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q-Version of Formulas Concerning Young Diagrams

Yoshiaki Ueno

We give here a survey of some recent results on some formulas concerning Young diagrams and their q-analogue formulas. The following exposition divides into two parts. In Part 1, we discuss the relative version of the generating functions of ordinary plane partitions, which go back to MacMahon [4]. In Part 2, we introduce the concept of the "reverse matching", which is so to say the "surjective counterpart" of the classical (complete) matching of an incidence structure in combinatorial theory. We give an incidence structure between two finite sets by a Young diagram λ, and count the number of reverse matchings and see how its q-analogue can be constructed.

Complete proofs and other details will be published elsewhere [7, 13].

§ 1. Enumeration of ordinary plane partitions

1.1. Ordinary plane partitions

Let λ be a Young diagram.

Definition. By an "ordinary plane partition (opp) T of shape λ and largest part $\leqq r$" we mean an array of integers $0, 1, \cdots, r$ placed in each of the squares of λ, subject to the restrictions that

(1) these numbers must be non-increasing along each row;

(2) and down each column.

For example:

$$\begin{array}{lllll} T= 3 & 1 & 1 & 0 & 0. \\ 2 & 1 & 0 & & \\ 2 & & & & \\ 0 & & & & \end{array}$$

Definition. For any given λ and r, we put

$$f_\lambda^{(r)}(q) := \sum_T q^{|T|} \in Z[q]$$

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where the sum is to be taken over all opp's T of shape λ and largest part $\leqq r$, and $|T|$ means the total sum of the integers placed in the boxes of λ. We call this polynomial the "generating function of opp's of shape λ and largest part $\leqq r$".

Then $f_\lambda^{(r)}(q)$ can be expressed by a determinant-type formula as follows:

Theorem ([1, Eq. (6.12)]).

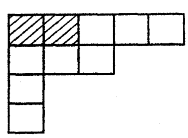
$$f_\lambda^{(r)}(q)=\left|q^{(i-j)(i-j-1)/2}\binom{\lambda_i+r}{i-j+r}_q\right|_{i,j=1,\cdots,l},$$

where $\lambda=(\lambda_1,\cdots,\lambda_l)$ *and* $\binom{m}{n}_q$ *denotes the Gaussian binomial coefficient or so-called q-analogue of the binomial coefficient.*

This formula was known to MacMahon up to $l=4$ [4, Section 495].

1.2. relative case

Let λ and μ be two Young diagrams such that μ is pictorially included in λ when both diagrams are up- and left-justified. For example, $\lambda=(5\ 3\ 1\ 1)$ and $\mu=(2)$:

$\lambda=$  .

Then by the skew diagram λ/μ we mean the crowd of boxes created by the set-theoretic difference $\lambda-\mu$ (i.e. the non-shadowed boxes in the above picture).

An opp of shape λ/μ and largest part $\leqq r$ can also be defined in the completely same way (see the picture below),

$$T=\begin{array}{ccccc} & & 1 & 0 & 0 \\ 2 & 1 & 0 & & \\ 2 & & & & \\ 0 & & & & \end{array}$$

and we also define the "generating function" $f_{\lambda/\mu}^{(r)}(q)$ of such opp's in the same fashion.

Then the function $f_{\lambda/\mu}^{(r)}(q)$ also allows a determinant-type formula:

Theorem ([7, § 1.10]).

$$f_{\lambda/\mu}^{(r)}(q)=\left|q^{(i-j)\mu_j+(i-j)(i-j-1)/2}\binom{\lambda_i-\mu_j+r}{i-j+r}_q\right|_{i,j=1,\cdots,l},$$

where $\lambda=(\lambda_1, \cdots, \lambda_l)$, $\lambda_1 \geqq \cdots \geqq \lambda_l > 0$, *and* $\mu=(\mu_1, \cdots, \mu_l)$, $\mu_1 \geqq \cdots \geqq \mu_l \geqq 0$ (*We add a string of* 0*'s to the end of* μ *if necessary*).

