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Littlewood's Formulas and their Application to Representations of Classical Weyl Groups

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Introduction

The reciprocity between the representations of the general linear groups and the symmetric groups is well known. For example, in I.G. Macdonald's book [M], this reciprocity is described as a ring isomorphism between the ring Λ of symmetric functions in countably many variables (see [M], [K–T]) and the graded ring $R=\oplus_n R(\mathfrak{S}_n)$, where $R(\mathfrak{S}_n)$ is the free $\boldsymbol{Z}$-module generated by the irreducible characters of the symmetric group of degree n and the multiplication in R is defined for $f \in R(\mathfrak{S}_n)$ and $g \in R(\mathfrak{S}_m)$ by $f \cdot g = \mathrm{ind}_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}}(f \times g)$. In an analogous manner, we define a graded ring $R_W=\oplus_n R(W(B_n))$ using the characters of the Weyl groups $W(B_n)$ of type B_n and a homomorphism from this ring R_W to Λ. This homomorphism clarifies the relationship between the representations of $GL(n)$ and the rule of decomposition (into irreducible constituents) of the representations of $\mathfrak{S}_{2n}$ induced by an irreducible representation of $W(B_n)$. In this procedure, Littlewood's formulas play a crucial role. Here, Littlewood's formulas mean the expansion formulas of the following four symmetric rational functions into Schur functions:

(1) $\prod_{1 \leq i < j \leq n} (1-z_i z_j)^{-1}$,

(2) $\prod_{1 \leq i \leq j \leq n} (1-z_i z_j)^{-1}$,

(3) $\prod_{1 \leq i < j \leq n} (1-z_i z_j)$,

(4) $\prod_{1 \leq i \leq j \leq n} (1-z_i z_j)$.

These formulas are also essential in describing the relations between the representations of $GL(n)$ and those of $Sp(2n)$ and $SO(n)$ (see [K-T]).

§ 1. Littlewood's formulas

The four rational functions listed in the introduction are all $\mathfrak{S}_n$-invariant (where $\mathfrak{S}_n$ acts by the permutations of variables $\{z_i\}_{i=1}^n$), There-

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fore if we embed the rational functions (1), (2), (3), and (4) into the formal power series ring $C[[z_1, z_2, \cdots, z_n]]$, they can be expressed as linear combinations (finite or infinite) of Schur functions $\chi_{GL(n)}(\lambda)(z)$'s. Here, $\chi_{GL(n)}(\lambda)(z)$ $(z=(z_1, z_2, \cdots, z_n))$ is the irreducible character of $GL(n, C)$ corresponding to the Young diagram (or equivalently partition) λ, restricted to the standard maximal torus $T=\{\operatorname{diag}(z_1, z_2, \cdots, z_n)\}$. We must prepare a few notations first.

For a partition $\kappa=(k_1, k_2, \cdots, k_n)$, 2κ denotes the even partition $2\kappa=(2k_1, 2k_2, \cdots, 2k_n)$. For a distinct partition $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_s)$ $(\alpha_1>\alpha_2>\cdots>\alpha_s\geqq 1)$, $\Gamma(\alpha)$ denotes the partition $\Gamma(\alpha)=(\alpha_1-1, \alpha_2-1, \cdots, \alpha_s-1 \,|\, \alpha_1, \alpha_2, \cdots, \alpha_s)$, using the Frobenius notation. The Frobenius notation $(\alpha_1, \alpha_2, \cdots, \alpha_r \,|\, \beta_1, \beta_2, \cdots, \beta_r)$ expresses the Young diagram $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_n)$ whose diagonal consists of r squares and the α_i, β_i $(1\leqq i\leqq r)$ and the λ_i $(1\leqq i\leqq n)$ are combined with the relations:

$$\alpha_i=\lambda_i-i, \quad \beta_i=\lambda'_i-i, \quad 1\leqq i\leqq r,$$

where we put ${}^t\lambda=(\lambda'_1, \lambda'_2, \cdots, \lambda'_l)$. Here, ${}^t\lambda$ denotes the transposed Young diagram of λ. In terms of Young diagrams, $(\alpha_1, \alpha_2, \cdots, \alpha_r \,|\, \beta_1, \cdots, \beta_r)$ is the diagram illustrated in Figure 1a. For example, $\Gamma(3, 2)$ is the one in Figure 1b.

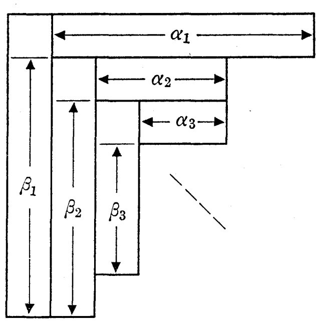

Fig. 1a

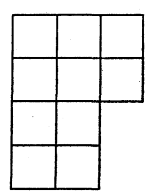

Fig. 1b

The following Lemma 1.1, 1)–4) was found by D.E. Littlewood (see [L, p. 238]). Under the setting of modern terminology, I.G. Macdonald [M, p. 45] gave the detailed proof of 1) and 2). But in [M, p. 46], he gave only an outline of the proof of 3) and 4). In view of the importance of this lemma, here we give the complete proof of 3) and 4).

Lemma 1.1 (D.E. Littlewood).

