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Linear Representations of the Galois Group over Local Fields

Hiroaki Hijikata

0.

Let F be a p-field, i.e. a complete discrete valuation field with a finite residue field of characteristic p, F^{alg} be an algebraic closure of F, F^{sep} be the separable closure of F in F^{alg}. Let $G=\mathrm{Gal}(F^{\mathrm{alg}}|F)=\mathrm{Gal}(F^{\mathrm{sep}}|F)$. It is a profinite group with the Krull topology. For a profinite group H, let $R(H)$ (resp. $\hat{H}$) denote the set of the equivalence classes of the finite dimensional continuous (resp. irreducible) representations σ over the complex number field. We are concerned with the classification or the parametrization of the set $\hat{G}$.

0.1. In [4], we gave a parametrization of a certain family of irreducible supercuspidal representations of $GL_n(F)$ induced from a certain class of representations (=très cuspidal representations of Carayol [1]) in terms of multiplicatively generic elements in F^{sep} (cf. § 5). In view of local Langlands conjecture (cf. [8]), it is desirable to have a similar parametrization for $\hat{G}$.

0.2. There are many interesting results on Galois representations, only a few of which I had occasions to study carefully. Primitive representations are studied by Weil [11], and finally classified by Koch [5]. Tame representations (i.e. $(\deg\sigma, p)=1$) are classified by Howe, Koch-Zink [7] or Moy [9]. Representations of degree p are classified by Koch [6]. However, the method employed there does not seem to match the way I imagine. In my talk at the meeting, I reported partial results on the parametrization, and more generally on the study of Galois representations based on the natural filtration of G by the absolute (upper) ramification groups G^v. But, later I noticed that, in the latter respect, a substantial work had been done by Deligne-Henniart [2]. Therefore in this note, confining ourselves to the former respect, we give a criterion of irreducibility of induced representations in Section 2, some reduction by r-invariant in Section 3, the description of the dual group of G^v/G^{v+} in Section 4, and its relation to

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the multiplicatively generic elements in Section 5.

0.3. For any finite subextension L of F^{alg}, let

$$\nu_L\colon L \twoheadrightarrow \boldsymbol{Z}\cup\{\infty\}$$

denote the normalized valuation and for $v \in \boldsymbol{R}$,

$$A_L(v) \text{ (resp. } A_L(v+)) = \{x \in L \mid \nu_L(x) \geqq v \text{ (resp. } > v)\},$$
$$U_L(0) = A_L(0)^{\times}, \quad U_L(v) = 1 + A_L(v) \quad \text{for } v > 0,$$
$$U_L(v+) = 1 + A_L(v+) \qquad \text{for } v \geqq 0,$$

$e_{L|F}$ (resp. $f_{L|F}$), be the index of ramification (resp. residual degree). If $L|F$ is separable, let $\delta_{L|F}$ be the exponent of the different i.e. $\delta_{L|F} = \nu_L(D_{L|F})$ and $c_{L|F} = \delta_{L|F} - (e_{L|F} - 1)$.

Let $\nu = \nu_F$ also denote its unique extension to F^{alg}, and set

$$A(v) \text{ (resp. } A(v+)) = \{x \in F^{\mathrm{alg}} \mid \nu(x) \geqq v \text{ (resp. } > v)\},$$
$$\Pi(v) = A(v) - A(v+),$$
$$U(v+) = 1 + A(v+) \qquad \text{for } v \geqq 0.$$

0.4. For the fundamental of the theory of local fields, we refer to Serre [10].

1.

For a real number $v \geqq -1$, let G^v be the upper ramification group of G ([10] IV §3 Remark 1]).

For convenience we put

(1) $$G^v = G \qquad \text{for } v \leqq -1$$

and for any $v \in \boldsymbol{R}$,

(2) $$G^{v+} = \text{the closure of } \bigcup_{\varepsilon > 0} G^{v+\varepsilon}.$$

Thus, $G^{-1+} = G^0$ is the ramification group, G^{0+} is the wild ramification group.

