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Degenerations of Surfaces

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§ 1. Introduction

In this paper, we study degenerations of surfaces with the nef canonical bundle. Here is a summary of our results. Let $\pi: \mathscr{X} \to \varDelta$ be a projective degeneration of nonsingular projective algebraic surfaces with nef canonical bundles. Then we find a modification of $\mathscr{X}$ whose canonical bundle is relatively nef.

Recently, Mori, Shokurov and Kawamata, independently, stated results more general than ours.

The proof is based on Mori's theory [5] and Kawamata's contraction theorem [2]. The remaining part of the proof is to construct a so-called flip [2], although we do not state the existence of a flip explicitly. Since this part of the proof is very elementary but might be hard to read, the readers are recommended to read this paper drawing pictures. Thanks are due to Professors M. Miyanishi, and D. Morrison for helpful advice.

§ 2. Main theorem

Let $\mathscr{X}$ be a nonsingular projective 3-fold over the complex number field $\boldsymbol{C}$ and $\varDelta$ a nonsingular curve. Suppose that we have a morphism $\pi: \mathscr{X} \to \varDelta$ satisfying the following conditions:

(i) π is surjective,

(ii) for each $p \in \varDelta$, the scheme-theoretic inverse image $\pi^{-1}(p)$ is a reduced divisor with only simple normal crossings,

(iii) if $\pi^{-1}(p)$ is nonsingular for some $p \in \varDelta$, then $\pi^{-1}(p)$ is a minimal surface, in other words, the canonical bundle of $\pi^{-1}(p)$ is nef,

(iv) the genus of $\varDelta$ is positive.

If $\pi: \mathscr{X} \to \varDelta$ (or $\mathscr{X}$ for short) satisfies these conditions, we call it an S-degeneration.

Next, we define an S-regular 3-fold as follows: Let $\mathscr{X}$ be an S-degeneration. Assume that there exists a tower of birational morphisms

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$$\mathscr{Y}_n = \mathscr{X} \xrightarrow{f_n} \mathscr{Y}_{n-1} \longrightarrow \cdots \longrightarrow \mathscr{Y}_1 \xrightarrow{f_1} \mathscr{Y}_0 = \mathscr{Y}$$

such that

(i) the rational map from $\mathscr{Y}$ to Δ induced by π is actually a morphism,

(ii) $\mathscr{Y}_i$ is Q-factorial,

(iii) f_i is the contraction of an irreducible divisor D_i to a point p_i,

(iv) $-K(\mathscr{Y}_i)|_{D_i}$ is an ample Q-divisor, where $K(\mathscr{Y}_i)$ is the canonical divisor of $\mathscr{Y}_i$,

(v) rank Im (Pic $(\mathscr{Y}_i) \otimes_Z Q \to$ Pic $(D_i) \otimes_Z Q) = 1$,

(vi) if X is an irreducible component of a fiber F of the morphism induced by π on $\mathscr{Y}_i$, the proper transform of X on $\mathscr{X}$ $(=\mathscr{Y}_n)$ is the minimal resolution of X.

We call the above $\mathscr{Y}$ an *S-regular* 3*-fold*. Also, we call f_i an *S-contraction* and $\mathscr{X} \to \mathscr{Y}_i$ an *S-resolution* of $\mathscr{Y}_i$.

Remark. Conditions (iv) and (v) indicate the fact that the contraction f_i is nothing but the contraction of an extremal rational curve.

Theorem 1. *Let $\mathscr{X}$ be an S-degeneration. Then we can find an S-regular* 3*-fold $\mathscr{Y}$ which is birational to $\mathscr{X}$ such that the rational map induced by π on $\mathscr{Y}$ is a morphism and that $K(\mathscr{Y})$ is nef.*

Remark. An S-resolution of $\mathscr{Y}$ might be different from $\mathscr{X}$. In order to prove the theorem, we use the so-called Kawamata contraction theorem.

Theorem 2 [2]. *Let $\mathscr{Y}$ be an S-regular* 3*-fold and H be a nef divisor on $\mathscr{Y}$. If $H-K(\mathscr{Y})$ is nef and big, then some multiple of H has no base points.*

Remark. Kawamata's results are actually stronger than the above.