(1) $$\frac{1}{\prod_{1\leqq i<j\leqq n}(1-z_i z_j)}=\sum_{f=0}^{\infty}\sum_{\substack{|\kappa|=f\\ d(\kappa)\leqq n}}\chi_{GL(n)}({}^t(2\kappa))(z),$$

(2) $$\frac{1}{\prod_{1\leqq i\leqq j\leqq n}(1-z_iz_j)}=\sum_{f=0}^{\infty}\sum_{\substack{|\kappa|=f\\ d(\kappa)\leqq n}}\chi_{GL(n)}(2\kappa)(z),$$

(3) $$\prod_{1\leqq i<j\leqq n}(1-z_iz_j)=\sum_{f=0}^{n(n-1)/2}\sum_{\substack{|\alpha|=f\\ \alpha=(\alpha_1,\alpha_2,\cdots,\alpha_s)\\ \alpha_1>\alpha_2>\cdots>\alpha_s\geqq 1\\ d(\Gamma(\alpha))\leqq n}}(-1)^{|\alpha|}\chi_{GL(n)}(\Gamma(\alpha))(z),$$

(4) $$\prod_{1\leqq i\leqq j\leqq n}(1-z_iz_j)=\sum_{f=0}^{n(n+1)/2}\sum_{\substack{|\alpha|=f\\ \alpha=(\alpha_1,\alpha_2,\cdots,\alpha_s)\\ \alpha_1>\alpha_2>\cdots>\alpha_s\geqq 1\\ d({}^t\Gamma(\alpha))\leqq n}}(-1)^{|\alpha|}\chi_{GL(n)}({}^t\Gamma(\alpha))(z).$$

Proof of (3). We shall use the denominator formula of H. Weyl for the Lie algebra $\mathfrak{so}(2n, C)$. We define $\mathfrak{so}(2n, C)$ to be $\{X \in \mathfrak{sl}(2n, C) \mid XJ_O+J_O{}^tX=0\}$, where J_O is the following matrix:

$$J_O=\overbrace{\begin{pmatrix} & & & & 1\\ & 0 & & 1 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & & & 0 & \\ 1 & & & & \end{pmatrix}}^{2n}\Bigg\updownarrow 2n$$

As a Cartan subalgebra we take $\mathfrak{h}=\{H=\mathrm{diag}(h_1, \cdots, h_n, -h_n, \cdots, -h_1)\}$. Let $\varepsilon_i : \mathfrak{h}\to C$ be defined by $\varepsilon_i(H)=h_i$, and Δ_D be the root system of $\mathfrak{so}(2n, C)$ with respect to h. Fix a set of positive roots $\Delta_D^+=\{\varepsilon_i\pm\varepsilon_j;\ i<j\}$ and let ρ_D denote the half sum of the positive roots, i.e.

$$\rho_D=\frac{1}{2}\sum_{\alpha\in\Delta_D^+}\alpha=(n-1)\varepsilon_1+(n-2)\varepsilon_2+\cdots+\varepsilon_{n-1}.$$

Then the denominator formula of H. Weyl is given by

$$\sum_{w\in W(D_n)}\det(w)\, e^{w\rho_D}=\prod_{\alpha\in\Delta_D^+}(e^{\alpha/2}-e^{-\alpha/2})=e^{\rho_D}\prod_{\alpha\in\Delta_D^+}(1-e^{-\alpha}),$$

where $W(D_n)$ is the Weyl group of type D_n and "det" denotes the linear character of $W(D_n)$ taking the determinant of the representation of $W(D_n)$ on $\mathfrak{h}_R^*$. We shall define $\phi_i \in GL(\mathfrak{h}_R^*)$, $i=1, 2, \cdots, n$ by $\phi_i(\varepsilon_j)=\varepsilon_j$ if $j\neq i$, $\phi_i(\varepsilon_i)=-\varepsilon_i$. Then $W(D_n)=\langle\mathfrak{S}_n, \phi_i\phi_j\ (1\leqq i<j\leqq n)\rangle$ where $\mathfrak{S}_n$ acts on $\mathfrak{h}_R^*$ by the permutations of the base elements ε_i of $\mathfrak{h}_R^*$. $W(D_n)$ has the following coset decomposition with respect to $\mathfrak{S}_n$:

$$W(D_n)=\sum_{1\leqq i_1<i_2<\cdots<i_{2t}\leqq n}\mathfrak{S}_n\phi_{i_1}\phi_{i_2}\cdots\phi_{i_{2t}}.$$

We put

$$\rho_{D,i_1,i_2\cdots,i_{2t}}=\phi_{i_1}\phi_{i_2}\cdots\phi_{i_{2t}}\rho_D$$
$$=\rho_D-2(n-i_1)\varepsilon_{i_1}-2(n-i_2)\varepsilon_{i_2}-\cdots-2(n-i_{2t})\varepsilon_{i_{2t}}.$$

Since $\phi_n(\rho_D)=\rho_D$, we have

$$\sum_{w\in W(D_n)}\det(w)\,e^{w\rho_D}=\sum_{1\leqq i_1<\cdots<i_k\leqq n-1}\ \sum_{w\in\mathfrak{S}_n}\det(w)e^{w\rho_{D,i_1,i_2,\cdots,i_k}}.$$

We put $e^{-\varepsilon_i}=z_i$ $(i=1, 2, \cdots, n)$ in the denominator formula. Since

$$e^{\rho_D}\prod_{\alpha\in\Delta_D^+}(1-e^{-\alpha})=z_1^{-(n-1)}z_2^{-(n-2)}\cdots z_{n-1}^{-1}\prod_{1\leqq i<j\leqq n}(1-z_iz_j)(1-z_iz_j^{-1})$$
$$=\prod_{1\leqq i<j\leqq n}(1-z_iz_j)(z_i^{-1}-z_j^{-1})$$
$$=\prod_{1\leqq i<j\leqq n}(1-z_iz_j)\times|z^{-(n-1)}, z^{-(n-2)}, \cdots, z^{-1}, 1|$$

and

$$\sum_{w\in\mathfrak{S}_n}\det(w)\,e^{w(\rho_{D,i_1,i_2,\cdots,i_k})}$$
$$=|z^{-(n-1)}, z^{-(n-2)}, \cdots, \overset{(i_1)}{z^{n-i_1}}, \cdots, \overset{(i_2)}{z^{n-i_2}}, \cdots, \overset{(i_k)}{z^{n-i_k}}, \cdots, z^{-1}, 1|$$

(the numbers i_l above the determinant signify the positions of the corresponding columns), we have

$$\prod_{1\leqq i<j\leqq n}(1-z_iz_j)$$
$$=\sum_{1\leqq i_1<\cdots<i_k\leqq n-1}\frac{|z^{-(n-1)}, \cdots, z^{n-i_1}, \cdots, z^{n-i_k}, \cdots, z^{-1}, 1|}{|z^{-(n-1)}, z^{-(n-2)}, \cdots, z^{-1}, 1|}=**$$