1.1. Since G^v is normal in G, G acts on $\hat{G}^v$ by $g\tau = \tau \circ I(g^{-1})\colon x \mapsto \tau(g^{-1}xg)$, $x \in G^v$. Let $\mathrm{Nor}_G(\tau)$ denote the normalizer of τ by this action:

(3) $$\mathrm{Nor}_G(\tau) = \{g \in G \mid \tau \circ I(g^{-1}) = \tau\}.$$

If $\sigma \in \hat{G}$, the restriction $\mathrm{Res}(\sigma, G\downarrow G^v)$ is a sum $\oplus_i \tau_i$ of G-conjugate irreducible representations τ_i of G^v. The map $\sigma \mapsto$ conjugate class of τ ($\tau = \tau_1$) induces the surjection $\theta = \theta_F^v$,

$$(4) \qquad \theta: \hat{G} \twoheadrightarrow G\backslash\hat{G}^v.$$

1.2. If $\sigma \in R(G)$ is not a trivial representation, i.e. $\sigma(G)\neq 1$, there exists a unique real number $v\geqq -1$ such that $\sigma(G^v)\neq 1, \sigma(G^{v+})=1$. Hence we can define an invariant $\alpha(\sigma)=\alpha_F(\sigma)$ of σ by

$$(5) \qquad \alpha(\sigma)=\sup\{v|\sigma(G^v)\neq 1\}=\inf\{v|\sigma(G^{v+})=1\}.$$

We put $\alpha(\sigma)=-\infty$ if $\sigma(G)=1$. If $\sigma(G^0)\neq 1$, this invariant $\alpha(\sigma)$ coincides with the $\alpha(\sigma)$ defined in [2] (for the Weil group).

Similarly one can define three more invariants.

$$(6) \qquad c(\sigma): =\sup\{v|\sigma(G^v)\not\subset \text{Center of } \sigma(G)\},$$

$$(7) \qquad \mathrm{ab}(\sigma): =\sup\{v|\sigma(G^v) \text{ is not abelian}\},$$

$$(c(\sigma)=\mathrm{ab}(\sigma)=-\infty \text{ if } \sigma(G) \text{ itself is abelian.})$$

$$(8) \qquad r(\sigma): =\sup\{v|\sigma(G^v) \text{ is irreducible}\},$$

$$r(\sigma)=\begin{cases}-\infty & \text{if } \sigma \text{ is reducible,}\\ \infty & \text{if } \deg\sigma=1.\end{cases}$$

When $v\leqq 0$, the third invariant ab(σ) is introduced in [7] in terms of lower ramification groups. Indeed they classified the irreducible representations with ab$(\sigma)\leqq 0$.

1.3. Let $a(\sigma)=a_F(\sigma)$ (resp. $\mathrm{sw}(\sigma)=\mathrm{sw}_F(\sigma)$) denote the Artin (resp. Swan) conductor of σ.

(i) If $\sigma \in \hat{G}$,

$$(9) \qquad \alpha(\sigma)=a(\sigma)/\deg\sigma-1 \qquad \text{([10] VI §2 Proposition 5)}$$

$$(10) \qquad =\mathrm{sw}(\sigma)/\deg\sigma \qquad \text{if } \alpha(\sigma)\geqq 0.$$

$$(11) \qquad \mathrm{sw}(\sigma)=\begin{cases}a(\sigma)-\deg\sigma & \text{if } \alpha(\sigma)\geqq 0,\\ 0 & \text{if } \alpha(\sigma)=-1.\end{cases}$$

(ii) If $\sigma=\oplus\sigma_i$ is a finite sum of $\sigma_i \in \hat{G}$,

$$(12) \qquad \alpha(\sigma)=\operatorname*{Min}_i \alpha(\sigma_i),$$

$$(13) \qquad \alpha(\sigma)+1\leqq a(\sigma)/\deg\sigma.$$

In (13), the equality occurs if and only if σ is *α-homogeneous* i.e. $\alpha(\sigma_i)=\alpha(\sigma_j)$ for any $i.j$,

