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Torus Embeddings and de Rham Complexes

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Introduction

An n-dimensional normal algebraic variety is said to be a torus embedding or a toric variety if it has an effective regular action of the split algebraic torus of dimension n. In [I1] and [IO], we studied reduced closed subschemes of a torus embedding which are partial unions of the orbits of the torus action, which we call *toric polyhedra* in this paper. In [I1] we gave dualizing complexes of affine toric polyhedra consisting of coherent sheaves, and as a corollary we gave criteria for the schemes to be Gorenstein or Cohen-Macaulay.

In this paper, we study the algebraic de Rham complexes of toric polyhedra. Then, we generalize the notion of toric polyhedra and define *semi-toroidal varieties* which are the varieties with singularities locally isomorphic to those of toric polyhedra in the étale topology. For the motivation to consider such varieties, see also the introduction of [IO]. Although the results in [I1] are local, the complex constructed in [I1] is generalized for semi-toroidal varieties with a good filtration, and we will show that it is a dualizing complex in a global sense.

By using this dualizing complex, we define the de Rham complex $\tilde{\Omega}^{\cdot}_X$ of a *semi-toroidal variety* X *with filtration.* Our de Rham complex consists of coherent sheaves and is a generalization of that of Danilov [Da], which is defined for normal varieties with toroidal singularities. For an arbitrary $\boldsymbol{C}$-scheme of finite type, du Bois [dB] defined a de Rham complex in a derived category by using the simplicial resolution of the scheme introduced by Deligne in his mixed Hodge theory [Del]. We show that our de Rham complex is equal to du Bois's for these varieties. In particular, if X is complete, the natural spectral sequence $E_1^{p,q}=H^q(X, \tilde{\Omega}^p_X)\Rightarrow H^{p+q}(X, \boldsymbol{C})$ degenerates at the E_1-terms and converges to the Hodge filteration.

Notation. For subsets A, B of a set S, we denote $A\backslash B=\{a\in A; a\notin B\}$. If S is an additive group, then we denote $A+B=\{a+b; a\in A, b\in B\}$.

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The tensor products of modules are taken over $\boldsymbol{Z}$ if the coefficient ring is not specified.

Rings are always commutative rings with unity.

For a ring R, and for complexes $A^{\cdot}$ and $B^{\cdot}$ of R-modules, $\mathrm{Hom}_R(B^{\cdot}, A^{\cdot})$ denotes the double complex whose (i, j)-component is $\mathrm{Hom}_R(B^{-j}, A^i)$ and $d_1^{i,j}(f) = d_A^i \circ f$ and $d_2^{i,j}(f) = (-1)^{j+1} f \circ d_B^{-j-1}$ for $f \in \mathrm{Hom}_R(B^{-j}, A^i)$. For a double complex $D^{\cdot\cdot}$ of R-modules bounded below in both indices, the associated single complex $D^{\cdot}$ is the complex defined by $D^n = \bigoplus_{i+j=n} D^{i,j}$ and $d^n(x) = d_1^{i,j}(x) + (-1)^i d_2^{i,j}(x)$ for $x \in D^{i,j}$.

We denote by $\mathrm{Hom}_R^{\cdot}(B^{\cdot}, A^{\cdot})$ the associated single complex of $\mathrm{Hom}_R(B^{\cdot}, A^{\cdot})$ in case it is well-defined.

We write a triangle of objects in a derived category as if it is an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

For the properties of triangles, see [RD, Chap. I].

§ 1. Fans and complexes

Let N be a free $\boldsymbol{Z}$-module of rank $r \geq 0$, and let M be its dual $\boldsymbol{Z}$-module. The natural pairing $\langle\ ,\ \rangle : M \times N \to \boldsymbol{Z}$ is extended to the bilinear form $\langle\ ,\ \rangle : M_{\boldsymbol{R}} \times N_{\boldsymbol{R}} \to \boldsymbol{R}$ where $M_{\boldsymbol{R}} = M \otimes \boldsymbol{R}$ and $N_{\boldsymbol{R}} = N \otimes \boldsymbol{R}$.

Definition 1.1. A nonempty subset σ of $N_{\boldsymbol{R}}$ is said to be a *rational polyhedral cone* if there exists a finite subset $\{n_1, \cdots, n_s\}$ of N such that $\sigma = \boldsymbol{R}_0 n_1 + \cdots + \boldsymbol{R}_0 n_s$, where $\boldsymbol{R}_0 = \{c \in \boldsymbol{R};\ c \geq 0\}$. σ is said to be a *strongly convex rational polyhedral cone* (*s.c.r.p. cone* for short) if, furthermore, it satisfies the condition $\sigma \cap (-\sigma) = \{0\}$, where $-\sigma = \{-a;\ a \in \sigma\}$. For a rational polyhedral cone $\sigma \subset N_{\boldsymbol{R}}$, we denote by $\mathrm{int}\,\sigma$ the interior of σ in the linear space $\sigma + (-\sigma)$. A subset ρ of a rational polyhedral cone σ is said to be a *face* of σ and we denote $\rho \prec \sigma$ if there exists an element x of $M_{\boldsymbol{R}}$ such that $\langle x, a \rangle \geq 0$ for every $a \in \sigma$ and ρ is equal to $\{a \in \sigma;\ \langle x, a \rangle = 0\}$.

Let σ be an s.c.r.p. cone. Then σ itself and $\boldsymbol{0} = \{0\}$ are faces of σ, and every face of σ is also an s.c.r.p. cone.

Definition 1.2. A set Σ of s.c.r.p. cones of $N_{\boldsymbol{R}}$ is said to be a *fan* if

(1) $\sigma \in \Sigma$ and $\rho \prec \sigma$ imply $\rho \in \Sigma$, and

(2) $\sigma, \tau \in \Sigma$ and $\rho = \sigma \cap \tau$ imply $\rho \prec \sigma$ and $\rho \prec \tau$.

A fan Σ is said to be *complete* if it is a finite set and the union $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$ is equal to $N_{\boldsymbol{R}}$.

Let π be an s.c.r.p. cone. Then the set $\Gamma(\pi)$ of the faces of π is a fan of $N_{\boldsymbol{R}}$.

From now on in this section, Σ is always a fan of $N_{\boldsymbol{R}}$.

Definition 1.3. For a subset Φ of Σ, we say

(1) Φ is *star closed* if $\sigma \in \Phi$, $\tau \in \Sigma$ and $\sigma \prec \tau$ imply $\tau \in \Phi$,

(2) Φ is *star open* if $\tau \in \Phi$ and $\sigma \prec \tau$ imply $\sigma \in \Phi$, and

(3) Φ is *locally star closed* if σ, $\rho \in \Phi$ and $\sigma \prec \tau \prec \rho$ imply $\tau \in \Phi$.

Let Φ be a locally star closed subset of Σ. Then, for elements σ, $\tau \in \Sigma$, we denote

$$\Phi(\sigma \prec)=\{\tau \in \Phi;\ \sigma \prec \tau\},\quad \Phi(\prec \rho)=\{\tau \in \Phi;\ \tau \prec \rho\}\quad \text{and}$$
$$\Phi(\sigma \mid \rho)=\{\tau \in \Phi;\ \sigma \prec \tau \prec \rho\}.$$

It is clear by definition that $\Sigma(\sigma \prec)$ is star closed, $\Sigma(\prec \rho)$ is star open and $\Sigma(\sigma \mid \rho)$ is locally star closed in Σ, respectively, for any $\sigma, \rho \in \Sigma$.

For a rational polyhedral cone σ, we set $\sigma^{\perp}=\{x \in M_{\boldsymbol{R}};\ \langle x, a\rangle=0$ for every $a \in \sigma\}$, $N[\sigma]=N/(N \cap (\sigma+(-\sigma)))$ and $M[\sigma]=M \cap \sigma^{\perp}$. Then $N[\sigma]$ and $M[\sigma]$ are mutually dual free $\boldsymbol{Z}$-modules of rank $\operatorname{codim} \sigma=r-\dim \sigma$. Hence, if we set $\boldsymbol{Z}(\sigma)=\Lambda^{\operatorname{codim} \sigma} M[\sigma]$, then $\boldsymbol{Z}(\sigma)$ is a free $\boldsymbol{Z}$-module of rank one. Here we understand $\boldsymbol{Z}(\sigma)=\boldsymbol{Z}$ if $\operatorname{codim} \sigma=0$.

For a locally star closed subset Φ of Σ and for an integer p, we set $\Phi(p)=\{\sigma \in \Phi;\ \operatorname{codim} \sigma=p\}$. For $\sigma \in \Sigma(p)$ and $\tau \in \Sigma(p-1)$ with $\sigma \prec \tau$, the isomorphism

$$q_{\sigma/\tau}\colon \boldsymbol{Z}(\sigma) \longrightarrow \boldsymbol{Z}(\tau)$$

is defined as follows. Let n_1 be an element of N such that the homomorphism $\langle\ , n_1\rangle\colon M \to \boldsymbol{Z}$ is zero on $M[\tau]$ and maps $M[\sigma] \cap \tau^{\vee}$ onto $\boldsymbol{Z}_0=\{c \in \boldsymbol{Z};\ c \geq 0\}$, where $\tau^{\vee}$ is the dual cone $\{x \in M_{\boldsymbol{R}};\ \langle x, a\rangle \geq 0$ for every $a \in \tau\}$. Then we define $q_{\sigma/\tau}(m_1 \wedge \cdots \wedge m_p)=\langle m_1, n_1\rangle m_2 \wedge \cdots \wedge m_p$ for $m_1 \in M[\sigma]$ and $m_2, \cdots, m_p \in M[\tau]$. This definition is independent of the choice of n_1.

Lemma 1.4 ([I1, Lemma 1.4]). *For any elements $\sigma \in \Sigma(p)$ and $\rho \in \Sigma(p-2)$ with $\sigma \prec \rho$, there exist exactly two elements $\tau \in \Sigma(p-1)$ with $\sigma \prec \tau \prec \rho$. If we let them τ_1 and τ_2, then the equality $q_{\tau_1/\rho} \circ q_{\sigma/\tau_1}+q_{\tau_2/\rho} \circ q_{\sigma/\tau_2}=0$ holds.*

Let $\mathscr{C}$ be an abelian category such that its objects are either additive groups or sheaves of additive groups on a topological space. We also regard a fan Σ as a category by defining that only morphisms between cones are inculsion maps. For a covariant functor $F\colon \Sigma \to \mathscr{C}$ and for a locally star closed subset $\Phi \subset \Sigma$, we define a complex $C^{\cdot}(\Phi, F)$ of objects of $\mathscr{C}$ as follows. For each integer p, we set

$$C^{p}(\Phi, F)=\bigoplus_{\sigma \in \Phi(-p)} F(\sigma) \otimes \boldsymbol{Z}(\sigma).$$

The component of the coboundary homomorphism $d^p\colon C^p(\Phi, F)\to C^{p+1}(\Phi, F)$ with respect to the direct summands $F(\sigma)\otimes Z(\sigma)$ and $F(\tau)\otimes Z(\tau)$ associated to $\sigma \in \Phi(-p)$ and $\tau \in \Phi(-p-1)$, respectively, is defined to be the zero map if σ is not a face of τ and to be $F(i_{\sigma/\tau})\otimes q_{\sigma/\tau}$ if $\sigma\prec\tau$ where $i_{\sigma/\tau}$ is the inclusion map. Then by Lemma 1.4, we know $d^{p+1}\circ d^p=0$ for every p, and $C^\cdot(\Phi, F)$ is a complex. It is clear by definition that $C^p(\Phi, F)=0$ for $p<-r$ or $p>0$.

We also define similar complexes for contravariant functors. We set $Z(\sigma)^*=\mathrm{Hom}_Z(Z(\sigma), Z)$ for every $\sigma \in \Sigma$, and we define

$$q^*_{\tau/\sigma}=(-1)^{\mathrm{codim}\,\sigma}(q_{\sigma/\tau})^*\colon Z(\tau)^*\longrightarrow Z(\sigma)^*.$$

Then, for a contravariant functor $F\colon \Sigma\to\mathscr{C}$, we define the complex $C^\cdot(\Phi, F)$ by

$$C^p(\Phi, F)=\bigoplus_{\sigma\in\Phi(p)} F(\sigma)\otimes Z(\sigma)^*$$

and by defining the coboundary homomorphism $d\colon C^p(\Phi, F)\to C^{p+1}(\Phi, F)$ so that its component with respect to $\tau \in \Phi(p)$ and $\sigma \in \Phi(p+1)$ is $F(i_{\sigma/\tau})\otimes q^*_{\tau/\sigma}$ if $\sigma\prec\tau$ and is zero otherwise. In this case, $C^p(\Phi, F)=0$ for $p<0$ or $p>r$.

We define complexes for one other case which we use later. Let Ψ be a subset of $\Sigma^2=\Sigma\times\Sigma$. We say Ψ is *locally skew closed* if $\tau_1\prec\tau\prec\tau_2$ and $\sigma_2\prec\sigma\prec\sigma_1$ imply $(\tau, \sigma) \in \Psi$ for any (τ_1, σ_1) and (τ_2, σ_2) in Σ^2. Let $F\colon \Sigma^2\to\mathscr{C}$ be a double functor which is covariant for the first variable and contravariant for the second. Then, for a locally skew closed subset $\Psi\subset\Sigma^2$, we define a double complex $C^{\cdot\cdot}(\Psi, F)$ by

$$C^{p,q}(\Psi, F)=\bigoplus_{(\tau,\sigma)\in\Psi(-p,q)} F(\tau, \sigma)\otimes Z(\tau/\sigma),$$

where $\Psi(i,j)=\{(\tau, \sigma) \in \Psi;\ \mathrm{codim}\ \tau=i,\ \mathrm{codim}\ \sigma=j\}$ and $Z(\tau/\sigma)=Z(\tau)\otimes Z(\sigma)^*$. The coboundary homomorphisms $d_1^{p,q}\colon C^{p,q}(\Psi, F)\to C^{p+1,q}(\Psi, F)$ and $d_2^{p,q}\colon C^{p,q}(\Psi, F)\to C^{p,q+1}(\Psi, F)$ are defined similarly as in the cases of covariant functors and contravariant functors, respectively. We denote by $C^\cdot(\Psi, F)$ the associated single complex of this double complex. Namely,

$$C^l(\Psi, F)=\bigoplus_{p+q=\ell} C^{p,q}(\Psi, F),$$

and, for integers p, q, p', q' with $p+q=\ell$ and $p'+q'=\ell+1$, the component of the homomorphism $d^\ell\colon C^\ell(\Psi, F)\to C^{\ell+1}(\Psi, F)$ with respect to the direct summands $C^{p,q}(\Psi, F)$ and $C^{p',q'}(\Psi, F)$ is equal to $d_1^{p,q}$ if $q=q'$, is equal to $(-1)^p d_2^{p,q}$ if $p=p'$ and is zero otherwise.

For nonnegative integers s, t, we define $Z_{s,t}$ to be the functor with the constant value Z from Σ^{s+t} which we regard covariant for the first s variables and contravariant for the last t variables.

For a star closed subset $\Phi \subset \Sigma$, we set $\Phi^{(2)} = \{(\tau, \sigma) \in \Phi^2;\ \sigma \prec \tau\}$. Then $\Phi^{(2)}$ is a locally skew closed subset of Σ^2. Hence we can consider the double complex $C^{\cdot\cdot}(\Phi^{(2)}, Z_{1,1})$. Since $C^{p,q}(\Phi^{(2)}, Z_{1,1}) = \bigoplus_{(\tau,\sigma)\in\Phi^{(2)}(-p,q)} Z(\tau/\sigma)$ and $\operatorname{codim}\tau \leq \operatorname{codim}\sigma$ for every $(\tau, \sigma) \in \Phi^{(2)}$, we know $C^{p,q}(\Phi^{(2)}, Z_{1,1}) = 0$ unless $-h \leq p \leq -d$, $d \leq q \leq h$ and $p+q \geq 0$, where $h = \operatorname{ht}\Phi = \max\{i;\ \Phi(i) \neq \phi\}$ and is called the *height* of Φ and $d = \min\{i;\ \Phi(i) \neq \phi\}$. Since $Z(\sigma/\sigma) = Z$ for every $\sigma \in \Sigma$, we know $C^0(\Phi^{(2)}, Z_{1,1}) = \bigoplus_{\sigma\in\Phi} Z = Z^{\Phi}$. We denote by $\lambda\colon Z \to C^0(\Phi^{(2)}, Z_{1,1})$ the diagonal homomorphism to the direct sum of Z's.

Lemma 1.5. *Let π be an s.c.r.p. cone, and let $\Sigma = \Gamma(\pi)$, i.e. the set of the faces of π. Then, for any nonempty star closed subset $\Phi \subset \Sigma$, the sequence of finitely generated free Z-modules*

$$0 \longrightarrow Z \xrightarrow{\lambda} C^0(\Phi^{(2)}, Z_{1,1}) \xrightarrow{d^0} \cdots \xrightarrow{d^{h-1}} C^h(\Phi^{(2)}, Z_{1,1}) \longrightarrow 0$$

is exact, where $h = \operatorname{ht}\Phi$.

Proof. In the proof, we denote this sequence by $A^{\cdot}$. In order to see that $A^{\cdot}$ is a complex, it is sufficient to show that $(d^0 \circ \lambda)(1) = 0$. Since $C^1(\Phi^{(2)}, Z_{1,1}) = \bigoplus_{p+q=1} \bigoplus_{(\tau,\sigma)\in\Phi^{(2)}(-p,q)} Z(\tau/\sigma)$, we have to show that each component of $(d^0 \circ \lambda)(1)$ in $Z(\tau/\sigma)$ is zero. Let a be a generator of $Z(\sigma)$. Then $b = q_{\sigma/\tau}(a)$ is a generator of $Z(\tau)$. Let $a^* \in Z(\sigma)^*$ and $b^* \in Z(\tau)^*$ be the generators dual to a and b, respectively. Then the components of $\lambda(1)$ at $Z(\sigma/\sigma)$ and $Z(\tau/\tau)$ are $a \otimes a^*$ and $b \otimes b^*$, respectively. Hence the component of $(d^0 \circ \lambda)(1)$ at $Z(\tau/\sigma)$ is equal to $q_{\sigma/\tau}(a) \otimes a^* + (-1)^p b \otimes q^*_{\tau/\sigma}(b^*) = b \otimes a^* + (-1)^{p+q} b \otimes a^* = 0$. Thus A is a complex. Let $d = \operatorname{codim}\pi$. We define a decreasing filtration $\{F^p\}$ on $A^{\cdot}$ by $F^p(A^{\cdot}) = A^{\cdot}$, for $p \leq d$, and $F^p(A^{\cdot}) = \bigoplus_{j\geq p} C^{i,j}(\Phi^{(2)}, Z_{1,1})$ for $p > d$. Then $\operatorname{Gr}^p_F(A^{\cdot}) = F^p(A^{\cdot})/F^{p+1}(A^{\cdot})$ is equal to zero for $p < d$, is equal to the sequence

$$\cdots \longrightarrow 0 \longrightarrow Z \xrightarrow{d^{-1}} Z(\pi/\pi) \xrightarrow{d^0} 0 \longrightarrow \cdots$$

for $p = d$ and is equal to $\bigoplus_{\sigma\in\Phi(p)} C_{\cdot}(\Sigma(\sigma\prec), Z_{1,0}) \otimes Z(\sigma)^*$ for $p > d$. The homomorphism d^{-1}, in the case $p = d$, is the identity since it is obtained by λ. The sequence in the case $p > d$ is equal to $\bigoplus_{\sigma\in\Phi(P)} C^{\cdot}(\Gamma(\pi[\sigma]), Z_{1,0})$, where $\pi[\sigma]$ is the image of π in $N[\sigma]_R = N_R/(\sigma + (-\sigma))$. Note that $\pi[\sigma]$ is an s.c.r.p. cone in $N[\sigma]_R$. It is also exact since $\Gamma(\pi[\sigma])$ is homologically trivial for $\sigma \neq \pi$ by [I1, Prop. 2.3]. Hence the E_2-terms of the spectral sequence associated to this filtration are all zero. This implies that all the cohomologies of the complex $A^{\bullet}$ vanish. q.e.d.