Multiplying both the denominator and the numerator on the right-hand side of the above equality by $(z_1z_2\cdots z_n)^{n-1}$ and permuting the columns, we have

$$**=\prod_{1\leqq i_1<\cdots<i_k\leqq n-1}\frac{|z^{n-1}, \cdots, z^{n-s_k}, \overset{(s_k+1)}{z^{n-1+s_k}}, \cdots, \overset{(s_1+1)}{z^{n-1+s_1}}, \cdots, z, 1|}{|z^{n-1}, z^{n-2}, \cdots, z, 1|}$$

where we have put $s_1=n-i_1$, $s_2=n-i_2$, $\cdots$, $s_k=n-i_k$. Since $i_k\leqq n-1$, $1\leqq s_k<s_{k-1}<\cdots<s_1\leqq n-1$.

Claim. $$\frac{|z^{n-1}, \cdots, z^{n-s_k}, \overset{(s_k+1)}{z^{n-1+s_k}}, \cdots, \overset{(s_1+1)}{z^{n-1+s_1}}, \cdots, z, 1|}{|z^{n-1}, z^{n-2}, \cdots, z, 1|}$$
$$=(-1)^{|s|}\chi_{GL(n)}(\Gamma(s)),$$

where $s=(s_1, s_2, \cdots, s_k)$ and $|s|=s_1+s_2+\cdots+s_k$.

Proof of (3). We use induction on k.

If $k=1$, the numerator on the left-hand side of the claim equals

$$|z^{n-1}, \cdots, \overset{(s_1)}{z^{n-1+s_1}}, \cdots, z, 1|.$$

If we exchange the columns, we have

$$\begin{aligned}&|z^{n-1}, \cdots, z^{n-1+s_1}, \cdots, z, 1|\\ &\qquad=(-1)^{s_1}|z^{n-1+s_1}, z^{n-1}, \cdots, z^{n-s_1}, z^{n-s_1-2}, \cdots, 1|.\end{aligned}$$

Owing to H. Weyl's character formula (see [W, p. 201, Theorem 7.5 B]) it follows that

$$\frac{|z^{n-1}, \cdots, z^{n-1+s_1}, \cdots, z, 1|}{|z^{n-1}, \cdots, z, 1|}=(-1)^{s_1}\chi_{GL(n)}(\Gamma(s_1)).$$

Assume that the claim holds for $k-1$. If we put $s'=(s_1, s_2, \cdots, s_{k-1})$ and exchange the columns, we have

$$\begin{aligned}&\text{(the numerator of the claim)}\\ &\qquad=(-1)^{|s'|}|z^{n-1+s_1}, \cdots, z^{n-1+s_{k-1}}, z^{n-1}, \cdots, \overset{(k+s_k)}{z^{n-1+s_k}}, \cdots, 1|.\end{aligned}$$

Moreover if we move the $(k+s_k)$-th column to just behind the column $z^{n-1+s_{k-1}}$, we have

$$\begin{aligned}&\text{(the numerator of the claim)}\\ &\qquad=(-1)^{|s|}|z^{n-1+s_1}, \cdots, z^{n-1+s_{k-1}}, z^{n-1+s_k}, z^{n-1}, \cdots, 1|.\end{aligned}$$

In the above determinant, we denote the set of exponents of z by

$$\begin{aligned}I_s=(n-1+s_1, \cdots, n-1+s_k, n-1, \cdots,&\\ \widehat{n-1-s_k}, \cdots, \widehat{n-1-s_1}, \cdots, 1, 0),&\end{aligned}$$

and also the exponents of z in the denominator of the claim by $\partial=(n-1, n-2, \cdots, 1, 0)$. Then if we put $\lambda=I_s-\partial$, according to the character formula, the left-hand side of the claim exactly expresses $(-1)^{|s|}\chi_{GL(n)}(\lambda)$. On the other hand, if we use the induction hypothesis for $s'=(s_1, s_2, \cdots, s_{k-1})$ we have

$$\begin{aligned}I_{s'}=(n-1+s_1, \cdots, n-1+s_{k-1}, n-1, \cdots,&\\ \widehat{n-1-s_{k-1}}, \cdots, \widehat{n-1-s_1}, \cdots, 1, 0)&\end{aligned}$$

and $I_{s'}-\delta=\Gamma(s')$. Comparing I_s with $I_{s'}$, the variation of exponents is exactly caused by exchanging the $(k+s_k)$-th exponent of $I_{s'}$ for $n-1+s_k$ and moving the $(k+s_k)$-th column z^{n-1+s_k} to right behind the column $z^{n-1+s_{k-1}}$. But if we refer to the case $k=1$, this variation corresponds to adding the hook of Fig. 2 diagonally to the Young diagram

$$\Gamma(s')=(s_1-1,\ s_2-1,\ \cdots,\ s_{k-1}-1\,|\,s_1,\ s_2,\ \cdots,\ s_{k-1}).$$

Hence the claim is proved.

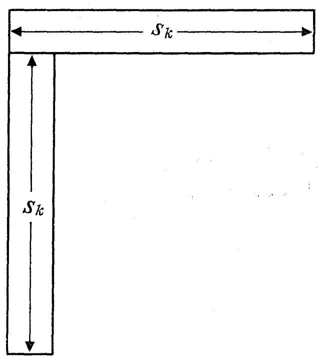

Fig. 2

(3) follows immediately from the above claim.