1.4. For any finite subextension L of F^{sep}, fixed by the closed subgroup G_L of G, there is associated ([10] IV §3 Remark 2) a function $\psi_{L|F}: \boldsymbol{R}\to\boldsymbol{R}$, characterized by the properties:

$$(14) \qquad \psi_{L|F} \text{ is continuous and } \psi_{L|F}(0)=0,$$

$$(15) \qquad D^-\psi_{L|F}(x)=e_{L|F}[G^{x+}G_L: G_L]^{-1},$$

(16) $$D^{-}\psi_{L|F}(x)=e_{L|F}[G^{x}G_{L}: G_{L}]^{-1}$$

where D^{+} (resp. D^{-}) denotes the right (resp. left) derivative. It is originally defined for $x\geqq -1$, we have extended it to $\boldsymbol{R}$ by our convention (1).

The function $\psi_{L|F}$ is piecewise linear, strictly increasing, hence has an inverse

(17) $$\varphi_{L|F}:=\psi_{L|F}^{-1}.$$

A point of discontinuity of $\psi'_{L|F}$ (resp. $\varphi'_{L|F}$) is called a jump of $\psi_{L|F}$ (resp. $\varphi_{L|F}$). Let $\alpha(L|F)$ denote the last jump of $\psi_{L|F}$ (and $\alpha(L|F)=-\infty$ if $L=F$), thus we have

(18) $$\alpha(L|F)=\inf\{v|G_{L}\supset G^{v+}\}.$$

This $\alpha(L|F)$ is again essentially equal to that of [2]. Following their notation, set

(19) $$\psi_{L|F}^{\infty}(x):=e_{L|F}x-c_{L|F}, \qquad \varphi_{L|F}^{\infty}:=(\psi_{L|F}^{\infty})^{-1}.$$

1.5. We write down the properties of $\psi_{L|F}$ which we shall use.

(i) (20) $$\psi_{L|F}(x)=\psi_{L|F}^{\infty}(x) \qquad \text{iff } x\geqq\alpha(L|F).$$

(21) $$N_{L|F}(U_{L}(v))\subset U_{F}(\varphi_{L|F}(v)) \qquad v\geqq 0.$$

([10] V § 6 Proposition 8)

(22) $$T_{L|F}(A_{L}(v))=A_{F}(\varphi_{L|F}^{\infty}(v)) \qquad v\in \boldsymbol{R}.$$

(23) $$G_{L}\cap G^{v}=G_{L}^{w} \quad \text{with} \quad w=\psi_{L|F}(v).$$

(21+), (22+), (23+) One can replace v, w by $v+$, $w+$.

(ii) Let M be an intermediate extension of $L|F$. Then

(24) $$\psi_{L|M}\circ\psi_{M|F}=\psi_{L|F},$$

(25) $$\varphi_{M|F}\circ\varphi_{L|M}=\varphi_{L|F}.$$

Assume $v>0$, $\alpha(L|F)$, $\alpha(M|F)<v$, and put $w=\psi_{L|F}(v)=\psi_{L|F}^{\infty}(v)$, $u=\psi_{M|F}(v)=\psi_{M|F}^{\infty}(v)$. The map $x\rightarrow 1+x$, $x\in A_{L}(w)$ (or $A_{M}(u)$) induces the following commutative diagram:

(26) $$\begin{array}{ccc} A_{L}(w)/A_{L}(w+) & \xrightarrow{T_{L|M}} & A_{M}(u)/A_{M}(u+) \\ \downarrow{\scriptstyle 1+x} & & \downarrow{\scriptstyle 1+x} \\ U_{L}(w)/U_{L}(w+) & \xrightarrow{N_{L|M}} & U_{M}(u)/U_{M}(u+). \end{array}$$

A proof of the above is given in [2] 3.8 as a corollary to the evaluation of the error term $N_{L|F}(E(x))-E(T_{L|F}(x))$ of the cut exponential map $E(x)=\sum_{i=0}^{p-1}x^{i}/i!$. The commutativity of (26) itself can be also derived from ([10] V § 3 Lemma 5) without recourse to $E(x)$.

2.