We first prove the following lemma:

Lemma 3. *Let $\mathscr{Y}$ be an S-regular* 3*-fold having an S-resolution $f: \mathscr{X} \to \mathscr{Y}$, ℓ an extremal rational curve with respect to $K(\mathscr{Y})$ and $g: \mathscr{Y} \to \mathscr{Z}$ the contraction of ℓ. Consider the following cases:*

Case 1. *g contracts a divisor to a point and gf is an S-resolution of $\mathscr{Z}$.*

Case 2. *g contracts a divisor to a point but gf is not an S-resolution.*

Case 3. *g contracts a divisor to a curve.*

Case 4. *g contracts only finitely many curves, one of which is a double curve on $\mathscr{Y}$.*

Case 5. *g contracts only finitely many curves and none of them is a double curve and one of them meets a double curve.*

Case 6. *g contracts only finitely many curves and none of them is a double curve or meets a double curve.*

Then we obtain an S-degeneration $\mathscr{X}'$ *such that*

(i) *there is a birational map* $\varphi: \mathscr{X} \to \mathscr{X}'$ *which is isomorphic outside the exceptional locus of* gf,

(ii) *for the Picard numbers,* $\rho(\mathscr{X}) \geqslant \rho(\mathscr{X}')$,

(iii) $\rho(\mathscr{X}) > \rho(\mathscr{X}')$ *in the cases* 2, 3, 4 *and* 5,

(iv) *if* $\rho(\mathscr{X}) = \rho(\mathscr{X}')$ *in the case* 6, *there exists an S-regular* 3-*fold* $\mathscr{Y}'$, *together with a morphism* $\mathscr{Y}' \to \mathscr{Z}$, *which has* $\mathscr{X}'$ *as an S-resolution, such that* (1) *the exceptional locus of* $\mathscr{X}' \to \mathscr{Y}'$ *is contained in the exceptional locus of* $\mathscr{X}' \to \mathscr{Z}$, (2) $\rho(\mathscr{Y}) = \rho(\mathscr{Y}')$, (3) $\tilde{\rho}(\mathscr{Y}) > \tilde{\rho}(\mathscr{Y}')$, *where* $\tilde{\rho}(\bullet)$ *denotes the sum of the Picard numbers of irreducible components of singular fibers of* $\bullet$.

Theorem 1 follows from Lemma 3. In the next section, we prove Lemma 3 only in the cases 5 and 6 which are the hardest to prove.

§3. The proof of Lemma 3 in the case 5

We use induction on $\rho(\mathscr{X}) - \rho(\mathscr{Y})$. As the first step, we must consider the case where $\rho(\mathscr{X}) - \rho(\mathscr{Y}) = 0$. However, in this case, $\mathscr{Y}$ is nonsingular and hence it is easy to prove Lemma 3. In fact, the case 5 or 6 does not occur.

Now, we fix notations.

Let $\ell_1, \cdots, \ell_r$ exhaust all irreducible curves on $\mathscr{Y}$ contracted by g. By assumption, some ℓ_i meets a double curve and hence there exists an irreducible component Y of a singular fiber such that Y meets ℓ_i and that $Y \not\supset \ell_i$. Then $(Y, \ell_i) > 0$. Therefore, (Y, ℓ_j) for any j, which means that any ℓ_j meets Y. Let F be the singular fiber containing ℓ_i and Y. Since $(Y, \ell_i) > 0$ and $(F, \ell_i) = 0$, there is an irreducible component X of F satisfying $(X, \ell_i) < 0$. Therefore, $(X, \ell_i) < 0$ for every i. Because any ℓ_j is not a double curve, such an X is unique. Let Y' (resp. X') be the proper transform of Y (resp. X) on $\mathscr{X}$ and let $\bar{Y}$ (resp. $\bar{X}$) be the image of Y (resp. X) on $\mathscr{Z}$. Let D be the exceptional locus of f and set $D' := D|_{X'}$. Let $D' = \sum D_i$ be the decomposition of D' into connected components. Because all ℓ_i's are contracted to so-called log-terminal singular points (see [4]), we have the following (see Figure):

(1) the dual graph of $D' + \ell_1' + \cdots + \ell_r'$ is a tree,

(2) the dual graph of D_j is a linear chain,

(3) all irreducible components contained in $D' + \ell_1' + \cdots + \ell_r'$ are nonsingular rational curves,

(4) $D' + \ell_1' + \cdots + \ell_r' + Y'|_{X'}$ has only simple normal crossings,