Proof of (4). We use Weyl's denominator formula for $\mathfrak{sp}(2n)=\{X\in\mathfrak{sl}(2n)\,|\,XJ_{Sp}+J_{Sp}{}^tX=0\}$, where J_{Sp} is the following matrix:

$$J_{Sp}=\begin{array}{c} \begin{array}{cc} \xleftarrow{\quad n \quad} & \xleftarrow{\quad n \quad} \end{array} \\ \left(\begin{array}{cccc|cccc} & & & & & & & 1 \\ & 0 & & & & & \cdot^{\cdot^{\cdot}} & \\ & & & & 1 & & & \\ \hline & & & -1 & & & & \\ & & \cdot^{\cdot^{\cdot}} & & & 0 & & \\ -1 & & & & & & & \end{array}\right) \end{array} \begin{array}{c} \updownarrow n \\ \updownarrow n \end{array}.$$

$\mathfrak{h}$ and the ε_i are defined in the same manner as in the proof of (3). Let $\Delta_C^+=\{\varepsilon_i\pm\varepsilon_j\ (i>j),\ 2\varepsilon_i\}$ be a set of positive roots of $\mathfrak{sp}(2n)$ and let $\rho_C=1/2\sum_{\alpha\in\Delta_C^+}\alpha$ be the half sum of the positive roots, then ρ_C is given by $\rho_C=n\varepsilon_1+(n-1)\varepsilon_2+\cdots+\varepsilon_n$. Let us recall Weyl's denominator formula:

$$\sum_{w\in W(C_n)}\det(w)e^{w\rho_C}=e^{\rho_C}\prod_{\alpha\in\Delta_C^+}(1-e^{-\alpha}),$$

where $W(C_n)=\langle\mathfrak{S}_n, \phi_i\ (1\leqq i\leqq n)\rangle$. As before $W(C_n)$ has the coset decomposition with respect to $\mathfrak{S}_n$ as follows:

$$W(C_n)=\bigcup_{1\leqq i_1<i_2<\cdots<i_k\leqq n}\mathfrak{S}_n\phi_{i_1}\phi_{i_2}\cdots\phi_{i_k}.$$

If we put $e^{-\varepsilon_i}=z_i$ $(1\leqq i\leqq n)$ and take the sum for every coset, we have

$$\prod_{1\leqq i\leqq j\leqq n}(1-z_iz_j)=\sum_{1\leqq i_1<\cdots<i_k\leqq n}\frac{|z^{-n},\cdots,\overset{(i_1)}{z^{n+1-i_1}},\cdots,\overset{(i_k)}{z^{n+1-i_k}},\cdots,z^{-1}|}{|z^{n-1},z^{n-2},\cdots,z,1|}.$$

Multiplying both the denominator and the numerator by $(z_1z_2\cdots z_n)^n$ and permuting the columns, we have

$$\prod_{1\leqq i\leqq j\leqq n}(1-z_iz_j)=\sum_{1\leqq i_1<\cdots<i_k\leqq n}\frac{|z^{n-1},\cdots,\overset{(s_k)}{z^{n+s_k}},\cdots,\overset{(s_1)}{z^{n+s_1}},\cdots,z,1|}{|z^{n-1},z^{n-2},\cdots,z,1|},$$

where $s_1=n+1-i_1, s_2=n+1-i_2, \cdots, s_k=n+1-i_k$ and $1\leqq s_k<s_{k-1}<\cdots<s_1\leqq n$.

Therefore we have only to prove the next claim.

Claim.
$$\frac{|z^{n-1},\cdots,\overset{(s_k)}{z^{n+s_k}},\cdots,\overset{(s_1)}{z^{n+s_1}},\cdots,z,1|}{|z^{n-1},z^{n-2},\cdots,z,1|}=(-1)^{|s|}\chi_{GL(n)}({}^t\Gamma(s))(z).$$

But the proof is similar to that of (3), so we omit it.

§ 2. Relations between the classical Weyl groups and the Universal Character Ring

In this section we deal with the relations between the Weyl group $W(B_n)=W(C_n)$, referred to as W_n hereafter, and the Universal Character Ring Λ. First, let us recall the definition of the ring Λ (cf. [M]).

Let $\Lambda_n=\boldsymbol{Z}[t_1, t_2, \cdots, t_n]^{\mathfrak{S}_n}=R_+(GL(n))$ be the graded algebra consisting of the symmetric polynomials in n variables and let $\tilde{\rho}_{m,n}\colon \boldsymbol{Z}[t_1, \cdots, t_m]\to\boldsymbol{Z}[t_1, \cdots, t_n]$ $(m\geqq n)$ be the homomorphism of graded algebras defined by $\tilde{\rho}_{m,n}(t_i)=t_i$ if $1\leqq i\leqq n$ and $\tilde{\rho}_{m,n}(t_i)=0$ if $n<i$. $\tilde{\rho}_{m,n}$ induces a homomorphism $\rho_{m,n}\colon \Lambda_m\to\Lambda_n$. Then $(\Lambda_n, \rho_{m,n})$ becomes a projective system and the projective limit of this system in the category of graded algebras is denoted by Λ, i.e. $\Lambda=\varprojlim \Lambda_n$. We call Λ the *Universal Character Ring.* By definition Λ is also a graded algebra: $\Lambda=\sum_{k\geqq 0}\Lambda^k$, where $\Lambda^k=\varprojlim \Lambda_n^k$. ($\Lambda_n^k$ is the homogeneous part of degree k of Λ_n). Note that Λ can be considered as the ring consisting of symmetric functions in countably

many variables $t_1, t_2, \cdots, t_n, \cdots$. Let $\pi_n\colon \Lambda\to\Lambda_n$ be the natural projection.

As is well known, $\{\chi_{GL(n)}(\lambda)\}_{\lambda:\text{partition}, d(\lambda)\leqq n}$ ($d(\lambda)$ denotes the depth of the Young diagram λ) is a $\boldsymbol{Z}$-base of $\Lambda_n=R_+(GL(n))$. (Here we are using $t_1, t_2, \cdots, t_n$ as variables of $\chi_{GL(n)}(\lambda)$, instead of $z_1, z_2, \cdots, z_n$.) It is known that for $m\geqq n\geqq d(\lambda)$ we have $\rho_{m,n}(\chi_{GL(m)}(\lambda))=\chi_{GL(n)}(\lambda)$ and for $d(\lambda)>k$ we have $\rho_{n,k}(\chi_{GL(n)}(\lambda))=0$. Hence the $\chi_{GL(n)}(\lambda)$'s form a projective system and we may define $\chi_{GL}(\lambda)\in\Lambda$, where $\pi_n(\chi_{GL}(\lambda))=\chi_{GL(n)}(\lambda)$ if $n\geqq d(\lambda)$ and $\pi_n(\chi_{GL}(\lambda))=0$ if $n<d(\lambda)$. $\{\chi_{GL}(\lambda)\}_{\lambda:\text{partition}}$ becomes a $\boldsymbol{Z}$-linear base of Λ.

