

## On the Burnside Rings of Finite Groups and Finite Categories

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### § 1. Introduction

The Burnside rings of finite groups introduced first by L. Solomon [So, 67] are getting more and more important in various fields in mathematics. The Burnside ring  $B(G)$  of a finite group  $G$  is simply the Grothendieck ring of the category  $\text{Set}_G^G$  of finite  $G$ -sets and  $G$ -maps with respect to disjoint unions and cartesian products, but it acts on many Grothendieck groups about the group  $G$ . In fact, if  $(\mathbf{a}, \tau, \rho, \sigma)$  is a  $G$ -functor, for example, the character ring  $R(G)$ , then  $B(G)$  acts on  $\mathbf{a}(G)$ . Combining this fact to the idempotent formula, we have some induction theorems ([Yo. 83]).

On the other hand, finite group theory can be regarded as theory of categories  $\text{Set}_G^G$  for various  $G$ . On this standing point, we can rewrite, for example, transfer theorems by categorical language. So if we pick another category like  $\text{Set}_G^G$ , we may expect to get another theory. Here it is best to choose categories called (elementary) toposes, for example, the functor category  $\text{Set}_G^A$  from a finite category to the category of finite sets. In fact we can develop “transfer-induction theory” in such categories.

However the existence of infinitely many connected objects in a locally finite topos obstructs our theory, so we need first to consider finite subcategories of this topos. Luckily we need little preparation to make abstract Burnside rings of finite categories. The following theorem is one of main theorems of this paper:

**Theorem A.** *Let  $A$  be a finite skeletal category with a unique epimono factorization property and with coequalizer of these identity and any automorphism of each object of  $A$ . Then  $\mathbf{Z}A$ , the free abelian group generated by  $\text{Ob}(A)$ , has a ring structure with identity element such that*

$$\phi: \mathbf{Z}A \longrightarrow \mathbf{Z}^A: a (\in A) \longmapsto (|A(i, a)|)_{i \in A}$$

*is a ring homomorphism.*

This ring  $ZA$  is called the *abstract Burnside ring*. If we take the category of transitive  $G$ -sets as  $A$ , we again get the Burnside ring  $B(G)$ . Furthermore this ring structure induces many other ring, for example, a large Hecke ring (Section 7).

The next purpose of this paper is to give an idempotent formula of the abstract Burnside rings. This is done in Section 5. We will construct posets corresponding to subgroup lattices for the ordinary Burnside rings. In Section 6, we give some examples. In Section 7, we state a “transfer theorem” for abstract Burnside rings which is a revised version of D. Higman’s focal subgroup theorem for finite groups.

**§ 2. The Burnside ring of a finite group**

In this section we collect some known results about Burnside rings of finite groups without proof. The details are found in, for example, tom Dieck’s book [Di. 79] and the author’s paper [Y. 83].

**2.1. Definition.** Let  $G$  be a finite group. The set of  $G$ -isomorphism classes of finite (right)  $G$ -sets becomes a commutative semi-ring  $B^+(G)$  with addition defined by disjoint unions and multiplications defined by cartesian products with diagonal  $G$ -actions. The Grothendieck ring of  $B^+(G)$  is denoted by  $B(G)$  and is called the *Burnside ring* of  $G$ .

We denote by  $C(G)$  the set of  $G$ -conjugate classes ( $H$ ) of subgroups  $H$  of  $G$ . Then the Burnside ring  $B(G)$  is a free abelian group with basis  $[G/H]$ , where ( $H$ ) is taken over the set  $C(G)$ .

**2.2.** Let  $H$  be a subgroup of  $G$ . Since the map  $X \mapsto |X^H|$ , where  $X^H$  is the  $H$ -fixed point set, preserves sums and products, it extends to a ring homomorphism  $\phi_H$  of  $B(G)$  to  $Z$ . Let  $\tilde{B}(G)$  be the set of integral valued functions  $\chi: H(\leq G) \rightarrow \chi(H)$  which are constant on every conjugate class, so that  $\tilde{B}(G)$  is a ring by pointwise multiplications. Then the product of  $\phi_H$  defines a ring homomorphism

$$\phi = (\phi_H): B(G) \longrightarrow \tilde{B}(G); [X] \longrightarrow (|X^H|)_{H \leq G}.$$

**2.3. Proposition** ([Di. 79; 1.3] [Yo. 83]). *There is an exact sequence*

$$0 \longrightarrow B(G) \xrightarrow{\phi} \tilde{B}(G) \xrightarrow{\psi} \prod_{(H)} (Z/|WH|Z) \longrightarrow 0,$$

where ( $H$ ) runs over  $C(G)$  and  $WH = N_G(H)/H$ .

By this proposition, which is suited to be a fundamental theorem for Burnside rings, we can decide prime ideals ([Dr. 69]) and primitive idempotents ([Gl. 81]).