The following lemma is essentially due to the contractibility of convex sets (cf. [TE, Chap. 1, § 3] and [Dem, § 4, Prop. 6]).

Lemma 1.6. *Assume Σ is finite and $\pi \subset |\Sigma|$ $\pi+(-\pi)$ is a rational polyhedral cone of codimension $d \geq 0$. Then, for the star closed subset $\Phi=\{\sigma \in \Sigma;\ \sigma \cap \operatorname{int} \pi \neq \phi\}$ of Σ, we have*

$$H^{i}(C^{\bullet}(\Phi, Z_{1,0}))=0 \quad \textit{for } i \neq -d \textit{ and } H^{-d}(C^{\bullet}(\Phi, Z_{1,0}))=Z(\pi).$$

In other words, $C^{\bullet}(\Phi, Z_{1,0})$ is naturally quasi-isomorphic to $Z(\pi)[d]$. In particular,

$$\sum_{\sigma \in \Phi}(-1)^{\dim \sigma}=(-1)^{r} \sum_{\sigma \in \Phi}(-1)^{\operatorname{codim} \sigma}=(-1)^{r-d}.$$

Proof. By replacing Σ by the fan $\{\sigma \cap \pi;\ \sigma \in \Sigma\}$, we may assume $|\Sigma|=\pi$. For $\sigma \in \Phi(d)$, we have $M[\sigma]=M[\pi]$ and $Z(\sigma)=Z(\pi)$. Hence $C^{-d}(\Phi, Z_{1,0})=\bigoplus_{\sigma \in \Phi(d)} Z(\sigma)=Z(\pi)^{\Phi(d)}$. Let ε: $C^{-d}(\Phi, Z_{1,0}) \rightarrow Z(\pi)$ be the trace homomorphism. We denote by $\tilde{C}^{\bullet}(\Phi, Z_{1,0})$ the augmented complex

$$\cdots \rightarrow 0 \longrightarrow C^{-r}(\Phi, Z_{1,0}) \rightarrow \cdots \rightarrow C^{-d}(\Phi, Z_{1,0}) \longrightarrow Z(\pi) \longrightarrow 0 \rightarrow \cdots.$$

It is sufficient to show the cohomologies of this complex to vanish, which we prove by induction on rank N and the number of elements in Φ. If $d>0$, then by replacing N by its submodule $N \cap (\pi+(-\pi))$, we can reduce it to the case of lower rank. Hence we assume $\dim \pi=r$. Since the assertion is obvious if $r=0$, we assume $r>0$. Set $\eta=\bigcap_{\sigma \in \Phi} \sigma$.

Assume $\eta \notin \Phi$. Then, there exist $\sigma, \tau \in \Phi$ such that $\sigma \cap \tau \cap \operatorname{int} \pi=\phi$. Hence there exists a rational hyperplane H such that, if we denote the two open half spaces with the common boundary H by H_{+} and H_{-}, we have $\sigma \cap \operatorname{int} \pi \subset H_{+}$ and $\tau \cap \operatorname{int} \pi \subset H_{-}$ as well as the fact that H contains no element of Φ. Then we get the exact sequence

$$0 \longrightarrow \tilde{C}^{\bullet}(\Phi_{0}, Z_{1,0}) \longrightarrow \tilde{C}^{\bullet}(\Phi_{+}, Z_{1,0}) \oplus \tilde{C}^{\bullet}(\Phi_{-}, Z_{1,0}) \longrightarrow \tilde{C}^{\bullet}(\Phi, Z_{1,0}) \longrightarrow 0$$

of complexes, where $\Phi_{+}=\{\sigma \in \Phi;\ \sigma \cap H_{+} \neq \phi\}$, $\Phi_{-}=\{\sigma \in \Phi;\ \sigma \cap H_{-} \neq \phi\}$ and $\Phi_{0}=\{\sigma \in \Phi;\ \sigma \cap H \neq \phi\}$. Since both Φ_{+} and Φ_{-} have less elements than Φ we know that the second complex of the exact sequence is exact by induction assumption. Since $\{\rho \cap H;\ \rho \in \Sigma\}$ is a fan of H, the first complex is also exact by assumption for the case N has lower rank. Thus $\tilde{C}^{\bullet}(\Phi, Z_{1,0})$ is also exact.

Next, assume $\eta \in \Phi$ and $\eta \neq \mathbf{0}$. In this case, by replacing every cone in Φ by its image in $N[\eta]_{\boldsymbol{R}}=N_{\boldsymbol{R}}/(\eta+(-\eta))$, we are reduced to the case of lower rank. Hence it is exact by induction assumption.

Finally, assume $\mathbf{0}\in\Phi$. In this case, we have $\pi=N_{\boldsymbol{R}}$ and $\Phi=\Sigma$. Since clearly $H^{-r}(\tilde{C}^{\cdot}(\Sigma, Z_{1,0}))=0$, it is sufficient to show $H^{i}(\tilde{C}^{\cdot}(\Sigma_{+}, Z_{1,0}))=0$ for $i>-r$, where $\Sigma_{+}=\Sigma\backslash\{\mathbf{0}\}$. We can show this similarly as in the case $\eta\notin\Phi$, by taking a general rational hyperplane H. q.e.d.

§ 2. Normal semigroup rings and Danilov's de Rham complex

In this section, we fix an s.c.r.p. cone π in $N_{\boldsymbol{R}}$ and we set $\Sigma=\Gamma(\pi)$, i.e. the set of all faces of π.

Let k be an arbitrary field, and let S be the semigroup ring $k[M\cap\pi^{\vee}]$, where $\pi^{\vee}\subset M_{\boldsymbol{R}}$ is the dual cone of π. We denote by $e(m)$ the element of S which corresponds to $m\in M\cap\pi^{\vee}$. Then $S=\bigoplus_{m\in M\cap\pi^{\vee}}ke(m)$, and $e(m)e(m')=e(m+m')$ for $m, m'\in M\cap\pi^{\vee}$. For a subset U of $M\cap\pi^{\vee}$, we denote $k[U]=\bigoplus_{m\in U}ke(m)$. Here note that $k[U]$ does not necessarily contain k in our notation. The subset U is said to be an *ideal* if $a\in U$ and $b\in M\cap\pi^{\vee}$ imply $a+b\in U$. If U is an ideal, then $k[U]$ is an ideal of the ring S. For the complement $E=(M\cap\pi^{\vee})\backslash U$, we identify $k[E]$ with the quotient ring $S/k[U]$. In particular, for a face σ of π, $P(\sigma)=k[M\cap(\pi^{\vee}\backslash\sigma^{\perp})]$ is a prime ideal of S, and $k[M\cap\pi^{\vee}\cap\sigma^{\perp}]$ is the quotient ring $S/P(\sigma)$. We denote $k[M\cap\pi^{\vee}\cap\sigma^{\perp}]$ by $A(\sigma)$ or $S[\sigma]$ according as whether we regard it as an S-module or a ring, respectively. The ring $S[\sigma]$ is an affine ring which has same quotient field as $k[M[\sigma]]$. Since rank $M[\sigma]=\operatorname{codim}\sigma$, we know dim $S[\sigma]=i$ for $\sigma\in\Sigma(i)$.

Let $\{\gamma_1, \cdots, \gamma_s\}$ be the set of one-dimensional faces of π. Let J be the ideal $P(\gamma_1)\cap\cdots\cap P(\gamma_s)=k[M\cap\operatorname{int}\pi^{\vee}]$ of S. We set

$$\Theta_S(-\log J)=\{\alpha\in\operatorname{Der}_k(S);\ \alpha(J)\subset J\},$$

where $\operatorname{Der}_k(S)$ is the set of k-derivations from S to itself. For each element n of N, the k-derivation δ_n which is defined by $\delta_n(e(m))=\langle m, n\rangle e(m)$ for every $m\in M\cap\pi^{\vee}$ is clearly an element of $\Theta_S(-\log J)$. By [IO, Prop. 1.12], this correspondence $n\mapsto\delta_n$ induces an isomorphism of S-modules $S\otimes N\simeq\Theta_S(-\log J)$. Let $\Omega^1(\log J)$ be the dual S-module

$$\operatorname{Hom}_S(\Theta_S(-\log J), S).$$

By the above isomorphism, we also have an isomorphism $S\otimes M\simeq\Omega^1(\log J)$. It is natural to denote by $de(m)/e(m)$ the element of $\Omega^1(\log J)$ which corresponds to $1\otimes m\in S\otimes M$ by this isomorphism. For a $\boldsymbol{Z}$-basis $\{m_1, \cdots, m_r\}$ of M, $\Omega^1(\log J)$ is the free S-module with the basis $\{de(m_1)/e(m_1), \cdots, de(m_r)e(m_r)\}$. For each integer p, we set $\Omega^p(\log J)=\Lambda^p\Omega^1(\log J)$. Clearly, $\Omega^0(\log J)=S$ and $\Omega^p(\log J)=0$ for $p<0$ or $p>r$.

The k-algebra S has a natural M-grading. Namely, we set $\deg(e(m)) = m$ for every $m \in M \cap \pi^\vee$. By defining $\deg(de(m)/e(m))=0$ for every $m \in M$, the S-module $\Omega^p(\log J)$ also has an M-grading for every p. Since $\Omega^p(\log J)$ is naturally isomorphic to $S \otimes \Lambda^p M$, its component $\Omega^p(\log J)(m)$ of degree m is equal to $ke(m)\otimes \Lambda^p M$ if m is in $M \cap \pi^\vee$ and is zero otherwise.

Danilov's S-module $\tilde{\Omega}_S^p$ is given as an S-submodule of $\Omega^p(\log J)$. For each element $m \in M \cap \pi^\vee$, the intersection $\rho(m) = \pi \cap m^\perp$ is a face of π, where $m^\perp = \{a \in N_{\mathbf{R}};\ \langle m, a\rangle = 0\}$. As we see easily,

$$\rho(m+m') = \rho(m) \cap \rho(m') \qquad \text{for } m, m' \in M \cap \pi^\vee.$$

Now, we set

$$\tilde{\Omega}_S^p = \bigoplus_{m \in M \cap \pi^\vee} ke(m) \otimes \Lambda^p M[\rho(m)] \subset \Omega_S^p(\log J).$$

Since $M[\rho(m)] \subset M[\rho(m+m')]$ for any $m, m' \in M \cap \pi^\vee$, we have

$$e(m')(ke(m)\otimes \Lambda^p M[\rho(m)]) \subset ke(m+m') \otimes \Lambda^p M[\rho(m+m')].$$

This implies $\tilde{\Omega}_S^p$ is an S-submodule of $\Omega_S^p(\log J)$. Note that Danilov [Da] denotes this module simply by Ω_S^p, but, in this paper, we denote it by $\tilde{\Omega}_S^p$ to avoid confusion. For each integer p, the exterior derivative

$$d\colon \Omega_S^p(\log J) \longrightarrow \Omega_S^{p+1}(\log J)$$

is defined to be the k-linear homomorphism with $d(e(m)\otimes(m_1 \wedge \cdots \wedge m_p)) = e(m)\otimes(m \wedge m_1 \wedge \cdots \wedge m_p)$ for any elements $m, m_1, \cdots, m_p \in M$. d is a differential operator of rank one which preserves the M-grading. Since $m \in M[\rho(m)]$ for every $m \in M \cap \pi^\vee$, we have $d(\tilde{\Omega}_S^p) \subset \tilde{\Omega}_S^{p+1}$ for every p. Since $d^2=0$ is clear, we get a complex

$$\tilde{\Omega}_S^{\cdot} = (\cdots \longrightarrow 0 \longrightarrow S \xrightarrow{d} \tilde{\Omega}_S^1 \xrightarrow{d} \cdots \xrightarrow{d} \tilde{\Omega}_S^r \longrightarrow 0 \longrightarrow \cdots)$$

for which Danilov showed the following "Poincaré lemma".

Proposition 2.1 ([Da]). *Assume* $\dim \pi$ *is equal to* $r = \dim N_{\mathbf{R}}$ *and the characteristic of* k *is equal to zero. Then the isomorphism* $k \simeq S(0)$ *defined by* $a \mapsto ae(0)$ *induces the exact sequence*

$$0 \longrightarrow k \longrightarrow S \xrightarrow{d} \tilde{\Omega}_S^1 \xrightarrow{d} \cdots \xrightarrow{d} \tilde{\Omega}_S^r \longrightarrow 0.$$

The cone π is said to be *nonsingular if* $\pi = \mathbf{R}_0 n_1 + \cdots + \mathbf{R}_0 n_t$ for a $\mathbf{Z}$-basis $\{n_1, \cdots, n_r\}$ of N and for an integer $0 \le t \le r$. If π is nonsingular, then S is a regular ring and $\tilde{\Omega}_S^p$ is equal to the S-module of the regular p-forms on the nonsingular affine scheme Spec(S). Actually, if we set $z_1=e(m_1), \cdots, z_r=e(m_r)$ for the basis $\{m_1, \cdots, m_r\}$ of M dual to $\{n_1, \cdots, n_r\}$, then $\tilde{\Omega}_S^p$ is the free S-module with the basis $\{dz_1, \cdots, dz_r\}$. For general π, $\tilde{\Omega}_S^p$ is equal to the S-module of regular p-forms on the complement of the singular locus of Spec(S) (see [Da, Chap. I, § 4]).

Now, let Φ be a star closed subset of $\Sigma=\Gamma(\pi)$. We set $I(\Phi)=\bigcap_{\sigma\in\Phi}P(\sigma)$, and denote the quotient ring $S/I(\Phi)$ by $B(\Phi)$. Since $I(\Phi)$ is a semiprime ideal, $B(\Phi)$ has no nilpotent element. We define the de Rham complex $\tilde{\Omega}^{\cdot}_{B(\Phi)}$ for the ring $B(\Phi)$ as follows. Since

$$I(\Phi)=k[M\cap(\pi^\vee\backslash(\bigcup_{\sigma\in\Phi}\sigma^\perp))]$$

and $\Omega_S^p(\log J)$ is a free S-module with a basis consisting of homogeneous elements of degree 0, the homogeneous part of degree m of $I(\Phi)\Omega_S^p(\log J)$ is equal to $\Omega_S^p(\log J)(m)=ke(m)\otimes\Lambda^pM$ if m is in $\pi^\vee\backslash(\bigcup_{\sigma\in\Phi}\sigma^\perp)$ and is equal to zero otherwise. We set

$$\tilde{\Omega}^p_{B(\Phi)}=(\tilde{\Omega}_S^p+I(\Phi)\Omega_S^p(\log J))/I(\Phi)\Omega_S^p(\log J).$$

Then $\tilde{\Omega}^p_{B(\Phi)}(m)$ is equal to $\tilde{\Omega}_S^p(m)=ke(m)\otimes\Lambda^pM[\rho(m)]$ if $m\in\pi^\vee\cap(\bigcup_{\sigma\in\Phi}\sigma^\perp)$ and $\tilde{\Omega}^p_{B(\Phi)}=0$ otherwise. Since $m\in\pi^\vee\cap(\bigcup_{\sigma\in\Phi}\sigma^\perp)$ implies $\rho(m)\in\Phi$ and rank $M[\sigma]=\operatorname{codim}\sigma$ for every $\sigma\in\Sigma$, we know that $\Lambda^pM[\sigma]=0$ for every $\sigma\in\Phi$ if $p>h=\operatorname{ht}\Phi$. Thus $\tilde{\Omega}^p_{B(\Phi)}=0$ for $p>h$. On the other hand, we have $\tilde{\Omega}^0_{B(\Phi)}=B(\Phi)$ easily by definition. Let Φ' be another star closed subset of Σ with $\Phi'\subset\Phi$. Then, since $I(\Phi)\subset I(\Phi')$, there exists a natural surjection $\tilde{\Omega}^p_{B(\Phi)}\to\tilde{\Omega}^p_{B(\Phi')}$.

For each element $\eta\in\Sigma$, $\Sigma(\eta\prec)=\{\sigma\in\Sigma;\eta\prec\sigma\}$ is a star closed subset of Σ. Since $I(\Sigma(\eta\prec))=P(\eta)$, we have $B(\Sigma(\eta\prec))=S[\eta]$. On the other hand, since $N[\eta]=N/N\cap(\eta+(-\eta))$ and $M[\eta]$ are mutually dual $\boldsymbol{Z}$-modules, we have $k[M[\eta]\cap\pi[\eta]^\vee]=k[M\cap\pi^\vee\cap\eta^\perp]$. Hence the ring $S[\eta]$ is also defined by $k[M[\eta]\cap\pi[\eta]^\vee]$ for the pair $(N[\eta], \pi[\eta])$ similarly as we defined S for (N, π). Danilov's $S[\eta]$-module $\tilde{\Omega}^p_{S[\eta]}$ defined for $S[\eta]$ is equal to

$$\bigoplus_{m\in M[\eta]\cap\pi[\eta]^\vee} ke(m)\otimes\Lambda^pM[\eta][\rho[\eta](m)],$$

where $\rho[\eta](m)=\pi[\eta]\cap m^\perp$ for $m\in M[\eta]$, and $M[\eta][\rho]=\{m\in M[\eta];\langle m, a\rangle=0$ for every $a\in\rho\}$ for $\rho\in\Gamma(\pi[\eta])$. Since, clearly $\eta\prec\rho(m)$ and $\rho[\eta](m)=\rho(m)[\eta]$ for $m\in M[\eta]\cap\pi[\eta]^\vee$ and since $M[\eta][\rho[\eta]]=M[\rho]$ for $\rho\in\Gamma(\pi)$ with $\eta\prec\rho$, we have

$$\tilde{\Omega}^p_{S[\eta]}=\bigoplus_{m\in M\cap\pi^\vee\cap\eta^\perp} ke(m)\otimes\Lambda^pM[\rho(m)].$$

Thus we have the following proposition which shows the compatibility of the notation.