If we take $\lambda=(f)$, we also denote $\chi_{GL}(\lambda)=\chi_{GL}((f))$ by p_f. $\pi_n(p_f)$ is the sum of all monomials with coefficient 1 in $t_1, \cdots, t_n$ of degree f. If we take $\lambda=(1^f)=(1, 1, \cdots, 1)$ (f times), then we also denote $\chi_{GL}(\lambda)=\chi_{GL}((1^f))$ by e_f. If $n\geqq f$, $\pi_n(e_f)$ is the f-th elementary symmetric polynomial in $t_1, \cdots, t_n$.

Our arguments here are based on the following theorem due to H. Weyl. Let $V=\boldsymbol{C}^m$ be the natural $GL(m)$-space. The symmetric group $\mathfrak{S}_k$ naturally acts on $\otimes^k V$, that is, $\sigma\in\mathfrak{S}_k$ acts on $x_1\otimes x_2\otimes\cdots\otimes x_k\in\otimes^k V$ by

$$\sigma(x_1\otimes x_2\otimes\cdots\otimes x_k)=x_{\sigma^{-1}(1)}\otimes x_{\sigma^{-1}(2)}\otimes\cdots\otimes x_{\sigma^{-1}(k)}.$$

On the other hand, $A\in GL(m)$ acts on $\otimes^k V$ by

$$A\cdot(x_1\otimes x_2\otimes\cdots\otimes x_k)=Ax_1\otimes Ax_2\otimes\cdots\otimes Ax_k,$$

and this action commutes with that of $\mathfrak{S}_k$ defined above.

Theorem 2.1 (H. Weyl's reciprocity). *If we regard $\otimes^k V$ as a $GL(m)\times\mathfrak{S}_k$-module, it decomposes as*

$$\otimes^k V=\sum_{\substack{\lambda:\text{partition}\\ d(\lambda)\leqq m\\ |\lambda|=k}} V_\lambda^{GL(m)}\otimes V_\lambda^{\mathfrak{S}_k}\ \textit{(direct sum)}.$$

Here $V_\lambda^{GL(m)}$ is the irreducible $GL(m)$-module corresponding to the character $\chi_{GL(m)}(\lambda)$, and $V_\lambda^{\mathfrak{S}_k}$ is the irreducible $\mathfrak{S}_k$-module corresponding to the Young diagram λ. (For the parametrization of the irreducible representations of $\mathfrak{S}_k$, see [J-K, Chap. 2]).

Since the equivalence classes of irreducible representations of $\mathfrak{S}_k$ are parametrized by the partitions of size k, we denote by $\chi_{\mathfrak{S}_k}(\lambda)$ the irreducible representation of $\mathfrak{S}_k$ or its character corresponding to a partition λ with $|\lambda|=k$.

Let R^k denote the character ring of $\mathfrak{S}_k$ (over $\boldsymbol{Z}$). Their module direct sum $R=\oplus_{k\geqq 0} R^k$ (where $R^0=\boldsymbol{Z}$) can be made into a graded algebra over $\boldsymbol{Z}$ with the multiplication $\cdot$ defined by

$$f\cdot g=\mathrm{Ind}_{\mathfrak{S}_m\times\mathfrak{S}_n}^{\mathfrak{S}_{m+n}}(f\times g)\in R^{m+n}\qquad \text{for } f\in R^m,\ g\in R^n.$$

The $\boldsymbol{Z}$-linear map defined by

$$\begin{array}{rccc} \mathrm{ch}\colon & R & \longrightarrow & \Lambda \\ & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ & \chi_{\mathfrak{S}_n}(\lambda) & \longrightarrow & \chi_{GL}(\lambda) \end{array}$$

gives an isomorphism of graded algebras, in virtue of the above theorem (H. Weyl's reciprocity). (See [M, p. 61, (7.3)])

W_n is embedded into $\mathfrak{S}_{2n}$ as the centralizer:

$$W_n=C_{\mathfrak{S}_{2n}}((1, 2)\,(3, 4)\cdots(2n-1, 2n)).$$

$C_{\mathfrak{S}_{2n}}(1, 2)(3, 4)\cdots(2n-1, 2n)$. More precisely, if we define an injective homomorphism $\Delta\colon \mathfrak{S}_n\to\mathfrak{S}_{2n}$ by

$$\Delta(\tau)\colon \begin{cases} 2i-1 \longrightarrow 2\tau(i)-1 \\ 2i \quad\;\;\, \longrightarrow 2\tau(i) \end{cases} \quad (i=1, 2, \cdots, n) \qquad \text{for } \tau\in\mathfrak{S}_n$$

and put $\sigma_i=(2i-1, 2i)$, $i=1, 2, \cdots, n$, then we have

$$W_n=\langle\Delta((\mathfrak{S}_n), \sigma_1, \cdots, \sigma_n\rangle=\Delta(\mathfrak{S}_n))\ltimes H,$$

where $H=\langle\sigma_1, \sigma_2, \cdots, \sigma_n\rangle\simeq\boldsymbol{Z}_2^n$.