(5) there is a unique connected component of D', say D_1, which meets Y',

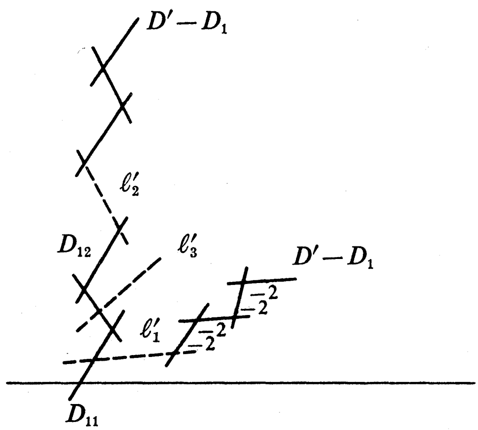
(6) all ℓ_i''s meet D_1,

(7) one of the edge components of D_1, say D_{11}, meets Y',

(8) for each i, a connected component of $D'-D_1$ meeting ℓ_i' is unique and it meets ℓ_i' at an edge component of it,

(9) if ℓ_i' does not meet at edge component D_{12} different from D_{11} and the ℓ_i' meets a connected component of D', say D_2, then each irreducible component of D_2 has self-intersection number -2,

(10) if more than one of ℓ_i''s meet D_{12}, then each irreducible component of a connected component of $D'-D_1$ meeting one of them has self-intersection number -2.



Figure

We need the following:

Lemma 4. *Let $\mathscr{W}$ be an S-regular 3-fold and A a partial sum of irreducible components of fibers with* Sing $(\mathscr{W})\cap A=\phi$. *Let M be an extremal rational curve with respect to $K(\mathscr{W})+A$. Then one of the following cases occur:*

(1) *there exists a curve M' such that $[M']\in R:=R_+[M]$ and that $M'\subseteq A$,*

(2) *the exceptional locus of the contraction of M is disjoint from A.*

Proof. Suppose that $M'\not\subseteq A$ for any curve M' with $[M']\in R$, and that $(M'', A)>0$ for some M'' with $[M'']\in R$. We derive contradictions. By assumption, $(M', A)>0$ for any curve M' with $[M']\in R$. Since $(K(\mathscr{W})+A, M')<0$, $(K(\mathscr{W}), M')<-1$. Hence we may assume that the contraction of M corresponds to one of the cases in Lemma 3. We consider each case separately.

Cases 1 and 2. Trivial.

Case 3. $(K(\mathscr{W}), f)=-1$, a contradiction, where f is a fiber of the contracted ruled surface.

Cases 4 and 5. Let B be an irreducible component with $(B, M)<0$. Then, M must also be an extremal rational curve in B with respect to $K(\mathscr{W})|_B$. Since $K(\mathscr{W})|_B=K(B)+\sum$ d.c. and $A|_B\subseteq\sum$ d.c., where d.c. is an abbreviation for double curves, we have $(K(\mathscr{W})|_B, M)\geqslant(K(B)+M+A, M)\geqslant-2+1=-1$, and this is a contradiction.

Case 6. Trivial.

By making use of the above lemma, we inductively construct an S-degeneration $\mathscr{X}'$ such that $X''\to\bar{X}$ is a minimal resolution in the following way: We start with $\mathscr{X}$. Since $K(\mathscr{X})+X'+Y'$ is not relatively nef to $\mathscr{Z}$, there is an extremal rational curve M with respect to $K(\mathscr{X})+X'+Y'$, which is contracted to a point in $\mathscr{Z}$.

Apply the lemma to this M. If the case (1) in Lemma 4 occurs, then we may assume $M\subseteq X'$ because $(K(\mathscr{X})+Y')|_{Y'}$ is relatively nef to $\mathscr{Z}$. Since a curve contained in X' and contracted to a point in $\mathscr{Z}$ is contained in $D'+\ell_1'+\cdots+\ell_r'$, M must coincide with one of ℓ_i''s. Let $\ell_1', \cdots, \ell_s'$ exhaust all ℓ_i''s with $[\ell_i']\in R:=\boldsymbol{R}_+[M]$. As observed before, the normal bundle of each ℓ_i' is of type $(-1, -1)$ or $(-1, -2)$. Furthermore, if the normal bundle of ℓ_1' is of type $(-1, -1)$, then all the other ℓ_i''s $(2\leq i\leq s)$ have normal bundles of the same type $(-1, -1)$ and if the normal bundle of ℓ_1' is of type $(-1, -2)$, then $s=1$. If the first case occurs, we make elementary transformations along all ℓ_i''s simultaneously. If the second case occurs, we first blow $M(=\ell_1')$ up and have a $(-1, -1)$-curve. Along this curve, we make an elementary transformation.