Let L be a finite subextension of F^{sep}, $e=e_{L|F}$, $f=f_{L|F}$, $\delta=\delta_{L|F}$ and $c=c_{L|F}$. Let $\rho \in R(G_L)$, $\sigma=\text{Ind}(\rho, G_L\uparrow G)$ and

$$(0) \qquad w=\alpha_L(\rho), \qquad v=\varphi^{\infty}_{L|F}(w)=\frac{1}{e}(v+c).$$

2.1. If ρ is α-homogeneous i.e. $\alpha_L(\rho)=a_L(\rho)/\deg\rho-1$, then

$$(1) \qquad \frac{a_F(\sigma)}{\deg\sigma}=\frac{1}{e}\frac{a_L(\rho)}{\deg\rho}+\delta$$

([10] VI § 2, Corollary to Proposition 4)

i.e. $$(1') \qquad \frac{a_F(\sigma)}{\deg\sigma}-1=\varphi^{\infty}_{L|F}(\alpha_L(\rho))=v.$$

2.2. If σ is irreducible, then (of course ρ is irreducible and)

$$(2) \qquad \alpha(L|F)\leqq c_F(\sigma)\leqq\alpha_F(\sigma)=v$$

2.3. If $\alpha(L|F)\leqq v$, then

$$(3) \qquad \alpha_F(\sigma)=v$$

Hence by (1′), σ is α-homogeneous.

2.4. If $\alpha(L|F)\leqq v$, $\rho\in\hat{G}$ and moreover

$$(4) \qquad (a_F(\sigma), \deg\sigma)=1,$$

then σ is irreducible.
The condition (4) is equivalent to (4′) & (4″).

$$(4') \qquad f=1,$$

$$(4'') \qquad (a_L(\rho)+\delta\deg\rho,\ e\deg\rho)=1.$$

Further if $\alpha_L(\rho)\geqq 0$, (4″) is equivalent to (4*).

$$(4^*) \qquad (\text{sw}_L(\rho)+c\deg\rho,\ e\deg\rho)=1.$$

2.5. Proposition. *If* $\deg\rho=1$, $\alpha_L(\rho)=\text{sw}_L(\rho)=w\geqq 0$,

$$(5) \qquad \alpha(L|F)\leqq\frac{1}{e}(w+c) \quad \textit{and} \quad (e, w+c)=1,$$

then $\sigma=\text{Ind}(\rho, G_L\uparrow G)$ *is irreducible*, $\alpha_F(\sigma)=\text{sw}_F(\sigma)/\deg\sigma=e^{-1}(w+c)=\varphi_{L|F}(w)$ *and*

$$(6) \qquad \text{ab}_F(\sigma)=\alpha(L|F).$$

2.6. *Proof of* 2.1 ~ 2.5.

1. The quoted proof of [10] is given for deg $\rho=1$; it is valid without any change.

2. If $c=c_F(\sigma)<\alpha(L|F)$, then $G_L \not\supset G^{c+}$. Since $\sigma(G^{c+})\subset$ Center of $\sigma(G)$, ρ lifts to $G^{c+}G_L$, hence Ind $(\rho, G_L\uparrow G^{c+}G_L)$ and consequently σ can not be irreducible.

3. In general $\alpha(\sigma)\leqq a(\sigma)/\deg \sigma - 1$ (1.2(13)), hence $\alpha(\sigma)\leqq v$ by (1′).

If $\alpha(L|F)\leqq v$ i.e. $G_L\subset G^{v+}$, then $G_L^{w+}=G_L\cap G^{v+}=G^{v+}$. Since $\rho\in R(G_L/G_L^{w+})$, $\sigma\in R(G/G^{v+})$, i.e. $\sigma(G^{v+})=1$ and $\alpha(\sigma)\geqq v$.

4. Let $\sigma=\oplus\sigma_i$ with $\sigma_i\in\hat{G}$. $\alpha(L|F)\leqq v$ implies by 2.3 that σ is α-homogeneous, i.e. $a(\alpha)/\deg\sigma=a(\sigma_i)/\deg \sigma_i$ for any i. Now (4) implies $\deg\sigma|\deg\sigma_i$ i.e. $\sigma=\sigma_i$.