**2.4. Definition.** For a finite poset (=partially ordered set)  $P$ , the Möbius function  $\mu: P \times P \rightarrow \mathbf{Z}$  is inductively defined by

$$\begin{aligned} \mu(a, a) &= 1; \mu(a, b) = 0 && \text{unless } a \leq b; \\ \sum_{a \leq x \leq b} \mu(a, x) &= \sum_{a \leq x \leq b} \mu(x, b) = \delta_{a,b} && \text{if } a \leq b. \end{aligned}$$

The Euler characteristic of  $P$  is defined by

$$\chi(P) := \sum_{x, y \in P} \mu(x, y).$$

**2.5.** Let  $p$  be a prime. Define a subring and a subgroup:

$$\begin{aligned} \mathbf{Z}_{(p)} &:= \{a/b \mid a \in \mathbf{Z}, b \in \mathbf{Z} - p\mathbf{Z}\} \subseteq \mathbf{Q}, \\ O^p(G) &:= \langle p'\text{-element of } G \rangle \leq G. \end{aligned}$$

A finite group  $G$  is  $p$ -perfect if  $O^p(G) = G$ , that is, if  $G$  has no normal subgroup of index  $p$ . For an natural number  $n$ , we denote its  $p$ -part by  $n_p$ .

**2.6. Proposition** ([Gl. 81], [Yo. 83]).

(i) Any primitive idempotent of  $\mathbf{Q} \otimes B(G)$  has the form

$$e_{G,H} = \frac{1}{|N_G(H)|} \sum_{D \leq H} |D| \mu(D, H) [G/D]$$

for a subgroup  $H$  of  $G$ .

(ii) Any primitive idempotent of  $\mathbf{Z}_{(p)} \otimes B(G)$  has the form

$$e_{G,H}^p = \sum_{(H): O^p(H) \sim Q} e_{G,H}$$

for a  $p$ -perfect subgroup  $Q$  of  $G$ .

**2.7. Corollary.** Let  $D$  be a subgroup of  $G$  and let  $\mathbf{S}_{>D}^p$  be the subposet of subgroup lattice of  $G$  consisting of all  $p$ -subgroup properly containing  $D$ . Then the Euler characteristic

$$\chi(\mathbf{S}_{>D}^p) \equiv 1 \pmod{|N_G(D): D|_p}.$$

The case where  $D = 1$  is well-known ([Br. 75]).

**2.8. Remark.** The conclusions of 2.6 (ii) and 2.7 hold even for “ $p = 1$ ”, where “1-perfect”, “1-subgroup”,  $n_1$  means “perfect”, “solvable subgroup”,  $n$ , respectively.

**2.9.** There are many other applications of the theory of Burnside rings in various fields. For example, the idempotent formula as above is

applied to induction theorems ([Dr. 71], [Yo. 83] in representation theory of finite groups, Wielandt order formula, class number relations for algebraic number fields ([RS. 83]), equations for Dedekind zeta functions ([Hu. 79]), equivariant cohomology theory ([Di. 79], [Ar. 82]), etc.

**Open Problem.** We list some open problems about Burnside rings of finite groups.

(1) Determine the structure of the unit group  $B(G)^*$  and the Picard group  $\text{Pic } B(G)$ .

(2) Does  $B(G) \cong B(H)$  implies  $G \cong H$ ?

(3) What structure does the local ring  $e_{GH}^p \mathbf{Z}_{(p)} \otimes B(G)$  have?

(4) Let  $I(H)$  be the augmentation ideal of  $\mathbf{Z}_{(p)} \otimes B(H)$ , where  $H \leq G$ . Then what can we say about the modules  $I(H)^n / I(H)^{n+1}$ ,  $n \geq 0$ , on the Hecke ring  $\mathbf{Z}_{(p)}[H \backslash G / H]$  (and also on the center of  $\mathbf{Z}_{(p)}G$ ) defined by the operation

$$(HgH): x \mapsto \text{ind}^H \text{res}_{g^{-1}Hg \cap H} \text{con}^g(x).$$

(5) What about Problems 1, 3, 4 on the character ring  $R(G)$ ?

On Problem (1), refer [Di. 79], [Ma. 82], [Yo. 86]. The local ring in Problem 3 is a CM ring of one dimensional but not Gorenstein in general. It is regular if and only if  $(p, |G|) = 1$ . See [Kr. 81]. The reason why Hecke rings acts on factor groups of a filtrations of  $B(H)$ ,  $B(H)^*$ ,  $R(H)$ , etc. is found in [Yo. 81].

### § 3. Abstract Burnside rings

**3.1. Prerequisite to the category A.** In this section, we will be concerned with a category  $A$ . The composition  $a \xrightarrow{f} b \xrightarrow{g} c$  of morphisms in  $A$  is written  $fg$ . The set of morphisms of  $a$  to  $b$  is simply written  $A(a, b)$ . The class of objects (resp. morphisms) of  $A$  is denoted by  $A$  itself (resp.  $\text{Mor}(A)$ ). The class of isomorphism classes of  $A$  is denoted by  $A/\cong$ . We say that

$A$  is *finite* if  $|\text{Mor}(A)| < \infty$ ;

$A$  is *locally finite* if  $|A(a, b)| < \infty$  for each  $a, b$ ;

$A$  is *skeletal* if  $a \cong b$  implies  $a = b$ .