Putting $q=1$ here, we obtain

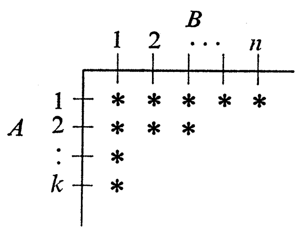
$$f^{(r)}_{\lambda/\mu}(1)=\left|\binom{\lambda_i-\mu_j+r}{i-j+r}\right|_{i,j=1,\cdots,l},$$

which gives the total number of opp's of shape λ/μ and largest part $\leqq r$, and was known to Kreweras [3, § 2.5.3.].

§ 2. Counting the number of reverse matchings on a Young diagram and its q-analogue

2.1. reverse matchings

Let $A=\{1, 2, \cdots, k\}$ and $B=\{1, 2, \cdots, n\}$, where $k, n \geqq 1$. Let $\lambda=(\lambda_1, \cdots, \lambda_k)$ be a partition such that $n=\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_k > 0$. In what follows, we regard the corresponding Young diagram λ as a subset of the direct product set $A \times B$ as the following picture indicates:

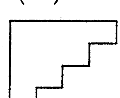


Definition. (a) ψ is a (complete) matching on λ if $\psi: A \to B$ is an injective mapping of A into B such that graph $(\psi) \subset \lambda$.

(b) φ is a reverse matching on λ if $\varphi: A \leftarrow B$ is a surjective mapping of B onto A and graph $(\varphi) \subset \lambda$.

We will be concerned mainly with reverse matchings. Let $e(\lambda)$ be the number of reverse matchings on λ. Then we observe that $e(\lambda)$ has the following properties:

1) If $\lambda=(n, n, \cdots, n)$ (k times), i.e. the $k \times n$-rectangle, then $e(\lambda)=k!\, S^k_n$, where S^k_n is the Stirling number of the second kind. Similarly, the number of complete matchings is equal to $k!\binom{n}{k}$.

2) $e(\lambda) \neq 0$ iff λ contains the staircase  of size k.

Property 2) is proved by using Philip Hall's marriage theorem. If we take λ to be the $k \times n$-rectangle, then Property 2) implies

2′) $S_n^k \neq 0$ iff $k \leqq n$.

Most of the properties enjoyed by Stirling numbers can be extended into a form including $e(\lambda)$'s in a similar way, so that they re-appear in a new fashionable style, in a "partition version", so to speak. The next two formulas are of such kind:

3) (I. Amemiya [14]). Let $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_k)$ be a partition such that $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_k > 0$, and put

$$\mu := \lambda - (\text{the last column});$$
$$\vartheta := \lambda - (\text{the last column and the first row});$$
$$i := \text{the length of the last column of } \lambda.$$

Then

$$e(\lambda) = \begin{cases} e(\mu) & \text{if } \lambda_1 > \lambda_2 + 1; \\ i\,(e(\mu) + e(\vartheta)) & \text{otherwise.} \end{cases}$$

Putting $\lambda=(n, n, \cdots, n)$ (k times), we have

$$S_n^k = S_{n-1}^{k-1} + kS_{n-1}^k \qquad \text{for } k \geqq 2.$$

4) Let $\lambda=(\lambda_1, \cdots, \lambda_k)$, $\lambda_1 \geqq \cdots \geqq \lambda_k > 0$, and $k \geqq 2$, and put

$$\lambda^{(i)} = \lambda - (\text{the last row and the 1st} \sim i\text{-th columns}).$$

Then

$$e(\lambda) = \sum_{i=1}^{\lambda_k} \binom{\lambda_k}{i} e(\lambda^{(i)}).$$

Putting $\lambda=(n, \cdots, n)$ (k times), we have

$$S_n^k = \frac{1}{k} \sum_{i=1}^{n} \binom{n}{i} S_{n-i}^{k-1}.$$

But there still seem to be some properties which don't come into picture among Stirling numbers. This stems from their being parametrized by partitions. Among such properties is:

5) (T. Imai's duality [10, Section 2, B-1]). Let $\lambda=(\lambda_1, \cdots, \lambda_k)$ be a "strict" partition, i.e. $n=\lambda_1 > \lambda_2 > \cdots > \lambda_k > 0$. Define the "complementary strict partition" $\lambda^* = \mu = (\mu_1, \cdots, \mu_l)$ of λ by

(a) $n=\mu_1 > \mu_2 > \cdots > \mu_l > 0$;

(b) $\{1, 2, \cdots, n-1\} \backslash \{\lambda_2, \cdots, \lambda_k\} = \{n-\mu_2, \cdots, n-\mu_l\}$.