In both cases, the 3-fold thus obtained is projective because it is a flip or its resolution. Hence we have a new S-degeneration $\mathscr{X}_1$. Let X_1 be the proper transform of X' on $\mathscr{X}_1$. The number of irreducible curves on X_1 which are contracted in $\mathscr{Z}$ is less than that on X'. In this case, we set $\mathscr{Y}_1=\mathscr{X}_1$. If the case (2) in Lemma 4 occurs, then M is an extremal curve with respect to $K(\mathscr{X})$ and disjoint from X'. Since $\rho(\mathscr{X})-\rho(\mathscr{Y})>\rho(\mathscr{X})-\rho(\mathscr{X})=0$, by induction, we find an S-degeneration $\mathscr{X}_1$ in Lemma 3 or an S-contraction $\mathscr{X}\to\mathscr{Y}_1$ in the case 1 of Lemma 3. Set $\mathscr{X}=\mathscr{X}_1$ in this case.

Now, we consider $(\mathscr{X}_1, \mathscr{Y}_1)$, where we should note that $\mathscr{X}_1$ is an S-degeneration and $\mathscr{Y}_1$ is an S-regular 3-fold having $\mathscr{X}_1$ as an S-resolution. If $K(\mathscr{Y}_1)+\bar{X}_1+\bar{Y}_1$ is relatively nef to $\mathscr{Z}$, then we stop here, where $\bar{X}_1$ (resp. $\bar{Y}_1$) is the proper transform of X' (resp. Y') on $\mathscr{Y}_1$. Assume that $K(\mathscr{Y}_1)+\bar{X}_1+\bar{Y}_1$ is not nef. According to Lemma 4, we have two cases. Let M_1

be an extremal rational curve with respect to $K(\mathscr{Y}_1)+\bar{X}_1+\bar{Y}_1$. If we have the case (1), the normal bundle of M_1 is of type $(0, -1)$, $(-1, -1)$ or $(-1, -2)$. If the normal bundle of M_1 is of type $(0, -1)$, then M_1 is also an extremal curve with respect to $K(\mathscr{Y}_1)$. Therefore, consider this case later. If the normal bundle of M_1 is of type $(-1, -1)$, then M_1 coincides with the image of some ℓ_i' or M_1 is a double curve. We make elementary transformations along curves whose classes are in $R_1=\boldsymbol{R}_+[M_1]$. If the normal bundle of M_1 is of type $(-1, -2)$, then we make the same transformation as before. Let this new S-regular 3-fold be $\mathscr{Y}_2$ and let $\mathscr{X}_2$ be a natural S-resolution of $\mathscr{Y}_2$. Now, we assume that M_2 is also an extremal rational curve with respect to $K(\mathscr{Y}_2)$. Since $\rho(\mathscr{X})-\rho(\mathscr{Y})>\rho(\mathscr{X}_1)-\rho(\mathscr{Y}_1)$, we find an S-degeneration $\mathscr{X}_2$ and an S-regular 3-fold $\mathscr{Y}_2$. We apply these procedures until (the canonical divisor)+(the proper transform of X')+ (the proper transform of Y') is relatively nef to $\mathscr{X}$. Let $(\mathscr{X}_n, \mathscr{Y}_n)$ be a pair obtained by the above procedure. We calculate $\rho(\mathscr{X}_n)-\rho(\mathscr{X})$. Let $C_1+\cdots+C_k$ be a linear chain of double curves in $\mathscr{X}_n$ satisfying the following conditions:

(0) $C_1+\cdots+C_k$ goes to a point in $\mathscr{X}$,

(1) C_k meets the proper transform X_n of X' and has the normal bundle of type $(-1, -1)$,

(2) C_i meets only C_{i+1} and C_{i-1} among C_i's,

(3) each normal bundle of $C_2, \cdots, C_{k-1}$ is of type $(0, -2)$,

(4) the normal bundle of C_1 is of type $(1, -2)$,

(5) all C_i's are nonsingular rational curves,

(6) the chain originally comes from some ℓ_i' of type $(-1, -2)$.