For each $i=0, 1, 2, \cdots, n$, define a representation ρ_i of H by

$$\rho_i(\sigma_j)=\begin{cases} \;\;1 & \text{if } 1\leqq j\leqq i, \\ -1 & \text{if } i+1\leqq j\leqq n. \end{cases}$$

Noting that $W_n/H\simeq\mathfrak{S}_n$, we denote by $\chi_{W_n}(\lambda, \phi)$ the pull-back of the character $\chi_{\mathfrak{S}_n}(\lambda)$ to W_n (ϕ denotes the empty diagram). On the other hand, the representation ρ_0 of H can be extended to that of W_n by letting $\Delta(\mathfrak{S}_n)$ act trivially, since ρ_0 is $\Delta(\mathfrak{S}_n)$-invariant. Denote this character by $\chi_{W_n}(\phi, (n))$ and put $\chi_{W_n}(\phi, \lambda):=\chi_{W_n}(\lambda, \phi)\otimes\chi_{W_n}(\phi, (n))$. Corresponding to each representation ρ_i of H, a subgroup $W_i\times W_{n-i}$ is defined by

$$W_i=\langle\Delta(\mathfrak{S}_i), \sigma_1, \sigma_2, \cdots, \sigma_i\rangle \quad\text{and}\quad W_{n-i}=\langle\Delta(\mathfrak{S}_{n-i}), \sigma_{i+1}, \sigma_{i+2}, \cdots, \sigma_n\rangle,$$

where
$$\mathfrak{S}_i=\langle(1, 2), (2, 3), \cdots, (i-1, i)\rangle \quad\text{and}$$
$$\mathfrak{S}_{n-i}=\langle(i+1, i+2), (i+2, i+3), \cdots, (n-1, n)\rangle$$

are subgroups of $\mathfrak{S}_{2n}$. Then, according to so-called "Mackey-Wigner's little group method" (See [S, p. 62, Proposition 25]), we have an irreducible representation $\chi_{W_n}(\mu, \nu)$ by putting

$$\chi_{W_n}(\mu, \nu) = \mathrm{Ind}_{W_i \times W_{n-i}}^{W_n}(\chi_{W_i}(\mu, \phi) \times \chi_{W_{n-i}}(\phi, \nu)).$$

Then the $\chi_{W_n}(\xi, \psi)$ (ξ, ψ are partitions with $|\xi|+|\psi|=n$) constitute a complete set of representatives of the equivalence classes of irreducible representations of W_n. Just as we did for $\mathfrak{S}_n$, we shall write $V_{(\xi,\psi)}^{W_n}$ for the irreducible W_n-module with character $\chi_{W_n}(\xi, \psi)$.

If we denote the character ring of W_n (over $\boldsymbol{Z}$) by R_W^n, then their module dirct sum $R_W = \bigoplus_{n\geqq 0} R_W^n$, with $R_W^0 = \boldsymbol{Z}$, can be made into a graded algebra over $\boldsymbol{Z}$ with the multiplication defined by

$$f \cdot g = \mathrm{Ind}_{W_n \times W_m}^{W_{n+m}}(f \times g) \in R_W^{n+m} \qquad \text{for } f \in R_W^n,\ g \in R_W^m.$$

Now we define another (noncommutative) multiplication in Λ. Recall that an element of Λ can be regarded as a certain infinite $\boldsymbol{Z}$-linear combination of monomials in countably many variables $t_1, t_2, \cdots$. Let $f, g \in \Lambda$ and put $g = \sum_\lambda u_\lambda t^\lambda$, where $u_\lambda \in \boldsymbol{Z}$ and λ runs through multi-indices, namely the infinite sequences of nonnegative integers with finitely many nonzero terms. Suppose all $u_\lambda \geqq 0$. If we bring u_λ copies of each t^λ, then altogether we have a collection of countably many monomials. Arrange them in one sequence and label them $s_1, s_2, \cdots$ (the order is arbitrary). If we substitute s_i for each variable t_i in f, we get a new symmetric function in Λ, which is denoted by $f \circ g$. This multiplication $\circ$ is extended for all $g \in \Lambda$ by $\boldsymbol{Z}$-linearity. This notion has been introduced and called *plethysm* by D.E. Littlewood (see [L]).

$\pi_n(\chi_{GL}(\lambda) \circ \chi_{GL}(\mu))$ is the character of the representation of $GL(n)$ obtained as the composite of the following two homomorphisms:

$$GL(n) \xrightarrow{\rho_{\mu, GL(n)}} GL(V_\mu) \xrightarrow{\rho_{\lambda, GL(V_\mu)}} GL(V_\lambda^{GL(V_\mu)}).$$

We define a $\boldsymbol{Z}$-linear map $\mathrm{ch}_W\colon R_W \to \Lambda$ by

$$\mathrm{ch}_W(\chi_{W_n}(\lambda, \mu)) = (\chi_{GL}(\lambda) \circ p_2)(\chi_{GL}(\mu) \circ e_2).$$

It is easy to see that ch_W is an algebra homomorphism.

Caution. In general, ch_W is not injective. However, the significance of ch_W lies in the following fact.

Proposition 2.2. *The decomposition coefficients of*

$$(\chi_{GL}(\lambda) \circ p_2)(\chi_{GL}(\mu) \circ e_2)$$

into Schur functions coincide with those of $\chi_{W_n}(\lambda, \mu)\uparrow_{W_n}^{\mathfrak{S}_{2n}}$ *into irreducible constituents. That is, if we put*

$$(\chi_{GL}(\lambda) \circ p_2)(\chi_{GL}(\mu) \circ e_2) = \sum_\nu d^\nu_{\lambda\mu} \chi_{GL}(\nu)$$

and

$$\chi_{W_n}(\lambda, \mu) \Big\uparrow_{W_n}^{\mathfrak{S}_{2n}} = \sum_\nu \tilde{d}^\nu_{\lambda\mu} \chi_{\mathfrak{S}_{2n}}(\nu),$$

then we have $d^\nu_{\lambda\mu} = \tilde{d}^\nu_{\lambda\mu}$ *for all* λ, μ, *and* ν.