Proposition 2.2. *For each element* $\eta \in \Sigma$ *and for each integer* p, *there exists a natural isomorphism*

$$\tilde{\Omega}^p_{S[\eta]} \simeq \tilde{\Omega}^p_{B(\Sigma(\eta\prec))}.$$

Since the exterior derivative $d\colon \tilde{\Omega}^p_S \to \tilde{\Omega}^{p+1}_S$ preserves the M-grading, this induces a differential operator $d\colon \tilde{\Omega}^p_{B(\Phi)} \to \tilde{\Omega}^{p+1}_{B(\Phi)}$, and we get a complex $\tilde{\Omega}^{\cdot}_{B(\Phi)}$. Since the homogeneous part $\tilde{\Omega}^{\cdot}_{B(\Phi)}(m)$ of this complex of degree $m \in M$, is equal to $\tilde{\Omega}^{\cdot}_S(m)$ if $m \in M \cap (\pi^\vee \cap (\bigcup_{\sigma\in\Phi}\sigma^\perp))$, and is equal to the zero complex otherwise, we have the following proposition by Proposition 2.1.

Proposition 2.3. *If* $\dim \pi = r$ *and* $\operatorname{char} k = 0$, *then the sequence*

$$0 \longrightarrow k \longrightarrow B(\Phi) \xrightarrow{d} \Omega^1_{B(\Phi)} \xrightarrow{d} \cdots \xrightarrow{d} \tilde{\Omega}^h_{B(\Phi)} \longrightarrow 0$$

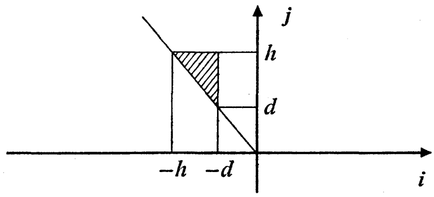
is exact, where $h = \operatorname{ht}\Phi$.

We are going to give another description of the $B(\Phi)$-module $\tilde{\Omega}^p_{B(\Phi)}$ for each integer p. Actually, it turns out to be the Grothendieck dual of the complex which consists of modules of logarithmic differential forms and Poincaré residue maps.

We understand the symbol A to be the covariant functor from Σ to the category of S-modules defined by $A(\sigma) = S/P(\sigma)$ for every $\sigma \in \Sigma$. Let Λ^p be the contravariant functor from Σ to the category of additive groups defined by $\Lambda^p(\sigma) = \Lambda^p M[\sigma]$. For elements $\sigma, \tau \in \Sigma$ with $\sigma \prec \tau$, the homomorphism $A(\sigma) \to A(\tau)$ is the natural surjection and the homomorphism $\Lambda^p M[\tau] \to \Lambda^p M[\sigma]$ is induced by the inclusion $M[\tau] \subset M[\sigma]$. We define the double functor $A \otimes \Lambda^p$ from Σ^2 to the category of S-modules by $A\otimes\Lambda^p(\tau, \sigma) = A(\tau)\otimes\Lambda^p M[\sigma]$. This functor is covariant for the first variable and contravariant for the second. For a star closed subset Φ of Σ, we denote $\Phi^{(2)} = \{(\tau, \sigma) \in \Phi^2; \sigma \prec \tau\}$ as in the previous section. Then $\Phi^{(2)}$ is locally skew closed, and we get the double complex $C^{\cdot\cdot}(\Phi^{(2)}, A\otimes\Lambda^p)$. Since

$$C^{i,j}(\Phi^{(2)}, A\otimes\Lambda^p) = \bigoplus_{(\tau,\sigma)\in\Phi^{(2)}(-i,j)} A(\tau)\otimes\Lambda^p M[\sigma]\otimes \mathbf{Z}(\tau/\sigma),$$

this component is not zero only for i, j with $-h \le i \le -d$, $d \le j \le h$, and $i + j \ge 0$, where $h = \operatorname{ht}\Phi$ and $d = \operatorname{codim}\pi$. Hence, the component $C^\ell(\Phi^{(2)}, A\otimes\Lambda^p)$ of the associated single complex $C^{\cdot}(\Phi^{(2)}, A\otimes\Lambda^p)$ is zero unless $0 \le \ell \le h - d$.



Since $Z(\eta/\eta)=Z$ for every $\eta \in \Sigma$, we have

$$C^0(\Phi^{(2)}, A\otimes\Lambda^p)=\bigoplus_{\eta\in\Phi} A(\eta)\otimes\Lambda^p M[\eta].$$

We define the $B(\Phi)$-homomorphism $\omega^p(\Phi)$: $\tilde{\Omega}^p_{B(\Phi)}\to C^0(\Phi^{(2)}, A\otimes\Lambda^p)$ as follows: For each $\eta \in \Phi$, we have $\Sigma(\eta\prec)\subset\Phi$. Hence there exists a natural surjection $\tilde{\Omega}^p_{B(\Phi)}\to\tilde{\Omega}^p_{B(\Sigma(\eta\prec))}$. By Proposition 2.2, we have $\tilde{\Omega}^p_{B(\Sigma(\eta\prec))}\simeq\tilde{\Omega}^p_{S[\eta]}$ and this is contained in $\Omega^p_{S[\eta]}(\log J[\eta])=S[\eta]\otimes\Lambda^p M[\eta]$, where $J[\eta]\subset S[\eta]$ is the ideal defined similarly as $J\subset S$. Since $S[\eta]=A(\eta)$, we get the composed homomorphism $\tilde{\Omega}^p_{B(\Phi)}\to A(\eta)\otimes\Lambda^p M[\eta]$. We define $\omega^p(\Phi)$ to be the homomorphism $\tilde{\Omega}^p_{B(\Phi)}\to C^0(\Phi^{(2)}, A\otimes\Lambda^p)=\bigoplus_{\eta\in\Phi}A(\eta)\otimes\Lambda^p M[\eta]$ which induces this homomorphism for each $\eta \in \Phi$.

Proposition 2.4. *Let Φ be a star closed subset of Σ, and let p be an integer. Then we have*

$$H^i(C^\cdot(\Phi^{(2)}, A\otimes\Lambda^p))=0 \qquad \textit{for } i\neq 0,$$

and the homomorphism $\omega^p(\Phi)$ gives an isomorphism

$$\tilde{\Omega}^p_{B(\Phi)}\simeq H^0(C^\cdot(\Phi^{(2)}, A\otimes\Lambda^p)).$$

Proof. Note that $\omega^p(\Phi)$ and every homomorphism in the complex $C^\cdot(\Phi^{(2)}, A\otimes\Lambda^p))$ preserve the grading of the modules. In the proof, we denote by $L(m)^\cdot$ the component of degree m of the sequence

$$0\longrightarrow\tilde{\Omega}^p_{B(\Phi)}\xrightarrow{\omega^p(\Phi)}C^\cdot(\Phi^{(2)}, A\otimes\Lambda^p),$$

for each $m \in M$. It is sufficient to show the exactness of $L(m)^\cdot$ for every $m \in M$. If m is not in $M\cap\pi^\vee$, then this is true, since all the components of $L(m)^\cdot$ are zero. We assume $m \in M\cap\pi^\vee$, and we set $\rho=\rho(m)$. Then $\tilde{\Omega}^p_{B(\Phi)}(m)=ke(m)\otimes\Lambda^p M[\rho]$ if $\rho \in \Phi$ and $\tilde{\Omega}^p_{B(\Phi)}(m)=0$ otherwise. On the other hand, since $C^{i,j}(\Phi^{(2)}, A\otimes\Lambda^p)=\bigoplus_{(\tau,\sigma)\in\Phi^{(2)}(-i,j)}A(\tau)\otimes\Lambda^p M[\sigma]$ and $A(\tau)(m)=ke(m)$ if $\tau\prec\rho$ and $A(\tau)(m)=0$ otherwise, we have

$$C^{\cdot\cdot}(\Phi^{(2)}, A\otimes\Lambda^p)(m)=ke(m)\otimes C^{\cdot\cdot}(\Phi(\prec\rho)^{(2)}, Z_{1,0}\otimes\Lambda^p).$$

This is also zero if ρ is not in Φ. Hence we may assume $\rho \in \Phi$. We are going to show that $L(m)^{\cdot}$ is a complex. Since $\Lambda^p M[\rho] \subset \Lambda^p M[\sigma]$ if $\sigma \prec \rho$, we know that $ke(m) \otimes C^{\cdot}(\Phi(\prec\rho)^{(2)}, Z_{1,0} \otimes \Lambda^p)$ contains the complex $ke(m) \otimes \Lambda^p M[\rho] \otimes C^{\cdot}(\Phi(\prec\rho)^{(2)}, Z_{1,1})$. Since the second complex makes an exact sequence from $\tilde{\Omega}^p_{B(\Phi)}(m) = ke(m) \otimes \Lambda^p M[\rho]$ by Lemma 1.5, $d^0 \circ \omega^p(\Phi)$ is zero at degree m and the sequence $L(m)^{\cdot}$ is a complex. We define a decreasing filtration F on $L(m)^{\cdot}$ by $F^q(L(m)^{\cdot}) = L(m)^{\cdot}$ for $q \leq d'$, and $F^q(L(m)^{\cdot}) = \bigoplus_{j \geq q} C^{i,j}(\Phi^{(2)}, A \otimes \Lambda^p)(m)$ for $q > d'$, where $d' = \operatorname{codim} \rho$, similarly as in the proof of Lemma 1.5. Then, we have $\operatorname{Gr}^q_F(L(m)^{\cdot}) = 0$ for $q < d'$,

$$\begin{aligned} \operatorname{Gr}^q_F(L(m)^{\cdot}) = (\cdots &\to 0 \to ke(m) \otimes \Lambda^p M[\rho] \\ &\to ke(m) \otimes \Lambda^p M[\rho] \otimes Z(\rho/\rho) \to 0 \to \cdots) \qquad \text{for } q = d' \end{aligned}$$

and

$$\operatorname{Gr}^q_F(L(m)^{\cdot}) = \bigoplus_{\sigma \in \Phi(q)} ke(m) \otimes \Lambda^p M[\sigma] \otimes C^{\cdot}(\Sigma(\sigma \mid \rho), Z_{1,0}) \qquad \text{for } q > d'.$$

Similarly as in the proof of Lemma 1.5, all the cohomologies of these complexes vanish and $L(m)^{\cdot}$ is also exact. q.e.d.

This proposition means that the $B(\Phi)$-module $\tilde{\Omega}^p_{B(\Phi)}$ is quasi-isomorphic to $C^{\cdot}(\Phi^{(2)}, A \otimes \Lambda^p)$.

Let $\Omega^{\vee}_p(A) \colon \Sigma \to \{S\text{-modules}\}$ be the covariant functor defined by $\Omega^{\vee}_p(A)(\sigma) = A(\sigma) \otimes \operatorname{Hom}_Z(\Lambda^p M[\sigma], Z)$. For elements $\sigma, \tau \in \Sigma$ with $\sigma \prec \tau$, the homomorphism $\Omega^{\vee}_p(A)(\sigma) \to \Omega^{\vee}_p(A)(\tau)$ is the tensor product of the natural homomorphisms $A(\sigma) \to A(\tau)$ and

$$\operatorname{Hom}_Z(\Lambda^p M[\sigma], Z) \longrightarrow \operatorname{Hom}_Z(\Lambda^p M[\tau], Z).$$

Let Φ be a star closed subset of Σ. By this covariant functor $\Omega^{\vee}_p(A)$, we get a complex $C^{\cdot}(\Phi, \Omega^{\vee}_p(A))$. For an integer d, and for an element $\sigma \in \Phi(d)$, we give an identification

$$\Lambda^{d-p} M[\sigma] = \operatorname{Hom}_Z(\Lambda^p M[\sigma], Z) \otimes Z(\sigma)$$

by the perfect pairing

$$\Lambda^{d-p} M[\sigma] \times \Lambda^p M[\sigma] \longrightarrow Z(\sigma) = \Lambda^d M[\sigma]$$

sending (x, y) to $x \wedge y$. Hence for each integer i, we have

$$\begin{aligned} C^i(\Phi, \Omega^{\vee}_p(A)) &= \bigoplus_{\sigma \in \Phi(-i)} A(\sigma) \otimes \operatorname{Hom}_Z(\Lambda^p M[\sigma], Z) \otimes Z(\sigma) \\ &= \bigoplus_{\sigma \in \Phi(-i)} S[\sigma] \otimes \Lambda^{-i-p} M[\sigma] \\ &= \bigoplus_{\sigma \in \Phi(-i)} \Omega^{-i-p}_{S[\sigma]}(\log J[\sigma]). \end{aligned}$$

It is easy to see that the (σ, τ)-component of the homomorphism

$$d\colon \bigoplus_{\sigma\in\Phi(-i)} S[\sigma]\otimes\Lambda^{-i-p}M[\sigma] \longrightarrow \bigoplus_{\tau\in\Phi(-i-1)} S[\tau]\otimes\Lambda^{-i-1-p}M[\tau]$$

is given by $m_1\wedge\cdots\wedge m_{-i-1-p}\to\langle m_1, n_1\rangle m_2\wedge\cdots\wedge m_{-i-p}$ if $\sigma\prec\tau$, where n_1 is an element of N which is used in the definition of $q_{\sigma/\tau}$ in Section 1. We call this homomorphism $\Omega^{-i-p}_{S[\sigma]}(\log J[\sigma])\to\Omega^{-i-1-p}_{S[\tau]}(\log J[\tau])$ the *Poincaré residue map*. The (σ, τ)-component is the zero homomorphism, if σ is not a face of τ.

Proposition 2.5. *Let Φ be a star closed subset of Σ. Then there exists a natural isomorphism*

$$C^{\cdot\cdot}(\Phi^{(2)}, A\otimes\Lambda^p)\simeq\operatorname{Hom}_{B(\Phi)}(C^{\cdot}(\Phi, \Omega_p^{\vee}(A)), C^{\cdot}(\Phi, \Omega_0^{\vee}(A)))$$

of double complexes.

Proof. For integers i, j, we have

$$C^i(\Phi, \Omega_0^{\vee}(A))=\bigoplus_{\sigma\in\Phi(-i)} A(\tau)\otimes Z(\tau)$$

and

$$C^{-j}(\Phi, \Omega_p^{\vee}(A))=\bigoplus_{\sigma\in\Phi(j)} A(\sigma)\otimes\operatorname{Hom}_Z(\Lambda^p M[\sigma], Z)\otimes Z(\sigma).$$

Since $\operatorname{Hom}_{B(\Phi)}(A(\sigma), A(\tau))$ is equal to $A(\tau)$ if $(\tau, \sigma)\in\Phi^{(2)}$ and is zero otherwise for any $\sigma, \tau\in\Phi$, the (i, j)-component of both sides are equal. We see easily that this identification gives the isomorphism of double complexes. q.e.d.

The complex $C^{\cdot}(\Phi, \Omega_0^{\vee}(A))$ is equal to the complex $K^{\cdot}$ in [I1] shifted r places to the left. Hence by [I1, Th. 3.3], we have the following.

Proposition 2.6. *For every star closed subset Φ of Σ, $C^{\cdot}(\Phi, \Omega_0^{\vee}(A))$ is a dualizing complex of the ring $B(\Phi)$.*

For the definition of the dualizing complexes, see [RD, Chap. V]. Note that this is only a local property of the complex $C^{\cdot}(\Phi, \Omega_0^{\vee}(A))$. In Section 5, we will see that this complex is a dualizing complex in a global sense.

Theorem 2.7. *Let Φ be a star closed subset of Σ, and let p be an integer. Then, the coherent sheaf $\tilde{\Omega}^p_{B(\Phi)}$ and the complex $C^{\cdot}(\Phi, \Omega_p^{\vee}(A))$ are mutually dual with respect to the dualizing complex $C^{\cdot}(\Phi, \Omega_0^{\vee}(A))$.*

Proof. By Propositions 2.4 and 2.5, it is sufficient to show that the natural homomorphism

$$\varepsilon_\Phi\colon \operatorname{Hom}^\cdot_{B(\Phi)}(C^\cdot(\Phi, \Omega_p^\vee(A)), C^\cdot(\Phi, \Omega_0^\vee(A)))$$
$$\longrightarrow \boldsymbol{R}\operatorname{Hom}^\cdot_{B(\Phi)}(C^\cdot(\Phi, \Omega_p^\vee(A)), C^\cdot(\Phi, \Omega_0^\vee(A)))$$

is an isomorphism in the derived category. Since each $C^i(\Phi, \Omega_p^\vee(A))$ is a direct sum of $B(\Phi)$-modules of type $A(\sigma)\otimes \boldsymbol{Z}^s$, it is reduced to proving the following lemma.

Lemma 2.8. *Let Φ be a star closed subset of Σ, and let σ be an element of Φ. Then the natural homomorphism*

$$\varepsilon_\sigma\colon \operatorname{Hom}^\cdot_{B(\Phi)}(A(\sigma), C^\cdot(\Phi, \Omega_0^\vee(\Sigma, A)))\longrightarrow \boldsymbol{R}\operatorname{Hom}^\cdot_{B(\Phi)}(A(\sigma), C^\cdot(\Phi, \Omega_0^\vee(\Sigma, A)))$$

is an isomorphism.