**3.2.** Suppose  $A$  is a locally finite category with small skeleton. Let  $\mathbf{Z}A$  denote the free abelian group generated by  $A/\cong$ . Let  $\mathbf{Z}^A$  denote the set of mappings of  $A$  to  $\mathbf{Z}$  which is constant on each isomorphism class, so that  $\mathbf{Z}^A$  is regarded as the ring of  $A/\cong$ -indexed integers. Then there is a linear map

$$(3.2.1) \quad \phi: \mathbf{Z}A \longrightarrow \mathbf{Z}^A; a(\in A) \longmapsto (|A(i, a)|)_{i \in A}.$$

**Definition.** The abelian group  $ZA$  is called an *abstract Burnside ring* provided  $ZA$  possesses a (unique) ring structure which makes  $\phi$  an injective ring homomorphism.

**3.3. Example.** (a) The Burnside ring  $B(G)$  of a finite group  $G$ . Here  $A$  is (a skeleton of) the category of transitive  $G$ -sets and  $G$ -maps.

(b) The rings  $G_0(E)$ ,  $K_0(E)$  for a locally finite topos  $E$  with small skeleton. Here  $A$  is the full subcategory of irreducible (resp. connected) objects. These rings are, as abelian groups, generated by all isomorphism classes of  $E$  together with relations

$$\begin{aligned} [X \cup Y] + [X \cap Y] &= [X] + [Y], & [0] &= 0 & \text{for } G_0; \\ [X + Y] &= [X] + [Y] & & & \text{for } K_0. \end{aligned}$$

(c) The character ring  $R(S_n)$  of the symmetric group  $S_n$ . The category  $A$  consists of transitive  $S_n$ -sets which have Young subgroups as stabilizers.

(d) The Möbius ring of a finite poset. See Section 5.

**3.4. Hypothesis FAC.** The category  $A$  has a factorization system  $(E, M)$  in which

(3.4.1) each morphism in  $E$  (resp.  $M$ ) is an epimorphism (resp. monomorphism).

A *factorization system*  $(E, M)$  on  $A$  consists of two classes  $E$  and  $M$  of morphisms satisfying

(3.4.2) each of  $E$  and  $M$  contains all isomorphisms and is closed under composition;

(3.4.3) every morphism  $f: a \rightarrow b$  has an  $(E, M)$ -factorization

$$f = (a \xrightarrow{e} \text{im}(f) \xrightarrow{m} b) \quad \text{with } e \in E, m \in M.$$

(3.4.4) The above factorization is unique up to isomorphism, that is, if  $f = (a \xrightarrow{e'} i' \xrightarrow{m'} b)$  is another  $(E, M)$ -factorization, then there is a unique isomorphism  $h: \text{im}(f) \rightarrow i'$  such that  $eh = e'$ ,  $m = hm'$ .

We then put

$$E(a, b) := A(a, b) \cap E \subseteq \text{Epi}(a, b),$$

$$M(a, b) := A(a, b) \cap M \subseteq \text{Mon}(a, b).$$

In this category, the following holds:

$$(3.4.5) \quad E(a, a) = M(a, a) = \text{Aut } a.$$

All results in this section hold for a category with a factorization system satisfying (3.4.5).

**3.5. Hypothesis CEQ.** For any object  $i$  of  $A$  and any automorphism  $\sigma$  of  $i$ , there exists a coequalizer diagram of  $1_i$  and  $\sigma$ :

$$(3.5.1) \quad i \begin{array}{c} \xrightarrow{1} \\ \xrightarrow{\sigma} \end{array} i \longrightarrow i/\sigma.$$

The condition that (3.5.1) is a coequalizer diagram means that there is a canonical bijection

$$(3.5.2) \quad A(i/\sigma, a) \cong A(i, a)^{\langle \sigma \rangle} := \{f \in A(i, a) \mid \sigma f = f\},$$

where the hom-set  $A(i, a)$  is viewed as a left  $\text{Aut}(i)$ -set by composition.

**Example.** Let  $G$  be a finite group and let  $H$  be a family of subgroups of  $G$  closed under conjugation. Let  $A$  be the category of transitive  $G$ -sets of which stabilizers belong to  $H$ . Then CEQ holds if and only if for any  $D \in H$  and  $n \in N_G(D)$ , there exists a unique minimal subgroup in  $H$  containing  $D$  and  $n$ . In this category, FAC always holds because all morphisms are epimorphisms.

**3.6. Main theorem of this paper.** Let  $A$  be a finite and skeletal category.

**Theorem A.** *If  $A$  satisfies Hypothesis FAC and CEQ, then  $\mathbf{Z}A$  becomes an abstract Burnside ring with identity.*

**Theorem B.** *If  $A$  satisfies Hypothesis FAC, then the map*

$$\phi: \mathbf{Z}A \longrightarrow \mathbf{Z}^A: a (\in A) \longmapsto (|A(i, a)|)_i$$

*is injective and its cokernel*

$$(3.6.1) \quad \text{Coker } \phi \cong \prod_{i \in A} (\mathbf{Z}/|\text{Aut } i| \mathbf{Z}).$$

**Theorem C.** *If  $A$  satisfies FAC and CEQ, then there is an exact sequence of abelian groups:*

$$(3.6.2) \quad 0 \longrightarrow \mathbf{Z}A \xrightarrow{\phi} \mathbf{Z}^A \xrightarrow{\psi} \prod_{i \in A} (\mathbf{Z}/|\text{Aut } i| \mathbf{Z}) \longrightarrow 0,$$

*where*

$$(3.6.3) \quad \psi: \chi \longmapsto \left( \sum_{\sigma \in \text{Aut } i} \chi(i/\sigma) + |\text{Aut } i| \mathbf{Z} \right)_i$$

**3.7. Corollary.** *Let  $A$  be a locally finite category satisfying FAC. Then the following statements are equivalent:*

- (a)  $x \cong y$  in  $A$ .
- (b)  $|A(i, x)| = |A(i, y)|$  for every  $i \in A$ .
- (c)  $|A(x, i)| = |A(y, i)|$  for every  $i \in A$ .