Let α be the number of linear chains as above. Then $\rho(\mathscr{X}_n)\leqslant\rho(\mathscr{X})+\alpha$, by our construction, because, by a transformation in Lemma 3, the number of the above chains decreases at most by one (This is easily checked case by case.). Starting with $\mathscr{X}_n$, contract S-regularly $\mathscr{X}_n$ as far as possible. Let $\tilde{\mathscr{Y}}$ be the S-regular 3-fold thus obtained. Let $\tilde{X}$ (resp. $\tilde{Y}$) be the proper transform of X' (resp. Y'). Since $K(\tilde{\mathscr{Y}})$ is not nef, we have an extremal rational curve with respect to $K(\tilde{\mathscr{Y}})$. Note that $\rho(\mathscr{X})-\rho(\mathscr{Y})>\rho(\mathscr{X}_n)-\rho(\tilde{\mathscr{Y}})$ by the existence of the above $(-1, -1)$-curves. If some extremal rational curve is different from the image of each $(-1, -1)$-curve on $\mathscr{X}_n$ contributing to α, apply a transformation as in Lemma 3. We have a pair $(\mathscr{X}_{n+1}, \mathscr{Y}_{n+1})$ which is better than $(\mathscr{X}_n, \tilde{\mathscr{Y}})$ preserving the condition $\rho(\mathscr{X}_{n+1})\leq\rho(\mathscr{X})+\alpha(\mathscr{X}_{n+1})$. If every extremal rational curve coincides with one of the images of the $(-1, -1)$-curves appearing in the definition of α, then we have a contradiction in the following fashion: Let $E_1, \cdots, E_\alpha$ exhaust the $(-1, -1)$-curves in the definition of α and let β be a number in $[0, 1]$ such that

(i) $(K(\tilde{\mathscr{Y}})+\beta(\tilde{X}+\tilde{Y}), E_i)\geq 0$, for any i,

(ii) $(K(\tilde{\mathscr{Y}})+\beta(\tilde{X}+\tilde{Y}), E_j)=0$ for some j,
where $\tilde{E}_i$ is the image of E_i.

We claim $K(\tilde{\mathscr{Y}})+\beta(\tilde{X}+\tilde{Y})$ is relatively nef to $\mathscr{Z}$. Suppose the contrary. Let M be an extremal rational curve with respect to $K(\tilde{\mathscr{Y}})+(\tilde{X}+\tilde{Y})$. If $(K(\tilde{\mathscr{Y}}), M)<0$, then M must be one of E_i by assumption. This contradicts the choice of β. Hence $(K(\tilde{\mathscr{Y}}), M)\geq 0$. Therefore, $(M, X+Y)<0$. Then since $(K(\tilde{\mathscr{Y}})+\beta(\tilde{X}+\tilde{Y}), M)<0$, $(K(\tilde{\mathscr{Y}})+\tilde{X}+\tilde{Y}, M)<0$. Note $(K(\tilde{\mathscr{Y}})+\tilde{Y}, M)\geq 0$ because $K(\tilde{Y})$ is relatively nef to Y. Therefore $M\subseteq\tilde{X}$ and M is a (-1)-curve in X and this is a contradiction. Then, by Kawamata's theorem (cf. Theorem 2), $m(K(\tilde{\mathscr{Y}})+\beta(\tilde{X}+\tilde{Y})+A)$ has no base point for a suitable $m>0$, where A is the pull-back of an ample divisor on $\mathscr{Z}$. Let L be a general member contained in $|m(K(\tilde{\mathscr{Y}})+\beta(\tilde{X}+\tilde{Y})+A)|$. Then $\tilde{L}\cap E_j=\phi$. Therefore, the pull-back of $\tilde{L}$ on $\mathscr{X}_n$ is disjoint from a chain associated with E_i. This means that the proper transform L' of $\tilde{L}$ on $\mathscr{X}$ does not contain some ℓ_i' corresponding to E_i. Since f is isomorphic on a general point of ℓ_i', the image L of L' on $\mathscr{Y}$ does not contain ℓ_i. However, on the other hand, L is linearly equivalent to $m(K(\mathscr{Y})+\beta(X+Y)+A)$ which is nothing but $mK(\mathscr{Y})$ around ℓ_i. Therefore, $0\leq(L, \ell_i)=m(K(\mathscr{Y}), \ell_i)<0$, which is absurd.