Then it holds that $e(\lambda) = e(\lambda^*)$.

Property 5), originally formulated as a puzzle using a square of size $n+1$ and n pebbles following the traditional style of Japanese Yedo period mathematics, was dedicated to the Suwa Shrine, Yamanashi Prefecture and put to the public [10].

2.2. the q-analogue of $e(\lambda)$.

Let us consider the q-analogue of $e(\lambda)$. This means it is required, and in fact it turns out to be possible, to attach to each partition λ a polynomial $e(\lambda)_q$ in a variable q:

$$\lambda \mapsto e(\lambda)_q \in Z[q]$$

in such a way that

(a) The value of $e(\lambda)_q$ at $q=1$ coincides with $e(\lambda)$;

(b) All formulas enjoyed by $e(\lambda)$'s have an extension to fit $e(\lambda)_q$'s;

(b′) (I am tempted to add here that) these q-formulas must satisfy one's sense of beauty.

It seems that such an extension is unique in a sense, although the last requirements may sound rather vague. Anyway, we can construct "a q-analogue" of $e(\lambda)$.

Now it is observed that the coefficients of the polynomial $e(\lambda)_q$ are always non-negative integers. Since $e(\lambda)_{q=1}=e(\lambda)$ is the number of reverse matchings on the Young diagram λ, it suggests that we can define a "weight" $r(\varphi) \in Z_{\geqq 0}$ of φ for each reverse matching φ on λ, such that

$$e(\lambda)_q = \sum_{\varphi} q^{r(\varphi)},$$

where φ runs through all reverse matchings on λ. A natural way of doing this is to define $r(\varphi)$ to be the natural extension of the number of inversions in a permutation, which one can read off easily from the graph of φ.

The q-analogue of the duality 5):

$$5)_q \qquad e(\lambda)_q = e(\lambda^*)_q$$

can be proved by a combinatorial correspondence using this weight function [13, § 1.14.].

There is an effective algorithm to calculate $e(\lambda)$ by filling in the boxes of the diagram λ with numbers successively. $e(\lambda)_q$ was first captured by extending this algorithm q-analogously [13, § 4.3.].

2.3. the weight function and some examples.

Let us define the weight function $r(\varphi)$ mentioned above and show briefly how it works.

Let λ, A and B be as in 2.1.

Definition. For any subset σ of $A\times B$ included in λ, we define the weight $r(\sigma)$ of σ as follows:

(1) Let the set σ be graphically presented on the Young diagram λ e.g. by filling in the boxes with 1's,

(2) Fill in the blank boxes of λ with 0's and $*$'s according to the following rule:

In successive rows, starting with the bottom row,

(2a) mark with $*$'s all the blank boxes in the row which have no 1's in the row to the left of them (if there is no mark 1 in the row, then fill up the row with $*$'s);

(2b) if there are some 1's in the row, then for the leftmost 1, fill in all the boxes above it in the same column with 0's and for all the other 1's, fill in all the boxes above them in the same columns with $*$'s.

(3) Finally, $r(\sigma)$ is the number of $*$'s drawn in the picture.