Proof. Fix an $m \geqq 2n$, and put $|\lambda| = i$. We shall show that $d^\nu_{\lambda\mu}$ and $\tilde{d}^\nu_{\lambda\mu}$ are both equal to the multiplicity of the irreducible $GL(m) \times W_i \times W_{n-i}$-module

$$(*) \qquad V_\nu^{GL(m)} \otimes V_{(\lambda,\phi)}^{W_i} \otimes V_{(\phi,\mu)}^{W_{n-i}}$$

in $\otimes^{2n} \boldsymbol{C}^m$. Here $\otimes^{2n} \boldsymbol{C}^m$ is regarded as a $GL(m, \boldsymbol{C}) \times \mathfrak{S}_{2n}$-modules as in H. Weyl's reciprocity, and $W_i \times W_{n-i} \subset W_n \subset \mathfrak{S}_{2n}$.

By H. Weyl's theorem, $\otimes^{2n} \boldsymbol{C}^m$ decomposes as

$$\otimes^{2n} \boldsymbol{C}^m = \sum_{|\nu| = 2n} V_\nu^{GL(m)} \otimes V_\nu^{\mathfrak{S}_{2n}}.$$

On the other hand, the definition of $\tilde{d}^\nu_{\lambda\mu}$ can be read as follows. Since $\chi_{W_n}(\lambda, \mu) = \mathrm{Ind}_{W_i \times W_{n-i}}^{W_n} (\chi_{W_i}(\lambda, \phi) \times \chi_{W_{n-i}}(\phi, \mu))$, we have

$$\chi_{W_n}(\lambda, \mu) \Big\uparrow_{W_n}^{\mathfrak{S}_{2n}} = \chi_{W_i}(\lambda, \phi) \times \chi_{W_{n-i}}(\phi, \mu) \Big\uparrow_{W_i \times W_{n-i}}^{\mathfrak{S}_{2n}}.$$

Frobenius' reciprocity shows that $V_\nu^{\mathfrak{S}_{2n}}$, regarded as a $W_i \times W_{n-i}$-module, contains $\tilde{d}^\nu_{\lambda\mu}$ times of $V_{(\lambda,\phi)}^{W_i} \otimes V_{(\phi,\mu)}^{W_{n-i}}$. So the multiplicity of (*) in $\otimes^{2n} \boldsymbol{C}^m$ is also $\tilde{d}^\nu_{\lambda\mu}$.

Next we consider $d^\nu_{\lambda\mu}$. Any submodule of $\otimes^{2n} \boldsymbol{C}^m$ isomorphic to (*) is contained in the ρ_i-isotypical component of $\otimes^{2n} \boldsymbol{C}^m$ with respect to $H \subset W_n$, because $\chi_{W_i}(\lambda, \phi) \times \chi_{W_{n-i}}(\phi, \mu) \downarrow_H$ is a multiple of ρ_i. The ρ_i-isotypical component is:

$$(\sharp) \qquad (\otimes^i S_2(\boldsymbol{C}^m)) \otimes (\otimes^{n-i} \Lambda^2(\boldsymbol{C}^m)),$$

where S_2 [resp. Λ^2] denotes the symmetric [resp. alternating] product of rank 2. $W_i \times W_{n-i}$ stabilizes this space, since it stabilizes ρ_i. We are going to take out its $\chi_{W_i}(\lambda, \phi) \times \chi_{W_{n-i}}(\phi, \mu)$-isotypical component.

$\Delta(\mathfrak{S}_i)$ acts on ($\sharp$) as the permutations of the i tensor factors of $\otimes^i S_2(\boldsymbol{C}^m)$. So if we regard it as a $GL(S_2(\boldsymbol{C}^m)) \times \Delta(\mathfrak{S}_i)$-module, then we have

$$\otimes^i S_2(C^m) = \sum_{|\lambda'|=i} V_{\lambda'}^{GL(S_2(C^m))} \otimes V_{\lambda'}^{\Delta(\mathfrak{S}_i)}.$$

Taking the action of $\langle \sigma_1, \sigma_2, \cdots, \sigma_i \rangle$ into account, we see that $V_{\lambda'}^{\Delta(\mathfrak{S}_n)}$ becomes a W_i -module $V_{(\lambda', \phi)}^{W_i}$.

On the other hand, $\Delta(\mathfrak{S}_{n-i})$ acts on ($\sharp$) as the permutations of the $n-i$ tensor factors of $\otimes^{n-i} \Lambda^2(C^m)$. So if we regard it as a $GL(\Lambda^2(C^m)) \times \Delta(\mathfrak{S}_{n-i})$-module, we have

$$\otimes^{n-i} \Lambda^2(C^m) = \sum_{|\mu'|=n-i} V_{\mu'}^{GL(\Lambda^2(C^m))} \otimes V_{\mu'}^{\Delta(\mathfrak{S}_{n-i})}.$$

Taking the action of $\langle \sigma_{i+1}, \sigma_{i+2}, \cdots, \sigma_n \rangle$ into account, we see that $V_{\mu'}^{\Delta(\mathfrak{S}_{n-i})}$ becomes a W_{n-i}-module $V_{(\phi, \mu')}^{W_{n-i}}$.

Hence the $\chi_{W_i}(\lambda, \phi) \times \chi_{W_{n-i}}(\phi, \mu)$-isotypical component of $\otimes^{2n} C^m$ is

$$V_{\lambda}^{GL(S_2(C^m))} \otimes V_{\mu}^{GL(\Lambda^2(C^m))} \otimes V_{(\lambda, \phi)}^{W_i} \otimes V_{(\phi, \mu)}^{W_{n-i}}.$$

As a $GL(m, C)$-module, $V_{\lambda}^{GL(S_2(C^m))} \otimes V_{\mu}^{GL(\Lambda^2(C))}$ affords $(\chi_{GL}(\lambda) \circ p_2)(\chi_{GL}(\mu) \circ e_2)$. By the definition of $d_{\lambda\mu}^{\nu}$, it contains $d_{\lambda\mu}^{\nu}$ times of $V_{\nu}^{GL(m)}$. So the multiplicity of (*) in $\otimes^{2n} C^m$ is also $d_{\lambda\mu}^{\nu}$. $\square$