Therefore, there is an extremal rational curve on $\tilde{\mathscr{Y}}$ which is different from E_i. We apply the same procedure to $(\mathscr{X}_{n+1}, \mathscr{Y}_{n+1})$. If we have a pair $(\mathscr{X}_m, \mathscr{Y}_m)$ at some stage satisfying

$$\rho(\mathscr{X}_m)<\rho(\mathscr{X})+\alpha(\mathscr{X}_m),$$

we are done because α decreases at most by one and we apply the above procedure as long as $\alpha>0$ which assures the existence of an extremal curve.

If the induced maps $X_n\rightarrow X_m$ and $Y_n\rightarrow Y_m$ are not morphisms, we have

$$\rho(\mathscr{X}_m)<\rho(\mathscr{X})+\alpha(\mathscr{X}_m),$$

where X_m (resp. Y_m) is the proper transform of X (resp. Y). In fact, under such a transformation, α does not change but the Picard number decreases.

Now, as the remaining case, we have the case where $\alpha(\mathscr{X}_m)=0$ and $\rho(\mathscr{X})=\rho(\mathscr{X}_m)$. We contract $\mathscr{X}_m$ S-regularly as far as possible and call such an S-regular 3-fold $\tilde{\mathscr{Y}}_m$. We shall prove that $\tilde{\mathscr{Y}}_m$ has a divisor contracted to a point on $\mathscr{Z}$. Suppose the contrary. Then, all exceptional divisors in $\mathscr{X}_m$ are contracted to points in $\tilde{\mathscr{Y}}_m$. Therefore, all exceptional curves in X_m and Y_m are also contracted to points. Hence, by the Zariski main theorem, $\tilde{\mathscr{Y}}_m\simeq\mathscr{Z}$ which is a contradiction. Therefore, in particular, $K(\tilde{\mathscr{Y}}_m)$ is not nef and we find an extremal rational curve. Since $\rho(\mathscr{X})-\rho(\mathscr{Y})>\rho(\mathscr{X}_m)-\rho(\tilde{\mathscr{Y}}_m)$, we can use Lemma 3. This concludes the proof of Lemma 3 in the case 5.

§ 4. The proof of Lemma 3 in the case 6

This proof is essentially the same as in the case 5. We point out only different points. A priori, we do not determine the configuration of the exceptional locus of gf (compare this with the case 5). However, restricted on a surface, such an exceptional locus should be contracted to quotient singular points. Let $X_1, \cdots, X_s$ exhaust all irreducible components containing some ℓ_i, set $X=\sum X_i$, let $\bar{X}_i$ be the image of X_i by g on $\mathscr{X}$ and let ℓ_i' (resp. X_i') be the proper transform of ℓ_i (resp. X_i) on $\mathscr{X}$. Let D be the exceptional locus of f and set $D'=D|_{X_i'}$ where $X'=\sum X_i'$. The conditions (1), (2), (3) and (4) in Section 3 hold concerning the configuration of $D'+\ell_1'+\cdots+\ell_r'$. Therefore we apply the same procedure. A major difference is the following: Assume that there are two $(-1,-1)$ curves E_1 and E_2 satisfying the condition in the definition of α and that E_1 and E_2 meet the same double curve E_3 on the proper transform of X which is also a $(-1,-1)$-curve and extremal. Then if we make an elementary transformation along E_3, then the Picard numbers of S-degenerations do not change but we lose one $(-1,-1)$-curve. However, in this case, we have a surface with two double curves whose normal bundles are of type $(0,-2)$ and which meet each other at one point. Therefore, in this case 6, we use $\tilde{\alpha}$ which is defined in the following way instead of α: $\tilde{\alpha}$ is the sum of the number α_1 of surfaces described above and the number α_2 of $(-1,-1)$-curves meeting the proper transform of X and coming from some ℓ_i' on $\mathscr{X}$. Note that $\alpha_2<\#\{\ell_i'\}$ if $\alpha_1>0$. As one finds in [3], such a surface does not appear in an S-degeneration. Then, we apply completely the same procedure as in the case 5. We have $(\mathscr{X}_n, \mathscr{Y}_n)$ such that $K(X_n)$ is relatively nef to $\mathscr{X}$ and that