Proof. We have to recall the injective resolution of the dualizing complex in the category of M-graded $B(\Phi)$-modules which we used in [I1, § 4]. We define a covariant functor

$$I_\Phi\colon \Gamma(\pi)\longrightarrow (B(\Phi)\text{-modules})$$

by $I_\Phi(\tau)=\underline{\operatorname{Hom}}_k(B_\tau^{-1}B(\Phi), k)$ if $\tau\in\Phi$, and $I_\Phi(\tau)=0$ otherwise, where B_τ is the set of M-homogeneous elements of $B(\Phi)$ not in the prime ideal $P(\tau)/I(\Phi)$, and $\underline{\operatorname{Hom}}_k(B_\tau^{-1}B(\Phi), k)$ is the submodule of $\operatorname{Hom}_k(B_\tau^{-1}B(\Phi), k)$ generated by M-homogeneous homomorphisms. (See [I1, § 4].) Then the complex $C^\cdot(\Phi, I_\Phi)$ is equal to $I^\cdot$ in [I1, § 4] shifted r places to the left. The results in [I1, § 4] say that there exists a natural homomorphism $\Omega_0^\vee(A)|_\Phi \to I_\Phi|_\Phi$ of the functors restricted to Φ, and the induced homomorphism

$$C^\cdot(\Phi, \Omega_0^\vee(A))\longrightarrow C^\cdot(\Phi, I_\Phi)$$

is a quasi-isomorphism (see [I1, Proposition 4.8]). Furthermore, since $I_\Phi(\tau)$ is injective in the category of M-graded $B(\Phi)$-modules by [I1, Lemma 4.5] for every τ, and since $A(\sigma)$ is a finitely generated M-graded $B(\Phi)$-module, we know that the complex $\operatorname{Hom}_{B(\Phi)}(A(\sigma), C^\cdot(\Phi, I_\Phi))$ represents $\boldsymbol{R}\operatorname{Hom}_{B(\Phi)}(A(\sigma), C^\cdot(\Phi, \Omega_0^\vee(A)))$ in the derived category. Since

$$\operatorname{Hom}_{B(\Phi)}(A(\sigma), \underline{\operatorname{Hom}}_k(B_\tau^{-1}B(\Phi), k))=\underline{\operatorname{Hom}}_k(A(\sigma)\underset{B(\Phi)}{\otimes} B_\tau^{-1}B(\Phi), k)$$

is equal to $\underline{\operatorname{Hom}}_k(B_\tau^{-1}A(\sigma), k)$ if $\tau\succ\sigma$ and is zero otherwise, and since $A(\sigma)=B(\Phi(\sigma\prec))$, we have $\operatorname{Hom}_{B(\Phi)}(A(\sigma), C^\cdot(\Phi, I_\Phi))=C^\cdot(\Phi(\sigma\prec), I_{\Phi(\sigma\prec)})$. On the other hand, since $\operatorname{Hom}_{B(\Phi)}(A(\sigma), A(\tau))$ is equal to $A(\tau)$ if $\sigma\prec\tau$ and is zero otherwise, we have $\operatorname{Hom}_{B(\Phi)}(A(\sigma), C^\cdot(\Phi, \Omega_0^\vee(A)))=C^\cdot(\Phi(\sigma\prec), \Omega_0^\vee(A))$. Since the homomorphism $C^\cdot(\Phi(\sigma\prec), \Omega_0^\vee(A))\to C^\cdot(\Phi(\sigma\prec), I_{\Phi(\sigma\prec)})$ is a quasi-isomorphism by [I1], the homomorphism ε_σ is an isomorphism in the derived category. q.e.d.

Let Φ be a star closed subset of Σ. We denote by $\min \Phi$ the set of minimal elements of Σ. Since $\{P(\sigma); \sigma \in \min \Phi\}$ is the set of minimal prime divisors of the semiprime ideal $I(\Phi)$, and since each $S[\sigma]=S/P(\sigma)$ is a normal ring, the normalization $\tilde{B}(\Phi)$ of $B(\Phi)$ is equal to $\bigoplus_{\sigma \in \min \Phi} S[\sigma]$. Let λ: $B(\Phi) \rightarrow \tilde{B}(\Phi)$ be the natural injective homomorphism. For $h=\operatorname{ht} \Phi$, we set $\Phi'=\{\sigma \in \Phi: \operatorname{codim} \sigma<h\}$. We will use the following lemma in Section 5.

Lemma 2.9. *In the above notation, λ induces an isomorphism $I(\Phi')/I(\Phi) \rightarrow \bigoplus_{\sigma \in \Phi(h)} J[\sigma]$ of ideals of $B(\Phi)$ and $\tilde{B}(\Phi)$.*

Proof. Since

$$I(\Phi)=k[M \cap (\pi^{\vee} \setminus \bigcup_{\sigma \in \Phi} \sigma^{\perp})] \quad \text{and} \quad I(\Phi')=k[M \cap (\pi^{\vee} \setminus \bigcup_{\sigma \in \Phi'} \sigma^{\perp})],$$

we have $I(\Phi')/I(\Phi)=k[M \cap (\bigcup_{\sigma \in \Phi(h)} \operatorname{int}(\pi^{\vee} \cap \sigma^{\perp}))]$, where $\operatorname{int}(\pi^{\vee} \cap \sigma^{\perp})$ is the interior of $\pi^{\vee} \cap \sigma^{\perp}$ in the linear space $\sigma^{\perp}$.

For each $\sigma \in \Phi(h)$ and each $m \in M \cap \operatorname{int}(\pi^{\vee} \cap \sigma^{\perp})$, the annihilator of $e(m) \in B(\Phi)$ is equal to $P(\sigma)/I(\Phi)$, since $\Phi(h) \subset \min \Phi$. Hence the projected image of $k[M \cap \operatorname{int}(\pi^{\vee} \cap \sigma^{\perp})]$ in the direct summand $S[\tau]$ of $\tilde{B}(\Phi)$ is equal to zero if $\tau \neq \sigma$ while it is equal to the ideal $J[\sigma]=k[M[\sigma] \cap \operatorname{int} \pi[\sigma]^{\vee}]$ of $S[\sigma]$ if $\tau=\sigma$. Hence the image of $I(\Phi')/I(\Phi)$ is equal to the ideal $\bigoplus_{\sigma \in \Phi(h)} J[\sigma]$ of $\tilde{B}(\Phi)$. q.e.d.

§ 3. Toric polyhedra

In this section, we fix a field k of an arbitrary characteristic.

As in the previous section, let N be a free $\mathbf{Z}$-module of rank $r \geq 0$, and let M be the dual $\mathbf{Z}$-module. For each fan Σ of $N_{\mathbf{R}}$, we denote by $Z(N, \Sigma)$ or simply $Z(\Sigma)$ the T_N-embedding over k associated to the fan Σ which is defined in [TE, Chap. 1] or [MO, Chap. 1], where T_N is the algebraic torus $\operatorname{Spec}(k[M])$. If $\Sigma=\Gamma(\pi)$ for an s.c.r.p. cone π, then $Z(\Sigma)$ is equal to the affine T_N-embedding $U(N, \pi)=\operatorname{Spec}(k[M \cap \pi^{\vee}])$. We denote by $\operatorname{orb}(N, \pi)$ the closed subvariety of $U(N, \pi)$ defined by the ideal $P(\pi)=k[M \cap (\pi^{\vee} \setminus \pi^{\perp})]$. $\operatorname{orb}(N, \pi)$ is a T_N-orbit of $U(N, \pi)$ under the action of T_N. We might abbreviate $U(N, \pi)$ and $\operatorname{orb}(N, \pi)$ as $U(\pi)$ and $\operatorname{orb}(\pi)$, respectively. For a general Σ, the T_N-embedding $Z(\Sigma)$ has the covering $\{U(\sigma); \sigma \in \Sigma\}$ by T_N-invariant affine open sets. We denote by $V(N, \Sigma, \sigma)$ the reduced subvariety which is the closure in $Z(\Sigma)$ of the locally closed subvariety $\operatorname{orb}(\sigma)$ of $Z(\Sigma)$. When N and Σ are obvious, we abbreviate it as $V(\Sigma, \sigma)$ or $V(\sigma)$. For each element $\rho \in \Sigma$, we set $\Sigma[\rho]=\{\sigma[\rho]; \sigma \in \Sigma(\rho \prec)\}$. Then $\Sigma[\rho]$ is a fan of $N[\rho]_{\mathbf{R}}$, and we have the following:

Lemma 3.1 ([MO, Th. 4.2, (iii)]). *For any $\rho \in \Sigma$, we have natural identifications $T_{N[\rho]}=\mathrm{orb}(\sigma)$ and $Z(N[\rho], \Sigma[\rho])=V(N, \Sigma, \rho)$.*

Let N' be another free $\boldsymbol{Z}$-module of finite rank, and let Σ, Σ' be fans of $N_{\boldsymbol{R}}$ and $N'_{\boldsymbol{R}}$, respectively. Then each homomorphism $h\colon N'\to N$ induces the homomorphism $T_{N'}\to T_N$ of the algebraic tori. This morphism is extended to a morphism of the torus embeddings $Z(N', \Sigma')\to Z(N, \Sigma)$ if and only if, for each element σ of Σ', there exists τ in Σ with $h_{\boldsymbol{R}}(\sigma)\subset\tau$ [MO, Th. 4.1], where $h_{\boldsymbol{R}}\colon N'_{\boldsymbol{R}}\to N_{\boldsymbol{R}}$ is the coefficient extension of h. Although h is assumed to have a finite cokernel in [MO], we see easily that this condition is not necessary. If this condition is satisfied, we denote by $\varphi(h, N'/N, \Sigma'/\Sigma)$ the morphism of torus embeddings. This might also be abbreviated as $\varphi(h, \Sigma'/\Sigma)$, $\varphi(h, N'/N)$ or $\varphi(h)$.

In the above case, for each element $\tau \in \Sigma$, we denote by Σ'_τ the set of elements σ of Σ' such that τ is the minimal element of Σ with $h_{\boldsymbol{R}}(\sigma)\subset\tau$. Σ'_τ is a locally star closed subset of Σ'. For an element $\sigma \in \Sigma'_\tau$, the restriction of $\varphi(h, \Sigma'/\Sigma)$ induces the morphism $V(N', \Sigma', \sigma)\to V(N, \Sigma, \tau)$. It is easy to see that this is equal to the morphism

$$\varphi(\bar{h}, \Sigma'[\sigma]/\Sigma[\tau])\colon Z(N'[\sigma], \Sigma'[\sigma])\longrightarrow Z(N[\tau], \Sigma[\tau])$$

for the naturally induced homomorphism $\bar{h}\colon N'[\sigma]\to N[\tau]$ with respect to the identification in Lemma 3.1.

The morphism $\varphi(h, \Sigma'/\Sigma)\colon Z(\Sigma')\to Z(\Sigma)$ is proper if and only if, for every $\tau \in \Sigma$, the set $\{\sigma \in \Sigma'\colon h_{\boldsymbol{R}}(\sigma)\subset\tau\}$ is finite and the union of its elements is equal to $h_{\boldsymbol{R}}^{-1}(\tau)$ [MO, Th. 4.4]. Assume $\varphi(h, \Sigma'/\Sigma)$ is proper. Then, if we set $N''=h(N')\subset N$ and $\Sigma''=\{N''_{\boldsymbol{R}}\cap\tau;\ \tau \in \Sigma\}$, then the homomorphism $\varphi(h, N'/N, \Sigma'/\Sigma)$ is decomposed as $\varphi(h', N''/N, \Sigma''/\Sigma)\circ\varphi(h'', N'/N'', \Sigma'/\Sigma'')$, where $h'\colon N''\to N$ and $h''\colon N'\to N''$ are the natural injection and surjection, respectively. This is nothing but the Stein factorization [EGA III, 4.3] of the proper morphism $\varphi(h, \Sigma'/\Sigma)$. In particular, $\varphi(h, \Sigma'/\Sigma)_*\mathcal{O}_{Z(\Sigma')}=\mathcal{O}_{Z(\Sigma)}$ if and only if h is surjective.

Let Σ, Σ' be two fans of $N_{\boldsymbol{R}}$. We say Σ' is a *subdivision* of Σ, if (1) for every $\sigma \in \Sigma'$, there exists $\tau \in \Sigma$ with $\sigma\subset\tau$, and (2) for every $\tau \in \Sigma$, the set $\{\sigma \in \Sigma';\ \sigma\subset\tau\}$ is finite and the union of the elements is equal to τ. In other words, Σ' is a subdivision of Σ if and only if the birational morphism $\varphi(\Sigma'/\Sigma)=\varphi(1_N, \Sigma'/\Sigma)\colon Z(\Sigma')\to Z(\Sigma)$ is defined and is proper.

For a locally Noetherian scheme X, we denote by $D^+_{\mathrm{coh}}(X)$ the derived category of the category of complexes of $\mathcal{O}_X$-modules bounded below with coherent cohomologies.

Theorem 3.2. *Let $h\colon N'\to N$ be a surjective homomorphism, and let Σ' and Σ be fans of $N'_{\boldsymbol{R}}$ and $N_{\boldsymbol{R}}$, respectively. If $\varphi(h, N'/N, \Sigma'/\Sigma)$ is well-*

defined and is proper, then we have $\boldsymbol{R}\varphi(h, \Sigma'/\Sigma)_*\mathcal{O}_{Z(\Sigma')}=\mathcal{O}_{Z(\Sigma)}$ *in the derived category* $D^+_{\mathrm{coh}}(Z(\Sigma))$.

Proof. If h is an isomorphism, or equivalently if $\varphi(h, \Sigma'/\Sigma)$ is a birational morphism, this follows from the results of [TE, Chap. 1, § 3]. We consider the general case. If we take a submodule $N^{(2)}$ of N' which is mapped isomorphically to N by h, then we get the decomposition $N'=N^{(1)}\oplus N^{(2)}$ for $N^{(1)}=\mathrm{Ker}(h)$. Let Σ_2 be the fan of $N^{(2)}_{\boldsymbol{R}}$ which is isomorphic to Σ by h, and let Σ_1 be an arbitrary complete fan of $N^{(1)}_{\boldsymbol{R}}$. Then we have $Z(N', \Sigma_1\times\Sigma_2)=Z(N^{(1)}, \Sigma_1)\times Z(N^{(2)}, \Sigma_2)$ and $Z(N^{(2)}, \Sigma_2)\simeq Z(N, \Sigma)$, where $\Sigma_1\times\Sigma_2=\{\sigma\times\tau;\ \sigma\in\Sigma_1,\ \tau\in\Sigma_2\}$. Since $Z(N^{(1)}, \Sigma_1)$ is a complete torus embedding, $H^i(Z(N^{(1)}, \Sigma_1), \mathcal{O}_{Z(N^{(1)},\Sigma_1)})=0$ for $i>0$ by [Da, Cor. 7.4]. Hence, by the base change theorem, we have $\boldsymbol{R}\varphi(h, \Sigma_1\times\Sigma_2/\Sigma)_*\mathcal{O}_{Z(\Sigma_1\times\Sigma_2)}=\mathcal{O}_{Z(\Sigma)}$. Let $\tilde{\Sigma}$ be a common subdivision of Σ' and $\Sigma_1\times\Sigma_2$. Then, since $\varphi(1_{N'}, \tilde{\Sigma}/\Sigma')$ and $\varphi(1_{N'}, \tilde{\Sigma}/\Sigma_1\times\Sigma_2)$ are birational morphisms, we have $\boldsymbol{R}\varphi(1_{N'}, \tilde{\Sigma}/\Sigma')_*\mathcal{O}_{Z(\tilde{\Sigma})}=\mathcal{O}_{Z(\Sigma')}$ and $\boldsymbol{R}\varphi(1_{N'}, \tilde{\Sigma}/\Sigma_1\times\Sigma_2)_*\mathcal{O}_{Z(\tilde{\Sigma})}=\mathcal{O}_{Z(\Sigma_1\times\Sigma_2)}$. Thus

$$\begin{aligned}\boldsymbol{R}\varphi(h, \Sigma'/\Sigma)_*\mathcal{O}_{Z(\Sigma')}&=\boldsymbol{R}\varphi(h, \Sigma'/\Sigma)_*\boldsymbol{R}\varphi(1_{N'}, \tilde{\Sigma}/\Sigma')_*\mathcal{O}_{Z(\tilde{\Sigma})}\\&=\boldsymbol{R}\varphi(h, \tilde{\Sigma}/\Sigma)_*\mathcal{O}_{Z(\tilde{\Sigma})}\\&=\boldsymbol{R}\varphi(h, \Sigma_1\times\Sigma_2/\Sigma)_*\boldsymbol{R}\varphi(1_{N'}, \tilde{\Sigma}/\Sigma_1\times\Sigma_2)_*\mathcal{O}_{Z(\tilde{\Sigma})}\\&=\boldsymbol{R}\varphi(h, \Sigma_1\times\Sigma_2/\Sigma)_*\mathcal{O}_{Z(\Sigma_1\times\Sigma_2)}\\&=\mathcal{O}_{Z(\Sigma)}.\end{aligned}$$

q.e.d.

Corollary 3.3. *Let* Σ *be a fan of* $N_{\boldsymbol{R}}$, *and let* Σ' *be a subdivision of* Σ. *Then for each element* $\sigma\in\Sigma'$, *we have* $\boldsymbol{R}\varphi(\Sigma'/\Sigma)_*\mathcal{O}_{V(\Sigma',\sigma)}=\mathcal{O}_{V(\Sigma,\bar{\sigma})}$, *where* $\bar{\sigma}$ *is the minimal element of* Σ *with* $\sigma\subset\bar{\sigma}$.

Proof. In view of Lemma 3.1, the morphism $V(\Sigma', \sigma)\to V(\Sigma, \bar{\sigma})$ obtained as the restriction of $\varphi(\Sigma'/\Sigma)$ to $V(\Sigma', \sigma)$ is equal to

$$\varphi(h, \Sigma'[\sigma]/\Sigma[\bar{\sigma}])\colon Z(N[\sigma], \Sigma'[\sigma])\longrightarrow Z(N[\bar{\sigma}], \Sigma[\bar{\sigma}])$$

for the natural surjection $h\colon N[\sigma]\to N[\bar{\sigma}]$. Hence the corollary follows from the above theorem. q.e.d.

We denote by $D(\Sigma)$ the reduced Weil divisor $Z(\Sigma)\backslash T_N$. If $\Sigma=\Gamma(\pi)$ for an s.c.r.p. cone π, then $D(\Sigma)=\mathrm{Spec}\,(S/J)$ for the ideal J in Section 2. We define the $\mathcal{O}_{Z(\Sigma)}$-module $\Omega^1_{Z(\Sigma)}(\log D(\Sigma))$ to be $\mathcal{O}_{Z(\Sigma)}\otimes M$. This is the globalization of $\Omega^1_S(\log J)$ in Section 2, and is a free $\mathcal{O}_{Z(\Sigma)}$-module of rank r. For each $\sigma\in\Sigma$, we denote by $D(\Sigma, \sigma)$ the reduce divisor $V(\Sigma, \sigma)\backslash\mathrm{orb}\,(\sigma)$ of the torus embedding $V(\Sigma, \sigma)$. Then $\Omega^1_{V(\Sigma,\sigma)}(\log D(\Sigma, \sigma))=\mathcal{O}_{V(\Sigma,\sigma)}\otimes M[\sigma]$ is a free $\mathcal{O}_{V(\Sigma,\sigma)}$-module of rank codim σ.