By (1), $a(\sigma)=f(a_L(\rho)+\delta\deg\rho)$, and $\deg\sigma=ef\deg\rho$, hence (4)$\Leftrightarrow$(4′) & (4″).

5. It remains to see (6). Let $v=\alpha(L|F)$. If $\sigma(G^v)$ is abelian, then ρ lifts to $G^vG_L\supsetneqq G_L$, and σ cannot be irreducible. Hence $\sigma(G^v)$ is not abelian. Let K be the Galois closure of $L|F$. Res $(\sigma, G\downarrow G_K)$ is the sum of the conjugates of Res $(\rho, G_L\downarrow G_K)$, hence $\sigma(G_K)$ is abelian. Since $G_L\supset G^{v+}$ and G^{v+} is normal in G, $G_K\supset G^{v+}$ and $\sigma(G^{v+})$ is abelian.

3.

Let $\sigma\in\hat{G}$, and $r(\sigma)=r$, i.e. $\sigma(G^r)$ is irreducible and $\sigma(G^{r+})$ is reducible. We call σ to be *r-isotypic* iff Res $(\sigma, G\downarrow G^{r+})$ is isotypic i.e. $\mathrm{Nor}_G(\theta(\sigma))=G$ by the map $\theta=\theta_F^{r+}$ of 1.1.

3.1. Proposition. *σ is not r-isotypic if and only if there exists (L, ρ) satisfying the folloiwng* (0)$\sim$(3).

(0) $\rho\in\hat{G}_L$, $\sigma=\mathrm{Ind}(\rho, G_L\uparrow G)$.

(1) $\psi_{L|F}$ *has only one jump r,*

consequently, $\psi_{L|F}(r)=r$ and $G_L\cap G^{r+}=G_L^{r+}$.

(2) *One of the following two holds*:

(2.1) *$r_L(\rho)=r$ and ρ is r-isotypic,*

(2.2) $r_L(\rho)>r$.

(3) $\mathrm{Nor}_G(\theta_L^{r+}(\rho))=G_L$ *by θ of* 1.1.

3.2. Proposition. *If (L, ρ), (L', ρ') satisfy* (0)$\sim$(3) *of the above,* Ind $(\rho, G_L\uparrow G)$ *is equivalent to* Ind $(\rho', G_{L'}\uparrow G)$ *if and only if (L, ρ) is conjugate to (L', ρ') by some $g\in G$, i.e. $gL=L'$ and $\rho'=\rho\circ I(g^{-1})$.*

3.3. Proposition. *Let $r=-1$ or* 0.

(i) *There is no r-isotypic σ with $r(\sigma)=r$. Hence any σ with $r(\sigma)=r$*

is induced from ρ *of* 3.1 *with* $r_L(\rho)>r$.

(ii) *In the statement of* (3.1) *the condition* (3) *can be replaced by the apparently weaker*

$$(3') \qquad \mathrm{Nor}_G(\theta_L^r(\rho))=G_L.$$

(iii) *If* $r=-1$, $\theta_L^r(\rho)=\rho$ *and* G_L *is normal in* G. *Hence* (3′) *is simply the condition of* $\mathrm{Ind}(\rho, G_L\uparrow G)$ *to be irreducible.*

(iv) *Thus, the reduction with respect to* $r(\sigma)$ *is fairly complete for* $r=0, -1$, *really complete for* $r=-1$.

It is also easy to observe that $r(\sigma)=-1$ if and only if σ is induced from some unramified G_L.

3.4. Proof of 3.1, 3.2 is straightforward. We take L as $G_L=\mathrm{Nor}_G(\theta_F^{r+}(\sigma))$. 3.3 follows from the fact that G^r/G^{r+} is procyclic for $r=0$, -1.

4.

Let $v\geqq 0$, and Ω be the fixed field of G^v. Let DG^v be the closure of the commutator subgroup of G^v, hence the fixed field of DG^v is the maximum abelian extension Ω^{ab} of Ω.

If $L\subset\Omega$, then $\psi_{L|F}(v)=\psi_{L|F}^{\infty}(v)$. Hence by (19)$\sim$(22) of Section 1, the family of groups $\{A_L(\psi_{L|F}(v))|L\subset\Omega\}$ (resp. $\{U_L(\psi_{L|F}(v))|L\subset\Omega\}$) together with the trace (resp. norm) map, is an inverse system of profinite groups.