The proof is easy. This corollary is applied to the categories of finite  $G$ -sets, finite (di-)graphs, finite posets, finite  $R$ -modules with  $R$  finite. See [Be. 84; 2.18.6].

**3.8.** Assume that  $x$  and  $y$  have a direct product  $x \times y$  in  $A$ . Then  $xy$  represents the products  $xy$  in the abstract Burnside ring  $ZA$ , and so if  $A$  has finite products, then  $ZA$  is the semigroup ring of  $A$ .

If  $A$  is a full subcategory of a locally finite topos  $E$  and each object of  $A$  is connected (resp. irreducible), then there exists a split epimorphism of  $K_0(E)$  (resp.  $G_0(E)$ ) to  $ZA$ .

**3.9. Problem.** The abstract Burnside ring  $ZA$  is clearly CM-ring of one dimensional. When is it Gorenstein?

#### § 4. Proof of the main theorems

In this section, we prove Theorems A, B and C. Throughout this section,  $A$  denotes a finite skeletal category.

**4.1. Lemma.** *Define  $A \times A$ -matrices  $H, L, D, U$  by*

$$\begin{aligned} H_{ab} &:= |A(a, b)|, \\ L_{ab} &:= |E(a, b)| / |\text{Aut } b|, \\ D_{ab} &:= |\text{Aut } a| \delta_{ab}, \\ U_{ab} &:= |M(a, b)| / |\text{Aut } a|. \end{aligned}$$

*Then these are integral matrices and*

$$(4.1.1) \quad H = LDU.$$

*Proof.* This follows from the uniqueness of the factorizations.

**4.2. Proof of Theorem B.** By the lemma, it will suffice to show that

$$(4.2.1) \quad \det L = \det U = 1.$$

To prove (4.2.1) introduce a partial order  $\leq_E$  on  $A$  by

$$(4.2.2) \quad a \leq_E b \text{ if } E(b, a) \text{ is not empty.}$$

Since  $L_{aa} = 1$  and  $a \geq_E b$  if  $L_{ab} \neq 0$ , we have that  $L$  is conjugate to a lower triangular matrix of which diagonal constituents are all 1. Hence  $\det L = 1$ . By the duality principal,  $\det U = 1$ , proving (4.2.1). The theorem is proved.

**4.3. Remark.** If  $A$  is a full subcategory of a locally finite topos  $E$  closed under image of morphism, then  $H = LDU$  in Lemma 4.1 is a so-called *LDU-decomposition*. Use the partial order generated by

(4.3.1)  $a \leq b$  if both of  $E(e, a)$  and  $M(e, b)$  are nonempty for some  $e \in E$  instead of (4.2.2).

**4.4. Proof of Theorem C.** Define a linear map

$$(4.4.1) \quad \psi'_i: ZA \longrightarrow Z; \chi_i \longrightarrow \sum_{\sigma \in \text{Aut } i} \chi(i/\sigma)$$

for  $i \in A$ . Next make the product of them:

$$(4.4.2) \quad \psi' := (\psi'_i): Z^A \longrightarrow Z^A; \chi_i \longrightarrow (\psi'_i(x))_i.$$

Then  $\psi$  is factored as

$$\psi = \text{proj} \circ \psi': Z^A \longrightarrow \prod_{i \in A} (Z / |\text{Aut } i| Z).$$

By Theorem B,  $\phi$  is injective and

$$\text{Coker } \phi \cong \prod_{i \in A} (Z / |\text{Aut } i| Z).$$

So in order to prove the exactness, it will suffice only to show the following:

$$(4.4.3) \quad \psi \circ \phi = 0;$$

$$(4.4.4) \quad \psi \text{ is surjective.}$$

First we will show (4.4.3). Let  $x, i \in A$ . Then

$$\psi'_i \circ \phi(x) = \sum_{\sigma \in \text{Aut } i} |A(i/\sigma, x)|$$

Let  $\pi$  be the permutation character afforded by the left  $\text{Aut } i$ -set  $A(i, x)$ , so that by (3.5.2), we have that

$$\pi(\sigma) = |\text{Aut}(i/\sigma, x)| \quad \text{for } \sigma \in \text{Aut } i.$$

Since

$$\frac{1}{|\text{Aut } i|} \sum_{\sigma \in \text{Aut } i} \pi(\sigma) = \text{the number of orbits in } A(i, x)$$



by a lemma of Burnside, we have that

$$\psi'_i \circ \phi(x) \equiv 0 \pmod{|\text{Aut } i|},$$

which proves (4.4.3).