By the definition of plethysm, the following four equalities hold:

i) $$\frac{1}{\prod_{1 \leq i < j \leq n} (1 - z_i z_j)} = \sum_{f=0}^{\infty} \pi_n(p_f \circ e_2),$$

ii) $$\frac{1}{\prod_{1 \leq i \leq j \leq n} (1 - z_i z_j)} = \sum_{f=0}^{\infty} \pi_n(p_f \circ p_2),$$

iii) $$\prod_{1 \leq i < j \leq n} (1 - z_i z_j) = \sum_{f=0}^{\infty} \pi_n(e_f \circ e_2),$$

iv) $$\prod_{1 \leq i \leq j \leq n} (1 - z_i z_j) = \sum_{f=0}^{\infty} \pi_n(e_f \circ p_2).$$

Comparing these with D.E. Littlewood's Lemma 1.1, we have:

Proposition 2.3.

i) $$p_f \circ e_2 = \sum_{\substack{\kappa:\text{partition} \\ |\kappa|=f}} \chi_{GL}({}^t(2\kappa)),$$

ii) $$p_f \circ p_2 = \sum_{\substack{\kappa:\text{partition} \\ |\kappa|=f}} \chi_{GL}(2\kappa),$$

iii) $$e_f \circ e_2 = \sum_{\substack{\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_s) \\ \alpha_1 > \alpha_2 > \cdots > \alpha_s > 0 \\ |\alpha|=f}} \chi_{GL}(\Gamma(\alpha)),$$

iv) $$e_f \circ p_2 = \sum_{\substack{\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_s) \\ \alpha_1 > \alpha_2 > \cdots > \alpha_s > 0 \\ |\alpha|=f}} \chi_{GL}({}^t\Gamma(\alpha)).$$

Applying Proposition 2.2 to $\chi_{W_n}(\phi, (n))$, $\chi_{W_n}((n), \phi)$, $\chi_{W_n}(\phi, (1^n))$, and $\chi_{W_n}((1^n)), \phi)$, we have:

Proposition 2.3′.

i) $\chi_{W_n}(\phi, (n)) \Big\uparrow_{W_n}^{\mathfrak{S}_{2n}} = \sum_{|\kappa|=n} \chi_{\mathfrak{S}_{2n}}({}^t(2\kappa))$,

ii) $\chi_{W_n}((n), \phi) \Big\uparrow_{W_n}^{\mathfrak{S}_{2n}} = \sum_{|\kappa|=n} \chi_{\mathfrak{S}_{2n}}(2\kappa)$,

iii) $\chi_{W_n}(\phi, (1^n)) \Big\uparrow_{W_n}^{\mathfrak{S}_{2n}} = \sum_{\substack{\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_s)\\ \alpha_1>\alpha_2>\cdots>\alpha_s>0\\ |\alpha|=n}} \chi_{\mathfrak{S}_{2n}}(\Gamma(\alpha))$,

iv) $\chi_{W_n}((1^n), \phi) \Big\downarrow_{W_n}^{\mathfrak{S}_{2n}} = \sum_{\substack{\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_s)\\ \alpha_1>\alpha_2>\cdots>\alpha_s>0\\ |\alpha|=n}} \chi_{\mathfrak{S}_{2n}}({}^t\Gamma(\alpha))$.

The formula ii) has been applied to the projective geometry over finite fields by J.G. Thompson [T].

Moreover, it should be noted that we can derive an algorithm to decompose the representation of $\mathfrak{S}_{2n}$ induced by an arbitrary irreducible representation of W_n. For a partition $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_k)$ of depth k, we have

$$\chi_{GL}(\lambda)=\det\begin{bmatrix} p_{\lambda_1} & p_{\lambda_1+1} & \cdots & p_{\lambda_1+(k-1)} \\ p_{\lambda_2-1} & p_{\lambda_2} & \cdots & p_{\lambda_2+(k-2)} \\ \vdots & \vdots & \ddots & \vdots \\ p_{\lambda_k-(k-1)} & p_{\lambda_k-(k-2)} & \cdots & p_{\lambda_k} \end{bmatrix}.$$

Hence, by ii) of Proposition 2.3, $\chi_{GL}(\lambda)\circ p_2$ is given by

$$\chi_{GL}(\lambda)\circ p_2=\det\begin{bmatrix} \sum_{|\kappa|=\lambda_1}\chi_{GL}(2\kappa) & \cdots & \sum_{|\kappa|=\lambda_1+(k-1)}\chi_{GL}(2\kappa) \\ \sum_{|\kappa|=\lambda_2-1}\chi_{GL}(2\kappa) & \cdots & \sum_{|\kappa|=\lambda_2+(k-2)}\chi_{GL}(2\kappa) \\ \vdots & \ddots & \vdots \\ \sum_{|\kappa|=\lambda_k-(k-1)}\chi_{GL}(2\kappa) & \cdots & \sum_{|\kappa|=\lambda_k}\chi_{GL}(2\kappa) \end{bmatrix}.$$

Similarly,

$$\chi_{GL}(\lambda)\circ e_2=\det\begin{bmatrix} \sum_{|\kappa|=\lambda_1}\chi_{GL}({}^t(2\kappa)) & \cdots & \sum_{|\kappa|=\lambda_1+(k-1)}\chi_{GL}({}^t(2\kappa)) \\ \sum_{|\kappa|=\lambda_2-1}\chi_{GL}({}^t(2\kappa)) & \cdots & \sum_{|\kappa|=\lambda_2+(k-2)}\chi_{GL}({}^t(2\kappa)) \\ \vdots & \ddots & \vdots \\ \sum_{|\kappa|=\lambda_k-(k-1)}\chi_{GL}({}^t(2\kappa)) & \cdots & \sum_{|\kappa|=\lambda_k}\chi_{GL}({}^t(2\kappa)) \end{bmatrix}.$$

For partitions μ and ν, $\chi_{GL}(\mu)\chi_{GL}(\nu)$ can be computed using Littlewood-Richardson's rule. Combining these facts with Proposition 2.2, we obtained an algorithm to write down the irreducible constituents of the representation of $\mathfrak{S}_{2n}$ induced by any irreducible representation of W_n.

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