4.1. Let ω_L: $L^\times\to\mathrm{Gal}(L^{\mathrm{ab}}|L)$ be the reciprocity homomorphism. The map

$$(0) \qquad gDG^v\longmapsto\omega_L^{-1}(g|_{L^{\mathrm{ab}}})$$

induces the following G-module isomorphisms.

$$(1) \qquad G^v/DG^v\simeq\varprojlim U_L(\psi_{L|F}(v)),$$

$$(2) \qquad G^v/G^{v+}\simeq\varprojlim U_L(\psi_{L|F}(v))/U_L(\psi_{L|F}(v)+),$$

$$(3) \qquad \simeq\varprojlim A_L(\psi_{L|F}(v))/A_L(\psi_{L|F}(v)+),$$

$$(4) \qquad G^v\widehat{/}G^{v+}\simeq\varinjlim A_L(\psi_{L|F}(v))\widehat{/}A_L(\psi_{L|F}(v)+),$$

where in the right, L runs over all finite subextensions of Ω. In the proof of (3), we have used the commutativity of (26) Section 1.

4.2. Fix one additive character τ: $F\to C^\times$, such that $\tau(A_F(0))\neq 1$, $\tau(A_F(0+))=1$, then identify L with its dual group $\hat{L}$ by

$$(5) \qquad x\longmapsto\hat{x}: y\longmapsto\tau(T_{L|F}(xy)).$$

Then the annihilator $A_L(\psi_{L|F}(v))^\perp$ is given by

(6) $$A_L(\psi_{L|F}(v))^{\perp}=A_L(-ev)=A(-v)\cap L.$$

Hence (4) turns into

(7) $$\widehat{G^v/G^{v+}}\simeq A(-v)\cap\Omega/A((-v)+)\cap\Omega$$

and

(8) $$\widehat{G^v/G^{v+}}-1\simeq\Pi(-v)\cap\Omega^{\times}/U(0+)\cap\Omega^{\times}.$$

4.3. Theorem. *The map* $\theta=\theta_F^v$ *of* 1.1 *induces the map* $\bar{\theta}$:

$$\begin{array}{ccc}\{\sigma\in G\,|\,\alpha(\sigma)=v\} & \xrightarrow{\bar{\theta}} & G\backslash\Pi(-v)\cap\Omega^{\times}/U(0+)\cap\Omega^{\times}\\ & & \big\downarrow\\ G\backslash(F^{\mathrm{alg}})^{\times}/U(0+) & \hookleftarrow & G\backslash\Pi(-v)/U(0+).\end{array}$$

Thus we have associated for any $\sigma\in\hat{G}$ with $\alpha(\sigma)\geqq 0$, an invariant $\bar{\theta}(\sigma)$, taking the value in $G\backslash(F^{\mathrm{alg}})^{\times}/U(0+)$, and

$$\mathrm{Nor}_G(\theta(\sigma))=\mathrm{Nor}_G(\bar{\theta}(\sigma)).$$

4.4. If v is not a rational number, $G^v/G^{v+}=1$ by (2). Hence we assume $v\in\boldsymbol{Q}$, and write as

(9) $$v=m/n,\quad n=n'p^s,\quad (m,n)=(n',p)=1.$$

Let π be a prime of F, $\mu\subset(F^{\mathrm{alg}})^{\times}$ be the group of the roots of unity of order prime to p. Fix one γ such that $\gamma^n=\pi^{-m}$. Then $\gamma\mu$ is a set of representatives of $\Pi(-v)/U(0+)$.