For an integer p, a covariant functor $\Omega_p^\vee(\Sigma, \mathcal{O})\colon \Sigma \to \{\mathcal{O}_{Z(\Sigma)}\text{-module}\}$ is defined by $\Omega_p^\vee(\Sigma, \mathcal{O})(\sigma) = \mathcal{O}_{V(\Sigma,\sigma)} \otimes \mathrm{Hom}_Z(\Lambda^p M[\sigma], Z)$. Similarly as in Section 2, we have $\Omega_p^\vee(\Sigma, \mathcal{O})(\sigma) \otimes Z(\sigma) = \Omega_{V(\Sigma,\sigma)}^{d-p}(\log D(\Sigma, \sigma))$ for $d = \mathrm{codim}\, \sigma$.

Proposition 3.4. *Let Σ be a fan of $N_{\boldsymbol{R}}$, and let Σ' be a subdivision of Σ. Then, there exists a natural isomorphism*

$$\boldsymbol{R}\varphi(\Sigma'/\Sigma)_* C^\cdot(\Sigma', \Omega_0^\vee(\Sigma', \mathcal{O})) \simeq C^\cdot(\Sigma, \Omega_0^\vee(\Sigma, \mathcal{O}))$$

in the derived category $D_{\mathrm{coh}}^+(Z(\Sigma))$.

Proof. Since $C^i(\Sigma', \Omega_0^\vee(\Sigma', \mathcal{O})) = \bigoplus_{\sigma \in \Sigma'(-i)} \mathcal{O}_{V(\Sigma',\sigma)} \otimes Z(\sigma)$, we get

$$\begin{aligned} \boldsymbol{R}\varphi(\Sigma'/\Sigma)_* C^i(\Sigma', \Omega_0^\vee(\Sigma', \mathcal{O})) &= \varphi(\Sigma'/\Sigma)_* C^i(\Sigma', \Omega_0^\vee(\Sigma', \mathcal{O})) \\ &= \bigoplus_{\sigma \in \Sigma'(-i)} \mathcal{O}_{V(\Sigma,\bar{\sigma})} \otimes Z(\sigma) \end{aligned}$$

by Corollary 3.3, where $\bar{\sigma}$ is the unique minimal element of Σ with $\sigma \subset \bar{\sigma}$. Recall that for a subset Φ of a fan Σ and for an integer i, we denote $\Phi(i) = \{\sigma \in \Phi;\ \mathrm{codim}\, \sigma = i\}$. In particular, $\dim V(\Sigma, \sigma) = i$ for $\sigma \in \Phi(i)$. We define the homomorphism

$$\bigoplus_{\sigma \in \Sigma'(-i)} \mathcal{O}_{V(\Sigma,\bar{\sigma})} \otimes Z(\sigma) \longrightarrow \bigoplus_{\tau \in \Sigma(-i)} \mathcal{O}_{V(\Sigma,\tau)} \otimes Z(\tau)$$

as follows. Let (σ, τ) be in $\Sigma'(-i) \times \Sigma(-i)$. If $\bar{\sigma} = \tau$, then $Z(\sigma) = Z(\tau)$, since $\mathrm{codim}\, \sigma = \mathrm{codim}\, \tau = -i$ and $M[\sigma] = M[\tau]$. We define the (σ, τ)-component of this homomorphism to be the identity if $\bar{\sigma} = \tau$ and to be zero otherwise. Then, as we see easily, we get a homomorphism

$$f^\cdot\colon \varphi(\Sigma'/\Sigma)_* C^\cdot(\Sigma', \Omega_0^\vee(\Sigma', \mathcal{O})) \longrightarrow C^\cdot(\Sigma, \Omega_0^\vee(\Sigma, \mathcal{O}))$$

of complexes. We define a decreasing filtration $\{F^q\}_{q \in Z}$ on the complex $A^\cdot = \varphi(\Sigma'/\Sigma)_* C^\cdot(\Sigma', \Omega_0^\vee(\Sigma', \mathcal{O}))$ by

$$F^q(A^i) = \bigotimes_{\substack{\sigma \in \Sigma'(-i) \\ \mathrm{codim}\, \bar{\sigma} + q \leq 0}} \mathcal{O}_{V(\Sigma,\bar{\sigma})},$$

as well as $\{G^q\}_{q \in Z}$ on the complex $B^\cdot = C^\cdot(\Sigma, \Omega_0^\vee(\Sigma, \mathcal{O}))$ by $G^q(B^i) = B^i$ if $i \geq q$ and $G^q(B^i) = 0$ for $i < q$. Then the homomorphism $f^\cdot$ preserves these filtrations and the induced homomorphism $\mathrm{Gr}_F^q(A^\cdot) \to \mathrm{Gr}_G^q(B^\cdot)$ is equal to the natural homomorphism

$$\bigoplus_{\tau \in \Sigma(-q)} \mathcal{O}_{V(\Sigma,\tau)} \otimes C^\cdot(\Sigma'_\tau, Z_{1,0}) \longrightarrow \bigoplus_{\tau \in \Sigma(-q)} \mathcal{O}_{V(\Sigma,\tau)} \otimes Z(\tau)[-q]$$

which is induced by the homomorphisms

$$C^{\cdot}(\Sigma'_{\tau}, Z_{1,0}) \longrightarrow Z(\tau)[\operatorname{codim} \tau]$$

in Lemma 1.6. Hence this is a quasi-isomorphism by the same lemma. Thus the homomorphism $f^{\cdot}$ is a quasi-isomorphism. q.e.d.

The functor $\Omega_p^{\vee}(\Sigma, \mathcal{O})$ is abbreviated as $\Omega_p^{\vee}(\mathcal{O})$ if the fan Σ is clear from the context.

Definition 3.5. For a star closed subset Φ of a fan Σ, we denote by $Y(\Phi)$ the reduced subscheme $\bigcup_{\sigma \in \Phi} V(\sigma)$ of $Z(\Sigma)$, and we call it the *toric polyhedron associated to* Φ.

Let $Y(\Phi)$ be the toric polyhedron associated to $\Phi \subset \Sigma$. Then we see easily that $Y(\Phi)$ has the open covering $\{Y(\Phi) \cap U(\pi); \pi \in \Phi\}$, and each $Y(\Phi) \cap U(\pi)$ is equal to $\operatorname{Spec}(B(\Phi \cap \Gamma(\pi)))$ where $B(\Phi \cap \Gamma(\pi))$ is the quotient ring of $S = k[M \cap \pi^{\vee}]$ defined in Section 2.

By Proposition 2.6, $C^{\cdot}(\Phi, \Omega_0^{\vee}(\mathcal{O}))$ is the (local) dualizing complex of the toric polyhedron $Y(\Phi)$.

Lemma 3.6. *Let Φ be a star closed subset of Σ, and let $L^{\cdot}$ be a finite complex of coherent $\mathcal{O}_{Y(\Phi)}$-modules. Assume that each L^i is decomposed as a direct sum of free $\mathcal{O}_{V(\sigma)}$-modules for $\sigma \in \Phi$. Then the natural homomorphism*

$$\mathcal{H}om^{\cdot}_{\mathcal{O}_{Y(\Phi)}}(L^{\cdot}, C^{\cdot}(\Phi, \Omega_0^{\vee}(\mathcal{O})) \longrightarrow \boldsymbol{R}\mathcal{H}om^{\cdot}_{\mathcal{O}_{Y(\Phi)}}(L^{\cdot}, C^{\cdot}(\Phi, \Omega_0^{\vee}(\mathcal{O}))$$

is an isomorphism in the derived category. Furthermore, if we regard $C^{\cdot}(\Phi, \Omega_0^{\vee}(\mathcal{O}))$ as a subcomplex of $C^{\cdot}(\Sigma, \Omega_0^{\vee}(\mathcal{O}))$, then the natural homomorphism

$$\mathcal{H}om^{\cdot}_{\mathcal{O}_{Y(\Phi)}}(L^{\cdot}, C^{\cdot}(\Phi, \Omega_0^{\vee}(\mathcal{O})) \longrightarrow \mathcal{H}om^{\cdot}_{\mathcal{O}_{Z(\Sigma)}}(L^{\cdot}, C^{\cdot}(\Sigma, \Omega_0^{\vee}(\mathcal{O})))$$

is an isomorphism.

Proof. By the standard spectral sequence, the first assertion is reduced to showing that the natural homomorphism

$$\mathcal{H}om^{\cdot}_{\mathcal{O}_{Y(\Phi)}}(\mathcal{O}_{V(\sigma)}, C^{\cdot}(\Phi, \Omega_0^{\vee}(\mathcal{O})) \longrightarrow \boldsymbol{R}\mathcal{H}om^{\cdot}_{\mathcal{O}_{Y(\Phi)}}(\mathcal{O}_{V(\sigma)}, C^{\cdot}(\Phi, \Omega_0^{\vee}(\mathcal{O}))$$

is an isomorphism for every $\sigma \in \Phi$. This is true by Lemma 2.8. The second assertion follows from the fact that

$$\mathcal{H}om_{\mathcal{O}_{Y(\Phi)}}(\mathcal{O}_{V(\sigma)}, \mathcal{O}_{V(\tau)}) = \mathcal{H}om_{\mathcal{O}_{Z(\Sigma)}}(\mathcal{O}_{V(\sigma)}, \mathcal{O}_{V(\tau)})$$

if $\sigma, \tau \in \Phi$, and $\mathcal{H}om_{\mathcal{O}_{Z(\Sigma)}}(\mathcal{O}_{V(\sigma)}, \mathcal{O}_{V(\tau)}) = 0$ if $\tau \in \Sigma \setminus \Phi$ and $\sigma \in \Phi$. q.e.d.

We denote $\Phi^{(2)}=\{(\tau, \sigma) \in \Phi^2; \sigma \prec \tau\}$ for a star closed subset Φ of Σ. The double functor $\mathcal{O}\otimes\Lambda^p$: $\Sigma^2 \to \{\mathcal{O}_{Z(\Sigma)}\text{-modules}\}$ is defined by $\mathcal{O}\otimes\Lambda^p(\tau, \sigma) = \mathcal{O}_{V(\tau)}\otimes\Lambda^p M[\sigma]$. This functor is covariant for the first variable and contravariant for the second. It is the globalization of the functor $A\otimes\Lambda^p$ in Section 2. We get the double complex $C^{\cdot\cdot}(\Phi^{(2)}, \mathcal{O}\otimes\Lambda^p)$ and the single complex $C^{\cdot}(\Phi^{(2)}, \mathcal{O}\otimes\Lambda^p)$ associated to it.

By globalizing Proposition 2.5, we have the following.

Proposition 3.7. *Let Φ be a star closed subset of Σ. Then, for each integer p, we have*

$$C^{\cdot\cdot}(\Phi^{(2)}, \mathcal{O}\otimes\Lambda^p) = \mathscr{H}om_{\mathcal{O}_{Y(\Phi)}}(C^{\cdot}(\Phi, \Omega_p^{\vee}(\mathcal{O})), C^{\cdot}(\Phi, \Omega_0^{\vee}(\mathcal{O}))).$$

Let Σ' be a subdivision of Σ, Φ a star closed subset of Σ and $\Phi' = \{\sigma \in \Sigma'; \bar{\sigma} \in \Phi\}$. We define the homomorphism

$$\begin{aligned}\beta\colon \varphi_*\mathscr{H}om_{\mathcal{O}_{Y(\Phi')}}&(C^{\cdot}(\Phi', \Omega_p^{\vee}(\mathcal{O})), C^{\cdot}(\Phi', \Omega_0^{\vee}(\mathcal{O})))\\ &\longrightarrow \mathscr{H}om_{\mathcal{O}_{Y(\Phi)}}(\varphi_* C^{\cdot}(\Phi', \Omega_p^{\vee}(\mathcal{O})), C^{\cdot}(\Phi, \Omega_0^{\vee}(\mathcal{O})))\end{aligned}$$

of double complexes as follows. Since $\varphi_*\mathcal{O}_{V(\sigma)} = \mathcal{O}_{V(\bar{\sigma})}$ for $\sigma \in \Phi'$, we have

$$\begin{aligned}\varphi_*\mathscr{H}om_{\mathcal{O}_{Y(\Phi')}}&(C^{-j}(\Phi', \Omega_p^{\vee}(\mathcal{O})), C^{i}(\Phi', \Omega_0^{\vee}(\mathcal{O})))\\ &= \bigoplus_{(\xi,\zeta)\in\Phi'^{(2)}(-i,j)} \mathcal{O}_{V(\bar{\xi})}\otimes\Lambda^p M[\zeta]\otimes \boldsymbol{Z}(\xi/\zeta)\end{aligned}$$

and

$$\begin{aligned}\mathscr{H}om_{\mathcal{O}_{Y(\Phi)}}&(\varphi_* C^{-j}(\Phi', \Omega_p^{\vee}(\mathcal{O})), C^{i}(\Phi, \Omega_0^{\vee}(\mathcal{O})))\\ &= \bigoplus_{\substack{\tau\in\Phi(-i),\eta\in\Phi'(j)\\ \bar{\eta}\prec\tau}} \mathcal{O}_{V(\tau)}\otimes\Lambda^p M[\eta]\otimes \boldsymbol{Z}(\tau/\eta).\end{aligned}$$

If $\bar{\xi}=\tau$ and $\zeta=\eta$, then we have $\boldsymbol{Z}(\tau/\eta)=\boldsymbol{Z}(\xi/\zeta)$, since $\dim \xi = \dim \tau = -i$. Then the $((\xi, \zeta), (\tau, \eta))$-component of $\beta^{i,j}$ is defined to be the identity. Otherwise, the component of $\beta^{i,j}$ is defined to be zero. Then we see that this is a homomorphism of double complexes.

Proposition 3.8. *Let the notation be as above. Then, the homomorphism*

$$\begin{aligned}\varphi_*\mathscr{H}om^{\cdot}_{\mathcal{O}_{Y(\Phi')}}&(C^{\cdot}(\Phi', \Omega_p^{\vee}(\mathcal{O})), C^{\cdot}(\Phi', \Omega_0^{\vee}(\mathcal{O})))\\ &\longrightarrow \mathscr{H}om^{\cdot}_{\mathcal{O}_{Y(\Phi)}}(\varphi_* C^{\cdot}(\Phi', \Omega_p^{\vee}(\mathcal{O})), C^{\cdot}(\Phi, \Omega_0^{\vee}(\mathcal{O})))\end{aligned}$$

of the associated single complexes obtained from β is a quasi-isomorphism.

Proof. Let the first complex be $K^{\cdot}$ and the second $L^{\cdot}$. We define decreasing filtrations F and G on $K^{\cdot}$ and $L^{\cdot}$, respectively, as follows. For each integer q, $F^q(K^\cdot)$ is defined to be the sum of $\mathcal{O}_{V(\bar{\xi})}\otimes\Lambda^pM[\zeta]\otimes\boldsymbol{Z}(\xi/\zeta)$ for $\xi, \zeta \in \Phi'$ with $\zeta \prec \xi$ and $\operatorname{codim}\zeta - \operatorname{codim}\bar{\xi} \geq q$, and $G^q(L^\cdot)$ is the sum of $\mathcal{O}_{V(\tau)}\otimes\Lambda^pM[\eta]\otimes\boldsymbol{Z}(\tau/\eta)$ for $\tau \in \Phi$, $\eta \in \Phi'$ with $\eta\subset\tau$ and $\operatorname{codim}\eta - \operatorname{codim}\tau \geq q$. Then, these filtrations are compatible with the homomorphisms. By considering the associated graded complexes, we reduce the problem to showing that the homomorphism

$$C^\cdot(\Sigma'_\tau(\eta\prec), \boldsymbol{Z}_{1,0})\otimes\mathcal{O}_{V(\tau)}\otimes\Lambda^pM[\eta]\longrightarrow\mathcal{O}_{V(\tau)}\otimes\Lambda^pM[\eta]\otimes\boldsymbol{Z}(\tau/\eta)[\operatorname{codim}\tau]$$

is a quasi-isomorphism for all $\tau \in \Phi$ and $\eta \in \Phi'$ with $\eta\subset\tau$, where $\Sigma'_\tau(\eta\prec) = \{\xi \in \Sigma';\ \eta\prec\xi \text{ and } \bar{\xi}=\tau\}$. By the quotient map

$$N_{\boldsymbol{R}}\longrightarrow N[\eta]_{\boldsymbol{R}} = N_{\boldsymbol{R}}/(\eta+(-\eta)),$$

$\Sigma'_\tau(\eta\prec)$ is in natural one to one correspondence with $\{\rho \in \Sigma'';\ \rho\cap\operatorname{int}\tau[\eta] \neq\phi\}$, where $\Sigma''=\{\rho \in \Sigma'[\eta];\ \rho\subset\tau[\eta]\}$ and $\tau[\eta]$ is the image of τ in $N[\eta]_{\boldsymbol{R}}$. Since $\tau[\eta]$ is a rational polyhedral cone and $\tau[\eta]=|\Sigma''|$, the homomorphism is quasi-isomorphic by Lemma 1.6. q.e.d.

In Section 5, we will see that $C^\cdot(\Phi, \Omega_0^\vee(\mathcal{O}))$ is a dualizing complex in a global sense. In particular, $C^\cdot(\Phi', \Omega_0^\vee(\mathcal{O}))$ is equal to the twisted inverse image $\varphi^!C^\cdot(\Phi, \Omega_0^\vee(\mathcal{O}))$ and Proposition 3.8 gives an explicit description of the relative duality [RD, Chap. VII, Cor. 3.4] in our special case.

Let Σ' be a subdivision of Σ, and let $\varphi=\varphi(\Sigma'/\Sigma)$: $Z(\Sigma')\to Z(\Sigma)$. For a star closed subset Φ and for each integer p, there is a natural homomorphism $\varphi_*C^\cdot(\Phi', \Omega_p^\vee(\mathcal{O}))\to C^\cdot(\Phi, \Omega_p^\vee(\mathcal{O}))$, where $\Phi'=\{\sigma \in \Sigma';\ \bar{\sigma} \in \Phi\}$. Actually, since

$$\varphi_*C^\cdot(\Phi', \Omega_p^\vee(\mathcal{O})) = \bigoplus_{\sigma\in\Phi'(-i)} \mathcal{O}_{V(\Sigma,\bar{\sigma})}\otimes\Lambda^{-i-p}M[\sigma]$$

and

$$C^\cdot(\Phi, \Omega_p^\vee(\mathcal{O})) = \bigoplus_{\tau\in\Phi(-i)} \mathcal{O}_{V(\Sigma,\tau)}\otimes\Lambda^{-i-p}M[\tau]$$

we can define the homomorphism similarly as in the proof of Proposition 3.4.