Next we will show (4.4.4). It will suffice to show that  $\psi'$  is surjective. The matrix corresponding to  $\psi': Z^A \rightarrow Z^A$  is  $f = (f(a, b))_{a, b \in A}$ , where

$$f(a, b) = |\{\sigma \in \text{Aut } a \mid a/\sigma = b\}|.$$

We have that  $f(a, a) = 1$ . Define a partial order  $\leq_e$  on  $A$  by

$$a \geq_e b \text{ if there is an epimorphism of } a \text{ to } b.$$

If  $f(a, b) \neq 0$ , then  $a \geq_e b$ . Thus  $f$  is conjugate to a lower triangular matrix of which diagonal constituents are all 1, and so  $\det f = 1$ . Hence  $\psi'$  is an isomorphism, as required. Theorem B is proved.

**4.5. Proof of Theorem A.** Since the injectivity of  $\phi$  is proved in Theorem B, it remains to prove that

$$(4.5.1) \quad \text{if } x, y \in A, \text{ then } \zeta := \phi(x)\phi(y) \in \text{Im } \phi;$$

$$(4.5.2) \quad 1 \in \text{Im } \phi.$$

By Theorem C, it will suffice to show that  $\zeta, 1 \in \text{Ker } \psi$ . The value of  $\zeta$  at  $i \in A$  is

$$\zeta(i) = |A(i, x)| \cdot |A(i, y)|.$$

By the diagonal  $\text{Aut}(i)$ -action, the set  $Z := A(i, x) \times A(i, y)$  becomes a left  $\text{Aut } i$ -set. The number of fixed points in  $Z$  by  $\sigma \in \text{Aut } i$  equals to  $\zeta(i/\sigma)$ . Again by a lemma of Burnside,

$$\psi'_i(\zeta) = \sum_{\sigma \in \text{Aut } i} \zeta(i/\sigma) \equiv 0 \pmod{|\text{Aut } i|}.$$

(Refer 4.4.1 for notation). Hence  $\zeta \in \text{Ker } \psi$ . Finally the fact that 1 is in  $\text{Ker } \psi$  is obvious by the definition of  $\psi$ . The proof is completed.

## § 5. An idempotent formula

In this section, we give an idempotent formula for the abstract Burnside ring of a category.

**5.1. Hypothesis.** Throughout this section, the category  $A$  is assumed to satisfy the following conditions:

- (a)  $A$  is finite and skeletal;
- (b)  $A$  satisfies Hypothesis FAC in 3.4.
- (c)  $A$  satisfies Hypothesis CEQ in 3.5.

Let  $p$  be a prime (or 1). We define  $Z_{(p)}$  as in 2.5 and we put  $Z_{(1)} := Z$  for convenience. By Theorem A,  $Z_{(p)}A := Z_{(p)} \otimes Z A$  is a ring with injective ring homomorphisms

$$\phi: Z_{(p)}A \longrightarrow Z_{(p)}^A; a(\in A) \longrightarrow (|A(i, a)|)_i$$

**5.2.** For a prime  $p$  (or 1), let  $\sim_p$  be the equivalence relation on  $A$  generated by

(5.2.1)  $a \sim_p b$  if  $b \cong a/\sigma$  for some  $\sigma \in (\text{Aut } a)_p$ , where  $(\text{Aut } a)_p$  is a Sylow  $p$ -subgroup of  $\text{Aut } a$ . Here we put  $(\text{Aut } a)_1 := \text{Aut } a$ .

**5.3.** For a subset  $S$  of  $A$ , let  $\chi_s \in Z^A$  be the characteristic function for  $S$ :

$$\chi_s(a) = \begin{cases} 1 & \text{if } a \in S \\ 0 & \text{otherwise.} \end{cases}$$

Since  $\phi$  gives a ring isomorphism  $QA \cong Q^A$  by Theorem B, there exists a unique idempotent  $e_s$  of  $QA$  such that  $\phi(e_s) = \chi_s$ . Conversely any idempotent of  $QA$  has this form.

By Theorem C, we have the following result (refer to Proposition 2.6 (ii)):

**Theorem D.** *The elements  $e_c$ , where  $C$ 's are  $\sim_p$ -equivalence classes, are the primitive idempotents of  $Z_{(p)}A$ .*

**5.4. Lemma.** *The primitive idempotent of  $QA$  corresponding to  $a$  in  $A$  is given by*

$$e_a := \sum_{i \in A} H_{ia}^{-1}i,$$

where  $(H_{ia}^{-1})$  is the inverse matrix of the hom-set matrix  $H = (|A(i, j)|)_{i, j \in A}$ .

The proof is easy. So in order to get an idempotent formula, we need to know  $H^{-1}$ .

**5.5.** If we wish an idempotent formula as in Proposition 2.6 (i), we need to construct posets corresponding to the subgroup lattices. It is always possible under our hypothesis 5.1, but in this paper, we make them under further restriction in order to avoid introducing furthermore new concept.