For $\sigma\in G$, ε, $\varepsilon'\in\mu$, we have

(10) $$\sigma(\gamma\varepsilon)\equiv\gamma\varepsilon' \bmod^{\times} U(0+) \quad \text{if and only if} \quad \sigma(\gamma\varepsilon)^{p^s}=(\gamma\varepsilon')^{p^s}.$$

$T=F((\alpha\varepsilon)^{p^s})$ is the maximal tamely ramified subfield of $F(\alpha\varepsilon)$, and

(11) $$\mathrm{Nor}_G(\alpha\varepsilon \bmod^{\times} U(0+))=G_T.$$

4.5. Proposition. *Let* $\bar{\theta}(\sigma)=\gamma\varepsilon \bmod^{\times} U(0+)$.

(i) *The following three conditions are mutually equivalent.*

a) $\mathrm{Nor}_G(\bar{\theta}(\sigma))=G$.

b) $c(\sigma)<\alpha(\sigma)$.

c) $n=p^s$ *and* $\varepsilon\in F^{\times}$.

(ii) *If* σ *does not satisfy any of* a) b) c), σ *is induced from* $\mathrm{Nor}_G(\bar{\theta}(\sigma))$ $=G_T\subsetneqq G$,, *with tame* T, *hence* $r(\sigma)$ *is either* -1 *or* 0.

4.6. Remark. In the case of i) the invariant $\bar{\theta}(\sigma)$ takes the value in the G-fixed points set $((F^{\mathrm{alg}})^{\times}/U(0+))^{G}$, which is isomorphic to $F^{\times}/U_F(0+)\otimes\boldsymbol{Z}[1/p]$ by the map $\gamma_{\varepsilon}\mapsto\varepsilon\otimes m/p^{s}$, Hence our $\bar{\theta}(\sigma)$ coincides with the invariant g_{σ} of Henniart [3].

5.

Let λ: $A(0)\to A(0)/A(0+)$ be the reduction homomorphism. Let E be a finite extension of F. We call $x\in(F^{\mathrm{alg}})^{\times}$ to be multiplicatively generic over E, iff the multiplicative group $\langle E^{\times}, x\rangle$, generated by $E^{\times}$ and x, satisfies the following $(0)=(0.1)+(0.2)$.

(0.1) $$\nu(\langle E^{\times}, x\rangle)=\nu(E(x)^{\times})$$

(0.2) $$\lambda(\langle E^{\times}, x\rangle\cap A(0))$$

generates the residue field $\lambda(A_{E(x)}(0))$ over $\lambda(A_E(0))$.

Let $\mathcal{M}_E$ denote the set of all multiplicatively generic elements over E, and $\mathcal{M}_E^n$ denote the subset of $\mathcal{M}_E$ consisting of elements of degree n over E.

5.1. For a natural number n, an integer m such that $(n, m)=1$, and an unramified extension M over F, let $E_m^n(M)$ denote the set of all monic polynomials $f(X)=X^n+\Sigma a(i)X^i$ of degree n over M satisfying the following $(1)=(1.1)+(1.2)$

(1.1) $$\nu(a(i))+im/n\geq m \qquad 0<i<n$$

(1.2) $$\nu(a(0))-m.$$

Let $E_m^n(M)^{\mathrm{gn}}$ denote the subset of $E_m^n(M)$ consisting of $f(X)$ with the extra condition (1.3).

(1.3) $\lambda(\pi^{-m}a(0))$ generates the residue field $\lambda(A_M(0))$ over $\lambda(A_F(0))$.

If $u=m/n$, the map $x\mapsto$ 'the minimal polynomial of x over the maximal unramified subfield M of $F(x)$' induces the natural bijection:

(2) $$G\backslash(\mathcal{M}_F\cap\Pi(u))\simeq\coprod E_m^n(M)^{\mathrm{gn}}$$

where the righthand side is the disjoint union over all finite unramified extensions M over F.