Let Σ', Φ and Φ' be as above. For each integer p, we set

$$\tilde{\Omega}^p_{Y(\Phi)} = \mathscr{H}^0(C^\cdot(\Phi^{(2)}, \mathcal{O}\otimes\Lambda^p)).$$

Note that $\mathscr{H}^i(C^\cdot(\Phi^{(2)}, \mathcal{O}\otimes\Lambda^p))=0$ for $i\neq 0$ by Proposition 2.4. By Corollary 3.3, $\boldsymbol{R}\varphi_*\tilde{\Omega}^p_{Y(\Phi')}$ is represented by $\varphi_*C^\cdot(\Phi'^{(2)}, \mathcal{O}\otimes\Lambda^p)$. Furthermore, it is also represented by $\mathscr{H}om^\cdot_{\mathcal{O}_{Y(\Phi)}}(\varphi_*C^\cdot(\Phi', \Omega_p^\vee(\mathcal{O})), C^\cdot(\Phi, \Omega_0^\vee(\mathcal{O})))$ by Propositions 3.7 and 3.8. Since $\tilde{\Omega}^p_{Y(\Phi)}$ is quasi-isomorphic to

$$\mathscr{H}om^\cdot_{\mathcal{O}_{Y(\Phi)}}(C^\cdot(\Phi, \Omega_p^\vee(\mathcal{O})), C^\cdot(\Phi, \Omega_0^\vee(\mathcal{O})))$$

by Proposition 3.7, we get a homomorphism $\tilde{\Omega}^p_{Y(\Phi)} \to \boldsymbol{R}\varphi_* \tilde{\Omega}^p_{Y(\Phi')}$ in the derived category induced by the homomorphism $\varphi_* C^\cdot(\Phi', \Omega_p^\vee(\mathcal{O})) \to C^\cdot(\Phi, \Omega_p^\vee(\mathcal{O}))$.

Lemma 3.9. *Let Σ' be a subdivision of Σ, and let $\Sigma_+ = \Sigma \setminus \{\mathbf{0}\}$ and $\Sigma'_+ = \Sigma' \setminus \{\mathbf{0}\}$. Let φ: $Z(\Sigma') \to Z(\Sigma)$ be the morphism as before. Then there exists a triangle*

$$0 \longrightarrow \tilde{\Omega}^p_{Z(\Sigma)} \longrightarrow \tilde{\Omega}^p_{Y(\Sigma_+)} \oplus \boldsymbol{R}\varphi_* \tilde{\Omega}^p_{Z(\Sigma')} \longrightarrow \boldsymbol{R}\varphi_* \tilde{\Omega}^p_{Y(\Sigma'_+)} \longrightarrow 0$$

in the derived category $D^+_{\mathrm{coh}}(Z(\Sigma))$ for each integer p.

Proof. We have a diagram

$$\begin{array}{ccccccccc} 0 \to & \varphi_* C^\cdot(\Sigma'_+, \Omega_p^\vee(\mathcal{O})) & \to & \varphi_* C^\cdot(\Sigma', \Omega_p^\vee(\mathcal{O})) & \to & \varphi_* \Omega^{r-p}_{Z(\Sigma')}(\log D(\Sigma'))[r] & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & C^\cdot(\Sigma_+, \Omega_p^\vee(\mathcal{O})) & \longrightarrow & C^\cdot(\Sigma, \Omega_p^\vee(\mathcal{O})) & \longrightarrow & \Omega^{r-p}_{Z(\Sigma)}(\log D(\Sigma))[r] & \longrightarrow 0 \end{array}$$

of exact sequences of complexes. Since

$$\varphi_* \Omega^{r-p}_{Z(\Sigma')}(\log D(\Sigma')) = \Omega^{r-p}_{Z(\Sigma)}(\log D(\Sigma)) = \mathcal{O}_{Z(\Sigma)} \otimes \Lambda^{r-p} M,$$

we have an exact sequence

$$\begin{aligned} 0 \longrightarrow \varphi_* C^\cdot(\Sigma'_+, \Omega_p^\vee(\mathcal{O})) \longrightarrow \varphi_* C^\cdot(\Sigma', \Omega_p^\vee(\mathcal{O})) \oplus C^\cdot(\Sigma_+, \Omega_p^\vee(\mathcal{O})) \\ \longrightarrow C^\cdot(\Sigma, \Omega_p^\vee(\mathcal{O})) \longrightarrow 0 \end{aligned}$$

of complexes. By taking $\boldsymbol{R}\mathscr{H}om^\cdot_{\mathcal{O}_{Z(\Sigma)}}(\ , C^\cdot(\Sigma, \Omega_0^\vee(\mathcal{O})))$ of this sequence, we get the triangle in the lemma in view of Propositions 3.7 and 3.8 and Lemma 3.6. q.e.d.

A fan Σ of $N_{\boldsymbol{R}}$ is said to be *nonsingular* if it consists of nonsingular cones. Σ is nonsingular if and only if the toric variety $Z(\Sigma)$ is nonsingular [TE].

Proposition 3.10. *Let Σ be a nonsingular fan. Then $\mathscr{H}^0(C^\cdot(\Sigma^{(2)}, \mathcal{O} \otimes \Lambda^p))$ is canonically isomorphic to the sheaf $\Omega^p_{Z(\Sigma)}$ of regular p-forms on $Z(\Sigma)$.*

Proof. Since $\tilde{\Omega}^p_{Z(\Sigma)} = \mathscr{H}^0(C^\cdot(\Sigma^{(2)}, \mathcal{O} \otimes \Lambda^p))$, the proposition follows from the definition of $\tilde{\Omega}^p_S$ in Section 2.

For general Σ, we know the following.

Proposition 3.11 ([Da, Proposition 4.3]). *Let $U = Z(\Sigma) \setminus \mathrm{Sing}\,(Z(\Sigma))$,* *and let* $j\colon U\to Z(\Sigma)$ *be the open immersion. Then we have* $\tilde{\Omega}^p_{Z(\Sigma)}=j_*\Omega^p_U$ *for every integer* p.

Let X be an irreducible nonsingular variety, and let $f\colon X\to Z(\Sigma)$ be a morphism with $f(X)\cap T_N\neq\phi$. For $U=Z(\Sigma)\backslash\mathrm{Sing}(Z(\Sigma))$, let $\bar{f}\colon f^{-1}(U)\to U$ be the restriction of f. Since $\bar{f}$ is a morphism between nonsingular varieties, there exists a natural homomorphism $\bar{g}\colon \bar{f}^*\Omega^p_U\to\Omega^p_{f^{-1}(U)}$ for each p.

Proposition 3.12. *The above homomorphism* $\bar{g}$ *is extended uniquely to a homomorphism* $g\colon f^*\tilde{\Omega}^p_{Z(\Sigma)}\to\Omega^p_X$. *Furthermore, these homomorphisms commute with the exterior derivations.*

Proof. The uniqueness is clear, since Ω^p_X is a locally free $\mathcal{O}_X$-module and U is a dense open set. We have to show that, for every section $s\in\Gamma(W,\tilde{\Omega}^p_{Z(\Sigma)})$ over an open subset W of $Z(\Sigma)$, the pull-back $u=f^*(s|_{W\cap U})\in\Gamma(f^{-1}(W\cap U),\Omega^p_X)$ is extended to an element of $\Gamma(f^{-1}(W),\Omega^p_X)$. Since Ω^p_X is locally free and X is nonsingular, it is sufficient to show that u is extended to the generic point of every irreducible divisor Y which intersects $f^{-1}(W)$. Since $T_N\subset U$, we may assume $f(Y)\cap T_N=\phi$. Let $U(\pi)$ be the minimal T_N-invariant open subset of $Z(\Sigma)$ which intersects $f(Y)$, and let R be the local ring at the generic point of Y in X. If $\dim\pi\leq 1$, then $U(\pi)$ is nonsingular and the assertion is obvious. Hence we assume $\dim\pi\geq 2$. We get the homomorphism $\varphi\colon k[M\cap\pi^\vee]\to R$ of rings. The condition $f(X)\cap T_N\neq\phi$ implies that the generic point of X is mapped into $T_N=\mathrm{Spec}(k[M])$. Hence, for the quotient field K of R, the homomorphism φ is extended to $\varphi'\colon k[M]\to K$. Since R is a discrete valuation ring, the composite map $m\mapsto v(\varphi'(e(m)))$ with the valuation v of R is a homomorphism from M to $\boldsymbol{Z}$, and hence is equal to an element n_0 in N. Since $f(Y)$ does not intersect T_N, we know $n_0\neq 0$. Let $\gamma=\boldsymbol{R}_0 n_0$. Then since $\varphi'(e(m))\in R$ for $m\in M\cap\gamma^\perp$, φ is decomposed as

$$k[M\cap\pi^\vee]\hookrightarrow k[M\cap\gamma^\vee]\longrightarrow R.$$

Hence for the open set $G=f^{-1}(W\cap U(\pi))$, G intersects Y and the restriction of f to G is decomposed as

$$G\xrightarrow{f'}U(\gamma)\xrightarrow{\lambda}U(\pi).$$

Since $\tilde{\Omega}^p_{S[\pi]}\subset\tilde{\Omega}^p_{S[\gamma]}$ is obvious from the definition of them in Section 2, the pull-back from $\tilde{\Omega}^p_{U(\pi)}$ to $\tilde{\Omega}^p_{U(\gamma)}$ is welldefined for λ. On the other hand, f' has the pull-back of p-forms since $U(\gamma)$ is nonsingular. Hence the p-form u is extended to the generic point of Y which is contained in G. q.e.d.

By using Proposition 3.12, we can define the homomorphism $g: f^*\tilde{\Omega}^p_{Y(\Phi)} \to \Omega^p_X$ for every morphism $f: X \to Y(\Phi)$ from a nonsingular irreducible variety X to a toric polyhedron $Y(\Phi)$ as follows. Let η be the maximal element of Φ with $f(X) \subset V(\eta)$. Since $\Sigma(\eta\prec) \subset \Phi$, there exists a natural homomorphism $u: \tilde{\Omega}^p_{Y(\Phi)} \to \tilde{\Omega}^p_{Y(\Sigma(\eta\prec))}$. Recall that $\tilde{\Omega}^p_{Y(\Sigma(\eta\prec))} = \tilde{\Omega}^p_{Z(\Sigma[\eta])}$ by Proposition 2.2. Since $f(X)$ intersects the torus $T_{N[\eta]}$, the homomorphism $v: f^*\tilde{\Omega}^p_{Z(\Sigma[\eta])} \to \Omega^p_X$ is defined by Proposition 3.12. We define the homomorphism $g: f^*\tilde{\Omega}^p_{Y(\Phi)} \to \Omega^p_X$ to be the composite $v \circ u$.

§ 4. du Bois's de Rham complexes for toric polyhedra

In this section, all varieties and schemes are assumed to be defined over the complex number field $\boldsymbol{C}$.

du Bois [dB] constructed a de Rham complex $\underline{\underline{\Omega}}^{\cdot}_Y$ for every separated $\boldsymbol{C}$-scheme Y of finite type. $\underline{\underline{\Omega}}^{\cdot}_Y$ is an object of the derived category $D_{\mathrm{diff}}(Y)$ of the category $C_{\mathrm{diff}}(Y)$ which is defined as follows. Each object of $C_{\mathrm{diff}}(Y)$ is a triple $(K^{\cdot}, d, F)$ consisting of a complex $(K^{\cdot}, d)$ of $\mathcal{O}_Y$-modules and a decreasing filteration F on $K^{\cdot}$ such that (1) $K^{\cdot}$ is bounded below, (2) the filteration F is biregular, i.e. for each component K^i of $K^{\cdot}$, there exist integers $m, n \in \boldsymbol{Z}$ such that $F^m K^i = K^i$ and $F^n K^i = 0$, (3) d is a differential operator of order at most one and preserves the filteration F, and (4) $\mathrm{Gr}^p_F(d): \mathrm{Gr}^p_F(K^i) \to \mathrm{Gr}^p_F(K^{i+1})$ is $\mathcal{O}_X$-linear for any integers p and i.

We review briefly the construction of du Bois's de Rham complex $\underline{\underline{\Omega}}^{\cdot}_Y$. We take a smooth simplicial resolution $\alpha: Y_{\cdot} \to Y$ introduced by Deligne [Del]. In this case, $Y_{\cdot}$ is taken so that each component Y_n is nonsingular and the morphism $\alpha_n: Y_n \to Y$ is proper and $\boldsymbol{R}\alpha_* \boldsymbol{C}_{Y_{\cdot}} = \boldsymbol{C}_Y$. Since each component Y_n has the usual de Rham complex $\Omega^{\cdot}_{Y_n}$, we get the complex $\Omega^{\cdot}_{Y_{\cdot}}$ on $Y_{\cdot}$ which consists of $\Omega^{\cdot}_{Y_n}$'s. The complex $\underline{\underline{\Omega}}^{\cdot}_Y$ is given as the direct image $\boldsymbol{R}\alpha_* \Omega^{\cdot}_{Y_{\cdot}}$. It was shown in [dB], that this is independent of the choice of the simplicial resolution $Y_{\cdot}$. For each integer p, $\mathrm{Gr}^p_F(\underline{\underline{\Omega}}^{\cdot}_Y)$ is an element of $D^+_{\mathrm{coh}}(Y)$ and is denoted by $\underline{\underline{\Omega}}^p_Y$.

Now, let $Y(\Phi)$ be the toric polyhedron defined over $k = \boldsymbol{C}$ associated to a star closed subset Φ of a fan Σ of $N_{\boldsymbol{R}}$. We defined the complex $\tilde{\Omega}^{\cdot}_{Y(\Phi)}$ in Section 3. We know by Proposition 3.12, that for any morphism $f: X \to Y(\Phi)$ from a nonsingular variety X, there exists a natural homomorphism $f^*\tilde{\Omega}^{\cdot}_{Y(\Phi)} \to \Omega^{\cdot}_X$ of filtered complexes, where the filteration F on the complex $\tilde{\Omega}^{\cdot}_{Y(\Phi)}$ is given by $F^q(\tilde{\Omega}^p_{Y(\Phi)}) = \tilde{\Omega}^p_{Y(\Phi)}$ if $p \geq q$ and $F^q(\tilde{\Omega}^p_{Y(\Phi)}) = 0$ if $p < q$. Consequently, for a morphism $\alpha: X_{\cdot} \to Y(\Phi)$ from a simplicial variety $X_{\cdot}$ consists of nonsingular varieties, we get the homomorphism $\tilde{\Omega}^{\cdot}_{Y(\Phi)} \to \boldsymbol{R}\alpha_* \Omega^{\cdot}_{X_{\cdot}}$ in the derived category $D_{\mathrm{diff}}(Y(\Phi))$.

In this section, we assume that Σ is finite, i.e. consists of a finite number of s.c.r.p. cones. Hence the toric polyhedron $Y(\Phi)$ is of finite type, and we can consider du Bois's de Rham complex $\underline{\underline{\Omega}}^{\cdot}_{Y(\Phi)}$. The purpose of this section is to prove the following.

Theorem 4.1. *Let $Y(\Phi)$ be a toric polyhedron for a star closed subset Φ of a finite fan Σ, and let $\alpha\colon Y_{\cdot}\to Y(\Phi)$ be a simplicial resolution of $Y(\Phi)$. Then the natural homomorphism*

$$\psi\colon \tilde{\Omega}^{\cdot}_{Y(\Phi)}\longrightarrow \boldsymbol{R}\alpha_*\Omega^{\cdot}_{Y_{\cdot}}$$

is an isomorphism in the derived category $D_{\mathrm{diff}}(Y(\Phi))$. In other words, $\tilde{\Omega}^{\cdot}_{Y(\Phi)}$ is canonically isomorphic to du Bois's de Rham complex $\underline{\underline{\Omega}}^{\cdot}_{Y(\Phi)}$.

Since the homomorphism ψ preserve the filtrations, we get a homomorphism $\psi^p\colon \tilde{\Omega}^p_{Y(\Phi)}\to \boldsymbol{R}\alpha_*\Omega^p_{Y(\Phi)}$ induced from ψ for each integer p. In order to prove the theorem, it is sufficient to show the following proposition.

Proposition 4.2. *For each integer p, the homomorphism $\psi^p\colon \tilde{\Omega}^p_{Y(\Phi)}\to \underline{\underline{\Omega}}^p_{Y(\Phi)}$ is an isomorphism in $D^+_{\mathrm{coh}}(Y(\Phi))$.*

Proof. The assertion is trivially true if $\Phi=\phi$. Thus we assume $\Phi\neq\phi$. We prove the proposition by induction on $d=\mathrm{ht}\,\Phi\,(=\dim Y(\Phi))$ and the number k of elements of Φ, where we do not fix N and Σ. If $Y(\Phi)$ is nonsingular, then both $\tilde{\Omega}^p_{Y(\Phi)}$ and $\underline{\underline{\Omega}}^p_{Y(\Phi)}$ are equal to the ordinary sheaf $\Omega^p_{Y(\Phi)}$ of p-forms by Proposition 3.10 and [dB, Prop. 4.1], and hence ψ^p is isomorphic. In particular, if $k=1$ and $\Phi=\{\pi\}$, then $Y(\Phi)=T_{N[\pi]}$ is nonsingular, and ψ^p is an isomorphism. Now, assume the proposition is true for $0\leq d<h$ or $d=h$, $1\leq k<n$ for integers h, n. It is sufficient to prove the proposition for the case $\mathrm{ht}\,\Phi=h$ and when the number of the elements in Φ is equal to n. We devide the proof into two cases. The first is the case Φ has a unique minimal element, i.e. $\Phi=\Sigma(\rho\prec)$ for an element $\rho\in\Phi$, and the second is the case Φ has at least two minimal elements.

Case 1. Since $Y(\Phi)=Z(\Sigma[\rho])$, by replacing Σ by $\Sigma[\rho]$, we may assume $\Phi=\Sigma$. Let Σ' be a nonsingular subdivision of Σ whose existence is guaranteed by [TE, Chap. 1, Th. I1]. Set $\Sigma_+=\Sigma\backslash\{\mathbf{0}\}$ and $\Sigma'_+=\Sigma'\backslash\{\mathbf{0}\}$. Then since $\mathrm{ht}\,\Sigma_+=\mathrm{ht}\,\Sigma'_+=h-1$ and Σ' is nonsingular, the proposition is true for Σ_+, Σ'_+ and Σ' by the induction assumption. We get the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{\Omega}^p_{Y(\Sigma)} & \longrightarrow & \tilde{\Omega}^p_{Y(\Sigma_+)}\oplus \boldsymbol{R}\varphi_*\tilde{\Omega}^p_{Y(\Sigma')} & \longrightarrow & \boldsymbol{R}\varphi_*\tilde{\Omega}^p_{Y(\Sigma'_+)} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\psi^p} & & \downarrow{\scriptstyle\psi'^p} & & \downarrow{\scriptstyle\psi''^p} & & \\
0 & \longrightarrow & \underline{\underline{\Omega}}^p_{Y(\Sigma)} & \longrightarrow & \underline{\underline{\Omega}}^p_{Y(\Sigma_+)}\oplus \boldsymbol{R}\varphi_*\underline{\underline{\Omega}}^p_{Y(\Sigma')} & \longrightarrow & \boldsymbol{R}\varphi_*\underline{\underline{\Omega}}^p_{Y(\Sigma'_+)} & \longrightarrow & 0
\end{array}$$

of elements of $D^+_{\mathrm{coh}}(Y(\Sigma))$. The first row of this diagram is a triangle by Proposition 3.9, and the second row is also a triangle by [dB, Prop. 4.11]. Since ψ'^p and ψ''^p are isomorphisms ψ^p is also an isomorphism in the derived category.