Examples. (1) Let $k=2, n=3$ and $\lambda=(3, 3)$. There are $8=2^3$ mappings φ of B into A, $6=2!\times 3$ of which are surjective. The resultant pictures are illustrated below ("–" denotes blank box):

$$\begin{matrix} 1\ 1\ 1 & 1\ 1\ 0 & 1\ 0\ 1 & 1\ 0\ * \\ *\ *\ * & *\ *\ 1 & *\ 1\ - & *\ 1\ 1 \\ \\ 0\ 1\ 1 & 0\ 1\ * & 0\ *\ 1 & 0\ *\ * \\ 1\ -\ - & 1\ -\ 1 & 1\ 1\ - & 1\ 1\ 1 \end{matrix}$$

(2) Let $k=2, n=3$ and $\lambda=(3, 3)$. There are $6=3\times 2$ injective mappings φ of A into B. The resulting pictures are:

$$\begin{matrix} 0\ 1\ - & 0\ *\ 1 & 1\ 0\ - & *\ 0\ 1 & 1\ -\ 0 & *\ 1\ 0 \\ 1\ -\ - & 1\ -\ - & *\ 1\ - & *\ 1\ - & *\ *\ 1 & *\ *\ 1 \end{matrix}$$

(3) Let $k=3, n=3$ and $\lambda=(3, 3, 2)$. There are $4=2\times 2\times 1$ bijective mappings φ of A onto B, whose graphs lie within λ, with pictures:

$$\begin{matrix} 1\ 0\ 0 & 0\ 0\ 1 & 0\ 1\ 0 & 0\ 0\ 1 \\ *\ 0\ 1 & 1\ 0\ - & 0\ *\ 1 & 0\ 1\ - \\ *\ 1\ \ \ & *\ 1\ \ \ & 1\ -\ \ \ & 1\ -\ \ \ \end{matrix}$$

Adding up these examples for each weight, or representing each picture by $q^{r(\varphi)}$, one obtains the following polynomials:

(1) $\cdots\ 1+3q+3q^2+q^3=(1+q)^3$;

(2) $\cdots\ 1+2q+2q^2+q^3=(1+q)(1+q+q^2)$;

(3) $\cdots\ 1+2q+q^2=(1+q)^2$.

Moreover, taking only surjective ones in (1), one has:

$(1') \cdots 1+3q+2q^2=(1+q)(1+2q).$

These examples show that our definition of the weight function is plausible.

2.4. the geometric interpretation of $e(\lambda)_q$.

It is natural to ask what we are in fact counting by $e(\lambda)_q$. The creature we are counting has been captured recently. It is to the flag in a finite dimensional vector space over a finite field of q elements what the surjective map is to the injective map; a sequence of subspaces subject to a set of tree-type inclusion relations.

Definition. Let U be a finite dimensional vector space over the finite field $\boldsymbol{F}_q$ of q elements. A branching flag in U of length n is a sequence $(U_1, U_2, \cdots, U_n)$ of subspaces of U satisfying:

(1) $\dim U_n=1$;

(2) $\dim U_j-\dim U_{j+1}=0$ or 1 $(1\leqq j\leqq n-1)$;

(3) if $\dim U_j=\dim U_{j+1}=\cdots=\dim U_{j+s-1}>\dim U_{j+s}$, then $U_j, U_{j+1}, \cdots, U_{j+s-1}\supseteqq U_{j+s}$.

A branching flag $(U_1, U_2, \cdots, U_n)$ is called surjective if $U_1=U$, and injective if $\dim U_j-\dim U_{j+1}=1$ $(1\leqq j\leqq n-1)$.

Let U be a k-dimensional vector space over $\boldsymbol{F}_q$. Let λ, k and n be as in 2.1 and $\lambda'=(\lambda'_1, \lambda'_2, \cdots, \lambda'_n)$ be the conjugate partition of λ. We fix a flag $(V_1, V_2, \cdots, V_n)$ in U such that $\dim V_j=\lambda'_j$ $(1\leqq j\leqq n)$. Then

Theorem ([13, § 3.1]). *The number of surjective branching flags $(U_1, U_2, \cdots, U_n)$ in U of length n satisfying $U_j\subseteq V_j$ $(1\leqq j\leqq n)$ is a polynomial in q over $\boldsymbol{Z}$, and is equal to $q^{\deg e(\lambda)_q}\cdot e(\lambda)_{q^{-1}}$.*

The proof uses the q-analogue formula of the formula 3) in 2.1.

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Tokyo Institute of Polytechnics
1583, Iiyama, Atsugi
Kanagawa 243–02, Japan