Restricting the attention to degree n part,

(3) $$G\backslash(\mathcal{M}_F^n\cap\Pi(u))\simeq E_m^n(F).$$

If E is a finite Galois extension of F, and $e=e_{E|F}$ is prime to n, the definition (1) implies $E_m^n(F)\subset E_{em}^n(E)$. Hence we have a natural inclusion

(4) $$G\backslash(\mathcal{M}_F^n\cap\Pi(u))\hookrightarrow G\backslash(\mathcal{M}_E^n\cap\Pi(u)).$$

5.2. Since γ_{ε} of 4.4 is multiplicatively generic over F, together with (2), we have the diagram:

$$(5)\quad \begin{array}{ccc} \coprod E^n_{-m}(M)^{\mathrm{gn}} \simeq G\backslash \mathscr{M}_F \cap \Pi(-v) & \longrightarrow & G\backslash(\mathscr{M}_F \cap \Pi(-v))/U(0+) \\ \downarrow & & \wr\| \\ \coprod E^n_{-m}(M)^{\mathrm{gn}}/\sim & \simeq & G\backslash \Pi(-v)/U(0+) \end{array}$$

where $f(X)=X^n+\cdots+\pi^{-m}c(0)\sim G(X)=X^n+\cdots+\pi^{-m}d(0)$ iff $c(0)\equiv d(0) \bmod^\times U(0+)$.

5.3. Starting with $f(X)\in E^n_{-m}(M)^{\mathrm{gn}}$, let β be a root of $f(X)=0$, and $L=F(\beta)$. Assume $\alpha(L|F)<v$ and put $w=\psi_{L|F}(v)=\psi^\infty_{L/F}(v)$. By 4.2, β determines $\hat{\beta}\in U_L((w/2)+)\widehat{/}U_L(w+)$. Let $\chi\in\hat{L}^\times$ be a lift of $\hat{\beta}$, and let $\sigma=\mathrm{Ind}(\chi, G_L\uparrow G)$. Using (iii) 3.3, one can see that σ is irreducible, and $\alpha(\sigma)=v$, $\deg\sigma=[L: F]=n[M: F]$. Thus to a pair (f,χ), we have associated an irreducible representation σ of G, arriving to the situation (not quite but) somewhat similar to that of GL.

What will happen if $\alpha(L|F)\geqq v$? Can we get all representation σ with $\alpha(\sigma)=v$ (or at least the ones with $(\mathrm{sw}(\sigma), \deg\sigma)=1$) in this way?

5.4. In view of 3.3 and 4.5, the essential case is when $\mathrm{Nor}_G(\bar{\theta}(\sigma))=G$, i.e. $n=p^s$ and $\bar{\theta}(\sigma)\in\Pi(-v)\cap\mathscr{M}^n_F$. Since $c(\sigma)>v$ in this case, if we assume σ is of degree n and is induced from some (ρ, L) with $\deg\rho=1$, then $\alpha(L|F)<v$ by 2.2, and (ρ, L) is necessarily of the form given in 5.3.

In general (still assuming $(\mathrm{sw}(\rho), \deg\rho)=1$), $r(\sigma)>0$ by 4.5, and $\sigma'=\mathrm{Res}\,(\sigma, G\downarrow G^{0+})$ is irreducible. Since G^{0+} is a pro-p-group, σ' is induced from some 1-dimensional representation of some subgroup.

Hence there exists a tamely ramified finite Galois extension E over F, such that $\sigma_E=\mathrm{Res}\,(\sigma, G\downarrow G_E)$ is irreducible and σ_E is induced from some 1-dimensional representation. Let $e=e_{E|F}$; then $\alpha_E(\sigma_E)=ev=em/n$, $(em, n)=1$. Hence σ_E is induced as in 5.3, from some β' with $\nu_E(\beta')=-ev$, i.e. $\beta'\in\mathscr{M}^n_E\cap\Pi(-v)$.

This means, in the parameter space $G\backslash\Pi(-v)$, if $\beta\in\mathscr{M}^n_F\cap\Pi(-v)$ gives $L=F(\beta)$ with $\alpha(L|F)\geqq v$, we shall substitute β by β', which is sufficiently near to β in the inclusion (4), is lying in $\mathscr{M}^n_E\cap\Pi(-v)$ with $\alpha(E(\beta')|E)<ev$.

5.5. Thus, if one admit the restriction σ_E, all important representations are obtained from multiplicatively generic elements. The restriction should correspond to the base change lift in GL. Since the multiplicatively generic elements behave very well like (4) at the tamely ramified base change, we will investigate the relation in more detail in the future.

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Department of Mathematics
Kyoto University
Kyoto 606, Japan