Let  $(A, E)$  (resp.  $(A, M)$ ) be the subcategory of  $A$  with morphisms contained in  $E$  (resp.  $M$ ). Let  $\text{Epi}(a, b)$  (resp.  $\text{Mon}(a, b)$ ) be the set of epimorphisms (resp. monomorphisms) of  $a$  to  $b$  in  $A$ .

**Hypothesis.** There exists objects  $g, g'$  of  $A$  such that each of  $\text{Epi}(g, a)$  and  $\text{Mon}(a, g')$  is not empty for every  $a \in A$ .

5.6. By Hypothesis 5.5, we have functors

$$\begin{aligned} \text{Epi}(g, -): (A, E) &\longrightarrow \text{Set}_f; a \longmapsto \text{Epi}(g, a), \\ \text{Mon}(-, g'): (A, M)^{op} &\longrightarrow \text{Set}_f; a \longmapsto \text{Mon}(a, g'). \end{aligned}$$

Remember that morphisms in  $E$  (resp.  $M$ ) are all epimorphisms (resp. monomorphisms) by FAC.

The *discrete cofibration*  $\tilde{E}$  on  $(A, E)$  corresponding to  $\text{Epi}(g, -)$  is the category of which object-set is

$$\tilde{E} = \{(a, u) \mid a \in A, u \in \text{Epi}(g, a)\}$$

and a hom-set is

$$\tilde{E}((a, u), (b, v)) = \{h \in A(a, b) \mid uh = v\}.$$

Dually the discrete fibration  $\tilde{M}$  on  $(A, M)$  corresponding to  $\text{Mon}(-, g')$  is defined, that is,

$$\begin{aligned} \tilde{M} &= \{(a, u) \mid a \in A, u \in \text{Mon}(a, g')\}, \\ \tilde{M}((a, u), (b, v)) &= \{h \in A(a, b) \mid hv = u\}. \end{aligned}$$

There are functors

$$\begin{aligned} f: \tilde{E} &\longrightarrow (A, E); (a, u) \longmapsto a, h \longmapsto h, \\ g: \tilde{M} &\longrightarrow (A, M); (a, u) \longmapsto a, h \longmapsto h, \end{aligned}$$

which are both surjective on objects.

Clearly, the isomorphism classes  $\tilde{E}/\cong$  and  $\tilde{M}/\cong$  are posets.

**Remark.** Discret (co-) fibrations of which isomorphism classes make finite posets and which is surjective on objects suffice for obtaining the idempotent formula.

5.7. Because of surjections of  $f$  and  $g$ , there are maps

$$\begin{aligned} f': \text{Ob}(A) &\longrightarrow \text{Ob}(\tilde{E})/\cong \text{ such that } f(f'(a)) = a, \\ g': \text{Ob}(A) &\longrightarrow \text{Ob}(\tilde{M})/\cong \text{ such that } g(g'(a)) = a. \end{aligned}$$

Let  $\mu_e$  and  $\mu_m$  be the Möbius function of the posets  $\tilde{E}$  and  $\tilde{M}$ , respectively. Let  $P$  be the fiber product of  $f: \tilde{E} \rightarrow A$  and  $g: \tilde{M} \rightarrow A$ :

$$P := \{(u, v) \mid u \in \tilde{E}/\cong, v \in \tilde{M}/\cong, f(u) = g(v)\}$$

**Theorem E.**

$$H_{ab}^{-1} = \sum_{(u,v) \in P} \frac{\mu_e(f'(a), u)\mu_m(v, g'(b))}{|\text{Aut } f(u)|}$$

*Proof.* We use the notation in Lemma 4.1. Since  $H^{-1} = U^{-1}D^{-1}L^{-1}$ , we must know  $L_{ab}^{-1}$  and  $U_{ab}^{-1}$ . Define matrices  $P, Q$  and  $\tilde{L}$  by

$$P := (\delta(f'(a), u))_{a \in A, u \in \tilde{E}/\cong},$$

$$Q := (\delta(f(u), b))_{u \in \tilde{E}/\cong, b \in A},$$

$$\tilde{L} := (\tilde{E}(u, v))_{u, v \in \tilde{E}/\cong},$$

where  $\delta$  is Kronecker's delta. Then  $PQ = I$  and  $QL = \tilde{L}Q$ , and so we have that  $L^{-1} = P\tilde{L}^{-1}Q$ . By the definition of the Möbius function,  $\tilde{L}_{ab}^{-1} = \mu_e(u, v)$ . Hence we have that

$$(5.7.1) \quad L_{ab}^{-1} = \sum_{v \in f^{-1}(b)/\text{Aut } b} \mu_e(f'(a), v).$$

Taking the dual of (5.7.1), we have that

$$(5.7.2) \quad U_{ab}^{-1} = \sum_{v \in g^{-1}(a)/\text{Aut } a} \mu_m(v, g'(b)).$$

Now the theorem follows from (5.7.1), (5.7.2).

### § 6. Examples

**6.1.** Let  $A$  be the full subcategory of **Set**, consisting of

$$[r] := \{1, 2, \dots, r\}, \quad 0 \leq r \leq n.$$

Clearly  $A$  satisfies FAC and CEQ. Note that for a permutation  $\sigma$  on  $[r]$ , the coequalizer  $[r]/\sigma$  is the set of the orbits of  $\sigma$  on  $[r]$ , and so it is isomorphic to  $[s]$ , where  $s$  is the number of orbits.