$$\rho(\mathscr{X}_n)\leq\rho(\mathscr{X})+\tilde{\alpha}(\mathscr{X}_n),$$

where X_n is the proper transform of X on $\mathscr{X}_n$. Assume that $\rho(\mathscr{X}_n)=\rho(\mathscr{X})+\tilde{\alpha}(\mathscr{X}_n)$. We take $\tilde{\mathscr{Y}}_n$ in the same way as in the case 5 and let M be an extremal rational curve with respect to $K(\tilde{\mathscr{Y}}_n)$. If M is different from any one of the α_2 curves of type $(-1,-1)$, then we find an S-degeneration $\mathscr{X}_{n+1}$ with $\rho(\mathscr{X}_{n+1})<\rho(\mathscr{X})+\tilde{\alpha}(\mathscr{X}_n)$ or with $\rho(\mathscr{X}_{n+1})=\rho(\mathscr{X})+\tilde{\alpha}(\mathscr{X}_n)$ and $\tilde{\alpha}(\mathscr{X}_{n+1})<\tilde{\alpha}(\mathscr{X}_n)$. Assume that each extremal rational curve coincides with one of the $(-1,-1)$-curves. We take a rational number θ such that $K(\tilde{\mathscr{Y}}_n)+\theta\tilde{X}$ is not ample but nef, where $\tilde{X}$ is the proper transform of X on $\tilde{\mathscr{Y}}_n$. Let C be a curve with $(K(\tilde{\mathscr{Y}}_n)+\theta\tilde{X}, C)=0$. Then, by assumption, C must be a $(-1,-1)$-curve as above. If C is a curve corresponding to a configuration appearing in the definition of α in the case 5, then we have a contradiction in the same way. Therefore, we may assume that C corresponds to two of ℓ_i''s. Hence $\alpha_1>0$ and so $\alpha_2<\#\{\ell_i'\}$. We make a modification by C considered in the case 3 in Lemma 3. Then the Picard number of the proper transform of X increases only by the number of curves C' with $(K(\tilde{\mathscr{Y}}_n)+\theta X_n), C')=0$ (cf. the next section.). Let $\mathscr{X}_{n+1}$ be the S-degeneration obtained by the above modification. As usual, we contract $\mathscr{X}_{n+1}$ S-regularly, as far as possible and call $\mathscr{Y}_{n+1}$ the S-regular 3-fold thus obtained. If all irreducible divisors contracted to points in $\mathscr{X}$ are also contracted to points in $\mathscr{Y}_{n+1}$, the sum of the Picard numbers of the proper transforms of X_i on $\mathscr{Y}_{n+1}$ is less than that of $X=\sum X_i$ on $\mathscr{Y}$. Therefore, we are done. If $\mathscr{Y}_{n+1}$ has a divisor contracted to a point in $\mathscr{X}$, $K(\mathscr{Y}_{n+1})$ is not nef. We apply a procedure which makes use of an extremal rational curve and have the same situation as above. Then we repeat the above procedure. Finally, we obtain a desired pair $(\mathscr{X}_., \mathscr{Y}_.)$ as described in Lemma 3.

§ 5. The case 4 in Lemma 3

First, we fix the notations. Let $\ell_1, \cdots, \ell_r$ exhaust all irreducible curves whose classes are contained in $R=\boldsymbol{R}_+[\ell]$. Since ℓ_1 is a double curve, there are two distinct irreducible components, say X and Y, containing ℓ_1. Because ℓ_1 is contracted to a log-terminal singular point in X, an irreducible component Z touching ℓ_1 and not containing ℓ_1 is unique and a singular point in X lying on ℓ_1 is also unique if it exists. Let ℓ_i' (resp. X', Y', Z') be the proper transform of ℓ_i (resp. X, Y, Z) on $\mathscr{X}$ and let E be the exceptional locus of f. By the triple point formula,

$$1+(\ell_1')^2_{X'}+(\ell_1')^2_{Y'}+(Z', \ell_1')=0.$$

Since $(\ell_1')^2_{X'}$ and $(\ell_1')^2_{Y'}$ are negative and $(Z', \ell_1')=0$ or 1 (see [4]), we have $(Z', \ell_1')=1$ and $(\ell_1')^2_{X'}=(\ell_1')^2_{Y'}=-1$. Now, looking at the classification of S-regular points on a double curve, one knows that one of the double curves, say C, meeting ℓ_1' and being contained in E, has the normal bundle of type $(-2, 0)$ or $(-2, 1)$. Therefore, if one makes a modification along ℓ_1', then the proper transform of C has the normal bundle $(-1, -1)$ and is an extremal rational curve. We continue this process and reach an edge component of E which is isomorphic to $\boldsymbol{P}^2$. So, we are done.

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