i'}\ \sum_{\ddot{A}}\ \sum_{(\ddot{A}_{\circ}^{\nabla},\ddot{B}_{\circ}^{\nabla},\ddot{B}_{\circ})} \times GW_0^{-1}P$$
$$\times\begin{pmatrix} (U_1^{\#}\cdots U_4^{\#})^{-1/2}\times\cdots\times\tau(\varDelta^1(\ddot{A};Z)) \qquad \text{(as in 4.5.7.3)} \\ +Z^{-1}W_0W_*^{-1}(\log\log P)^2+G\nu(U_1\cdots U_4)\{T^{-1}W_0W_*^{-1}+1\}+(\log P)^{0.8}\end{pmatrix}.$$

This, is, owing to 4.4.22,

$$\ll \sum_{\varepsilon,a,b,t,t'}\ \sum_{\ddot{A}}\ \sum_{\ddot{A}_{\circ}^{\nabla}} GW_0^{-1}P\times(\cdots),$$

then

$$\ll\sum_{\ddot{A}}\tau(\varDelta^1(\ddot{A};Z))\times\sum_{\ddot{A}_{\circ}^{\nabla}}\prod_i(U_ia,A_i^{(0)})$$
$$\times GW_0^{-1}P(U_1^{\#}\cdots U_4^{\#})^{-1/2}[U_1^{\#},\cdots,U_4^{\#}]\cdot\{\varDelta^1(U_1;Z)\cdots\varDelta^1(U_4;Z)\}$$
$$\times\log(U_1\cdots U_4)$$
$$+P^2Z^{-1}G(\log\log P)^2+P^2T^{-1}G\nu(U_1\cdots U_4)$$
$$+P^2GW_0^{-1}W_*\{\nu(U_1\cdots U_4)+(\log P)^{0.8}\}.$$

4.5.7.6. We put as

$$T=G_0\nu(U_1\cdots U_4V_1\cdots V_4)$$

with a large positive constant G_0. We suppose that

$$Z\geqq G_0(\log\log P)^2 \qquad (\text{and } \geqq T).$$

We have

$$W_0^{-1}W_*\ll 2^{u_0}[V_1,\cdots,V_4]\cdot[U_1V_1,\cdots,U_4V_4]^{-1}$$
$$\ll\min(p_0^{-1},(G_0\log P)^{-0.8})\ll\min(Z^{-1},(G_0\log P)^{-0.8})$$

by the assumptions of the Theorem. It follows that the last three terms in 4.5.7.5 are

$$\ll\delta|\lambda_1\cdots\lambda_4|^{-1/2}P^2,$$

by choosing G_0 sufficiently large.

4.5.7.7. We treat the first term in the last estimate in 4.5.7.5. We have, by 4.4.15 (ii), that

$$\sum_{\ddot{A}_0^V} \prod_i (U_i a, A_i^{(0)}) \ll G W_* \prod_i \Delta_{G'}^1(U_i).$$

We have

$$\sum_{\ddot{A}} \tau(\Delta^1(\ddot{A}; Z)) \ll P (\log P)^{0.8}.$$

We have, then, the contribution of the first term is

$$\begin{aligned}
&\ll GP^2 \cdot W_0^{-1} W_* \cdot (\log P)^{0.8} \log(U_1 \cdots U_4) \times (U_1^\# \cdots U_4^\#)^{-1/2} [U_1^\#, \cdots, U_4^\#] \\
&\quad \times \Delta_{G'}^1(U_1) \cdots \Delta_{G'}^1(U_4) \times \Delta^1(U_1; Z) \cdots \Delta^1(U_4; Z) \\
&\ll GP^2 \cdot (\log P)^{0.8} \log(U_1 \cdots U_4) \times [U_1 V_1, \cdots, U_4 V_4]^{-1} [V_1, \cdots, V_4] \\
&\quad \times (U_1^\# \cdots U_4^\#)^{-1/2} [U_1^\#, \cdots, U_4^\#] \times \Delta_{G'}^1(U_1) \cdots \Delta_{G'}^1(U_4) \\
&\quad \times \Delta^1(U_1; Z) \cdots \Delta^1(U_4; Z).
\end{aligned}$$

This is

$$\ll \delta |\lambda \cdots \lambda_4|^{-1/2} P^2$$

by the assumptions of the Theorem.

4.5.7.8. Thus, as was proposed in 4.1.5.1, we have treated all of α's and obtain the Theorem, under so many restrictive assumptions on λ_i's.

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