We have that

$$H = (j^i),$$

$$L = (S_{ij}), \text{ the second kind Stirling number,}$$

$$D = (\delta_{ij} i!),$$

$$U = \left( \begin{bmatrix} j \\ i \end{bmatrix} \right), \text{ the binomial coefficient.}$$

Thus we have that

$$\begin{aligned} \det H &= \prod_{i=1}^n i!, \text{ Vandermonde's determinant,} \\ L^{-1} &= (s_{ij}), \text{ the Stirling numbers of first kind,} \\ U^{-1} &= \left( (-1)^{i+j} \begin{bmatrix} j \\ i \end{bmatrix} \right), \\ H_{ij}^{-1} &= \sum_{k=i}^n \frac{(-1)^{k-i}}{i!(k-i)!} s_{kj}. \end{aligned}$$

In the viewpoint of Theorem *E*, we have that

$$L_{ij}^{-1} = \sum_{\pi: \text{ht}(\pi)=j} \mu_{P(n)}(1^{i-1} \cdot (n-i+1), \pi),$$

where  $\mu_{P(n)}$  is the Möbius function of the partition lattice.

The product of  $[x]$  and  $[y]$  in the abstract Burnside ring  $\mathbf{Z}A$  is given by

$$[x] \cdot [y] = \sum_{z=0}^n \sum_{j=0}^n H_{zj}^{-1}(xy)^j [z].$$

For complex numbers  $a_0, \dots, a_n$ , the polynomial

$$f(t) = \sum_{r=0}^n a_r \sum_{i=0}^n H_{ri}^{-1} t^i$$

is the Lagrange interpolation, that is,  $f(r) = a_r$  for  $0 \leq r \leq n$ .

Let  $\psi'_i: \mathbf{Z}^A \rightarrow \mathbf{Z}^A$  be the map defined in (4.4.1). Then it maps  $\chi = (x^r)_r$ , where  $x$  is an integer, to

$$\sum_{\sigma \in S_r} x^{[r]/\langle \sigma \rangle} = r! Z(S_r; x, \dots, x) = x(x+1) \cdots (x+r-1),$$

where  $Z$  is the cycle indicator ([Ai. 79]; p. 209).

For a prime  $p$ , let  $\sim_p$  be the equivalence relation defined in 5.2. Then  $[x] \sim_p [y]$ , where  $0 \leq x, y \leq n$ , if and only if  $x = y = 0$ , or  $x, y \neq 0$  and  $x \equiv y \pmod{p-1}$ . Thus for any  $0 < i, j < n$ , the summation of  $H_{ir}^{-1}$ , where  $r$  runs over integers such that  $0 < r < n$  and  $r \equiv j \pmod{p-1}$ , is a  $p$ -local integer.

**6.2.** Let  $A$  be the category of  $F_q$ -vector spaces  $F_q^r$ ,  $0 \leq r \leq n$ , and linear maps. Then

$$\begin{aligned} H &= (q^{ij}), \quad L = U^t, \\ U &= \left( \begin{bmatrix} j \\ i \end{bmatrix}_q \right), \text{ the Gauss binomial coefficient,} \end{aligned}$$

$$U_{ij}^{-1} = (-1)^{j-i} \begin{bmatrix} j \\ i \end{bmatrix}_q q^{\lfloor \frac{j-i}{2} \rfloor},$$

$$H_{ij}^{-1} = (-1)^{i+j} \sum_{k=0}^n \frac{\begin{bmatrix} k \\ i \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\lfloor \frac{k-i}{2} \rfloor + \lfloor \frac{k-j}{2} \rfloor}}{|GL(k, q)|}.$$

The poset appeared in Theorem E is the subspace lattice of  $F_q^n$ . When  $p$  is the characteristic of  $F_q$ , the abstract Burnside ring  $Z_{(p)}A$  is a local ring.

**6.3.** Let  $A$  be the category of cyclic groups  $Z/rZ$ ,  $1 \leq r \leq n$ , and group homomorphisms. Then

$$H_{ij} = \text{GCD}(i, j),$$

$$L_{ij} = U_{ji} = \begin{cases} 1 & \text{if } j|i \\ 0 & \text{otherwise,} \end{cases}$$

$$D_{i,j} = \delta_{i,j} \phi(i), \quad \text{the Euler function,}$$

$$\det H = \phi(1)\phi(2) \cdots \phi(n),$$

$$U^{-1} = (\mu(j|i))_{i,j}, \quad L^{-1} = (\mu(i|j))_{i,j},$$

where  $\mu$  is the Möbius function in elementary number theory and  $\mu(x)$  is assumed to be 0 if  $x$  is not an integer.