Case 2. Let ρ be a minimal element of Φ. Set $\Psi=\Sigma(\rho\prec)$, $\Phi'=\Phi\backslash\{\rho\}$ and $\Psi'=\Psi\backslash\{\rho\}$. Clearly, these are star closed subsets of Σ. We have $Y(\Psi')=Y(\Psi)\cap Y(\Phi')$ and $Y(\Phi)=Y(\Psi)\cup Y(\Phi')$, and we get a sequence

$$0\longrightarrow C^{\cdot}(\Phi^{(2)}, \mathcal{O}\otimes\Lambda^p)\longrightarrow C^{\cdot}(\Phi'^{(2)}, \mathcal{O}\otimes\Lambda^p)\oplus C^{\cdot}(\Psi^{(2)}, \mathcal{O}\otimes\Lambda^p)$$
$$\longrightarrow C^{\cdot}(\Psi'^{(2)}, \mathcal{O}\otimes\Lambda^p)\longrightarrow 0$$

of complexes, which we see easily to be exact. On the other hand, by applying [dB, Th. 4.11] to the inclusion morphism $Y(\Phi')\to Y(\Phi)$ and the closed subscheme $Y(\Psi)\subset Y(\Phi)$, we get a triangle

$$0\longrightarrow \underline{\underline{\Omega}}^p_{Y(\Phi)}\longrightarrow \underline{\underline{\Omega}}^p_{Y(\Phi')}\oplus\underline{\underline{\Omega}}^p_{Y(\Psi)}\longrightarrow \underline{\underline{\Omega}}^p_{Y(\Psi')}\longrightarrow 0.$$

Similarly as in the first case, we see that $\psi^p\colon \tilde{\Omega}^p_{Y(\Phi)}\to\underline{\underline{\Omega}}^p_{Y(\Phi)}$ is isomorphic, since the number of elements in Φ', Ψ and Ψ' are smaller than n and the proposition is true for them by the induction assumption. q.e.d.

In [S], Steenbrink defined an algebraic singularity (X, x) over $\boldsymbol{C}$ to be a *du Bois singularity* if the natural homomorphism $\mathcal{O}_{X,x}\to\underline{\underline{\Omega}}^0_{X,x}$ is a quasi-isomorphism. $\tilde{\Omega}^0_{Y(\Phi)}=\mathcal{O}_{Y(\Phi)}$ for a toric polyhedron $Y(\Phi)$ by the definition of $\tilde{\Omega}^p_{B(\Phi)}$ in Section 2. Hence we have the following.

Corollary 4.3. *Every toric polyhedron defined over* $\boldsymbol{C}$ *has only du Bois singularities. In particular, toric singularities are du Bois singularities.*

§ 5. Semi-toroidal varieties and the dualizing complexes

Let k be a field of an arbitrary characteristic. In this section, we assume all schemes and morphisms among them are defined over k.

A pair (X, U) of a scheme X locally of finite type and its open subscheme U is said to be a *toroidal embedding* if, for each point $x\in X$, there exist a torus embedding $Z(x)$ with a torus $T(x)$ defined over k, a scheme $R(x)$ and étale morphisms $\psi_x\colon R(x)\to Z(x)$ and $\varphi_x\colon R(x)\to X$ such that $\psi_x^{-1}(T(x))=\varphi_x^{-1}(U)$ and $x\in\varphi_x(R(x))$. This definition is equivalent to that of [TE, Chap. II, § 1] by [A]. Let (X, U) be a toroidal embedding and let D be the reduced divisor $X\backslash U$. We define an $\mathcal{O}_X$-module $\Theta_X(-\log D)$ by $\Gamma(V, \Theta_X(-\log D))=\{\alpha\colon \mathcal{O}_V\to\mathcal{O}_V;\ \alpha\in\mathrm{Der}(\mathcal{O}_V)$ and $\alpha(\mathcal{I}(D)|_V)\subset\mathcal{I}(D)|_V\}$ for each open set $V\subset X$, where $\mathcal{I}(D)$ is the ideal of $\mathcal{O}_X$ defining D. We call a morphism $f\colon X'\to X$ of toroidal embeddings (X', U'), (X, U) *toroidal étale morphism* if it is étale and $f^{-1}(U)=U'$. Let $f\colon X'\to X$ be a toroidal étale morphism and let $D=X\backslash U$ and $D'=X'\backslash U'$. Then, there exists a natural identification $f^*\Theta_X(-\log D)=\Theta_{X'}(-\log D')$.

Lemma 5.1. *In the above notation, the $\mathcal{O}_X$-module $\Theta_X(-\log D)$ is a locally free sheaf. Its rank at each point x of X is equal to the dimension of X at x.*

Proof. Note that $(R(x), \psi_x^{-1}(T(x)))$ in the definition of the toroidal embeddings is also a toroidal embedding and ψ_x and φ_x are toroidal étale morphisms. Since $\Theta_{Z(x)}(-\log D(x))$ is a free $\mathcal{O}_X$-module of rank dim $Z(x)$ for $D(x)=Z(x)\backslash T(x)$, we know that $\Theta_X(-\log D)$ is a locally free $\mathcal{O}_X$-module of the same rank at x. q.e.d.

We define $\Omega_X^1(\log D)=\mathcal{H}om_{\mathcal{O}_X}(\Theta_X(-\log D), \mathcal{O}_X)$ and $\Omega_X^p(\log D)=\Lambda^p\Omega_X^1(\log D)$ for each integer p.

Let $Y=Y(\Phi)$ be the toric polyhedron defined for a star closed subset Φ of a fan Σ of $N_{\boldsymbol{R}}$. The k-scheme Y has a natural increasing filteration $\{Y_i\}$ defined by

$$Y_i=\bigcup_{\sigma\in\Phi(\leq i)} V(\sigma)$$

where $\Phi(\leq i)=\bigcup_{j=0}^{i}\Phi(j)$. Since dim $V(\sigma)=i$ for $\sigma\in\Phi(i)$, Y_i is a closed subscheme of Y of dimension i for every $0\leq i\leq \dim Y$. For each integer i, the locally closed subsheme $S^{(i)}=Y_i\backslash Y_{i-1}$ of Y is equal to $\bigcup_{\sigma\in\Phi(i)}\operatorname{orb}(\sigma)$ and hence is nonsingular of pure dimension i. We denote by $Y^{(i)}$ the normalization of the closure of $S^{(i)}$ in Y. Clearly, $Y^{(i)}$ is equal to the disjoint union $\coprod_{\sigma\in\Phi(i)} V(\sigma)$. We denote by $\lambda^{(i)}$ the natural homomorphism $Y^{(i)}\to Y$. Hence by Lemma 3.1, the pair $(Y^{(i)}, S^{(i)})$ is a toroidal embedding of pure dimension i. Hence, for $E^{(i)}=Y^{(i)}\backslash S^{(i)}=\coprod_{\sigma\in\Phi(i)}D(\sigma)$, the sheaf $\Theta_{Y^{(i)}}(-\log E^{(i)})$ is a locally free $\mathcal{O}_{Y^{(i)}}$-module of rank i. Let p be an integer. Note that $C^i(\Phi, \Omega_p^{\vee}(\mathcal{O}))$ of the complex $C^{\cdot}(\Phi, \Omega_p^{\vee}(\mathcal{O}))$ defined in Section 3 is equal to

$$\bigoplus_{\sigma\in\Phi(-i)}\Omega_{V(\sigma)}^{-i-p}(\log D(\sigma))=\lambda_*^{(-i)}\Omega_{Y^{(-i)}}^{-i-p}(\log E^{(-i)}).$$

Definition 5.2. A scheme X locally of finite type is said to be a *semi-toroidal variety* if, for each point $x\in X$, there exist a toric polyhedron $Y(x)$, a scheme $W(x)$ and étale morphisms $\psi_x\colon W(x)\to Y(x)$ and $\varphi_x\colon W(x)\to X$ such that $x\in\varphi_x(W(x))$. We call it a *filtered semi-toroidal variety* if, furthermore, X has an increasing filtration $\{X_i\}$ and $\varphi_x^{-1}(X_i)=\psi_x^{-1}(Y(x)_i)$ for each i and for the natural filtration $\{Y(x)_i\}$ of the toric polyhedron $Y(x)$.

Let X be a filtered semi-toroidal variety. For each integer i, let $U^{(i)}=X_i\backslash X_{i-1}$ and let $X^{(i)}$ be the normalization of the closure of $U^{(i)}$. We know $(X^{(i)}, U^{(i)})$ is a toroidal embedding of pure dimension i since it is locally isomorphic to $(Y^{(i)}, S^{(i)})$ for some toric polyhedron Y at each point of X. Hence for each integer p, we get a locally free $\mathcal{O}_{X^{(i)}}$-module $\Omega^p_{X^{(i)}}(\log D^{(i)})$ for the reduced divisor $D^{(i)}=X^{(i)}\backslash U^{(i)}$. We define the complex $\mathscr{C}^{\cdot}(X, \Omega_p^{\vee})$ of coherent $\mathcal{O}_X$-modules as follows. Let $\phi^{(i)}: X^{(i)}\to X$ be the natural homomorphism. For each integer i, we set

$$\mathscr{C}^i(X, \Omega_p^{\vee})=\phi_*^{(-i)}\Omega_{X^{(-i)}}^{-i-p}(\log D^{(-i)}).$$

Clearly we have $\varphi_x^*\mathscr{C}^i(X, \Omega_p^{\vee})=\psi_x^*C^i(\Phi(x), \Omega_p^{\vee}(\mathcal{O}))$ in the notation in Definition 5.2, where $\Phi(x)$ is the star closed subset of a fan which defines the toric polyhedron $Y(x)$. Since the homomorphism d: $C^i(\Phi(x), \Omega_p^{\vee}(\mathcal{O}))\to C^{i+1}(\Phi(x), \Omega_p^{\vee}(\mathcal{O}))$ is defined naturally by the Poincaré residue map for every $x\in X$, we get a homomorphism d: $\mathscr{C}^i(X, \Omega_p^{\vee})\to\mathscr{C}^{i+1}(X, \Omega_p^{\vee})$ such that $\varphi_x^*\mathscr{C}^{\cdot}(X, \Omega_p^{\vee})=\psi_x^*C^{\cdot}(\Phi(x), \Omega_p^{\vee}(\mathcal{O}))$ at each point $x\in X$ by descent theory [SGA1]. Thus the complex $\mathscr{C}^{\cdot}(X, \Omega_p^{\vee})$ is defined for each integer p.

Remark 5.3. It is clear by definition that any toric polyhedron $Y(\Phi)$ is a filtered semi-toroidal variety and the complex $\mathscr{C}^{\cdot}(Y(\Phi), \Omega_p^{\vee})$ is equal to $C^{\cdot}(\Phi, \Omega_p^{\vee}(\mathcal{O}))$.

Let S be a scheme of finite type. We call a complex $R^{\cdot}$ of $\mathcal{O}_X$-modules a *global dualizing complex* of S if $R^{\cdot}$ represents the twisted inverse image $f_S^!k$ in the derived category $D^+_{\mathrm{coh}}(S)$, where f_S: $S\to\mathrm{Spec}(k)$ is the structure morphism and $k=\mathcal{O}_{\mathrm{Spec}(k)}$. Recall that, for a morphism f: $S'\to S$ of schemes of finite type, the functor of twisted inverse image $f^!$: $D^+_{\mathrm{coh}}(S)\to D^+_{\mathrm{coh}}(S')$ is defined and, if f is proper, satisfies the relative duality [RD], [V].

The global dualizing complex of a filtered semi-toroidal variety is explicitly given as a complex of coherent modules as follows.

Theorem 5.4. *Let X be a filtered semi-toroidal variety. Then $\mathscr{C}^{\cdot}(X, \Omega_0^{\vee})$ is a global dualizing complex.*

Let S be a scheme of finite type. Then by [RD, Chap. VI], the twisted inverse image $f_S^!k$ is represented by a residual complex, which we denote by $f_S^{\nabla}k$ as in [RD, Chap. IV] and is defined as follows. We take an open covering $\{S_\lambda\}$ of S such that each S_λ has a closed immersion $S_\lambda\to P_\lambda$ to a nonsingular irreducible variety P_λ. The local chart $f_{S_\lambda}^{\nabla}k$ on S_λ of the residual complex $f_S^{\nabla}k$ is given by $f_{S_\lambda}^{\nabla}k=\mathscr{H}om^{\cdot}_{\mathcal{O}_{P_\lambda}}(\mathcal{O}_{S_\lambda}, E^{\cdot}(\omega_{P_\lambda}[n_\lambda]))$, where $E^{\cdot}(A^{\cdot})$ is the Cousin complex of a complex $A^{\cdot}$ [RD, Chap. III], $n_\lambda=\dim P_\lambda$ and $\omega_{P_\lambda}=\Omega^{n_\lambda}_{P_\lambda}$. Then, for any two open sets S_λ, S_μ the restrictions $f_{S_\lambda}^{\nabla}k|_{S_\lambda\cap S_\mu}$ and $f_{S_\mu}^{\nabla}k|_{S_\lambda\cap S_\mu}$ are naturally isomorphic as complexes of $\mathcal{O}_{S_\lambda\cap S_\mu}$-

modules. By gluing the charts, we get the residual complex $f_S^\nabla k$ which is a bounded complex of quasi-coherent injective $\mathcal{O}_S$-modules. Hence, in order to prove the theorem, it is sufficient to show that there exists a quasi-isomorphism $\mathscr{C}^{\boldsymbol{\cdot}}(X, \Omega_0^\vee) \to f_X^\nabla k$ for the filtered semi-toroidal variety X.

We have to recall the Cousin complex in more detail.

Let P be a nonsingular irreducible variety of dimension $n \geq 0$, and let $\mathscr{L}[n]$ be invertible sheaf $\mathscr{L}$ shifted n places to the left as a complex. Then, there exists a natural injective resolution $E^{\boldsymbol{\cdot}}(\mathscr{L}[n])$ of $\mathscr{L}[n]$ which is called the Cousin complex. It is a complex of the relative local cohomology sheaves

$$\cdots \longrightarrow \mathscr{H}^p_{Z_{-p}/Z_{-p-1}}(\mathscr{L}[n]) \longrightarrow \mathscr{H}^{p+1}_{Z_{-p-1}/Z_{-p-2}}(\mathscr{L}[n]) \longrightarrow \cdots$$

where $Z_i = \{x \in P;\ \dim \{x\}^- \leq i\}$. The complex $E^{\boldsymbol{\cdot}}(\mathscr{L}[n])$ is obtained as the limit of an inductive system of complexes of coherent sheaves as follows. Let $\mathscr{S}$ be the set of decreasing filterations $I = (\mathscr{I}^1, \cdots, \mathscr{I}^{n+1})$ of $\mathcal{O}_P$ by its ideals with the condition $\dim \operatorname{supp}(\mathcal{O}_P/\mathscr{I}^i) < i$ for every $1 \leq i \leq n+1$. For $I = (\mathscr{I}^1, \cdots, \mathscr{I}^{n+1})$ and $K = (\mathscr{K}^1, \cdots, \mathscr{K}^{n+1})$ of $\mathscr{S}$, we define $I \leq K$ if $\mathscr{K}^1 \subset \mathscr{I}^1, \cdots, \mathscr{K}^{n+1} \subset \mathscr{I}^{n+1}$. By this ordering, $\mathscr{S}$ becomes a directed set. For each $I = (\mathscr{I}^1, \cdots, \mathscr{I}^{n+1})$, there exists a spectral sequence

$$\begin{aligned} E_1^{p,q}(I) &= \mathscr{E}xt^{p+q}_{\mathcal{O}_P}(\mathscr{I}^{-p}/\mathscr{I}^{1-p}, \mathscr{L}[n]) \\ &\Rightarrow \mathscr{E}xt^{p+q}_{\mathcal{O}_P}(\mathcal{O}_P/\mathscr{I}^{n+1}, \mathscr{L}[n]), \end{aligned}$$

where we understand $\mathscr{I}^i = \mathscr{I}^{n+1}$ for $i > n+1$ and $\mathscr{I}^i = \mathcal{O}_P$ for $i \leq 0$. For $q = 0$, we have a complex

$$\mathscr{D}^{\boldsymbol{\cdot}}(I, \mathscr{L}[n]) = (\cdots \to E_1^{p,0}(I) \xrightarrow{d_1^{p,0}} E_1^{p+1,0}(I) \to \cdots).$$

For each pair (I, K) with $I \leq K$, we get a natural homomorphism $\{E_r^{p,q}(I)\} \to \{E_r^{p,q}(K)\}$ of spectral sequences. In particular, we have a homomorphism $\alpha_{I/K}\colon \mathscr{D}^{\boldsymbol{\cdot}}(I, \mathscr{L}[n]) \to \mathscr{D}^{\boldsymbol{\cdot}}(K, \mathscr{L}[n])$ of the complexes. Since P is nonsingular and $\mathscr{L}$ is invertible, $\mathscr{L}$ has depth n at each point of P. In particular, for the shifted sheaf $\mathscr{L}[n]$, we have $\mathscr{E}xt^j_{\mathcal{O}_P}(\mathscr{F}, \mathscr{L}[n]) = 0$ for a coherent sheaf $\mathscr{F}$ and an integer j with $\dim \operatorname{supp}(\mathscr{F}) + j < 0$. By the diagram

$$\begin{array}{ccccccc}
\mathscr{E}xt^{p-1}_{\mathcal{O}_P}(\mathscr{I}^{1-p}/\mathscr{K}^{1-p}, \mathscr{L}[n]) = 0 & & & & 0 = \mathscr{E}xt^p_{\mathcal{O}_P}(\mathscr{I}^{-p}/\mathscr{K}^{-p}, \mathscr{L}[n]) \\
\downarrow & & & & \downarrow \\
0 \to \mathscr{E}xt^p_{\mathcal{O}_P}(\mathcal{O}_P/\mathscr{I}^{1-p}, \mathscr{L}[n]) & \to & \mathscr{E}xt^p_{\mathcal{O}_P}(\mathscr{I}^{-p}/\mathscr{I}^{1-p}, \mathscr{L}[n]) & \to & \mathscr{E}xt^{p+1}_{\mathcal{O}_P}(\mathcal{O}_P/\mathscr{I}^{-p}, \mathscr{L}[n]) \\
\downarrow & & \downarrow{\scriptstyle \alpha^p_{I/K}} & & \downarrow \\
0 \to \mathscr{E}xt^p_{\mathcal{O}_P}(\mathcal{O}_P/\mathscr{K}^{1-p}, \mathscr{L}[n]) & \to & \mathscr{E}xt^p_{\mathcal{O}_P}(\mathscr{K}^{-p}/\mathscr{K}^{1-p}, \mathscr{L}[n]) & \to & \mathscr{E}xt^{p+1}_{\mathcal{O}_P}(\mathcal{O}_P/\mathscr{K}^{-p}, \mathscr{L}[n])
\end{array}$$

the homomorphism $\alpha_{I/K}$ is injective. The Cousin complex $E^{\cdot}(\mathscr{L}[n])$ is equal to the inductive limit

$$\operatorname{ind.lim}_{\mathscr{S}} \mathscr{D}^{\cdot}(I, \mathscr{L}[n]),$$

and is a resolution of $\mathscr{L}[n]$ with respect to the natural homomorphism

$$\mathscr{L} \longrightarrow E^{-n}(\mathscr{L}[n]) = \operatorname{ind.lim}_{\mathscr{S}} \mathscr{E}xt^0(\mathscr{I}^n/\mathscr{I}^{n+1}, \mathscr{L}).$$

It is known that $E^p(\mathscr{L}[n])$ is isomorphic to $\bigoplus_{x \in Z_{-p} \setminus Z_{-p-1}} i_x(H^{-p}_{\mathfrak{m}_x}(\mathcal{O}_x))$ where $i_x(G)$ is the constant sheaf G on the closure $\{x\}^-$ of the point x and zero outside, and the local cohomology $H^{-p}_{\mathfrak{m}_x}(\mathcal{O}_x)$ is isomorphic to the injective hull of the residue field $k(x)$ as an $\mathcal{O}_{X,x}$-module. In particular, $\mathscr{H}om_{\mathcal{O}_P}(\mathscr{F}, E^p(\mathscr{L}[n])) = 0$ if $\dim \operatorname{supp}(\mathscr{F}) + p < 0$.