**6.4.** Let  $P$  be a finite poset. As usual, we consider  $P$  as a category with object set  $P$ , that is,  $|P(a, b)| \leq 1$  and  $x \leq y$  if and only if  $P(x, y)$  is not empty. Then the abstract Burnside ring  $ZA$  is just the Möbius ring  $\text{Möb}(P)$ . The multiplication is given by

$$pq := \sum_{r \in P} \sum_{i \leq p, q} \mu(r, i)r, \quad p, q \in P,$$

where  $\mu$  is the Möbius function. On the other hand, let  $\hat{P}$  be the category of contravariant functors  $P^{op} \rightarrow \mathbf{Set}_f$ . Then coproducts and products in  $\hat{P}$  make  $G_0(\hat{P})$  a commutative ring. See 3.3 (b) for the definition. The Yoneda functor  $p \mapsto P(-, p)$  induces the ring isomorphism of  $\text{Möb}(P)$  to  $G_0(P)$ . See [Yo. 84].

**6.5.** Let  $G$  be a finite group with  $(B, N)$ -pair. Let  $S$  be the set of generators of the Weyl group. Let  $A$  be the full subcategory of  $\mathbf{Set}_f^G$  consisting of  $G/P$ , where  $P$  is a parabolic subgroup. Then the abstract Burnside ring  $ZA$  is isomorphic to the Möbius ring  $\text{Möb}(B(S))$  of the Boolean lattice  $B(S)$  of  $S$ .

§ 7. Large Hecker rings

In this section, we argue on relations with some other rings. Throughout this section,  $A$  denotes a finite category satisfying Hypothesis FAC and CEQ in 3.4 and 3.5, so that  $ZA$  becomes an abstract Burnside ring (Theorem A).

**7.1. Lemma.** *Let  $X$  be a small and locally finite category and let  $f: A \rightarrow X$  be a functor preserving coequalizers.*

(i) *There exists a unique linear map  $f: ZA \rightarrow ZX$  such that*

$$\phi(f^*(x)) = (\phi(B(f(i), x)))_{i \in A}.$$

(ii) *If  $ZX$  is an abstract Burnside ring, then  $f^*$  is a ring homomorphism.*

(iii) *If  $f$  is fully faithful, then  $f^*$  is a split epimorphism, in fact its left inverse is the linear map  $f_*: ZA \rightarrow ZX$ .*

**7.2.** For an object  $x$  of  $A$ , the comma category  $A/x$  consists of all morphisms to  $x$  in  $A$ . Similarly, the categories  $A/xy$ ,  $A/xyz$ ,  $a/x^n$ , etc. are defined. We can apply Lemma 7.1 to the forgetful functors  $(a \rightarrow x) \rightarrow a$ ,  $(a \rightarrow x, a \rightarrow y, a \rightarrow z) \rightarrow (a \rightarrow x, a \rightarrow z)$ , etc.

For objects  $x, y$  and  $z$ , we have the linear mapping

$$(7.2.1) \quad ZA/xy \times ZA/yz \longrightarrow ZA/xyz \times ZA/xyz \xrightarrow{\mu} A/xyz \longrightarrow ZA/xz,$$

where the first map is the product of maps defined in Lemma 7.1, the map  $\mu$  is the multiplication map of the abstract Burnside ring  $ZA/xyz$  and the last map is the map induced by the forgetful functor.

Thus we obtain the large Hecke category  $\mathbf{Hec}_A$  of which objects are same as in  $A$  and hom-sets are the form  $ZA/xy$  with composition defined by (7.2.1).

An additive functor of  $\mathbf{Hec}_A$  to an additive category is called an abstract Mackey functor. For example,  $x \rightarrow ZA/x$  is an abstract Mackey functor. This means that the Mackey decomposition holds, that is, the following diagram is commutative:

$$\begin{array}{ccc} ZA/xy & \longrightarrow & ZA/x \\ \uparrow & & \uparrow \\ ZA/y & \longrightarrow & ZA \end{array}$$

**7.3.** By the ring homomorphism defined in Lemma 7.1, the modules  $Z_{(p)}A/x$ , etc. become modules over  $Z_{(p)}A$ . Let  $e_s$  be an idempotent of

$\mathbf{Z}_{(p)}A$ . We put  $M(x^n) := e_S \mathbf{Z}_{(p)}A/x^n$ . The following theorem is a version of D. Higman's focal subgroup theorem and the stable element theorem for cohomology of groups. See [Dr. 73; Theorem 1, Corollary, 1 in Section 3].

**Theorem F.** *Assume that  $x$  is an object of  $A$  such that  $A(s, x)$  is not empty for each  $s \in S$ . Then there is the following exact sequence of abelian groups:*

$$0 \longrightarrow e_S \mathbf{Z}_{(p)}A \longrightarrow M(x) \longrightarrow M(x^2) \longrightarrow \dots$$

**7.4. Example.** Suppose  $A$  is the category of transitive  $G$ -sets. Then abstract Mackey functors are in fact Mackey functor ([Dr. 73]). There is a surjective ring homomorphism of  $\mathbf{Z}A/x^2$  onto the Hecke ring  $\text{End}_{\mathbf{Z}G}(\mathbf{Z}x)$ . The large Hecke ring  $\mathbf{Z}A/x^2$  is isomorphic to the ring of stable  $G$ -cohomotopies ([Di. 79]).

**7.5.** We can further continue our construction, for example, abstract monomial rings, abstract incidence algebra, and so on, but the present paper is already too long.

**Added in Proof.** J. Thévenaz wrote me that there is a counterexample for Open Problem (2) in Section 2.

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