By construction, there exists a natural inclusion homomorphism $\mathscr{D}^{\cdot}(I, \mathscr{L}[n]) \to E^{\cdot}(\mathscr{L}[n])$ for each $I \in \mathscr{S}$. Actually, if we use $E^{\cdot}(\mathscr{L}[n])$ to obtain the extension sheaves, for $I = (\mathscr{I}^1, \cdots, \mathscr{I}^{n+1})$ we see easily that

$$\begin{aligned}\mathscr{E}xt^p_{\mathcal{O}_P}(\mathscr{I}^{-p}/\mathscr{I}^{1-p}, \mathscr{L}[n]) = \operatorname{Ker}(&\mathscr{H}om_{\mathcal{O}_P}(\mathcal{O}_P/\mathscr{I}^{1-p}, E^p(\mathscr{L}[n])) \\ &\longrightarrow \mathscr{H}om_{\mathcal{O}_P}(\mathscr{I}^{-p}/\mathscr{I}^{1-p}, E^{p+1}(\mathscr{L}[n]))) \subset E^p(\mathscr{L}[n])\end{aligned}$$

for every p. In particular, $\mathscr{D}^{\cdot}(I, \mathscr{L}[n])$ is a subcomplex of $\mathscr{H}om^{\cdot}_{\mathcal{O}_P}(\mathcal{O}_P/\mathscr{I}^{n+1}, E^{\cdot}(\mathscr{L}[n]))$ since $\mathscr{I}^{n+1} \subset \mathscr{I}^{1-p}$ and $E^p(\mathscr{L}[n])$ is injective for each p.

Lemma 5.5. *Let $I = (\mathscr{I}^1, \cdots, \mathscr{I}^{n+1})$ be in $\mathscr{S}$. If $\mathscr{E}xt^{p+q}_{\mathcal{O}_P}(\mathscr{I}^{-p}/\mathscr{I}^{1-p}, \mathscr{L}[n]) = 0$ for any p, q with $q \neq 0$, then the inclusion*

$$\mathscr{D}^{\cdot}(I, \mathscr{L}[n]) \longrightarrow \mathscr{H}om^{\cdot}_{\mathcal{O}_P}(\mathcal{O}_P/\mathscr{I}^{n+1}, E^{\cdot}(\mathscr{L}[n]))$$

is a quasi-isomorphism.

Proof. Let $A^{\cdot} = \mathscr{H}om^{\cdot}_{\mathcal{O}_P}(\mathcal{O}_P/\mathscr{I}^{n+1}, E^{\cdot}(\mathscr{L}[n]))$. The complex $A^{\cdot}$ has a decreasing filteration defined by

$$F^p(A^{\cdot}) = \mathscr{H}om_{\mathcal{O}_P}(\mathcal{O}_P/\mathscr{I}^{1-p}, E^{\cdot}(\mathscr{L}[n])),$$

and the spectral sequence $E_1^{p,q}(I) \Rightarrow \mathscr{E}xt^{p+q}_{\mathcal{O}_P}(\mathcal{O}_P/\mathscr{I}^{n+1}, \mathscr{L}[n])$ is obtained from this filtered complex. By assumption, the spectral sequence degenerates at the E_2-terms and $E_2^{p,0}(I) \simeq \mathscr{E}xt^p_{\mathcal{O}_P}(\mathcal{O}_P/\mathscr{I}^{n+1}, \mathscr{L}[n])$ for every p. Since the E_2-terms are the cohomologies of $\mathscr{D}^{\cdot}(I, \mathscr{L}[n])$, the inclusion is a quasi-isomorphism. q.e.d.

Let (S, U) be a toroidal embedding of pure dimension r, and let $\mathscr{I}(D) \subset \mathcal{O}_S$ be the ideal defining the reduced divisor $D = S \setminus U$. The variety S is Cohen-Macaulay by [Ho] and the dualizing sheaf ω_S is equal to

$\mathscr{I}(D)\Omega_S^r(\log D)$ by [TE, Chap. I, Th. 9 and Th. 14] or [MO, the remark after Prop. 6.6]. In particular, if $S \to P$ is a closed immersion in a nonsingular irreducible variety P of dimension n, then there exists a natural isomorphism

$$\mathscr{I}(D)\Omega_S^r(\log D)[r] \simeq \mathscr{E}xt_{\mathcal{O}_P}^{-r}(\mathcal{O}_S, \omega_P[n]).$$

Assume that the filtered semi-toroidal variety X is embedded in a nonsingular irreducible variety P of dimension n. We set $\mathscr{I}_X^i = \mathscr{I}(X_{i-1})$ for $i=1, \cdots, n+1$ and $I(X) = (\mathscr{I}_X^1, \cdots, \mathscr{I}_X^{n+1})$, where $\mathscr{I}(X_i) \subset \mathcal{O}_P$ is the ideal defining $X_i \subset P$. We understand $X_i = X$ for $i > r = \dim X$, and hence $\mathscr{I}_X^{r+1} = \cdots = \mathscr{I}_X^{n+1} = \mathscr{I}(X)$. For the canonical invertible sheaf $\omega_P = \Omega_P^n$ on P, we consider the Cousin complex $E^\cdot(\omega_P[n])$. By Lemma 2.9, $\mathscr{I}_X^i/\mathscr{I}_X^{i+1}$ is naturally isomorphic to $\phi_*^{(i)}\mathscr{I}(D^{(i)})$, where $\mathscr{I}(D^{(i)})$ is the ideal of $\mathcal{O}_{X^{(i)}}$ defining $D^{(i)} \subset X^{(i)}$. By the relative duality for the finite morphism $X^{(i)} \to P$, we have

$$\mathscr{E}xt_{\mathcal{O}_P}^j(\mathscr{I}_X^i/\mathscr{I}_X^{i+1}, \omega_P[n]) \simeq \mathscr{E}xt_{\mathcal{O}_{X^{(i)}}}^j(\mathscr{I}(D^{(i)}), \mathscr{I}(D^{(i)})\Omega_{X^{(i)}}^i(\log D^{(i)})[i]),$$

for every integer j. Since $\Omega_{X^{(i)}}^i(\log D^i)$ is equal to the invertible sheaf $\omega_{X^{(i)}}(D^{(i)})$, we have

$$\mathscr{E}xt_{\mathcal{O}_P}^j(\mathscr{I}_X^i/\mathscr{I}_X^{i+1}, \omega_P[n]) = 0 \quad \text{for } j \neq -i \quad \text{and}$$
$$\mathscr{E}xt_{\mathcal{O}_P}^{-i}(\mathscr{I}_X^i/\mathscr{I}_X^{i+1}, \omega_P[n]) = \phi_*^{(i)}\omega_{X^{(i)}}(D^{(i)}).$$

By Lemma 5.5, we have a natural quasi-isomorphism

$$\mathscr{D}^\cdot(I(X), \omega_P[n]) \longrightarrow \mathscr{H}om_{\mathcal{O}_P}^\cdot(\mathcal{O}_X, E^\cdot(\omega_P[n]))$$

We easily see that the homomorphism

$$d\colon \mathscr{E}xt_{\mathcal{O}_P}^i(\mathscr{I}_X^i/\mathscr{I}_X^{i+1}, \omega_P[n]) \longrightarrow \mathscr{E}xt_{\mathcal{O}_P}^{i+1}(\mathscr{I}_X^{i-1}/\mathscr{I}_X^i, \omega_P[n])$$

is equal to the Poincaré residue map. Hence $\mathscr{D}^\cdot(I(X), \omega_p[n])$ is equal to $\mathscr{C}^\cdot(X, \Omega_0^\vee)$.

In general case, we take an open covering $X = \bigcup X_\lambda$ so that each X_λ is embedded in a nonsingular irreducible variety P_λ of dimension n_λ. The restriction $\mathscr{C}^\cdot(X, \Omega_0^\vee)|_{X_\lambda}$ is equal to $\mathscr{D}^\cdot(I(X_\lambda), \omega_{P_\lambda}[n_\lambda])$ and there exist a natural injective quasi-isomorphism

$$\mathscr{D}^\cdot(I(X_\lambda), \omega_{P_\lambda}[n_\lambda]) \longrightarrow \mathscr{H}om_{\mathcal{O}_{P_\lambda}}^\cdot(\mathcal{O}_{X_\lambda}, E^\cdot(\omega_{P_\lambda}[n_\lambda])).$$

Since the charts $(X_\lambda, \mathscr{H}om_{\mathcal{O}_{P_\lambda}}(\mathcal{O}_{X_\lambda}, E^\cdot(\omega_{P_\lambda}[n_\lambda])))$ are naturally patched together to the residual complex $f_X^\nabla k$, we get an injective quasi-isomorphism

$$\mathscr{C}^{\cdot}(X, \Omega_0^{\vee}) \longrightarrow f_X^{\nabla} k.$$

Thus we proved Theorem 5.4.

By Remark 5.3, we have the following.

Corollary 5.6. *Let Φ be a star closed subset a fan Σ of $N_{\boldsymbol{R}}$ as Section 3. Then $C^{\cdot}(\Phi, \Omega_0^{\vee}(\mathscr{O}))$ is the global dualizing complex of the toric polyhedron $Y(\Phi)$.*

§ 6. de Rham complexes on semi-toroidal varieties

In this section, we assume again $k=\boldsymbol{C}$.
Let $(X, \{X_i\})$ be a filtered semi-toroidal variety of dimension $d \geq 0$.

Proposition 6.1. *Let p be an integer. Then we have*

$$\mathscr{E}xt^i_{\mathscr{O}_X}(\mathscr{C}^{\cdot}(X, \Omega_p^{\vee}), \mathscr{C}^{\cdot}(X, \Omega_0^{\vee}))=0$$

for every nonzero integer i.

Proof. Since the assertion is a local property in the étale topology, we are done by Theorem 2.7. q.e.d.

For each integer p, we set

$$\tilde{\Omega}_X^p = \mathscr{E}xt^0_{\mathscr{O}_X}(\mathscr{C}^{\cdot}(X, \Omega_p^{\vee}), \mathscr{C}^{\cdot}(X, \Omega_0^{\vee})).$$

By the above proposition, the coherent sheaf $\tilde{\Omega}_X^p$ is equal to

$$\boldsymbol{R}\mathscr{H}om^{\cdot}_{\mathscr{O}_X}(\mathscr{C}^{\cdot}(X, \Omega_p^{\vee}), \mathscr{C}^{\cdot}(X, \Omega_0^{\vee}))$$

in the derived category $D^+_{\mathrm{coh}}(X)$. Clearly, $\tilde{\Omega}_X^p$ is a natural globalization of $\tilde{\Omega}^p_{B(\Phi)}$ in Section 2. Hence the exterior derivatives $d\colon \tilde{\Omega}^p_{B(\Phi)} \to \tilde{\Omega}^{p+1}_{B(\Phi)}$ are also generalized for these sheaves and we get a complex

$$\tilde{\Omega}_X^{\cdot} = (\cdots \to 0 \longrightarrow \mathscr{O}_X \xrightarrow{d} \tilde{\Omega}_X^1 \xrightarrow{d} \cdots \xrightarrow{d} \tilde{\Omega}_X^d \longrightarrow 0 \to \cdots).$$

We define the decreasing filteration on $\tilde{\Omega}_X^{\cdot}$ by $F^q(\tilde{\Omega}_X^p)=\tilde{\Omega}_X^p$ if $p \geq q$ and zero otherwise.

Let V be an irreducible nonsingular variety, and let $f\colon V \to X$ be a morphism. Let i be the smallest integer with $f(V) \cap X_i$. Then, f factors as $\phi^{(i)} \circ f'$ for a morphism $f'\colon V \to X^{(i)}$ with $f'(V) \cap U^{(i)} \neq \phi$, where $X^{(i)}$, $\phi^{(i)}\colon X^{(i)} \to X$ and $U^{(i)} \subset X^{(i)}$ are the same as those in the previous section. Hence we have a natural homomorphism $f^*\tilde{\Omega}_X^p \to \Omega_V^p$ by Proposition 3.12 for each p which is compatible with the exterior derivatives similarly as we saw it for toric polyhedra in Section 4. Let $\alpha\colon X_{\bullet} \to X$ be a simplicial resolution of X as in Section 4. Then, similarly as in Section 4, we get a natural homomorphism

$$\psi\colon \tilde{\Omega}^{\bullet}_X \longrightarrow \boldsymbol{R}\alpha_* \Omega^{\bullet}_{X}.$$

of filtered complexes. By our definition of $\tilde{\Omega}^{\bullet}_X$ and [dB], these two complexes are compatible with pull-backs by étale morphisms. Hence by Theorem 4.1, we get the following generalization.

Theorem 6.2. *The complex $\tilde{\Omega}^{\bullet}_X$ is quasi-isomorphic to du Bois's de Rham complex $\underline{\underline{\Omega}}^{p}_X$ as filtered complexes.*

As a consequence of [dB, Th. 4.5], we get the following:

Corollary 6.3. *If the filtered semi-toroidal variety X is complete, then there exists a spectral sequence*

$$E_1^{p,q} = H^q(X, \tilde{\Omega}^p_X) \Rightarrow H^{p+q}(X, \boldsymbol{C}).$$

which degenerates at the E_1-terms. Furthermore, the induced filtration on $H^n(X, \boldsymbol{C})$ is equal to the Hodge filtration in the sense of [Del] *for each integer n.*

Remark 6.4. Danilov [Da, Chap. IV] conjectured this assertion for "toroidal varieties" which means, in our terminology, normal semi-toroidal varieties. We still have some difficulties to generalize Corollary 6.3 to semi-toroidal varieties without filtrations. However, since toroidal embeddings are normal filtered semi-toroidal varieties, this corollary implies that Danilov's conjecture is true for toroidal embeddings.

By definition, $\tilde{\Omega}^p_X$ and $\mathscr{C}^{\bullet}(X, \Omega^{\vee}_p)$ are mutually dual with respect to the global dualizing complex $\mathscr{C}^{\bullet}(X, \Omega^{\vee}_0)$. Hence by the duality theory [RD, Chap. VII], we have the following theorem.

Theorem 6.5. *If X is complete, then $H^q(X, \tilde{\Omega}^p_X)$ and $\boldsymbol{R}^{-q}\Gamma(X, \mathscr{C}^{\bullet}(X, \Omega^{\vee}_p))$ are mutually dual finite dimensional $\boldsymbol{C}$-vector spaces for every integer q.*

Remark 6.6. It seems easier to calculate $\boldsymbol{R}^{-q}\Gamma(X, \mathscr{C}^{\bullet}(X, \Omega^{\vee}_p))$ than $H^q(X, \tilde{\Omega}^p_X)$. Indeed, since $\mathscr{C}^{\ell}(X, \Omega^{\vee}_p) = \phi^{(-\ell)}_* \Omega^{-\ell-p}_{(-\ell)}(\log D^{(-\ell)})$ and $\phi^{(-\ell)}$ is a finite morphism for every ℓ, there exists a spectral sequence

$$E_1^{\ell,m} = H^m(X^{(-\ell)}, \Omega^{-\ell-p}_{X^{(-\ell)}}(\log D^{(-\ell)})) \Rightarrow \boldsymbol{R}^{\ell+m}\Gamma(X, \mathscr{C}^{\bullet}(X, \Omega^{\vee}_p)).$$

Note that $\Omega^p_{X^{(n)}}(\log D^{(n)})$ is a locally free sheaf on the toroidal embedding $(X^{(n)}, U^{(n)})$. Furthermore, it is known that any toroidal embedding (X, U)

has a resolution $f\colon \tilde{X}\to X$ of singularities which locally in the étale topology of X is an equivariant morphism of toric varieties. For the divisor $D=X\setminus U$, the divisor $\tilde{D}=f^{-1}(D)$ is a normal crossing divisor and $f^*\Omega_X^p(\log D)=\Omega_{\tilde{X}}^p(\log \tilde{D})$ for each integer p. By Theorem 3.2, we also have

$$\boldsymbol{R}f_*\Omega_{\tilde{X}}^p(\log \tilde{D})=\Omega_X^p(\log D).$$

In particular, we have a natural isomorphism

$$H^m(\tilde{X}, \Omega_{\tilde{X}}^p(\log \tilde{D}))\simeq H^m(X, \Omega_X^p(\log D))$$

for each integer m.

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